A proof system for contact relation algebras*

Ivo Düntsch
School of Information and Software Engineering
University of Ulster at Jordanstown
Newtownabbey, BT 37 0QB, N.Ireland
I.Duentsch@ulst.ac.uk

Ewa Orłowska
Institute of Telecommunications
Szachowa 1
04–894, Warszawa, Poland
orlowska@itl.waw.pl

Abstract

Contact relations have been studied in the context of qualitative geometry and physics since the early 1920s, and have recently received attention in qualitative spatial reasoning. In this paper, we present a sound and complete proof system in the style of Rasiowa & Sikorski (1963) for relation algebras generated by a contact relation.

1 Introduction

Contact relations arise in the context of qualitative geometry and spatial reasoning, going back to the work of de Laguna (1922), Nicod (1924), Whitehead (1929), and, more recently, of Clarke (1981), Cohn et al. (1997), Pratt & Schoop (1998, 1999) and others. They are a generalisation of the "overlap relation" , obtained from a "part of" relation, which for the first time was formalised by Leśniewski (1916), (see also Leśniewski, 1983). One of Leśniewski's main concerns was to build a paradox–free foundation of Mathematics, one pillar of which was mereology[1] or, as it was originally called, the general theory of manifolds or collective sets. Nowadays, mereology has become synonymous with the relational part of qualitative spatial reasoning.

The traditional example of a contact structure is the set of all nonempty regular closed sets of a connected regular T_0 space with contact defined by

$$xCy \iff x \cap y \neq \emptyset. \tag{1.1}$$

Another example is the set W of all nonempty closed disks of the Euclidean plane where the contact relation C is also defined by (1.1). We will see below that there are very different models of contact structures.

*The authors gratefully acknowledge support by the KBN/British Council Grant No WAR/992/151

[1] Τὸ μέρος = The part

For a discussion of the ontological issues we refer the reader to Simons (1987), Cohn et al. (1997), Pratt & Lemon (1997), and to the special edition on ontology of the *International Journal of Human–Computer Studies* **43** (1995); an overview of current development in mereology can be found in Varzi (1996).

Before we properly define contact relations, we should like to recall a few facts of binary relations.

Relations and their algebras have been studied since the latter half of the last century, e.g. by de Morgan (1864), Peirce (1870) and Schröder (1890 - 1905). Tarski (1941), who, incidentally, was Leśniewski's only doctoral student, gave a first formal introduction to the algebra of relations; his aim was to provide an algebraic semantics for first order logic – just as Boolean algebras were an adequate algebraisation of classical propositional logic.

Suppose that W is a non–empty set. A *binary relation on* W is a subset of $W \times W$. The collection of all binary relations on W will be denoted by $Rel(W)$. If $R, S \in Rel(W)$ and $x, y, z \in W$, we often write xRy for $\langle x, y\rangle \in R$, and $xRySz$ for $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in S$. Furthermore, $R(x)$ denotes the set $\{y \in W : xRy\}$, and I is the identity relation on W.

It is easy to see that $Rel(W)$ is a Boolean algebra under the set theoretic operations with smallest element $\emptyset$ and largest element $W \times W$, which we also denote by V. Other natural operations on binary relations are the "relative" operations, namely, *composition* ; and *converse* $\breve{}$: Relational composition is defined as

$$R; S = \{\langle x, y\rangle : (\exists z \in W) xRzSy\},$$

and converse is

$$R\breve{} = \{\langle x, y\rangle : yRx\}.$$

Any $A \subseteq Rel(W)$ which is closed with respect to the Boolean set operations, and the relative operations and contains the constants $\emptyset, V, I$ is called an *algebra of binary relations* (BRA). If $M \subseteq Rel(W)$, then $[M]$ denotes the smallest BRA containing M.

A decisive property of BRAs, which exhibits their expressive power is

Proposition 1.1. *(Tarski & Givant, 1987)*
If $M \subseteq Rel(W)$, then $[M]$ contains exactly those relations which are definable in the relational structure $\langle W, M\rangle$ by first order formulas containing at most three variables, two of which are free.

We note that any equation and any inequality between relations can be written as $T = V$ for some T, viz.

(1.2) $$R = S \iff -(R \otimes S) = V,$$

(1.3) $$R \neq S \iff V; (R \otimes S); V = V.$$

Here, $R \otimes S$ is the symmetric difference of R and S, i.e.

$$R \otimes S = (R \cap -S) \cup (S \cap -R).$$

Finite BRAs can be conveniently represented by composition tables, where the rows and the columns are labelled by the atoms, and a cell contains the relative product of the two atoms which point to it; to save space, we usually just list the atoms which make up the relative product. For example, the entry EC, DC in the cell $\langle TPP, EC\rangle$ means that $TPP; EC = EC \cup DC$. If I is an atom, then row and column I are usually omitted. An example of such an array is given in Table 1.

A *contact relation C on a set W (of regions)* has the properties

(1.4) C is reflexive.

(1.5) C is symmetric.

(1.6) If $xCz \iff yCz$ for all $z \in W$, then $x = y$.

By the extensionality axiom (1.6), a region is determined by all regions it is in contact to. The pair $\langle W, C\rangle$ will be called a *contact structure*. A *contact relation algebra* (CRA) is the BRA on W generated by a contact relation C. In working with CRAs we disregard any underlying algebraic structure on the set of regions, and consider only the relational part. Since the calculus handles relations, no knowledge about the concrete geometrical objects is necessary.

Relation algebras were introduced into spatial reasoning by Egenhofer & Sharma (1992) with additional results published in Egenhofer & Sharma (1993), Egenhofer (1994). Many well known spatial relations can be expressed by the relation operators and constants, $=$, and the single relation C, for example,

(1.7)	$P = -(C; -C),$	part of
(1.8)	$PP = P \cdot -I.$	proper part of
(1.9)	$O = P\breve{}; P$	overlap
(1.10)	$PO = O \cdot -(P + P\breve{})$	partially overlap
(1.11)	$EC = C \cdot -O$	external contact
(1.12)	$TPP = PP \cdot (EC; EC)$	tangential proper part
(1.13)	$NTPP = PP \cdot -TPP$	non–tangential proper part
(1.14)	$DC = -C.$	disconnected

Depending on the domain of interpretation, some of these may be empty; for example, if $EC = \emptyset$, then $P\breve{}; P = O = C$, and hence, C is RA definable by P as in classical mereology.

