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Generic variations and NTP_1

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Abstract We prove a preservation theorem for NTP_1 in the context of the generic variations construction. We also prove that NTP_1 is preserved under adding to a geometric theory a generic predicate.

Keywords Tree property of the first kind · Generic variations · Parametrized Fraïssé class

Mathematics Subject Classification 03C45

1 Introduction

The theorem of Shelah stating that a theory has the tree property (TP) if and only if it has the tree property of the first kind (TP_1) or the tree property of the second kind (TP_2) leads to two natural generalizations of the notion a of a simple theory (i.e., a theory without the tree property), namely NTP_1 and NTP_2 theories.

Initially, the NTP_1 property was studied mainly in the context of $\triangleleft^*$-maximality ([7,9]). Recently, several very interesting new results on NTP_1 have appeared in [4]. First, it has been proved there that the TP_1 property is always witnessed by a formula in a single variable, which gives chances to apply techniques from other contexts (e.g. stability, NIP, NTP_2) relying on analogous facts. Secondly, the authors studied a notion

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closely related to NTP_1, namely $NSOP_1$ (see for example [4, Definition 5.1])—this is a strengthening of NTP_1, which still holds in all simple theories. It is an open question whether it is equivalent to NTP_1.

Question 1.1 ([6]) *Is NTP_1 equivalent to $NSOP_1$?*

Chernikov and Ramsey characterized $NSOP_1$ in terms of some natural properties of certain ternary relations between small sets of parameters and also proved that existence of an abstract independence relation satisfying a certain list of natural axioms implies $NSOP_1$ (in particular, it implies NTP_1), which was used to provide natural examples of $NSOP_1$ theories: the ω-free PAC fields studied by Chatzidakis [9] and linear spaces with a generic bilinear form studied by Nicolas Granger in his Ph.D. thesis ([7]). Another class of such examples is provided by the "pfc" construction, thanks to the following fact from [4]:

Fact 1.2 *Suppose T is a simple theory which is the theory of a Fraïssé limit of a SAP Fraïssé class K. Then T_{pfc} is $NSOP_1$. Moreover, if the D-rank of T is at least 2, then T_{pfc} is not simple.*

The main motivation for this paper is proving the preservation of NTP_1 under certain constructions enriching a given theory, which yields new examples of NTP_1 theories from the already established ones.

We prove (in Theorem 3.1) the preservation of NTP_1 under the generic variations construction in the context considered in [1], which generalizes that of the pfc construction. This strengthens significantly [1, Theorem 4.4], where under a much stronger assumption of stability it was proved that the resulting theory is NSOP (which is a weaker property than NTP_1).

The paper is organized as follows: In Sect. 2, we review some definitions and outline the generic variations construction from [1] as well as the pfc construction from [4]. We notice that the theory obtained in the second construction is interpretable in the theory obtained in the first one. In Sect. 3, we prove the above-mentioned Theorem 3.1 (and conclude that NTP_1 is preserved under pfc). In Sect. 4, we prove that NTP_1 is preserved by the generic predicate construction from [2].

2 Preliminaries

Throughout the paper, unless stated otherwise, variables and parameters can have arbitrary (finite) length.

First, we give the definitions of NTP_1 and NTP_2.

Definition 2.1 A formula $\phi(x; y)$ has TP_1 (in a fixed theory T) if there is a collection of tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ such that:

(1) For all $\eta \in \omega^\omega$, the set $\{\phi(x; a_{\eta|n}) : n < \omega\}$ is consistent,
(2) If $\eta, \nu \in \omega^{<\omega}$ are incomparable (with respect to inclusion), then $\phi(x; a_\eta) \wedge \phi(x; a_\nu)$ is inconsistent.

A formula $\varphi(x, y)$ has TP_2 if there is $k < \omega$ and an array $\{a_{i,j} \mid i, j < \omega\}$ such that $\{\varphi(x, a_{i,f(i)}) \mid i < \omega\}$ is consistent for every function $f : \omega \to \omega$ and $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k-inconsistent for every $i < \omega$.
A theory T has TP_1 [TP_2] if some formula does. Otherwise, we say that T has NTP_1 [NTP_2].

