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Research Article

The Hierarchical Iterative Identification Algorithm for Multi-Input-Output-Error Systems with Autoregressive Noise

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This paper considers the identification problem of multi-input-output-error autoregressive systems. A hierarchical gradient based iterative (H-GI) algorithm and a hierarchical least squares based iterative (H-LSI) algorithm are presented by using the hierarchical identification principle. A gradient based iterative (GI) algorithm and a least squares based iterative (LSI) algorithm are presented for comparison. The simulation results indicate that the H-LSI algorithm can obtain more accurate parameter estimates than the LSI algorithm, and the H-GI algorithm converges faster than the GI algorithm.

1. Introduction

System identification studies mathematical models of dynamic systems by fitting experimental data to a suitable model structure [1, 2]. Many practical systems have multiple inputs and multiple outputs such as chemical processes [3, 4], automation devices [5–7], and network communication engineering [8–10]. For decades, much research has been performed on the multivariable systems [11, 12], and some typical approaches for the parameter estimation of the multivariable systems have been reported [13], such as the canonical approach [14], the iterative methods [15, 16], and the least squares methods [17]. Recently, Panda and Vijayaraghavan adopted the sequential relay feedback test to estimate the parameter of the linear multivariable systems [18]. Jafari et al. presented an iterative least squares algorithm to identify the multivariable nonlinear systems with colored noises [19].

The multivariable systems contain both parameter vectors and parameter matrices, and the systems inputs and system outputs are relevant and coupled [20–22]. For the sake of reducing the computational complexity, the hierarchical identification principle is utilized to transform a complex system into several subsystems and then to estimate the parameter vector of each subsystem [23, 24], respectively. In this literature, Schranz et al. proposed a feasible hierarchical identification process for identifying the viscoelastic model of respiratory mechanics [25]. Xu et al. developed the parameter estimation for dynamical response signals [26, 27].

The iterative methods have been widely applied in identifying the parameters of linear or nonlinear systems [28–30]. Many iterative algorithms for system identification are based on the gradient method [31]and the least squares method [32–35]. The basic idea of iterative methods is to update the parameter estimates using batch data.

This paper focuses on the parameter estimation for output-error autoregressive (OEAR) systems using the hierarchical identification principle and the iterative identification principle and presents a hierarchical gradient based iterative (H-GI) algorithm and a hierarchical least squares based iterative (H-LSI) algorithm. The key is to decompose a multi-input OEAR system into two subsystems and then to identify each subsystem. The work in [36, 37] discussed the single-input single-output systems, but many practical systems have multiple inputs and multiple outputs with the development of industrial technology. Compared with the work in [36, 37], this paper discusses the parameter estimation for multi-input OEAR systems and the presented H-LSI algorithm can achieve higher estimation accuracy than the LSI algorithm, and the H-GI algorithm also can achieve higher estimation accuracy than the GI algorithm.

The rest of this paper is organized as follows. Section 2 gives some definitions and the identification model of multi-input OEAR systems. Section 3 presents a gradient based iterative algorithm and a least squares based iterative algorithm for multi-input OEAR systems. Section 4 derives a hierarchical gradient based iterative algorithm. Section 5 derives a hierarchical least squares based iterative algorithm. Section 6 provides two illustrative examples to demonstrate the effectiveness of the proposed algorithms. Finally, concluding remarks are given in Section 7.

2. The Problem Formulation

Let us define some notation.

Symbols: meaning

$\mathbf{1}_n$: an n dimensional column vector whose entries are all 1

p_0: a large positive constant, for example, $p_0 = 10^6$

$\mathbf{X}^T$: the transpose of the vector or matrix $\mathbf{X}$

$\|\mathbf{X}\|^2$: $\|\mathbf{X}\|^2 = \mathrm{tr}[\mathbf{X}\mathbf{X}^T]$

$X := A$: A defined as X

$\lambda_{\max}[\mathbf{X}]$: the maximum eigenvalue of the symmetric real matrix $\mathbf{X}$.

Consider the following multi-input-output-error type models:

$$y(\tau) = \sum_{j=1}^{r} \frac{B_j(z)}{A_j(z)} u_j(\tau) + w(\tau), \tag{1}$$

where $y(\tau) \in \mathbb{R}$ is the system output, $u_j(\tau) \in \mathbb{R}$, $j = 1, 2, \ldots, r$, are the system inputs, and $w(\tau) \in \mathbb{R}$ is the colored noise with zero mean. $A_j(z)$ and $B_j(z)$ are polynomials in the unit backward shift operator z^{-1}, and

$$\begin{aligned} A_j(z) &:= 1 + a_{j1}z^{-1} + a_{j2}z^{-2} + \cdots + a_{jn_j}z^{-n_j}, \\ &\qquad\qquad a_{ji} \in \mathbb{R}, \\ B_j(z) &:= b_{j1}z^{-1} + b_{j2}z^{-2} + \cdots + b_{jn_j}z^{-n_j}, \quad b_{ji} \in \mathbb{R}. \end{aligned} \tag{2}$$

Assume that the orders n_j are known, $y(\tau) = 0$, $u_j(\tau) = 0$, and $w(\tau) = 0$ as $\tau \leqslant 0$. The colored noise $w(\tau)$ can be fitted by a moving average process

$$w(\tau) = D(z)v(\tau), \tag{3}$$

or an autoregressive process

$$w(\tau) = \frac{1}{C(z)} v(\tau), \tag{4}$$

or an autoregressive moving average process

$$w(\tau) = \frac{D(z)}{C(z)} v(\tau), \tag{5}$$

where $v(\tau)$ is the white noise with zero mean and $C(z)$, $D(z)$ are polynomials in the unit backward shift operator z^{-1}:

$$\begin{aligned} C(z) &:= 1 + c_1z^{-1} + c_2z^{-2} + \cdots + c_{n_c}z^{-n_c}, \quad c_j \in \mathbb{R}, \\ D(z) &:= 1 + d_1z^{-1} + d_2z^{-2} + \cdots + d_{n_d}z^{-n_d}, \quad d_j \in \mathbb{R}. \end{aligned} \tag{6}$$

This paper considers the colored noise to be an autoregressive process, so the models in (1) can be taken as the multi-input OEAR systems.