It may be worthy of mention that, in the presence of the other two axioms, the extensionality axiom (1.6) is equivalent to

(1.15) P is antisymmetric.

Figure 1: Circle relations

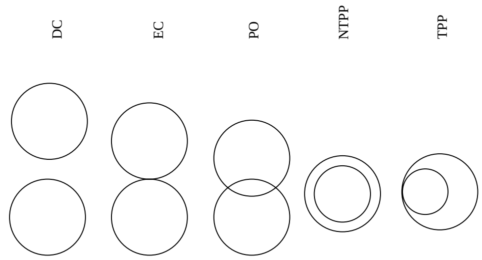


Table 1: Closed circle algebra $\mathcal{C}_c$

;	TPP	$TPP^{\smile}$	$NTPP$	$NTPP^{\smile}$	PO	EC	DC
TPP	PP	$-(NTPP \cup NTPP^{\smile})$	$NTPP$	$-P$	$-P^{\smile}$	EC, DC	DC
$TPP^{\smile}$	$I, TPP, TPP^{\smile}, PO$	$PP^{\smile}$	$PP^{\smile}, PO$	$NTPP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO, EC$	$-P$
$NTPP$	$NTPP$	$-P^{\smile}$	$NTPP$	1	$-P^{\smile}$	DC	DC
$NTPP^{\smile}$	$PP^{\smile}, PO$	$NTPP^{\smile}$	$-(EC \cup DC)$	$NTPP^{\smile}$	$PP^{\smile}, PO$	$PP^{\smile}, PO$	$-P$
PO	PP, PO	$-P$	PP, PO	$-P$	1	$-P$	$-P$
EC	PP, PO, EC	EC, DC	PP, PO	DC	$-P^{\smile}$	$-(NTPP \cup NTPP^{\smile})$	$-P$
DC	$-P^{\smile}$	DC	$-P^{\smile}$	DC	$-P^{\smile}$	$-P^{\smile}$	1

It is known (Pratt, 1992) that for a symmetric C, an element P of a relation algebra defined by (1.7) is always reflexive and transitive, so that P is indeed a partial order. The converse need not be true: There are domains, in which a "part of" P makes sense, but for which there is no contact relation C such that $P = -(C; -C)$. A case in point is the class of those regular double Stone algebras, which are not Boolean algebras (and which can be interpreted as algebras whose elements approximate regions); on such an algebra there is no contact relation $C \neq I$, for which $P = \leq$ (Düntsch et al., 1999a).

To give the reader an intuition of CRAs, we shall present a few examples from Düntsch et al. (1999c). The CRA generated on the set of all proper nonempty closed disks by the contact relation C of (1.1) is shown in Table 1. It has the eight atoms

$$I, TPP, TPP^{\smile}, NTPP, NTPP^{\smile}, PO, EC, DC,$$

with C being the union of the first seven of these. Some of the non–identity atoms are pictured in Figure 1.

The algebra of closed circles can be regarded as a translation into the plane of the interval algebra (Allen, 1983) by "forgetting" the direction, and it should not be confused with the example of all non-empty proper regular closed sets of the Euclidean plane; the algebra generated by C in this domain is much more complicated and has at least 25 atoms (Düntsch et al., 1999b). Neither should Figure 1 be used to exemplify relations on the domain of regular closed sets; this is a much too simplistic view. For example, if x is the disjoint union of two closed disks a and b, then $a TPP x$; however, this instance of TPP has quite different topological properties than the instance of TPP in Figure 1.

CRAs are by no means restricted to traditional interpretations of contact. The smallest CRA is the algebra known as $\mathcal{N}_1$ (Comer, 1983); it has four atoms, and its composition is given in Table 2. Here, $C = PP \cup PP^{\smile} \cup I$. An indication of what P looks like can be found in Figure 2; think of a fractal-like

Figure 2: An ordering for $\mathcal{N}_1$

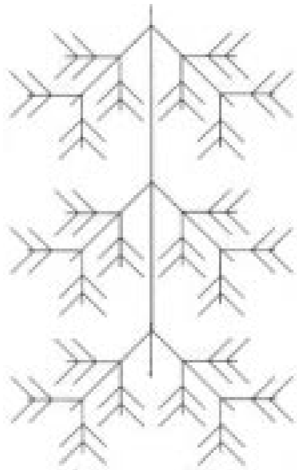


Table 2: The algebra $\mathcal{N}_1$

$\circ$	PP	$PP^{\smile}$	DC
PP	PP	1	DC
$PP^{\smile}$	$-DC$	$PP^{\smile}$	$PP^{\smile}, DC$
DC	PP, DC	DC	1

structure with a copy $\mathbb{Q}$ of the rational numbers as its "backbone", and ever branching at each point into two copies of $\mathbb{Q}$; details can be found in Düntsch (1991) and Andréka et al. (1994).

As a final example we look at the relation algebra $\mathcal{G}$ generated by the natural order of an atomless Boolean algebra B with the extreme elements removed (Table 3); we let $P = \leq$ and $PP = \lneq$. Observe that DD is Boolean complementation.

Table 3: The algebra $\mathcal{G}$

$\circ$	O				D	
	PP	$PP^{\smile}$	PON	POD	DN	DD
PP	PP	$-(POD \cup DD)$	PP, PON, DN	PP, PO, D	DN	DN
$PP^{\smile}$	I, O	$PP^{\smile}$	$PP^{\smile}, PO$	POD	$PP^{\smile}, PO, D$	POD
PON	PP, PO	$PP^{\smile}, PON, DN$	1	PP, PO	$PP^{\smile}, PON, DN$	PON
POD	POD	$PP^{\smile}, PO, D$	$PP^{\smile}, PO$	I, O	$PP^{\smile}$	$PP^{\smile}$
DN	PP, PO, D	DN	PP, PON, DN	PP	$-(POD \cup DD)$	PP
DD	POD	DN	PON	PP	$PP^{\smile}$	I

There are two possibilities to define a contact relation on $\mathcal{G}$ which satisfies (1.7): We can take either $C = O$ or $C = O \cup DD$.