It was observed in [4] that the TP_1 property of a formula is equivalent to the following SOP_2 property.

Definition 2.2 A formula $\phi(x; y)$ is said to have SOP_2 if, working in the monster model, there are tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following two properties:

1. For every $\xi \in 2^\omega$, the set $\{\phi(x, a_{\xi|n}) : n < \omega\}$ is consistent;
2. For every pair of incomparable elements $\eta, \nu \in 2^{<\omega}$, the formula $\phi(x; a_\eta) \wedge \phi(x; a_\nu)$ is inconsistent.

And a theory has SOP_2 if some formula has it.

Fact 2.3 ([4]) *A theory T has TP_1 if and only if there is some formula $\phi(x, y)$ with $|x| = 1$ witnessing this.*

Now, let us recall the modeling property on strongly indiscernible trees, which we will use repeatedly in the paper. Let L_0 be the language consisting of two binary relation symbols and a binary function symbol, which we interpret in subtrees of $\omega^{<\omega}$ as the inclusion relation, the lexicographic order and the function sending (η, ν) to $inf(\eta, \nu) = \eta \cap \nu$, respectively.

Definition 2.4 For any $S \subseteq \omega^{<\omega}$, we say that a tree $(a_\eta)_{\eta \in S}$ of compatible tuples of elements of a model M is strongly indiscernible over a set $C \subseteq M$, if

$$qftp_{L_0}(\eta_0, \ldots, \eta_{n-1}) = qftp_{L_0}(\nu_0, \ldots, \nu_{n-1})$$

implies $tp(a_{\eta_0}, \ldots, a_{\eta_{n-1}}/C) = tp(a_{\nu_0}, \ldots, a_{\nu_{n-1}}/C)$ for all $n < \omega$ and all tuples $(\eta_0, \ldots, \eta_{n-1})$, $(\nu_0, \ldots, \nu_{n-1})$ of elements of S.

The following fact comes from [10].

Fact 2.5 *Let $\mathfrak{C}$ be a monster model of a complete theory. Then for any tree of parameters $(a_\eta)_{\eta \in \omega^{<\omega}}$ from $\mathfrak{C}$ there is a strongly indiscernible tree $(b_\eta)_{\eta \in \omega^{<\omega}}$ locally based on the tree $(a_\eta)_{\eta \in \omega^{<\omega}}$, which means that for every finite set of formulas Δ and $\eta_0, \ldots, \eta_{n-1} \in \omega^{<\omega}$, there are $\mu_0, \ldots, \mu_{n-1} \in \omega^{<\omega}$ such that $qftp_{L_0}(\eta_0, \ldots, \eta_{n-1}) = qftp_{L_0}(\mu_0, \ldots, \mu_{n-1})$ and $tp_\Delta(b_{\eta_0}, \ldots, b_{\eta_{n-1}}) = tp_\Delta(a_{\mu_0}, \ldots, a_{\mu_{n-1}})$.*

The following observation comes from [5].

Fact 2.6 *A theory T has TP_1 if and only if there is a formula $\phi(x, y)$ and a strongly indicernible tree $(a_\eta)_{\eta \in 2^{<\omega}}$, such that $\{\phi(x, a_{0^n}) : n < \omega\}$ has infinitely many realizations, and the formula $\phi(x, a_0) \wedge \phi(x, a_1)$ has finitely many realizations.*

Also, by compactness, one easily gets:

Remark 2.7 If $(a_\eta)_{\eta\in\omega^{<\omega}}$ is a tree witnessing TP_1 of $\phi(x, y)$, then for any $\nu \in \omega^\omega$ the type $\{\phi(x, a_{\nu|_n}) : n < \omega\}$ is non-algebraic. The same holds for a tree $(a_\eta)_{\eta\in 2^{<\omega}}$ witnessing TP_1 of $\phi(x, y)$ and $\nu \in 2^\omega$.

Next, we outline the "generic variations" construction from [1].
We write $M \models \exists^\infty x_1, \dots, x_n\phi(x_1, \dots, x_n, b)$ (where x_i's are single variables) if there are infinitely many pairwise disjoint n-tuples in M satisfying the formula $\phi(x_1, \dots, x_n, b)$.