Define the intermediate variables:

$$x_j(\tau) := \frac{B_j(z)}{A_j(z)} u_j(\tau), \quad j = 1, 2, \ldots, r. \tag{7}$$

From (4) and (7), we have

$$\begin{aligned} w(\tau) &= [1 - C(z)]w(\tau) + v(\tau) \\ &= -\sum_{i=1}^{n_c} c_i w(\tau - i) + v(\tau), \\ x_j(\tau) &= \left[1 - A_j(z)\right] x_j(\tau) + B_j(z)u_j(\tau) \\ &= \sum_{i=1}^{n_j} \left[-a_{ji}x_j(\tau - i) + b_{ji}u_j(\tau - i)\right]. \end{aligned} \tag{8}$$

The output $y(\tau)$ in (1) can be written as

$$\begin{aligned} y(\tau) = &\sum_{j=1}^{r}\sum_{i=1}^{n_j} \left[-a_{ji}x_j(\tau - i) + b_{ji}u_j(\tau - i)\right] \\ &- \sum_{i=1}^{n_c} c_i w(\tau - i) + v(\tau). \end{aligned} \tag{9}$$

Define the parameter vectors as

$$\begin{aligned} \boldsymbol{\theta} &:= \left[\boldsymbol{\vartheta}^T, \mathbf{c}^T\right]^T \in \mathbb{R}^n, \quad n := n_0 + n_c, \ n_0 := \sum_{j=1}^{r} 2n_j, \\ \boldsymbol{\vartheta} &:= \left[\boldsymbol{\vartheta}_1^T, \boldsymbol{\vartheta}_2^T, \ldots, \boldsymbol{\vartheta}_r^T\right]^T \in \mathbb{R}^{n_0}, \\ \boldsymbol{\vartheta}_j &:= \left[a_{j1}, a_{j2}, \ldots, a_{jn_j}, b_{j1}, b_{j2}, \ldots, b_{jn_j}\right]^T \in \mathbb{R}^{2n_j}, \\ &\qquad\qquad j = 1, 2, \ldots, r, \\ \mathbf{c} &:= \left[c_1, c_2, \ldots, c_{n_c}\right]^T \in \mathbb{R}^{n_c} \end{aligned} \tag{10}$$

and the information vectors as

$$\begin{aligned}
\boldsymbol{\varphi}(\tau) &:= \left[\boldsymbol{\phi}^T(\tau), \boldsymbol{\psi}^T(\tau)\right]^T \in \mathbb{R}^n, \\
\boldsymbol{\phi}(\tau) &:= \left[\boldsymbol{\phi}_1^T(\tau), \boldsymbol{\phi}_2^T(\tau), \ldots, \boldsymbol{\phi}_r^T(\tau)\right]^T \in \mathbb{R}^{n_0}, \\
\boldsymbol{\phi}_j(\tau) &:= \left[-x_j(\tau-1), -x_j(\tau-2), \ldots,\right. \\
&\quad \left. -x_j\left(\tau-n_j\right), u_j(\tau-1), u_j(\tau-2), \ldots, u_j\left(\tau-n_j\right)\right]^T \\
&\quad \in \mathbb{R}^{2n_j}, \\
\boldsymbol{\psi}(\tau) &:= \left[-w(\tau-1), -w(\tau-2), \ldots, -w\left(\tau-n_c\right)\right]^T \\
&\quad \in \mathbb{R}^{n_c}.
\end{aligned} \tag{11}$$

According to the above definitions, (8) and (9) can be written as

$$w(\tau) = \boldsymbol{\psi}^T(\tau)\mathbf{c} + v(\tau), \tag{12}$$

$$x_j(\tau) = \boldsymbol{\phi}_j^T(\tau)\boldsymbol{\vartheta}_j, \tag{13}$$

$$\begin{aligned}
y(\tau) &= \sum_{j=1}^{r}\boldsymbol{\phi}_j^T(\tau)\boldsymbol{\vartheta}_j + \boldsymbol{\psi}^T(\tau)\mathbf{c} + v(\tau) \\
&= \boldsymbol{\phi}^T(\tau)\boldsymbol{\vartheta} + \boldsymbol{\psi}^T(\tau)\mathbf{c} + v(\tau) = \boldsymbol{\varphi}^T(\tau)\boldsymbol{\theta} + v(\tau).
\end{aligned} \tag{14}$$

Equation (14) is the identification model of the multi-input OEAR system.

3. The Gradient Based and Least Squares Based Iterative Algorithm

Consider the data $\{u_1(i), u_2(i), \ldots, u_r(i), y(i)\}$ from $i = \tau - p + 1$ to $i = \tau$ and define quadratic criterion function as

$$J(\boldsymbol{\theta}) := \sum_{i=\tau-p+1}^{\tau}\left[y(i) - \boldsymbol{\varphi}^T(i)\boldsymbol{\theta}\right]^2. \tag{15}$$

Let $k = 1, 2, 3, \ldots$ be an iteration variable and $\widehat{\boldsymbol{\theta}}_k(\tau)$ be the estimate of $\boldsymbol{\theta}$ at iteration k. Minimizing $J(\boldsymbol{\theta})$ by using the negative gradient search, we can obtain

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}_k(\tau) &= \widehat{\boldsymbol{\theta}}_{k-1}(\tau) - \frac{\mu_k(\tau)}{2}\operatorname{grad}\left[J\left(\widehat{\boldsymbol{\theta}}_{k-1}(\tau)\right)\right] \\
&= \widehat{\boldsymbol{\theta}}_{k-1}(\tau) \\
&\quad + \mu_k(\tau)\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\varphi}(i)\left[y(i) - \boldsymbol{\varphi}^T(i)\widehat{\boldsymbol{\theta}}_{k-1}(\tau)\right],
\end{aligned} \tag{16}$$

where $\mu_k(\tau) > 0$ is an iterative step-size. Because the information vector $\boldsymbol{\varphi}(\tau)$ contains the unknown variables $x_j(\tau-i)$ and $w(\tau-i)$, we use the estimates $\widehat{x}_{j,k-1}(\tau-i)$ and $\widehat{w}_{k-1}(\tau-i)$ at iteration $k-1$ to replace the unknown variables $x_j(\tau-i)$ and $w(\tau-i)$; we can obtain the gradient based iterative (GI) algorithm for estimating the parameter vector $\boldsymbol{\theta}$ of the multi-input OEAR systems:

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}_k(\tau) &= \widehat{\boldsymbol{\theta}}_{k-1}(\tau-1) + \mu_k\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\varphi}}_k(i)\left[y(i) - \widehat{\boldsymbol{\varphi}}_k^T(i)\right. \\
&\quad \left.\cdot\widehat{\boldsymbol{\theta}}_{k-1}(\tau)\right], \quad k = 1, 2, \ldots, \\
\widehat{\boldsymbol{\varphi}}_k(\tau) &= \left[\widehat{\boldsymbol{\phi}}_k^T(\tau), \widehat{\boldsymbol{\psi}}_k^T(\tau)\right]^T, \\
\widehat{\boldsymbol{\phi}}_k(\tau) &= \left[\widehat{\boldsymbol{\phi}}_{1,k}^T(\tau), \widehat{\boldsymbol{\phi}}_{2,k}^T(\tau), \ldots, \widehat{\boldsymbol{\phi}}_{r,k}^T(\tau)\right]^T, \\
\widehat{\boldsymbol{\phi}}_{j,k}(\tau) &= \left[-\widehat{x}_{j,k-1}(\tau-1), \ldots, -\widehat{x}_{j,k-1}\left(\tau-n_j\right),\right. \\
&\quad \left. u_j(\tau-1), \ldots, u_j\left(\tau-n_j\right)\right]^T, \\
\widehat{\boldsymbol{\psi}}_k(\tau) &= \left[-\widehat{w}_{k-1}(\tau-1), -\widehat{w}_{k-1}(\tau-2), \ldots,\right. \\
&\quad \left. -\widehat{w}_{k-1}\left(\tau-n_c\right)\right]^T, \\
\widehat{x}_{j,k}(\tau-i) &= \widehat{\boldsymbol{\phi}}_{j,k}^T(\tau-i)\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau), \\
\widehat{w}_k(\tau-i) &= y(\tau-i) - \widehat{\boldsymbol{\varphi}}_k^T(\tau-i)\widehat{\boldsymbol{\vartheta}}_k(\tau), \\
\mu_k &\leq 2\lambda_{\max}^{-1}\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\varphi}}_k(i)\widehat{\boldsymbol{\varphi}}_k^T(i)\right], \\
\widehat{\boldsymbol{\theta}}_k(\tau) &= \left[\widehat{\boldsymbol{\vartheta}}_k^T(\tau), \widehat{\mathbf{c}}_k^T(\tau)\right]^T, \\
\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \left[\widehat{\boldsymbol{\vartheta}}_{1,k}^T(\tau), \widehat{\boldsymbol{\vartheta}}_{2,k}^T(\tau), \ldots, \widehat{\boldsymbol{\vartheta}}_{r,k}^T(\tau)\right]^T, \\
\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau) &= \left[\widehat{a}_{j1,k}(\tau), \widehat{a}_{j2,k}(\tau), \ldots, \widehat{a}_{jn_j,k}(\tau), \widehat{b}_{j1,k}(\tau),\right. \\
&\quad \left. \widehat{b}_{j2,k}(\tau), \ldots, \widehat{b}_{jn_j,k}(\tau)\right]^T, \\
\widehat{\mathbf{c}}_k(\tau) &= \left[\widehat{c}_{1,k}(\tau), \widehat{c}_{2,k}(\tau), \ldots, \widehat{c}_{n_c,k}(\tau)\right]^T.
\end{aligned} \tag{17}$$

The convergence rate of the GI algorithm is slow. To improve the convergence speed, we derive a least squares based iterative (LSI) identification algorithm. Minimizing $J(\boldsymbol{\theta})$ and letting the derivative of $J(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ be zero give the LSI identification algorithm for the multi-input OEAR systems:

$$\begin{aligned}
\widehat{\boldsymbol{\theta}}_k(\tau) &= \left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\varphi}}_k(i)\widehat{\boldsymbol{\varphi}}_k^T(i)\right]^{-1}\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\varphi}}_k(i)y(i)\right], \\
&\qquad k = 1, 2, 3, \ldots, \\
\widehat{\boldsymbol{\varphi}}_k(\tau) &= \left[\widehat{\boldsymbol{\phi}}_k^T(\tau), \widehat{\boldsymbol{\psi}}_k^T(\tau)\right]^T, \\
\widehat{\boldsymbol{\phi}}_k(\tau) &= \left[\widehat{\boldsymbol{\phi}}_{1,k}^T(\tau), \widehat{\boldsymbol{\phi}}_{2,k}^T(\tau), \ldots, \widehat{\boldsymbol{\phi}}_{r,k}^T(\tau)\right]^T,
\end{aligned}$$

$$\begin{aligned}
\widehat{\boldsymbol{\phi}}_{j,k}(\tau) &= \left[-\widehat{x}_{j,k-1}(\tau-1),\ldots,-\widehat{x}_{j,k-1}\left(\tau-n_j\right),\right.\\
&\quad \left.u_j(\tau-1),\ldots,u_j\left(\tau-n_j\right)\right]^T,\\
\widehat{\boldsymbol{\psi}}_k(\tau) &= \left[-\widehat{w}_{k-1}(\tau-1),-\widehat{w}_{k-1}(\tau-2),\ldots,\right.\\
&\quad \left.-\widehat{w}_{k-1}(\tau-n_c)\right]^T,\\
\widehat{x}_{j,k}(\tau-i) &= \widehat{\boldsymbol{\phi}}_{j,k}^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau),\\
\widehat{w}_k(\tau-i) &= y(\tau-i)-\widehat{\boldsymbol{\varphi}}_k^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_k(\tau),\\
\widehat{\boldsymbol{\theta}}_k(\tau) &= \left[\widehat{\boldsymbol{\vartheta}}_k^T(\tau),\widehat{\mathbf{c}}_k^T(\tau)\right]^T,\\
\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \left[\widehat{\boldsymbol{\vartheta}}_{1,k}^T(\tau),\widehat{\boldsymbol{\vartheta}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\vartheta}}_{r,k}^T(\tau)\right]^T,\\
\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau) &= \left[\widehat{a}_{j1,k}(\tau),\widehat{a}_{j2,k}(\tau),\ldots,\widehat{a}_{jn_j,k}(\tau),\widehat{b}_{j1,k}(\tau),\right.\\
&\quad \left.\widehat{b}_{j2,k}(\tau),\ldots,\widehat{b}_{jn_j,k}(\tau)\right]^T,\\
\widehat{\mathbf{c}}_k(\tau) &= \left[\widehat{c}_{1,k}(\tau),\widehat{c}_{2,k}(\tau),\ldots,\widehat{c}_{n_c,k}(\tau)\right]^T.
\end{aligned} \tag{18}$$

4. The Hierarchical Gradient Based Iterative Algorithm

Define intermediate variables:

$$y_1(\tau) := y(\tau)-\boldsymbol{\psi}^T(\tau)\,\mathbf{c}, \tag{19}$$

$$y_2(\tau) := y(\tau)-\boldsymbol{\phi}^T(\tau)\,\boldsymbol{\vartheta}. \tag{20}$$

Using the hierarchical identification principle, the multi-input OEAR system in (14) can be decomposed into two fictitious subsystems:

$$y_1(\tau) = \boldsymbol{\phi}^T(\tau)\,\boldsymbol{\vartheta}+v(\tau), \tag{21}$$

$$y_2(\tau) = \boldsymbol{\psi}^T(\tau)\,\mathbf{c}+v(\tau). \tag{22}$$

Next, we identify the parameters $\boldsymbol{\vartheta}$ and $\mathbf{c}$ of each subsystem in (21) and (22), respectively. Define quadratic criterion functions as

$$\begin{aligned}
J_1(\boldsymbol{\vartheta},\mathbf{c}) &:= \sum_{i=\tau-p+1}^{\tau}\left[y_1(i)-\boldsymbol{\phi}^T(i)\,\boldsymbol{\vartheta}\right]^2,\\
J_2(\boldsymbol{\vartheta},\mathbf{c}) &:= \sum_{i=\tau-p+1}^{\tau}\left[y_2(i)-\boldsymbol{\psi}^T(i)\,\mathbf{c}\right]^2.
\end{aligned} \tag{23}$$