It is our aim to present in this communication a sound and complete logic for contact relation algebras in which we can prove general facts about contact relation algebras. The semantics of this logic is relational as introduced by Orłowska (1991, 1996), while the proof system is in the style of Rasiowa & Sikorski (1963). The rest of the paper is structured as follows: We start with a definition of the language $\mathcal{L}$ and its semantics, followed by the proof system. Before we embark on the proofs of soundness and completeness of the system, we shall give an example of a proof, namely, we show that P as defined by (1.7) is antisymmetric. Finally, we show that our logic is undecidable, and that the extensionality condition (1.6) is not definable by a modal formula.

2 The language $\mathcal{L}$ and its semantics

The alphabet of the language $\mathcal{L}$ consists of the union of the following sets:

1. A set $\{C, 1'\}$ of constants, representing, respectively, the contact and identity[2] relations.
2. A countably infinite set VI of individuum variables.
3. A set $\{\cup, \cap, -, ;, \breve{}\}$ of names for the relational operators.
4. A set $\{(,)\}$ of delimiters.

With some abuse of language, we use the same symbols as for the actual operations; it will be clear from the context which meaning is intended.

The set CE of terms ("contact expressions") is defined as follows:

1. C and $1'$ are terms.
2. If R and S are terms, so are

$$(R \cup S), (R \cap S), (-R), (R;S), (R\breve{}).$$

3. No other string is a term.

We will use the usual conventions of reducing brackets. Note that CE can be regarded as the absolutely free algebra of type $\langle 2, 2, 1, 2, 1\rangle$ over $\{C, 1'\}$.

The set of $\mathcal{L}$–formulas is

$$\{xRy : R \in CE, x, y \in VI\}.$$

A model of $\mathcal{L}$ is a pair $M = \langle W, m\rangle$, where W is a nonempty set, and $m : CE \to W \times W$ is a mapping such that

(2.1) $m(C)$ is a contact relation.

(2.2) $m(1')$ is the identity relation on W.

(2.3) m is a homomorphism from CE to $\langle Rel(W), \cup, \cap, -, ;, \breve{}\rangle$.

A valuation v is a mapping from VI to W. If xRy is a formula, then we say that M *satisfies* xRy *under* v, written as $M, v \models xRy$, if $\langle v(x), v(y)\rangle \in m(R)$. xRy is called *true in the model* M, if $M, v \models xRy$ for all valuations v, i.e. if $m(R) = W^2$. xRy is called *valid*, if it is true in all models.

[2]For historical reasons, it is customary for researchers in the area of relation algebras to use $1'$ for the identity; the superscript comma signifies that $1'$ is the "relative" identity, as opposed to the "absolute" identity 1. Similarly, composition, i.e. relative multiplication is denoted by ;, which is not a semicolon, but absolute multiplication $\cdot$ with a comma attached below.

Table 4: Decomposition rules

$(\cup)$	$\dfrac{K, x(R\cup S)y, H}{K, xRy, xSy, H}$	$(-\cup)$	$\dfrac{K, x-(R\cup S)y, H}{K, x(-R)y, H \mid K, x(-S)y, H}$
$(\cap)$	$\dfrac{K, x(R\cap S), H}{K, xRy, H \mid K, xSy, H}$	$(-\cap)$	$\dfrac{K, x-(R\cap S)y, H}{K, x(-R)y, x(-S)y, H}$
$(\breve{\ })$	$\dfrac{K, x\breve{R}y, H}{K, yRx, H}$	$(-\breve{\ })$	$\dfrac{K, x(-\breve{R})y, H}{K, y(-R)x, H}$
$(--)$	$\dfrac{K, x(--R)y, H}{K, xRy, H}$		
$(;)$	$\dfrac{K, x(R;S)y, H}{K, xRz, H, x(R;S)y \mid K, zSy, H, x(R;S)y}$	where z is any variable	
$(-;)$	$\dfrac{K, x-(R;S)y, H}{K, x(-R)z, z(-S)y, H}$	where z is a restricted variable	

3 The proof system

We will define a proof system in the style of Rasiowa & Sikorski (1963). The system consists of two types of rules: *Decomposition rules* enable us to break up formulas into an equivalent sequence of simpler formulas. *Specific rules* modify a sequence of formulas, and have the status of structural rules. More precisely, the rules are actually rule schemas in most instances. The role of axioms is played by *axiomatic sequences*.

Rasiowa–Sikorski (RS) proof systems are, in a way, dual to tableaux systems: Whereas in the latter one tries to refute the negation of a formula, the RS systems attempt to verify a formula by closing the branches of a decomposition tree with axiomatic sequences. Rules in RS systems go in both directions: we call a rule *admissible* if

The upper sequence is valid iff the lower sequence(s) is (are) valid.

Here, a sequence of formulas is valid if its meta-level disjunction is valid.

The rules of our system are given in Tables 4 and 5. A variable z is called *restricted in a rule* if it does not occur in the upper part of that rule. K and H are finite, possibly empty, sequences of formulas of the relational calculus.

The axiomatic sequences are

(3.1) $$xRy,\ x(-R)y,$$

(3.2) $$x1'x,$$

where $R \in CE$.

Proofs have the form of trees: Given a formula xRy, we successively apply decomposition or specific rules; in this way we obtain a tree whose root is xRy, and whose nodes consist of sequences of

Table 5: Specific rules

$(\mathrm{sym}\,1')$	$\dfrac{K, x1'y, H}{K, y1'x, H}$	
$(\mathrm{tran}\,1')$	$\dfrac{K, x1'y, H}{K, x1'z, H, x1'y \mid K, z1'y, H, x1'y}$,	where z is any variable
$(1'_1)$	$\dfrac{K, xRy, H}{K, x1'z, H, xRy \mid K, zRy, H, xRy}$,	where z is any variable
$(1'_2)$	$\dfrac{K, xRy, H}{K, xRz, H, xRy \mid K, z1'y, H, xRy}$,	where z is any variable
$(\mathrm{refl}\,C)$	$\dfrac{K, xCy, H}{K, x1'y, xCy, H}$	
$(\mathrm{sym}\,C)$	$\dfrac{K, xCy, H}{K, yCx, H}$	
$(\mathrm{ext}\,C)$	$\dfrac{K}{K, x(-C)z, yCz \mid K, y(-C)t, xCt \mid K, x(-1')y}$	where z and t are restricted variables
$(\mathrm{cut}\,C)$	$\dfrac{K}{K, xCy \mid K, x(-C)y}$	

formulas. A branch of tree is *closed* if it contains a node which contains an axiomatic sequence as a subsequence. A tree is called *closed* if all its branches are closed.