Definition 2.8 T eliminates the quantifier $\exists^\infty x_1, \dots, x_n$ if for every formula $\phi(x_1, \dots, x_n, y)$ (where x_i's are single variables) there is a formula $\psi_\phi(y)$ such that for every model M of T and every $b \in M$, we have

$$M \models \exists^\infty x_1, \dots, x_n\phi(x_1, \dots, x_n, b) \iff M \models \psi_{\phi(b)}.$$

Fact 2.9 (Winkler) *If T eliminates the quantifier $\exists^\infty x_1$ (where x_1 is a single variable) then T eliminates the quantifiers $\exists x_1, \dots, x_n$ for all n.*

For any language L, a new language L^+ is obtained by replacing each n-ary symbol $R(x_1, \dots, x_n)$, $f(x_1, \dots, x_n)$, or c (if $n = 0$) by an $(n + 1)$-ary symbol $R^+(x_0, x_1, \dots, x_n)$, $f^+(x_0, x_1, \dots, x_n)$, or $c^+(x_0)$, respectively. In the same manner, one defines an L^+-formula $\phi^+(x_0, x_1, \dots, x_n)$ for each L-formula $\phi(x_1, \dots, x_n)$.

For an L^+-structure M and $a \in M$, we denote by M^a the L-structure with universe M such that for each L-formula $\phi(x_1, \dots, x_n)$ we have $\phi((M^a)^n) = \phi^+(a, M^n)$. Let T^+ be the theory of all L^+-structures M such that for each $a \in M$, $M^a \models T$.

Let T be a model-complete theory in a language L eliminating the quantifier $\exists^\infty$.

Definition 2.10 ([1]) A suitable tuple is a sequence $(n, n+m, \phi_0(x, y), \dots, \phi_{n+m-1}(x, y))$, such that:

- for each $i < n + m$, the formula $\phi_i(x, y)$ is a conjuction of atomic formulas and negated atomic formulas in L, where

$$x = (x_0, \dots, x_{m-1}), \quad y = (y_0, \dots, y_{n-1}).$$

- for $i < n + m$,

$$T \models \phi_i(x, y) \to \bigwedge_{r<s<m} x_r \neq x_s \wedge \bigwedge_{r<s<n} y_r \neq y_s \wedge \bigwedge_{r<n, s<m} y_r \neq x_s.$$

- for $i < m$, $T \cup \exists x\exists y\phi_{n+i}(x, y)$ is consistent.

Then, we define:

$$T^1 = T^+ \cup \left\{\forall\, y \left(\bigwedge_{j<n} (\psi_{\phi_j})^+ (y_j, y)\right.\right.$$

$$\rightarrow \exists x \left(\bigwedge_{j<n} \phi_j^+(y_j, x, y) \wedge \bigwedge_{i<m} \phi_{n+i}^+(x_i, x, y) \right) \Bigg) :$$
$$(n, n+m, \phi_0(x, y), \ldots, \phi_{n+m-1}(x, y)) - suitable\} .$$

Fact 2.11 ([1]) *Let T be a model-complete theory eliminating the quantifier $\exists^\infty$. Then T^1 is consistent and is a model-companion of T^+.*

Fact 2.12 ([1]) *Assume T is model-complete and eliminates the quantifier $\exists^\infty$. Then T^1 is complete.*

Finally, we will outline the "pfc" construction from Subsection 6.3 of [4]. For the reader's convenience, we repeat the definitions used there.

Definition 2.13 Suppose K is a class of finite structures. We say K has the strong amalgamation property (SAP) if given $A, B, C \in K$ and embeddings $e : A \to B$ and $f : A \to C$ there is $D \in K$ and embeddings $g : B \to D$ and $h : C \to D$ such that

(1) $ge = hf$, and
(2) $im(g) \cap im(h) = im(ge)$ (and hence $= im(hf)$ as well).

We will say that a theory is SAP if it has a countable ultrahomogeneous model whose age (i.e. the family of all finite substructures) is SAP. The following criterion comes from [8].