Let $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ be the estimates of $\boldsymbol{\vartheta}$ and $\mathbf{c}$ at iteration k. Using the negative gradient search and minimizing $J_1(\boldsymbol{\vartheta},\mathbf{c})$ and $J_2(\boldsymbol{\vartheta},\mathbf{c})$, we can obtain

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)\\
&\quad+\mu_{1k}(\tau)\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)\left[y_1(i)-\boldsymbol{\phi}^T(i)\,\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)\right],
\end{aligned} \tag{24}$$

$$\begin{aligned}
\widehat{\mathbf{c}}_k(\tau) &= \widehat{\mathbf{c}}_{k-1}(\tau)\\
&\quad+\mu_{2k}(\tau)\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)\left[y_2(i)-\boldsymbol{\psi}^T(i)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right].
\end{aligned} \tag{25}$$

Here, $\mu_{1k}(\tau)$ and $\mu_{2k}(\tau)$ are the iterative step-sizes or convergence factors. Substituting (19) into (24) and (20) into (25), we can obtain

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)+\mu_{1k}(\tau)\\
&\quad\cdot\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)\left[y(i)-\boldsymbol{\psi}^T(i)\,\mathbf{c}-\boldsymbol{\phi}^T(i)\,\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)\right],
\end{aligned} \tag{26}$$

$$\begin{aligned}
\widehat{\mathbf{c}}_k(\tau) &= \widehat{\mathbf{c}}_{k-1}(\tau)+\mu_{2k}(\tau)\\
&\quad\cdot\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)\left[y(i)-\boldsymbol{\phi}^T(i)\,\boldsymbol{\vartheta}-\boldsymbol{\psi}^T(i)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right].
\end{aligned} \tag{27}$$

The parameter estimates $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ cannot be computed by (26) and (27), because the information vectors $\boldsymbol{\phi}(\tau)$ and $\boldsymbol{\psi}(\tau)$ contain unknown variables $x_j(\tau-i)$ and $w(\tau-i)$, and the parameter vectors $\boldsymbol{\vartheta}$ and $\mathbf{c}$ in (26) and (27) are unknown. We solve this problem by replacing the unknown variables $x_j(\tau-i)$ and $w(\tau-i)$ with their corresponding estimates $\widehat{x}_{j,k-1}(\tau-i)$ and $\widehat{w}_{k-1}(\tau-i)$ at iteration $k-1$ and define the estimates $\widehat{\boldsymbol{\phi}}_k(\tau)$ and $\widehat{\boldsymbol{\psi}}_k(\tau)$ at iteration k as

$$\begin{aligned}
\widehat{\boldsymbol{\phi}}_k(\tau) &:= \left[\widehat{\boldsymbol{\phi}}_{1,k}^T(\tau),\widehat{\boldsymbol{\phi}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\phi}}_{r,k}^T(\tau)\right]^T\in\mathbb{R}^{n_0},\\
\widehat{\boldsymbol{\phi}}_{j,k}(\tau) &:= \left[-\widehat{x}_{j,k-1}(\tau-1),\ldots,\right.\\
&\quad \left.-\widehat{x}_{j,k-1}\left(\tau-n_j\right),u_j(\tau-1),\ldots,u_j\left(\tau-n_j\right)\right]^T\\
&\quad \in\mathbb{R}^{2n_j},\\
\widehat{\boldsymbol{\psi}}_k(\tau) &:= \left[-\widehat{w}_{k-1}(\tau-1),-\widehat{w}_{k-1}(\tau-2),\ldots,\right.\\
&\quad \left.-\widehat{w}_{k-1}(\tau-n_c)\right]^T\in\mathbb{R}^{n_c}.
\end{aligned} \tag{28}$$

From (12) and (14), we have

$$\begin{aligned}
x_j(\tau-i) &= \boldsymbol{\phi}_j^T(\tau-i)\,\boldsymbol{\vartheta}_j,\\
w(\tau-i) &= y(\tau-i)-\boldsymbol{\phi}^T(\tau-i)\,\boldsymbol{\vartheta}.
\end{aligned} \tag{29}$$

Substituting $\boldsymbol{\phi}_j(\tau-i)$ and $\boldsymbol{\vartheta}_j$ with their estimates $\widehat{\boldsymbol{\phi}}_{j,k}(\tau-i)$ and $\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau)$, we can get the estimates $\widehat{x}_{j,k}(\tau-i)$ and $\widehat{w}_k(\tau-i)$ at iteration k:

$$\begin{aligned}\widehat{x}_{j,k}(\tau-i) &= \widehat{\boldsymbol{\phi}}_{j,k}^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau),\\ \widehat{w}_k(\tau-i) &= y(\tau-i)-\widehat{\boldsymbol{\phi}}_k^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_k(\tau).\end{aligned} \tag{30}$$

Replacing $\boldsymbol{\phi}(\tau)$, $\boldsymbol{\psi}(\tau)$ in (26) and (27) with their estimates $\widehat{\boldsymbol{\phi}}_k(\tau)$ and $\widehat{\boldsymbol{\psi}}_k(\tau)$, replacing $\mathbf{c}$ in (26) with its estimate $\widehat{\mathbf{c}}_{k-1}(\tau)$, and replacing $\boldsymbol{\vartheta}$ in (27) with its estimate $\widehat{\boldsymbol{\vartheta}}_k(\tau)$, we have

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)+\mu_{1k}(\tau)\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\\ &\quad\cdot\left[y(i)-\widehat{\boldsymbol{\phi}}_k^T(i)\,\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)-\widehat{\boldsymbol{\psi}}_k^T(i)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right],\\ \widehat{\mathbf{c}}_k(\tau) &= \widehat{\mathbf{c}}_{k-1}(\tau)+\mu_{2k}(\tau)\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\\ &\quad\cdot\left[y(i)-\widehat{\boldsymbol{\phi}}_k^T(i)\,\widehat{\boldsymbol{\vartheta}}_k(\tau)-\widehat{\boldsymbol{\psi}}_k^T(\tau)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right].\end{aligned} \tag{31}$$

In order to guarantee the convergence of $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$, a conservative choice is

$$\begin{aligned}\mu_{1k}(\tau) &= \mu_{2k}(\tau)=\overline{\mu}_k(\tau)\\ &\le \max\left[2\lambda_{\max}^{-1}\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\,\widehat{\boldsymbol{\phi}}_k^T(i),\right.\\ &\quad\left. 2\lambda_{\max}^{-1}\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\,\widehat{\boldsymbol{\psi}}_k^T(i)\right].\end{aligned} \tag{32}$$

At last, we can summarize the hierarchical gradient based iterative parameter estimation (H-GI) algorithm for estimating $\boldsymbol{\vartheta}$ and $\mathbf{c}$ of the multi-input OEAR systems:

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}}_k(\tau) &= \widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)+\overline{\mu}_k(\tau)\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\left[y(i)-\widehat{\boldsymbol{\phi}}_k^T(i)\right.\\ &\quad\left.\cdot\,\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)-\widehat{\boldsymbol{\psi}}_k^T(i)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right],\end{aligned} \tag{33}$$

$$\begin{aligned}\widehat{\mathbf{c}}_k(\tau) &= \widehat{\mathbf{c}}_{k-1}(\tau)+\overline{\mu}_k(\tau)\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\left[y(i)-\widehat{\boldsymbol{\phi}}_k^T(i)\right.\\ &\quad\left.\cdot\,\widehat{\boldsymbol{\vartheta}}_k(\tau)-\widehat{\boldsymbol{\psi}}_k^T(i)\,\widehat{\mathbf{c}}_{k-1}(\tau)\right],\end{aligned} \tag{34}$$

$$\widehat{\boldsymbol{\phi}}_k(\tau)=\left[\widehat{\boldsymbol{\phi}}_{1,k}^T(\tau),\widehat{\boldsymbol{\phi}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\phi}}_{r,k}^T(\tau)\right]^T, \tag{35}$$

$$\begin{aligned}\widehat{\boldsymbol{\phi}}_{j,k}(\tau) &= \left[-\widehat{x}_{j,k-1}(\tau-1),\ldots,-\widehat{x}_{j,k-1}\left(\tau-n_j\right),u_j(\tau\right.\\ &\quad\left.-1),\ldots,u_j\left(\tau-n_j\right)\right]^T,\end{aligned} \tag{36}$$

$$\begin{aligned}\widehat{\boldsymbol{\psi}}_k(\tau) &= \left[-\widehat{w}_{k-1}(\tau-1),-\widehat{w}_{k-1}(\tau-2),\ldots,\right.\\ &\quad\left.-\widehat{w}_{k-1}(\tau-n_c)\right]^T,\end{aligned} \tag{37}$$

$$\widehat{x}_{j,k}(\tau-i)=\widehat{\boldsymbol{\phi}}_{j,k}^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau), \tag{38}$$

$$\widehat{w}_k(\tau-i)=y(\tau-i)-\widehat{\boldsymbol{\phi}}_k^T(\tau-i)\,\widehat{\boldsymbol{\vartheta}}_k(\tau), \tag{39}$$

$$\begin{aligned}\overline{\mu}_k(\tau) &\le \max\left[2\lambda_{\max}^{-1}\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\,\widehat{\boldsymbol{\phi}}_k^T(i),\right.\\ &\quad\left.2\lambda_{\max}^{-1}\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\,\widehat{\boldsymbol{\psi}}_k^T(i)\right],\end{aligned} \tag{40}$$

$$\widehat{\boldsymbol{\vartheta}}_k(\tau)=\left[\widehat{\boldsymbol{\vartheta}}_{1,k}^T(\tau),\widehat{\boldsymbol{\vartheta}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\vartheta}}_{r,k}^T(\tau)\right]^T, \tag{41}$$

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau) &= \left[\widehat{a}_{j1,k}(\tau),\widehat{a}_{j2,k}(\tau),\ldots,\widehat{a}_{jn_j,k}(\tau),\widehat{b}_{j1,k}(\tau),\right.\\ &\quad\left.\widehat{b}_{j2,k}(\tau),\ldots,\widehat{b}_{jn_j,k}(\tau)\right]^T,\end{aligned} \tag{42}$$

$$\widehat{\mathbf{c}}_k(\tau)=\left[\widehat{c}_{1,k}(\tau),\widehat{c}_{2,k}(\tau),\ldots,\widehat{c}_{n_c,k}(\tau)\right]^T. \tag{43}$$

The steps of computing the parameter estimates $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ for the multi-input OEAR systems are as follows.

(1) Set the data length p, let $\tau = p$, and collect the input-output data $\{u_1(i),u_2(i),\ldots,u_r(i),y(i) : i = 0,1,\ldots,p-1\}$.

(2) Collect the input-output data $\{u_1(\tau),u_2(\tau),\ldots,u_r(\tau)\}$ and $y(\tau)$.

(3) To initialize, let $k = 1$, $\widehat{\boldsymbol{\vartheta}}_0(\tau) = \mathbf{1}_{n_0}/p_0$, $\widehat{\mathbf{c}}_0(\tau) = \mathbf{1}_{n_c}/p_0$, $\widehat{x}_{j,0}(\tau-i) = 1/p_0$, $\widehat{w}_0(\tau-i) = 1/p_0$ for $i = 1,2,\ldots,\max[n_j,n_c]$.

(4) Form $\widehat{\boldsymbol{\phi}}_{j,k}(\tau)$ and $\widehat{\boldsymbol{\psi}}_k(\tau)$ by (36) and (37), and form $\widehat{\boldsymbol{\phi}}_k(\tau)$ by (35).

(5) Choose a $\overline{\mu}_k(\tau)$ satisfying (40) and update the estimate $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ using (33) and $\widehat{\mathbf{c}}_k(\tau)$ using (34).

(6) Read $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau)$ using (41) and (42) and compute $\widehat{x}_{j,k}(\tau-i)$ using (38) and $\widehat{w}_k(\tau-i)$ using (39).

(7) Give a small positive $\varepsilon > 0$. If $\|\widehat{\boldsymbol{\vartheta}}_k(\tau)-\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)\| + \|\widehat{\mathbf{c}}_k(\tau)-\widehat{\mathbf{c}}_{k-1}(\tau)\| > \varepsilon$, increase k by 1 and go to Step (4); otherwise, obtain the parameters $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ and increase s by 1 and go to Step (2).

5. The Hierarchical Least Squares Based Iterative Algorithm

The H-GI algorithm can produce higher parameter estimation accuracy compared with the GI algorithm, but it converges slowly. In order to solve this short board, we derive a hierarchical least squares based iterative algorithm for the multi-input OEAR systems.