If Z is a branch and F a formula, we write $F\varepsilon\ Z$ if Z contains a sequence Γ in which F appears; similarly we write $F\varepsilon\ \Gamma$, if F appears in Γ.

We suppose without loss of generality that each branch Z of a proof tree is maximal in the sense that, whenever $F\varepsilon\ Z$ and there is a (decomposition or specific) rule which can be applied to F, then this rule is applied to F with an appropriate result appearing in Z. For example, if F is $x(R;S)y$, and K, F, H appear in Z, then, by decomposition rule $(;)$, for all z, the sequence

$$K, xRz, H, x(R;S)y$$

appears in Z, or the sequence

$$K, zSy, H, x(R;S)y$$

appears in Z. Similarly, if K appears in Z, then, by rule $\mathrm{ext}\,C$, (at least) one of

$$K, x(-C)z, yCz,$$
$$K, y(-C)t, xCt,$$
$$K, x(-1')y$$

appears in Z, where z, t do not occur in K.

4 A decomposition tree

Before we proceed to show that the proof system is sound and complete, we shall give an example of a derivation of a valid formula. We want to show that the relation P as defined by (1.7) is antisymmetric, i.e. that

$$P \cap P^{\smile} \subseteq 1',$$

that is, by definition of P,

$$(C; -C) \cup (-C; C) \cup 1' = V.$$

For this example only, we use the same symbols as in $\mathcal{L}$ with some abuse of notation. Thus, to prove our claim, we must find a closed proof tree for the formula

$$x((C; -C) \cup (-C; C) \cup 1')y. \tag{4.1}$$

Applying rule $(\cup)$ to (4.1) and again to the resulting formula, we obtain

$$x(C; -C)y,\ x(-C; C)y,\ x1'y. \tag{4.2}$$

Rule $(\mathrm{ext}\,C)$ with K given by (4.2) leads to three branches:

$$x(C; -C)y,\ x(-C; C)y,\ x1'y,\ x(-C)z,\ yCz \tag{4.3}$$
$$x(C; -C)y,\ x(-C; C)y,\ x1'y,\ y(-C)t,\ xCt, \tag{4.4}$$
$$x(C; -C)y,\ x(-C; C)y,\ x1'y,\ x(-1')y, \tag{4.5}$$

Node (4.5) is closed, and we look at node (4.3). Decomposing $x(C; -C)y$ with rule $(;)$ gives two more branches:

$$xCz,\ x(-C; C)y,\ x1'y,\ x(-C)z,\ yCz, x(C; -C)y, \tag{4.6}$$
$$z(-C)y,\ x(-C; C)y,\ x1'y,\ x(-C)z,\ yCz, x(C; -C)y. \tag{4.7}$$

Node (4.6) is closed. If we apply rule $(\mathrm{sym}\,C)$ to yCz in (4.7), we obtain

$$z(-C)y,\ x(-C; C)y,\ x1'y,\ x(-C)z,\ zCy, x(C; -C)y, \tag{4.8}$$

which is closed. Similarly, one shows that (4.4) leads to closed branches as well.

5 Soundness and completeness

The soundness of the proof system follows from the following

Proposition 5.1. *1. All decomposition rules are admissible.*

2. All specific rules are admissible.

3. The axiomatic sequences are valid.

Proof. The admissibility of the decomposition rules and the $(\mathrm{tran}\,1')$, $(\mathrm{sym}\,1')$, $(\mathrm{sym}\,C)$, $(\mathrm{refl}\,C)$ rules are proved, mutatis mutandis, in Orłowska (1996), and the admissibility of (cut) is obvious. Clearly, the axiomatic sequences are valid.

Next, we show that

(5.1) $$(1'_1) \text{ is admissible } \iff m(1'); m(R) \subseteq m(R)$$

for all terms $R \in CE$ and all models $\langle W, m\rangle$. The proof of the admissibility of rule $(1'_2)$ is analogous.

"$\Rightarrow$": Let $R \in CE$, and K be the sequence

$$x(-1')z,\ z(-R)y;$$

furthermore, set $H = \emptyset$. Then, the sequences

(5.2) $$K,\ x1'z,\ H,\ xRy$$

(5.3) $$K,\ zRy,\ H,\ xRy$$

are clearly valid, and thus, by our assumption, the upper sequence of $1'_1$

(5.4) $$x(-1')z,\ z(-R)y,\ xRy$$

is valid as well.

Recall that I is the identity relation on the set in question, and that $m(1') = I$ by (2.2). Assume that there are some $R \in CE$ and a model $M = \langle W, m\rangle$ such that $I; m(R) \not\subseteq m(R)$, i.e. that there are $a, b, c \in W$ for which

(5.5) $$aIc,\ c(m(R))b,\ a(-m(R))b,$$

in particular, $m(a) \neq \emptyset$.

If v is a valuation such that

$$v(x) = a,\ v(y) = b,\ v(z) = c,$$

then M, v satisfies none of the formulas of (5.4), contradicting the validity of (5.4).

"$\Leftarrow$": Name the sequences occurring in $1'_1$ by

Γ_1. K, xRy, H,

Γ_2. $K, x1'z, H, xRy$,

Γ_3. K, zRy, H, xRy.

Since Γ_1 is a subsequence of both Γ_2 and Γ_3, the validity of Γ_1 implies the validity of Γ_2 and Γ_3. Conversely, suppose that Γ_2 and Γ_3 are valid, i.e. for all models M, for all valuations $v : VI \to M$ there are F in Γ_2, G in Γ_3 such that

$$M, v \models F \text{ and } M, v \models G.$$

If F or G occur in Γ_1, there is nothing more to show. Otherwise, F is $x1'z$, G is zRy, and, by our hypothesis, $M, v \models xRy$.

Finally, we prove

(5.6) $$(\text{ext}\, C) \text{ is admissible } \iff m(C)v(x) = m(C)v(y) \text{ implies } v(x) = v(y)$$

for all models $M = \langle W, m\rangle$, all valuations $v : VI \to W$ and all $x, y \in VI$.