Fact 2.14 *Suppose K is the age of a countable structure M. Then, the following are equivalent:*

(1) *K has SAP.*
(2) *M has no algebraicity.*

Let K denote an SAP Fraïssé class in a finite relational language $\mathcal{L} = (R^i : i < k)$, where each R^i has arity n_i. Denote by T the theory of the Fraïssé limit of the class K. The language $\mathcal{L}_{pfc}$ is defined to be a two-sorted language, with the sorts denoted by O ("objects") and P ("parameters") and relation symbols $R_x^i(y_1, y_2, \ldots, y_{n_i})$ (of arity $n_i + 1$) where x is a variable of the sort P and y_i's are variables of the sort O. Given an $\mathcal{L}_{pfc}$-structure $M = (A, B)$ and $b \in B$, the $\mathcal{L}$-structure associated to b in M, denoted by A_b, is defined to be the $\mathcal{L}$-structure interpreted in M with domain A and each R^i interpreted as $R_b^i(A)$. Put

$$K_{pfc} = \{M = (A, B) \in Mod(L_{pfc}) : |M| < \omega, (\forall\, b \in B)(\exists D \in K)(A_b \simeq D)\}.$$

Fact 2.15 ([4]) *K_{pfc} is a Fraïssé class satisfying SAP.*

Thanks to the above fact, there is a unique countable ultrahomogeneous $\mathcal{L}_{pfc}$-structure with age K_{pfc}—the Fraïssé limit of K_{pfc}.

Definition 2.16 By T_{pfc} we denote the theory of the Fraïssé limit of the class K_{pfc} constructed above.

Note that T_{pfc} has quantifier elimination.

Proposition 2.17 *Let T be the theory of the Fraïssé limit of a SAP Fraïssé class in a finite relational language. Then the countable model M_{pfc} of T_{pfc} is interpretable in a model of T^1. (Since T has q.e. and, by ω-categoricity, it eliminates $\exists^\infty$, one can apply the generic variations construction to T.)*

Proof First, we prove the following:

Claim 1 *Let $M_{pfc} = (A, B)$ be the Fraïssé limit of K_{pfc}. Then, there is a bijection $f : A \to B$ for which the L^1 structure N with universe equal to A such that $(\forall\, a \in A)(N^a = A_{f(a)})$ is a model of T^1.*

Proof of Claim 1 Choose enumerations $A = \{a_k : k < \omega\}$ and $B = \{b_k : k < \omega\}$. Also, let $\{\Delta_k : k < \omega\}$ be the set of all suitable tuples for T. We will construct inductively injections $f_k : A_k \to B_k$ such that A_k and B_k are finite subsets of A and B, respectively, $a_k \in A_{k+1}$, $b_k \in B_{k+1}$, and for each suitable tuple

$$\Delta_s = (n, n+m, \phi_0(x, y), \dots, \phi_{n+m-1}(x, y)),$$

where $s < k$, and each tuple $d = (d_0, \dots, d_{n-1})$ of elements of A_k such that for each $j < n$ the formula $\phi_j(x, d)$ has infinitely many pairwise disjoint realizations in $A_{f_k(d_j)}$, there is a tuple $c = (c_0, \dots, c_{m-1})$ of elements of A_{k+1} such that for each $i < m$ and $j < n$, $A_{f_{k+1}(c_i)} \models \phi_{n+i}(c, d)$ and $A_{f_k(d_j)} \models \phi_j(c, d)$. The construction starts by choosing $f_0 = A_0 = B_0 = \emptyset$.