Minimizing $J_1(\boldsymbol{\vartheta},\mathbf{c})$ and letting the partial derivative of $J_1(\boldsymbol{\vartheta},\mathbf{c})$ with respect to $\boldsymbol{\vartheta}$ be zero and minimizing $J_2(\boldsymbol{\vartheta},\mathbf{c})$ and letting the partial derivative of $J_2(\boldsymbol{\vartheta},\mathbf{c})$ with respect to $\mathbf{c}$ be zero, respectively, we can obtain the least squares estimate $\widehat{\boldsymbol{\vartheta}}(\tau)$:

$$\widehat{\boldsymbol{\vartheta}}(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)\boldsymbol{\phi}^T(i)\right]^{-1}\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)y_1(i)\right], \quad (44)$$

$$\widehat{\mathbf{c}}(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)\boldsymbol{\psi}^T(i)\right]^{-1}\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)y_2(i)\right]. \quad (45)$$

Inserting (19) into (44) and (20) into (45) gives

$$\widehat{\boldsymbol{\vartheta}}(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)\boldsymbol{\phi}^T(i)\right]^{-1}\cdot\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\phi}(i)\left[y(i)-\boldsymbol{\psi}^T(i)\mathbf{c}\right]\right], \quad (46)$$

$$\widehat{\mathbf{c}}(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)\boldsymbol{\psi}^T(i)\right]^{-1}\cdot\left[\sum_{i=\tau-p+1}^{\tau}\boldsymbol{\psi}(i)\left[y(i)-\boldsymbol{\phi}^T(i)\boldsymbol{\vartheta}\right]\right]. \quad (47)$$

The above estimates $\widehat{\boldsymbol{\vartheta}}(\tau)$ and $\widehat{\mathbf{c}}(\tau)$ are impossible to compute, since the right-hand side of (46) contains the unknown parameter vector $\mathbf{c}$ and the unknown information vectors $\boldsymbol{\phi}(i)$ and $\boldsymbol{\psi}(i)$ and the right-hand side of (47) also contains the unknown parameter vector $\boldsymbol{\vartheta}$ and the unknown information vectors $\boldsymbol{\phi}(i)$ and $\boldsymbol{\psi}(i)$. We solve this difficulty by replacing $\boldsymbol{\phi}(i)$, $\boldsymbol{\psi}(i)$ with their estimates $\widehat{\boldsymbol{\phi}}_k(i)$, $\widehat{\boldsymbol{\psi}}_k(i)$ and replacing $\mathbf{c}$ in (46) and $\boldsymbol{\vartheta}$ in (47) with their estimates $\widehat{\mathbf{c}}_{k-1}(\tau)$ and $\widehat{\boldsymbol{\vartheta}}_k(\tau)$. Then, we can summarize the hierarchical least squares based iterative (LSI) algorithm of estimating the parameter vectors $\boldsymbol{\vartheta}$ and $\mathbf{c}$ as follows:

$$\widehat{\boldsymbol{\vartheta}}_k(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\widehat{\boldsymbol{\phi}}_k^T(i)\right]^{-1}\cdot\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\phi}}_k(i)\left[y(i)-\widehat{\boldsymbol{\psi}}_k^T(i)\widehat{\mathbf{c}}_{k-1}(\tau)\right]\right], \quad (48)$$

$$\widehat{\mathbf{c}}_k(\tau)=\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\widehat{\boldsymbol{\psi}}_k^T(i)\right]^{-1}\cdot\left[\sum_{i=\tau-p+1}^{\tau}\widehat{\boldsymbol{\psi}}_k(i)\left[y(i)-\widehat{\boldsymbol{\phi}}_k^T(i)\widehat{\boldsymbol{\vartheta}}_k(\tau)\right]\right], \quad (49)$$

$$\widehat{\boldsymbol{\phi}}_k(\tau)=\left[\widehat{\boldsymbol{\phi}}_{1,k}^T(\tau),\widehat{\boldsymbol{\phi}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\phi}}_{r,k}^T(\tau)\right]^T, \quad (50)$$

$$\widehat{\boldsymbol{\phi}}_{j,k}(\tau)=\left[-\widehat{x}_{j,k-1}(\tau-1),\ldots,-\widehat{x}_{j,k-1}(\tau-n_j),u_j(\tau-1),\ldots,u_j(\tau-n_j)\right]^T, \quad (51)$$

$$\widehat{\boldsymbol{\psi}}_k(\tau)=\left[-\widehat{w}_{k-1}(\tau-1),-\widehat{w}_{k-1}(\tau-2),\ldots,-\widehat{w}_{k-1}(\tau-n_c)\right]^T, \quad (52)$$

$$\widehat{x}_{j,k}(\tau-i)=\widehat{\boldsymbol{\phi}}_{j,k}^T(\tau-i)\widehat{\boldsymbol{\vartheta}}_{j,k}(\tau), \quad (53)$$

$$\widehat{w}_k(\tau-i)=y(\tau-i)-\widehat{\boldsymbol{\phi}}_k^T(\tau-i)\widehat{\boldsymbol{\vartheta}}_k(\tau), \quad (54)$$

$$\widehat{\boldsymbol{\vartheta}}_k(\tau)=\left[\widehat{\boldsymbol{\vartheta}}_{1,k}^T(\tau),\widehat{\boldsymbol{\vartheta}}_{2,k}^T(\tau),\ldots,\widehat{\boldsymbol{\vartheta}}_{r,k}^T(\tau)\right]^T. \quad (55)$$

The procedure for computing the parameter estimation $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ is as follows.

(1) Give the data length p, let $\tau = p$, collect the input-output data $\{u_1(i),u_2(i),\ldots,u_r(i),y(i) : i = 1,\ldots,p-1\}$, and give a small positive $\varepsilon>0$.

(2) Collect the input-output data $\{u_1(\tau),u_2(\tau),\ldots,u_r(\tau)\}$ and $y(\tau)$.

(3) To initialize, let $k=1$, $\widehat{x}_{j,0}(\tau-i)=1/p_0$, $\widehat{w}_0(\tau-i)=1/p_0$ for $i=1,2,\ldots,\max[n_j,n_c]$.

(4) Form $\widehat{\boldsymbol{\phi}}_{j,k}(\tau)$, $\widehat{\boldsymbol{\psi}}_k(\tau)$, and $\widehat{\boldsymbol{\varphi}}_k(\tau)$ by (51), (52), and (50), respectively.

(5) Update the estimates $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ by (48) and (49) and read $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ by (55).

(6) Compute $\widehat{x}_{j,k}(\tau)$ by (53) and $\widehat{w}_k(\tau)$ by (54).

(7) If $\|\widehat{\boldsymbol{\vartheta}}_k(\tau)-\widehat{\boldsymbol{\vartheta}}_{k-1}(\tau)\|+\|\widehat{\mathbf{c}}_k(\tau)-\widehat{\mathbf{c}}_{k-1}(\tau)\|>\varepsilon$, increase k by 1 and go to Step (4); otherwise, obtain the parameters $\widehat{\boldsymbol{\vartheta}}_k(\tau)$ and $\widehat{\mathbf{c}}_k(\tau)$ and increase s by 1 and go to Step (2).