"$\Rightarrow$": Suppose that $\langle W, m\rangle$ is a model, $v : VI \to W$ a valuation, and $x, y, z \in VI$. We have to show that

(5.7)
$$[(\forall z)(v(x)m(C)v(z) \Rightarrow v(y)m(C)v(z)) \text{ and } (\forall t)(v(y)m(c)v(t) \Rightarrow v(x)m(C)v(t))] \Rightarrow v(x) = v(y),$$

i.e. for all $z, t \in W$,

(5.8)
$$(M, v \models x(-C)z \text{ or } M, v \models yCz) \text{ and } (M, v \models y(-C)t \text{ or } M, v \models xCt) \Rightarrow M, v \models x1'y.$$

Set

(5.9) $$\Gamma_0 = \{K\},\ \Gamma_1 = \{K, x(-C)z, yCz\},\ \Gamma_2 = \{K, y(-C)t, xCt\},\ \Gamma_3 = \{K, x(-1')y\}.$$

By our hypothesis, we know that the joint validity of $\Gamma_1, \Gamma_2, \Gamma_3$ implies the validity of Γ_0. This implies, in particular, that $F_1 \in \Gamma_1, F_2 \in \Gamma_2, F_3 \in \Gamma_3$ with all $F_i \neq K$ cannot live together. Thus, if

1. $M, v \models x(-C)z$ or $M, v \models yCz$, and
2. $M, v \models y(-C)t$ or $M, v \models xCt$,

then we cannot have $M, v \models x(-1')y$. Thus, $M, v \models x1'y$, which was to be shown.

"$\Leftarrow$": Suppose that (5.8) is satisfied for all models $M = \langle W, m\rangle$, all valuations $v : VI \to W$, and all $x, y, z, t \in VI$; furthermore, suppose that $\Gamma_1 - \Gamma_3$ of (5.9) are valid. We need to show that $M, v \models K$. Assume not; then, since Γ_1 –Γ_3 are valid, we know that

1. $M, v \models x(-C)z$ or $M, v \models yCz$,
2. $M, v \models y(-C)t$ or $M, v \models xCt$,
3. $M, v \models x(-1')y$.

This contradicts (5.8). ☐

Proposition 5.2. *If a formula is valid then it has a closed proof tree.*

Proof. Suppose that F is valid formula, and assume that there is a non closed branch Z in a proof tree of F with root F. Recall that we assume Z to be maximal with respect to the application of rules as explained on page 8.

First we define a relation E on VI by

(5.10) $$xEy \iff x1'y \not\varepsilon Z.$$

Claim 1. E is an equivalence relation on VI.

Proof. (Claim 1). Let $x \in VI$; if $x1'x\varepsilon\ Z$, then Z is closed, contrary to our assumption. Thus, $x1'x \not\varepsilon Z$, and it follows that xEx. If xEy, then $x1'y \not\varepsilon Z$, and it follows from the maximality of Z and rule $(\mathrm{sym}\, 1')$ that $y1'x \not\varepsilon Z$; hence, yEx. Finally, let xEy and yEz; then, $x1'y \not\varepsilon Z,\ y1'z \not\varepsilon Z$. Assume that $x1'z\varepsilon\ Z$; by the maximality of Z and rule $(\mathrm{tran}\, 1')$ we obtain $x1'y\varepsilon\ Z$ or $y1'z\varepsilon\ Z$, a contradiction. ☐

Now, we will construct a model which will falsify F, contradicting our assumption. Its base set W is the set of equivalence classes of E. Let $f : VI \twoheadrightarrow W$ the canonical mapping which assigns to each $x \in VI$ its equivalence class with respect to E. Define $D \in Rel(W)$ by

(5.11) $$f(x)Df(y) \iff xCy \not\varepsilon Z,$$

The definition of D is independent of the choice of representative, as the following shows:

Claim 2. If $xCy \not\varepsilon Z,\ sEx,\ tEy$, then $sCt \not\varepsilon Z$.

Proof. (Claim 2)

Assume $sCt\varepsilon\ Z$. By the maximality of Z and rule $(\mathrm{refl}\, C)$ we have $s1't\varepsilon\ Z$, and with rule $(1'_1)$ for sCt with the new variable x, we obtain

$$s1'x\varepsilon\ Z, sCt\varepsilon\ Z \text{ or } xCt\varepsilon\ Z, sCt\varepsilon\ Z.$$

Since sEx, the first case is not possible, and hence, $xCt\varepsilon\ Z$. Similarly, using $(1'_2)$ for xCy with the new variable y, we have $sCy\varepsilon\ Z$. Finally, by maximality of Z and rule $(\mathrm{ext}\, C)$, we have

$$s(-C)y\varepsilon\ Z,\ tCy\varepsilon\ Z \text{ or } t(-C)x\varepsilon\ Z,\ sCx\varepsilon\ Z \text{ or } s(-1')t\varepsilon\ Z.$$

By our assumption, none of these is possible. ☐

Now, set $m(1') = I$ on W, $m(C) = D$, extend m over CE homomorphically, and let $M = \langle W, m\rangle$. Observe that by definition of E

(5.12) $$f(x) I f(y) \iff x1'y \not\varepsilon Z.$$

All that remains to show that M is a model, is to prove

Claim 3. D is a contact relation.

Proof. (Claim 3)

Assume that $f(x)(-D)f(x)$ for some $x \in VI$. Then, $xCx\varepsilon\, Z$, and by maximality with respect to $(\text{refl}\, C)$ we obtain $x1'x\varepsilon\, Z$; this contradicts the fact that Z is not closed.

Let $f(x)Df(y)$ and assume that $f(y)(-D)f(x)$; then $yCx\varepsilon\, Z$, and it follows from the maximality of Z and $(\text{sym}\, C)$ that $xCy\varepsilon\, Z$. Hence, $f(x)(-D)f(y)$, a contradiction.

Finally, suppose that $D(f(x)) = D(f(y))$ for some $x, y \in VI$, and let $z \in VI$. By maximality with respect to $(\text{ext}\, C)$ we have exactly one of the following three cases:

1. $x(-C)z\varepsilon\, Z, yCz\,\varepsilon\, Z.$
2. $y(-C)t\varepsilon\, Z, xCt\,\varepsilon\, Z.$
3. $x(-1')y\,\varepsilon\, Z.$

If $x(-C)z\varepsilon\;\, Z, yCz\,\varepsilon\, Z$, then $xCz\;\not\varepsilon\, Z$, since Z is not closed, and thus, $f(x)Df(z)$, which implies $f(z) \in D(f(x))$. Then, $f(y)Df(z)$, hence, $yCz\;\not\varepsilon\, Z$, contradicting our hypothesis. Thus, this case is not possible, and similarly, one shows that 2. cannot happen either. It follows that $x(-1')y\,\varepsilon\, Z$; since Z is not closed, $x1'y\;\not\varepsilon\, Z$, and therefore, xEy. We conclude that $f(x) = f(y)$. $\square$

Thus, M is indeed a model of $\mathcal{L}$.