Suppose we have already defined $f_k : A_k \to B_k$. Consider any $\Delta_s = (n, n + m, \phi_0(x, y), \dots, \phi_{n+m-1}(x, y))$, where $s < k$, and a tuple $d = (d_0, \dots, d_{n-1})$ of elements of A_k such that for each $j < n$ the formula $\phi_j(x, d)$ has infinitely many pairwise disjoint realizations in $A_{f_k(d_j)}$. Let $e = (e_0, \dots, e_{m-1})$ be a tuple of pairwise distinct "new" elements not belonging to A, and put $C = A_k e$. Since, for each $j < n$, the formula $\phi_j(x, d)$ has a realization in $A_{f(d_j)}$ disjoint from A_k, one can put an L_{pfc} structure on (C, B_k) in such a way that $C_{f_k(d_j)} \models \phi_j(e, d)$ for each $j < n$, and (C, B_k) belongs to K_{pfc}. Then, we can find its isomorphic over (A_k, B_k) copy $(C', B_k) = (A_k e', B_k)$ being a substructure of M. Now, choose a tuple of new (not belonging to B) elements $p = (p_0, \dots, p_{m-1})$, put $D = B_k p$ and define an L_{pfc}-structure on on (C', D) extending the structure on (C', B_k) (inherited from M) in such a way that for each $i < m$, $C'_{p_i} \models \phi_{n+i}(e, d)$ (we use here the assumption that $\phi_{n+i}(x, y) \cup T$ is consistent). Since $(C', B_k p) \in K_{pfc}$, we can find its isomorphic over (C', B_k) copy $(C', B_k p')$ being a substructure of M. Extend f_k to a function $f'_k : C' \to B_k p'$ by putting $f'_k(e'_j) = p'_j$. Repeating this for all (finitely many) tuples as above (with d contained in A_k and $s < k$) and, finally, extending the obtained function to a finite injection f_{k+1} such that $a_k \in dom(f_{k+1})$ and $b_k \in rng(f_{k+1})$, we complete the inductive construction.

Now, $f := \bigcup_{k<\omega} f_k$ clearly satisfies the conclusion of Claim 1. □

Let f and N be as in the claim. Then the L_{pfc}-structure $C := (A, A)$ in which $R_b(a)$ holds iff $N \models R^+(a, b)$ is interpretable in N, and, by the definition of N, we have:

$$C \models R_b(a) \iff M_{pfc} \models R_{f(b)}(a).$$

Hence, C is isomorphic to M_{pfc} via $id \times f$. This shows that M_{pfc} is interpretable in N. $\square$

3 Preservation of NTP_1 under the generic variation

The main goal of this section is to prove the Theorem 3.1 below, which strengthens [1, Theorem 4.4]. By Corollary 3.6 from [1], the assumptions are satisfied by any theory having quantifier elimination in which $acl(A) = A$ for all A. (See also Example 3.3 and the discussion in the paragraph preceding it.)

Theorem 3.1 *Suppose that T and T^1 both have elimination of quantifiers and T eliminates the quantifier $\exists^\infty$. If T has NTP_1, then so does T^1.*

Proof As TP_1 is equivalent to SOP_2, we will suppose for a contradiction that T^1 has SOP_2 witnessed by a formula $\psi(x, y)$ with $|x| = 1$ (which we can assume by Fact 2.3) and a strongly indiscernible tree of parameters $(a^\eta)_{\eta \in 2^{<\omega}}$ (which we can assume by Fact 2.5). By q.e., we can additionally assume that $\psi(x, y) = \vee_{i<M}\theta_i(x, y)$, where each $\theta_i(x, y)$ is a conjunction of atomic formulas and negations of atomic formulas. Now, by the pigeonhole principle, there is $i < M$ such that there are infinitely many formulas among $\psi(x, a^{0^n}), n < \omega$ with a common realization. By considering an appropriate subtree of $(a^\eta)_{\eta \in 2^{<\omega}}$, we can assume that $\psi(x, y) = \theta_i(x, y)$. Hence, $\psi(x, y)$ is of a form $\bigwedge_{i<n} \phi_i^+(y_i; x, y) \wedge \phi_n^+(x; x, y)$, where ϕ_i's are conjunctions of atomic formulas and negations of atomic formulas, and $y = (y_0, \dots, y_{n-1})$ with $|y_i| = 1$.

Notice that, by strong indiscernibility, for each $i < n$ either a_i^η are equal for all $\eta \in 2^{<\omega}$ or they are pairwise distinct. We can assume that there is some $l < n$ such that:

$$(\forall\, \eta, \nu \in 2^{<\omega})((\forall\, i < l)(a_i^\eta = a_i^\nu) \wedge (\forall\, i \geq l)(\eta \neq \nu \implies a_i^\eta \neq a_i^\nu)). \qquad (*)$$

Put $y' = (y'_l, \dots, y'_{n-1})$, where y'_i's are single variables. We aim to get a contradiction by showing that the formula $\psi(x, a^0) \wedge \psi(x, a^1)$ is consistent. This will follow from the following claim:

Claim 1 *Let $\alpha(x, y, y')$ be a formula expressing that the coordinates of y, y' and x are all pairwise distinct. Then*

(1) $(2n - l, 2n - l + 1, \phi_0(x, y) \wedge \phi_0(x, y_{<l}y') \wedge \alpha(x, y, y'), \dots, \phi_{l-1}(x, y) \wedge \phi_{l-1}(x, y_{<l}y') \wedge \alpha(x, y, y'), \phi_l(x, y) \wedge \alpha(x, y, y'), \phi_l(x, y_{<l}y') \wedge \alpha(x, y, y'), \phi_{l+1}(x, y) \wedge \alpha(x, y, y'), \phi_{l+1}(x, y_{<l}y') \wedge \alpha(x, y, y'), \dots, \phi_{n-1}(x, y) \wedge \alpha(x, y, y'), \phi_{n-1}(x, y_{<l}y') \wedge \alpha(x, y, y'), \phi_n(x, y) \wedge \phi_n(x, y_{<l}y') \wedge \alpha(x, y, y'))$ *is a suitable tuple of formulas in variables (x, yy'),*
(2) *For each $i < l$, the formula $\phi_i(x, a^0) \wedge \phi_i(x, a^1) \wedge \alpha(x, a^0, a_l^1, \dots, a_{n-1}^1)$ is non-algebraic in $\mathfrak{C}^{a_i^0}$, and*

(3) *For each* $i \in \{l, \dots, n-1\}$, *the formula* $\phi_i(x, a^0) \wedge \alpha(x, a^0, a^1_l, \dots, a^1_{n-1})$ *in non-algebraic in* $\mathfrak{C}^{a^0_i}$, *and the formula* $\phi_i(x, a^1) \wedge \alpha(x, a^0, a^1_l, \dots, a^1_{n-1})$ *in non-algebraic in* $\mathfrak{C}^{a^1_i}$.

Proof of Claim 1 (1) We only need to check that the formula $\phi_n(x, y) \wedge \phi_n(x, y_{<l}y') \wedge \alpha(x, y, y')$ is consistent with T. Since the formula $\psi(x, a^{\emptyset}) \wedge \psi(x, a^0)$ in non-algebraic (by Remark 2.7), we can find its realization c not occurring in $a^{\emptyset}$ and in a^0. In particular, c satisfies $\phi^+_n(x; x, a^{\emptyset}) \wedge \phi^+_n(x; x, a^0)$, so $\mathfrak{C}^c \models \phi_0(c, a^{\emptyset}) \wedge \phi_0(c, a^0)$. Also, since c does not occur in $a^{\emptyset}$ and in a^0, we get by $(*)$ that $\mathfrak{C}^c \models \alpha(c, a^{\emptyset}, a^0_l, \dots, a^0_{n-1})$. This suffices, as $\mathfrak{C}^c \models T$.

(2) Suppose $\phi_i(x, a^0) \wedge \phi_i(x, a^1) \wedge \alpha(x, a^0, a^1_l, \dots, a^1_{n-1})$ is algebraic for some $i < l$. Since a^{η}_i is the same for each $\eta \in 2^{<\omega}$, we see that the tree $(a^{\eta})_{\eta \in 2^{<\omega}}$ is strongly indiscernible over a^0_i in the sense of the model $\mathfrak{C}$ of T^1, so it is strongly indiscernible in the sense of the model $\mathfrak{C}^{a^0_i}$ of T. Since any realization of $\psi(x, a^{\eta})$ (in the sense of $\mathfrak{C}$) is a realization of $\phi_i(x, a^{\eta})$ in the sense of $\mathfrak{C}^{a^0_i}$, we get (by Remark 2.7) that $\phi_i(x, y)$ together with the tree $(a^{\eta})_{\eta \in 2^{<\omega}}$ (of tuples from $\mathfrak{C}^{a^0_i}$) satisfy the assumptions of Fact 2.6. A contradiction to the assumption that T has NTP$_1$.