6. Example

Example 1. Consider the following two-input OEAR system:

$$y(\tau)=\frac{B_1(z)}{A_1(z)}u_1(\tau)+\frac{B_2(z)}{A_2(z)}u_2(\tau)+\frac{1}{C(z)}v(\tau),$$

$$A_1(z)=1+a_{11}z^{-1}+a_{12}z^{-2}=1+0.35z^{-1}+0.27z^{-2},$$

$$B_1(z)=b_{11}z^{-1}+b_{12}z^{-2}=0.78z^{-1}-0.40z^{-2},$$

Table 1: The LSI parameter estimates and errors for Example 1.

k	a_{11}	a_{12}	b_{11}	b_{12}	a_{21}	a_{22}	b_{21}	b_{22}	c_1	δ (%)
1	−0.00174	0.00053	0.78866	−0.68631	0.00895	−0.00818	0.57027	0.61302	−0.00050	55.79849
2	0.35369	0.27184	0.78727	−0.40443	−0.17629	0.15694	0.55786	0.51059	0.27174	12.15009
3	0.35056	0.26746	0.78457	−0.40524	−0.19397	0.17507	0.56099	0.50495	0.38379	3.57156
4	0.35033	0.26820	0.78447	−0.40537	−0.18622	0.17255	0.56123	0.50724	0.38677	3.52725
5	0.35051	0.26838	0.78449	−0.40522	−0.18616	0.16982	0.56117	0.50718	0.38644	3.58654
6	0.35054	0.26843	0.78450	−0.40520	−0.18807	0.17162	0.56114	0.50618	0.38616	3.53048
7	0.35054	0.26839	0.78449	−0.40520	−0.18776	0.17156	0.56117	0.50634	0.38641	3.52131
8	0.35053	0.26838	0.78449	−0.40521	−0.18765	0.17141	0.56116	0.50639	0.38639	3.52730
9	0.35053	0.26839	0.78449	−0.40520	−0.18771	0.17145	0.56116	0.50636	0.38637	3.52652
10	0.35053	0.26839	0.78449	−0.40520	−0.18771	0.17146	0.56116	0.50636	0.38638	3.52588
True values	0.35000	0.27000	0.78000	−0.40000	−0.20000	0.18000	0.56000	0.50000	0.43000	

Table 2: The H-LSI parameter estimates and errors for Example 1.

k	a_{11}	a_{12}	b_{11}	b_{12}	a_{21}	a_{22}	b_{21}	b_{22}	c_1	δ (%)
1	−0.00165	0.00045	0.78903	−0.68649	0.00000	0.00000	0.57043	0.61368	0.04854	53.42619
2	0.33787	0.25774	0.78598	−0.41664	−0.17740	0.16201	0.56029	0.50972	0.22291	15.77773
3	0.35202	0.26929	0.78458	−0.40359	−0.18973	0.17231	0.56283	0.50476	0.39352	2.96102
4	0.35065	0.26837	0.78458	−0.40503	−0.18892	0.17175	0.56297	0.50464	0.44493	1.66301
5	0.35054	0.26835	0.78458	−0.40512	−0.18923	0.17183	0.56296	0.50444	0.44660	1.73488
6	0.35052	0.26832	0.78458	−0.40514	−0.18926	0.17188	0.56296	0.50442	0.44567	1.68306
7	0.35053	0.26833	0.78457	−0.40514	−0.18926	0.17187	0.56296	0.50443	0.44578	1.68902
8	0.35052	0.26833	0.78457	−0.40514	−0.18926	0.17187	0.56296	0.50443	0.44578	1.68914
9	0.35052	0.26833	0.78457	−0.40514	−0.18926	0.17187	0.56296	0.50443	0.44578	1.68894
10	0.35052	0.26833	0.78457	−0.40514	−0.18926	0.17187	0.56296	0.50443	0.44578	1.68898
True values	0.35000	0.27000	0.78000	−0.40000	−0.20000	0.18000	0.56000	0.50000	0.43000	

$$\begin{aligned} A_2(z) &= 1 + a_{21}z^{-1} + a_{22}z^{-2} = 1 - 0.20z^{-1} \\ &\quad + 0.18z^{-2}, \\ B_2(z) &= b_{21}z^{-1} + b_{22}z^{-2} = 0.56z^{-1} + 0.50z^{-2}, \\ C(z) &= 1 + c_1 z^{-1} = 1 + 0.43z^{-1}, \\ \boldsymbol{\theta} &= [a_{11}, a_{12}, b_{11}, b_{12}, a_{21}, a_{22}, b_{21}, b_{22}, c_1]^T \\ &= [0.35, 0.27, 0.78, -0.40, \\ &\quad -0.20, 0.18, 0.56, 0.50, 0.43]^T. \end{aligned} \tag{56}$$

The inputs $\{u_1(\tau), u_2(\tau)\}$ are taken as two persistent excitation signal sequences with zero mean and unit variance and $\{v(\tau)\}$ as a white noise sequence with zero mean and variance $\sigma^2 = 0.20^2$.

Take the data length $L = 2000$, applying the LSI algorithm and the H-LSI algorithm to estimate the parameters of this example system. The parameter estimates and their errors of the LSI algorithm are shown in Table 1, the parameter estimates and their errors of H-LSI algorithm are shown in Table 2, and the parameter estimation errors of the LSI and H-LSI algorithms versus k are shown in Figure 1.

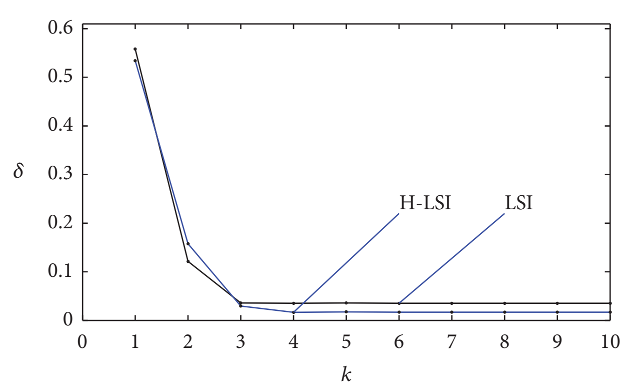

Figure 1: The GI estimation errors δ versus k with different σ^2.

From the simulation results in Tables 1 and 2 and Figure 1, we can draw the following conclusions.

(i) The estimation errors given by the LSI algorithm and H-LSI algorithm become smaller and smaller as iteration variable k increases.

(ii) Under the same noise variance, the estimation errors given by the H-LSI algorithm are lower than that given by the LSI algorithm.