Next, we define the *relational complexity* $rc(R)$ of a contact expression R:

1. If $R \in \{C, 1'\}$, then $rc(R) = 1$.
2. If $R \in \{-S, S\breve{}\}$, then $rc(R) = rc(S) + 1$.
3. If $R \in \{S \cup T, S \cap T, S;T\}$, then $rc(R) = rc(S) + rc(T) + 1$.

Claim 4. For all $x, y \in VI,\ R \in CE$,

$$f(x)m(R)f(y) \iff xRy \not\varepsilon\, Z.$$

Proof. (Claim4)

1. Let $rc(R) = 1$. Then, $R \in \{C, 1'\}$, and, by (5.12) and (5.11),

$$f(x)m(1')f(y) \iff x1'y \notin Z,$$
$$f(x)m(C)f(y) \iff xCy \notin Z.$$

2. Suppose the claim is true for $rc(S) \leq n$.

(a) R is $-S$: Then,

$$\begin{aligned} f(x)m(R)f(y) &\iff \text{not } f(x)Sf(y), && \text{since } m \text{ is a homomorphism} \\ &\iff xSy\varepsilon\, Z, && \text{by the induction hypothesis} \\ &\iff x(-S)y \notin Z. \end{aligned}$$

In the latter equivalence, the $\Rightarrow$ part holds by the fact that Z is not closed, and the $\Leftarrow$ direction by the maximality of Z and the rule (cut).

(b) R is $S\breve{}$: Then,

$$\begin{aligned} f(x)m(R)f(y) &\iff f(x)m(S)\breve{}f(y), && \text{since } m \text{ is a homomorphism} \\ &\iff f(y)m(S)f(x) \\ &\iff ySx \notin Z && \text{by the induction hypothesis} \\ &\iff xS\breve{}y \notin Z && \text{by maximality of } Z \text{ and rule } (\breve{}). \end{aligned}$$

(c) R is $S \cup T$: Then,

$$\begin{aligned} f(x)m(R)f(y) &\iff f(x)m(S)f(y) \text{ or } f(x)m(T)f(y), && \text{since } m \text{ is a homomorphism} \\ &\iff xSy \notin Z \text{ or } xTy \notin Z && \text{by the induction hypothesis} \\ &\iff x(S \cup T)y \notin Z && \text{by maximality of } Z \text{ and rule } (\cup). \end{aligned}$$

(d) R is $S \cap T$: Then,

$$\begin{aligned} f(x)m(R)f(y) &\iff f(x)m(S)f(y) \text{ and } f(x)m(T)f(y), && \text{since } m \text{ is a homomorphism} \\ &\iff xSy \notin Z \text{ and } xTy \notin Z && \text{by the induction hypothesis} \\ &\iff x(S \cap T)y \notin Z && \text{by maximality of } Z \text{ and rule } (\cap). \end{aligned}$$

(e) R is $S;T$: Then,

$$\begin{aligned} f(x)m(R)f(y) &\iff (\exists z \in VI)f(x)m(S)f(z)m(T)f(y) && \text{since } m \text{ is a homomorphism} \\ &\iff (\exists z \in VI)[xSz \notin Z \text{ and } zTy \notin Z] && \text{by the induction hypothesis} \\ &\iff x(S;T)y \notin Z && \text{by maximality of } Z \text{ and rule } (;). \end{aligned}$$

This proves Claim 4. $\square$

Now we can finish the proof of Proposition 5.2. Let F be xPy, and let $v : VI \to W$ be the valuation $x \overset{v}{\mapsto} f(x)$. Since F is the root of the tree, we have $xPy \varepsilon Z$. On the other hand,

$$\begin{aligned} M, v \models xPy &\Rightarrow v(x)m(P)v(y) && \text{by definition of } \models \\ &\Rightarrow f(x)m(P)f(y) && \text{by definition of } v \\ &\Rightarrow xPy \not\varepsilon Z && \text{by Claim 4,} \end{aligned}$$

a contradiction. □

6 Decidability and modal expressibility

First, we will show that the equational theory of contact relation algebras is undecidable; this is pertinent for our logic which axiomatises RA expressions of the form $\tau(C, 1') = 1$, where $\tau(C, 1') \in CE$. The result will follow from

Proposition 6.1. *(Andréka, Givant & Németi, 1997)*
If a class $\mathbb{K}$ of relation algebras contains a simple algebra with infinitely many elements below the identity relation, then $\mathbb{K}$ has an undecidable equational theory. □

Proposition 6.2. *The equational theory of CRA is undecidable.*

Proof. Let D_0, D_1 be two disjoint sets; we suppose that each D_i is the disjoint union of D_{i0} and D_{i1} such that each $\langle D_{ij}, \leq \rangle$ is order isomorphic to ω, the order type of the natural numbers, and we let $D_{ij} = \{0_{ij}, 1_{ij}, 2_{ij}, \dots\}$. Each D_i is ordered as follows:

$$x \preceq_i y \iff \begin{cases} x \in D_{i1} \text{ and } y \in D_{i0}, & \text{or} \\ x, y \in D_{i0} \text{ and } y \leq x, & \text{or} \\ x, y \in D_{i1} \text{ and } x \leq y. \end{cases}$$

Thus, $\langle D_i, \preceq_i \rangle$ has order type $\omega + \omega^*$, see Figure 3. Set $D = D_0 \cup D_1 \cup \{\top, \bot\}$, where $\{\top, \bot\} \cap (D_0 \cup D_1) = \emptyset$. Order D by

$$x \preceq y \iff \begin{cases} x = \bot & \text{or} \\ y = \top & \text{or} \\ x \preceq_i y & \text{for some } i \in \{0, 1\}. \end{cases}$$