(3) The assertion follows since the formulas $\psi(x, a^0)$ and $\psi(x, a^1)$ are non-algebraic (by Remark 2.7), and any realization of $\psi(x, a^k)$ is a realization of $\phi_i(x, a^k)$ in $\mathfrak{C}^{a^k_i}$ for $k = 0, 1$. □

By the above claim and the definition of T^1, we get in particular (forgetting about the formula α) that there is some c realizing formulas $\phi^+_0(a^0_0; x, a^0) \wedge \phi^+_0(a^1_0; x, a^1), \dots, \phi^+_{l-1}(a^0_{l-1}; x, a^0) \wedge \phi^+_{l-1}(a^1_{l-1}; x, a^1), \phi^+_l(a^0_l; x, a^0), \phi^+_l(a^1_l; x, a^1)$, $\phi^+_{l+1}(a^0_{l+1}; x, a^0), \phi^+_{l+1}(a^1_{l+1}; x, a^1), \dots, \phi^+_{n-1}(a^0_{n-1}; x, a^0), \phi^+_{n-1}(a^1_{n-1}; x, a^1)$ and the formula $\phi^+_n(x, a^0) \wedge \phi^+_n(x, a^1)$. Hence, c realizes $\psi(x, a^0) \wedge \psi(x, a^1)$, a contradiction. □

By Proposition 2.17, we get:

Corollary 3.2 *Assume* T *is as in Definition 2.16. If* T *is NTP*$_1$, *then so is* T_{pfc}.

Note that the class of theories T eliminating $\exists^{\infty}$ and such that both T and in T^1 have q.e. (i.e., theories as in the assumptions of Theorem 3.1) is strictly bigger than the class of theories to which one can apply the pfc construction:

Example 3.3 Let T be the theory of independent predicates, i.e. the theory in a language L consisting of unary predicates P_n, $n < \omega$, axiomatized by sentences $\exists x \bigwedge_{i \in I} P_i(x) \wedge \bigwedge_{i \in J} \neg P_i(x)$, I,J: finite disjoint subsets of ω. Then T is superstable, has q.e., and satisfies the condition $(\forall\ A) acl(A) = A$ (so it eliminates the quantifier $\exists^{\infty}$), hence also T^1 has q.e. by Corollary 3.6 from [1]. But T is not ω-categorical, so it cannot be obtained as a theory of the Fraïssé limit of a class of finite structures, so one cannot apply the pfc construction to T.

The following is Lemma 4.2 from [1]:

Fact 3.4 *Assume that* T *has quantifier elimination and eliminates* $\exists^{\infty}$. *If there is a formula* $\phi(x, y)$ *with* $|x| = 1$ *and tuples* $a_0, a_1, \dots$ *in a model* M *of* T *such that the sets* $\phi(M, a_i)$ *are pairwise disjoint and infinite, then* T^1 *is not simple.*

It is easy to see that in the above context T^1 actually has TP$_2$:

Remark 3.5 Let T be as in Fact 3.4. Then T^1 has TP$_2$.

Proof It is easy to see that, by q.e., we can find a formula $\phi(x, y)$ as above being a conjunction of atomic formulas and negations of atomic formulas. Put $\psi(x; yz) := \phi^+(z, x, y)$ (z is a single variable). Choose pairwise disctinct elements $b_0, b_1, \dots$ in a monster model of T^1 and, for each $i < \omega$, find $a_{i,0}, a_{i,1}, \dots$ such that the formulas $\phi(b_i; xa_{i,j})$ are pairwise inconsistent and non-algebraic (which we can do, since $\mathfrak{C}^{b_i} \models T$). Then the formula $\psi(x, yz) := \phi(z; x, y)$ has TP$_2$, witnessed by the array $(a_{i,j}, b_j)_{i,j<\omega}$ (the consistency condition follows by compactness from the axioms of T^1). □

4 Generic predicate

In this section, we show that NTP$_1$ is preserved by the generic predicate construction introduced in [2]. The idea of the proof is similar as in [3, Theorem 7.3], but we should point out that there seems to be a small gap in the proof from [3]. Namely, in the last paragraph of the proof, since one does not know whether a_{ij} has the same type over $acl(b)$ for various j's, one cannot conclude that there are colorings on $acl(a_{ij}b)$ agreeing on $acl(b)$ (for distinct j's) induced by sending $a_{i0}b_0$ to $a_{ij}b$ by an L-elementary map. One can, however, work with algebraically closed tuples, which we do below and which also yields a correct proof of [3, Theorem 7.3].