Table 3: The H-LSI parameter estimates and errors for Example 2.

k	a_{11}	a_{12}	b_{11}	b_{12}	a_{21}	a_{22}	b_{21}	b_{22}	c_1	δ (%)
1	0.00000	0.00000	0.55541	−0.78641	0.05229	−0.02811	0.59789	−0.92588	0.01269	34.42173
2	0.14148	−0.30660	0.55273	−0.70377	0.18346	−0.14142	0.60426	−0.82256	0.58395	21.10705
3	0.19132	−0.23272	0.55125	−0.67866	0.21589	−0.09534	0.60251	−0.80142	0.01985	19.82911
4	0.18954	−0.24197	0.55162	−0.67979	0.21649	−0.09730	0.60299	−0.80191	0.37416	5.32335
5	0.19059	−0.23897	0.55155	−0.67915	0.21666	−0.09655	0.60291	−0.80167	0.27810	1.93353
6	0.19032	−0.23977	0.55157	−0.67931	0.21662	−0.09670	0.60293	−0.80172	0.30098	1.11341
7	0.19039	−0.23956	0.55157	−0.67927	0.21663	−0.09667	0.60293	−0.80171	0.29514	1.17583
8	0.19037	−0.23961	0.55157	−0.67928	0.21662	−0.09668	0.60293	−0.80171	0.29664	1.14596
9	0.19038	−0.23960	0.55157	−0.67928	0.21662	−0.09667	0.60293	−0.80171	0.29625	1.15289
10	0.19038	−0.23960	0.55157	−0.67928	0.21662	−0.09668	0.60293	−0.80171	0.29635	1.15101
True values	0.18000	−0.25000	0.55000	−0.68000	0.22000	−0.10000	0.60000	−0.80000	0.30000	

Table 4: The H-GI parameter estimates and errors for Example 2.

k	a_{11}	a_{12}	b_{11}	b_{12}	a_{21}	a_{22}	b_{21}	b_{22}	c_1	δ (%)
1	0.06363	−0.03646	0.51672	−0.72852	0.01182	−0.03937	0.54579	−0.85581	0.00627	31.71160
2	0.17357	−0.19236	0.53839	−0.74274	0.14494	−0.16525	0.57604	−0.89230	0.09049	18.71188
5	0.18370	−0.23560	0.55059	−0.69456	0.15382	−0.14339	0.59693	−0.85356	0.09666	15.92395
10	0.18597	−0.24230	0.55243	−0.68240	0.17824	−0.12609	0.60144	−0.83140	0.10937	14.08831
20	0.18821	−0.24117	0.55225	−0.68049	0.20204	−0.10795	0.60193	−0.81289	0.13405	11.85943
50	0.18994	−0.23981	0.55225	−0.67953	0.21588	−0.09733	0.60184	−0.80231	0.18766	8.00236
True values	0.18000	−0.25000	0.55000	−0.68000	0.22000	−0.10000	0.60000	−0.80000	0.30000	

Table 5: The H-GI parameter estimates and errors for Example 2.

k	a_{11}	a_{12}	b_{11}	b_{12}	a_{21}	a_{22}	b_{21}	b_{22}	c_1	δ (%)
1	0.02264	−0.01297	0.18381	−0.25916	0.02264	−0.01297	0.19416	−0.30444	0.00295	68.31458
2	0.09666	−0.06071	0.34760	−0.48848	0.10932	−0.05447	0.36923	−0.57459	0.28748	34.49578
5	0.24231	−0.19630	0.53350	−0.64647	0.22618	−0.09991	0.57665	−0.77609	0.33914	7.35051
10	0.23004	−0.20855	0.55156	−0.65412	0.22289	−0.09268	0.59940	−0.79531	0.33141	5.44993
20	0.21210	−0.22233	0.55260	−0.66565	0.21857	−0.09528	0.60089	−0.80002	0.31890	3.45097
50	0.19366	−0.23682	0.55262	−0.67729	0.21663	−0.09677	0.60117	−0.80156	0.29411	1.47162
True values	0.18000	−0.25000	0.55000	−0.68000	0.22000	−0.10000	0.60000	−0.80000	0.30000	

(iii) The estimation accuracy of the H-LSI algorithm is close to their true values; this indicates that the proposed algorithm can effectively identify the multi-input OEAR systems.

Example 2. Consider the following another two-input OEAR system:

$$y(\tau) = \frac{B_1(z)}{A_1(z)}u_1(\tau) + \frac{B_2(z)}{A_2(z)}u_2(\tau) + \frac{1}{C(z)}v(\tau),$$

$$A_1(z) = 1 + a_{11}z^{-1} + a_{12}z^{-2} = 1 + 0.30z^{-1} - 0.20z^{-2},$$

$$B_1(z) = b_{11}z^{-1} + b_{12}z^{-2} = 0.55z^{-1} - 0.80z^{-2},$$

$$A_2(z) = 1 + a_{21}z^{-1} + a_{22}z^{-2} = 1 + 0.20z^{-1} - 0.10z^{-2},$$

$$B_2(z) = b_{21}z^{-1} + b_{22}z^{-2} = 0.40z^{-1} - 0.90z^{-2},$$

$$C(z) = 1 + c_1z^{-1} = 1 + 0.41z^{-1},$$

$$\boldsymbol{\theta} = [a_{11}, a_{12}, b_{11}, b_{12}, a_{21}, a_{22}, b_{21}, b_{22}, c_1]^T = [0.30, -0.20, 0.55, -0.80, 0.20, -0.10, 0.40, -0.90, 0.41]^T. \quad (57)$$

The simulation conditions are the same as that of Example 1, and the noise variance $\sigma^2 = 0.20^2$. Take the data length $L = 3000$. Applying the GI algorithm and the H-GI algorithm to estimate the parameters of this example system, the simulation results are shown in Tables 3–5 and Figure 2.

From the simulation results in Tables 3–5 and Figure 2, we can draw the following conclusions.

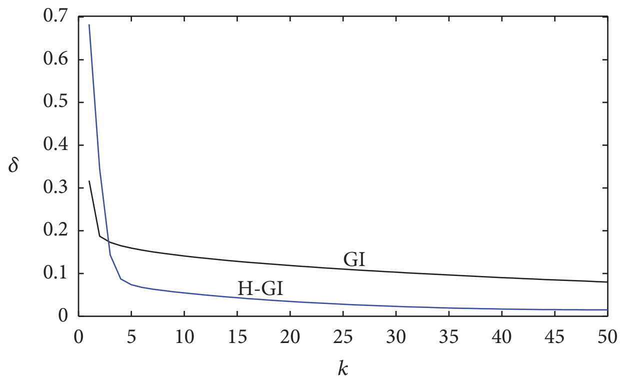

FIGURE 2: The GI and H-GI estimation errors δ versus k.

(i) Under the same noise variance and data length, the H-GI algorithm has less estimation errors than the GI algorithm. This shows that the H-GI estimation algorithm can obtain more accurate estimates than the GI algorithm.

(ii) As the iteration variable k increases, the H-GI parameter estimates are very close to their true values.

(iii) The proposed H-GI algorithm requires more iterations than the H-LSI algorithm to achieve almost same estimation accuracy.

7. Conclusions

Combining the iterative technique and the hierarchical identification principle, a H-GI algorithm and a H-LSI algorithm are derived for identifying the multi-input OEAR systems. Compared with the GI algorithm, the H-GI algorithm can generate more accurate parameter estimates. Compared with the H-GI algorithm, the H-LSI algorithm has faster convergence speed. The proposed methods can be extended to discuss the parameter estimation of the multi-input-output systems with colored noise [38–42] and time-delay systems [43, 44], such as network and signal processing [45–52].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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