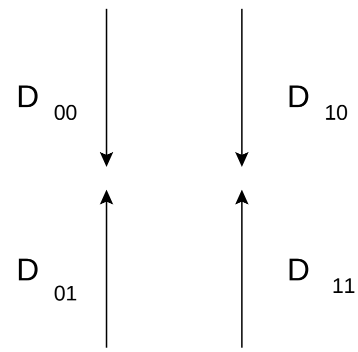
Let $m : D \to D$ be the order anti–isomorphism of D with

$$x_{ij} \overset{m}{\mapsto} x_{(i+1)(j+1)},$$

where addition in the indices is modulo 2. Now, it is not hard to see that $\langle D, \preceq \rangle$ is an orthocomplemented lattice, with $x^* = m(x)$ for all $x \not\in \{\top, \bot\}$, and $\top^* = \bot$, $\bot^* = \top$. Let the relation C be defined by

$$xCy \iff y \not\preceq x^*.$$

Figure 3: The ordering of D_i



Clearly, C is reflexive and symmetric. Suppose that $x \not\preceq y$; then, $y^* \not\preceq x^*$, and thus, xCy^*. On the other hand $y(-C)y^*$, and it follows that $C(x) \not\subseteq C(y)$, in particular, $C(x) \neq C(y)$ (see also Biacino & Gerla, 1991, Proposition 2). Hence, C is a contact relation. Let A be the BRA generated by C on D_0; then, A is simple (see e.g. Jónsson & Tarski, 1951).

For each $x \in \omega$, let $1'_x = \{x_{01}, x_{11}\}^2 \cap 1'$. We are going to show that each $1'_x$ is in A, and thus, the equational theory of A is undecidable by Proposition 6.1. Recall from (1.8) that PP is the relation $\preceq \cap -1'$ on D_0. Since both 0_{01} and 0_{11} are the only minimal elements, we have

$$1'_0 = 1' \cap -(PP^{\smile}; PP).$$

Next, suppose that $1'_n$ is defined for all $n \lneq k$. Let $1'_{<k} = \bigcup_{n \lneq k} 1'_n$, and $U = V \cap -(1'_{<k}; V; 1'_{<k})$; then, k_{01} and k_{11} are minimal in $U \cap PP$, and thus,

$$1'_k = 1' \cap -[(U \cap PP)^{\smile}; (U \cap PP)].$$

Applying Proposition 6.1 completes the proof. □

Finally, we want to show that the extensionality condition (1.6) is not expressible by a formula of modal logic.

A *bounded morphism* from a frame $K_0 = \langle W_0, R_0 \rangle$ to a frame $K_1 = \langle W_1, R_1 \rangle$ is a mapping $h : W_0 \to W_1$ for which

(6.1) $\langle x, y \rangle \in R_0$ implies $\langle h(x), h(y) \rangle \in R_1$,

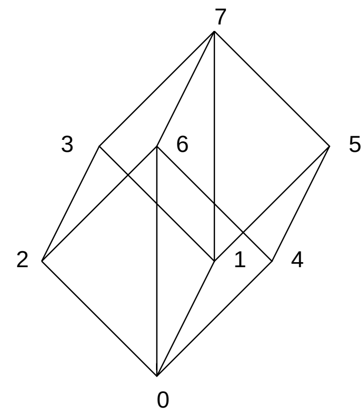
(6.2) $\langle x, h(y) \rangle \in R_1$ implies $(\exists z \in W_0)[\langle z, y \rangle \in R_0 \wedge h(z) = x]$.

for all $x, y \in W_0$.

A necessary condition for a first order condition to be expressible by a modal formula is invariance under bounded morphisms (see e.g. van Benthem, 1984, for the result, and further references).

Proposition 6.3. *The class of contact structures cannot be axiomatised by modal formulas.*

Figure 4: A contact frame



Proof. We will present a contact frame whose image under a bounded morphism is not a contact frame. Consider $W = \{0, 1, \dots, 7\}$, $U = \{1, 3, 5, 7\}$ and $C \in Rel(W)$ as depicted in Figure 4; there aCb iff $a = b$ or a and b are direct neighbours. It is not hard to check that C is a contact relation. Let S be the restriction of C to $U \times U$; then,

$$S(1) = \{1, 3, 5, 7\} = S(7),$$

and thus, S does not satisfy (1.6). On the other hand, the mapping $f : W \to U$ defined for all $a \leq 7$ by

$$f(a) = \begin{cases} a + 1, & \text{if } a \text{ is even,} \\ a, & \text{otherwise,} \end{cases}$$

is a bounded morphism. □

7 Summary and outlook

A contact relation C on a set W of regions is characterised by reflexivity, symmetry, and the extensionality axiom (1.6). The set of all relations which can be defined from C and the identity $1'$ on W by using the (absolute) Boolean operators and the relative operators ; and $\breve{}$ is a contact relation algebra (CRA) on W (Düntsch et al., 1999c). The relations in a CRA are present in any model of mereology which uses an extensional contact relation C; thus, CRAs are useful for various scenarios.

In this article, we have presented a sound and complete proof system for contact relation algebras with a Rasiowa & Sikorski – style proof system, and the relational semantics of Orłowska (1991, 1996). We have also shown that the equational theory of CRAs is undecidable.

Buszkowski & Orlowska (1997) present a logic for proving data dependencies and relationships among them. They show that the logic is undecidable, and exhibit decidable fragments of their theory. It would be interesting to know how far their method is applicable to the theory of CRAs.

One further task is to find a relational or "modal–style" logic for contact frames, i.e. a logic whose models are frames $\langle W, C \rangle$, where C is a contact relation. As shown above, such a logic cannot be a traditional modal logic.

Another task is to develop and implement complete proof systems for models of mereology which carry an additional algebraic structure such as the mereology of Clarke (1981) or the RCC of Randell et al. (1992). Some such systems have recently been presented, for example, Asher & Vieu (1995), Bennett (1996), Pratt & Schoop (1998), but the discussion in Lemon & Pratt (1997) shows that improvements can still be made.

Acknowledgement

We would like to thank the anonymous referees for their constructive and thoughtful remarks.

References

Allen, J. F. (1983). Maintaining knowledge about temporal intervals. *Communications of the ACM*, **26**, 832–843.

Andréka, H., Givant, S. & Németi, I. (1994). The lattice of varieties of representable relation algebras. *Journal of Symbolic Logic*, **59**, 631–661.

Andréka, H., Givant, S. & Németi, I. (1997). Decision problems for equational theories of relation algebras. No. 604 in Memoirs of the American Mathematical Society. Providence: Amer. Math. Soc.

Asher, N. & Vieu, L. (1995). Toward a geometry of common sense: A semantics and a complete axiomatization of mereotopology. In C. Mellish (Ed.), *IJCAI 95, Proceedings of the 14th International Joint Conference on Artificial Intelligence*.

Bennett, B. (1996). Modal logics for qualitative spatial reasoning. *Bulletin of the Interest Group in Pure and Applied Logic (IGPL)*, **4**, 23–45.

Biacino, L. & Gerla, G. (1991). Connection structures. *Notre Dame Journal of Formal Logic*, **32**, 242–247.

Buszkowski, W. & Orlowska, E. (1997). Indiscernibility–based formalization of dependencies in information systems. In E. Orłowska (Ed.), *Incomplete Information – Rough Set Analysis*, 293–315. Heidelberg: Physica – Verlag.

Clarke, B. L. (1981). A calculus of individuals based on 'connection'. *Notre Dame Journal of Formal Logic*, **22**, 204–218.

Cohn, A. G., Bennett, B., Gooday, J. & Gotts, N. M. (1997). Qualitative spatial representation and reasoning with the region connection calculus. *Geoinformatica*, **13**, pp 1–42.

Comer, S. (1983). A remark on chromatic polygroups. *Congressus Numerantium*, **38**, 85–95.

de Laguna, T. (1922). Point, line and surface as sets of solids. *The Journal of Philosophy*, **19**, 449–461.

de Morgan, A. (1864). On the syllogism: IV, and on the logic of relations. *Transactions of the Cambridge Philosophical Society*, **10**, 331–358. (read April 23, 1860) Reprinted in de Morgan (1966).

de Morgan, A. (1966). On the Syllogism, and Other Logical Writings. New Haven: Yale Univ. Press.

Düntsch, I. (1991). Small integral relation algebras generated by a partial order. *Period. Math. Hungar.*, **23**, 129–138.

Düntsch, I., Orłowska, E. & Wang, H. (1999a). An algebraic and logical approach to the approximation of regions. Preprint, `http://www.infj.ulst.ac.uk/˜cccz23/papers/apprea.html`.

Düntsch, I., Schmidt, G. & Winter, M. (1999b). A necessary relation algebra for mereotopology. Submitted for publication.

Düntsch, I., Wang, H. & McCloskey, S. (1999c). Relation algebras in qualitative spatial reasoning. *Fundamenta Informaticae*. To appear.

Egenhofer, M. (1994). Deriving the composition of binary topological relations. *Journal of Visual Languages and Computing*, **5**, 133–149.

Egenhofer, M. & Sharma, J. (1992). Topological consistency. In *Fifth International Symposium on Spatial Data Handling*, Charleston, SC.

Egenhofer, M. & Sharma, J. (1993). Assessing the consistency of complete and incomplete topological information. *Geographical Systems 1*, 47–68.

Jónsson, B. & Tarski, A. (1951). Boolean algebras with operators I. *Amer. J. Math.*, **73**, 891–939.

Lemon, O. & Pratt, I. (1997). On the incompleteness of modal logics of space: advancing complete modal logics of place. In M. Kracht, M. de Rijke, H. Wansing & M. Zakharyaschev (Eds.), *Advances in Modal Logic*, 113–130. Stanford: CSLI Publications.

Leśniewski, S. (1916). Podstawy ogólnej teoryi mnogosci.I. *Prace Polskiego Koła Naukowe w Moskwie, Sekcya matematycznoprzyrodnicza*, **2**.

Leśniewski, S. (1927 – 1931). O podstawach matematyki. *Przeglad Filozoficzny*, **30–34**.

Leśniewski, S. (1983). On the foundation of mathematics. *Topoi*, **2**, 7–52.

Nicod, J. (1924). Geometry in a sensible world. Doctoral thesis, Sorbonne, Paris. English translation in *Geometry and Induction*, Routledge and Kegan Paul, 1969.

Orłowska, E. (1991). Relational interpretation of modal logics. In H. Andréka, J. D. Monk & I. Németi (Eds.), *Algebraic Logic*, vol. 54 of *Colloquia Mathematica Societatis János Bolyai*, 443–471. Amsterdam: North Holland.

Orłowska, E. (1996). Relational proof systems for modal logics. In H. Wansing (Ed.), *Proof theory of modal logic*, 55–78. Kluwer.

Peirce, C. S. (1870). Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic. *Memoirs of the American Academy of Sciences*, **9**, 317–378. Reprint by Welch, Bigelow and Co., Cambridge, Mass., 1870, pp. 1–62.

Pratt, I. & Lemon, O. (1997). Ontologies for plane, polygonal mereotopology. *Notre Dame Journal of Formal Logic*, **38**, 225–245.

Pratt, I. & Schoop, D. (1998). A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, **27**, 621–658.

Pratt, I. & Schoop, D. (1999). Expressivity in polygonal, plane mereotopology. *Journal of Symbolic Logic*. To appear.

Pratt, V. (1992). Dynamic algebras: Examples, constructions, applications. Dept of Computer Science, Stanford.

Randell, D. A., Cohn, A. & Cui, Z. (1992). Computing transitivity tables: A challenge for automated theorem provers. In *Proc CADE 11*, 786–790. Springer Verlag.

Rasiowa, H. & Sikorski, R. (1963). The Mathematics of Metamathematics, vol. 41 of *Polska Akademia Nauk. Monografie matematyczne*. Warsaw: PWN.

Schröder, E. (1890 - 1905). Vorlesungen über die Algebra der Logik, Volumes 1 to 3. Leipzig: Teubner. Reprinted by Chelsea, New York, 1966.

Simons, P. (1987). Parts. A Study in Ontology. Oxford: Clarendon Press.

Tarski, A. (1941). On the calculus of relations. *J. Symbolic Logic*, **6**, 73–89.

Tarski, A. & Givant, S. (1987). A formalization of set theory without variables, vol. 41 of *Colloquium Publications*. Providence: Amer. Math. Soc.

van Benthem, J. (1984). Correspondence theory. In D. M. Gabbay & F. Guenthner (Eds.), *Extensions of classical logic*, vol. 2 of *Handbook of Philosophical Logic*, 167–247. Dordrecht: Reidel.

Varzi, A. C. (1996). Parts, wholes, and part–whole relations: The prospect of mereotopology. *Data & Knowledge Engineering*, **20**, 259–286.

Whitehead, A. N. (1929). Process and reality. New York: MacMillan.