First, we outline the random predicate construction. Consider a theory T in a language L. For $S(x) \in L$, we let L_P denote the language obtained by adding to L a unary predicate $P(x)$ and we put $T^0_{P,S} = T \cup \{\forall\, x(P(x) \to S(x))\}$.

Fact 4.1 ([2]) *Let T be a theory eliminating quantifiers and eliminating the quantifier $\exists^\infty$. Then:*

(1) *$T^0_{P,S}$ has a model companion $T_{P,S}$ which is axiomatized by T together with*

$$\forall\, z\left[(\exists x)\left(\phi(x,z)\wedge(x\cap acl_L(z)=\emptyset)\wedge\bigwedge_{i<n}s(x_i)\wedge\bigwedge_{i\neq j<n}x_i\neq x_j\right)\right]$$
$$\to(\exists x)\left(\phi(x,z)\wedge\bigwedge_{i\in I}P(x_i)\wedge\bigwedge_{i\notin I}\neg P(x_i)\right),$$

where $x = (x_0, x_1, \dots, x_{n-1})$ and I ranges over all subsets of the set $\{0, 1, \dots, n-1\}$.

(2) *$acl_L(a) = acl_{L_P}(a)$*

(3) *$a \equiv^{L_P} b \iff$ there is an isomorphism between L_P-structures $f : \mathrm{acl}(a) \to acl(b)$ such that $f(a) = b$.*

(4) *Modulo $T_{P,S}$, every formula $\phi(x)$ is equivalent to a disjunction of formulas of the form $\exists z\phi(x, z)$, where $\phi(x, z)$ is a quantifier-free L_P-formula, and for any a, b, if $\models \phi(a, b)$, then $b \in acl(a)$.*

Proposition 4.2 *Suppose T is geometric (i.e., it eliminates $\exists^\infty$ and acl satisfies the exchange principle) and NTP$_1$. Then T_P is NTP$_1$.*

Proof By independence we shall mean the relation of algebraic independence (in particular, by geometricity, it is symmetric). Suppose for a contradiction that T_P has NTP$_1$ witnessed by a formula $\phi(x, y)$ with $|x| = 1$ and a strongly indiscernible tree of parameters $(a^\eta)_{\eta\in 2^{<\omega}}$.

Claim 1 *The set $A := \{a_\eta : \eta \in 2^{<\omega}\}$ is algebraically independent and disjoint from $acl(\emptyset)$.*

Proof of Claim 1 It is enough to show that for any $\eta \in 2^{<\omega}$, a_η is not in the algebraic closure of the set $B := \{a_\nu : \nu \in 2^{<\omega}, |\nu| \leq |\eta|, \nu \neq \eta\}$. But this follows from the fact that (by strong indiscernibility) for each $\mu \supseteq \eta$, we have that $tp_{L_P}(a_\mu/B) = tp_{L_P}(a_\eta/B)$. □

Since (by Remark 2.7) the type $\bigcup_{i<\omega}\{\phi(x, a_{0^i}) : i < \omega\}$ is not algebraic, we can choose an A-indiscernible sequence $c_0, c_1, \ldots$ such that each c_i is an enumeration of the algebraic closure of some b_i realizing the above type and the b_i's are pairwise distinct. Put

$$p(z, a_0) = tp_L(c_0/a_0)$$

(where z is possibly infinite and extends x) and

$$p_N(z_0, \ldots, z_{N-1}, a_0) = tp_L(c_0, \ldots, c_{N-1}/a_0)$$

for any $N < \omega$ (so $p(z, a_0) = p_1(z, a_0)$). Since T is NTP$_1$, $\cup_{i<\omega} p_N(z_0, \ldots, z_{N-1}, a_{0^i})$ is consistent, and $(a^\eta)_{\eta\in 2^{<\omega}}$ is strongly indiscernible, we get that $p_N(z_0, \ldots, z_{N-1}, a_0) \cup p_N(z_0, \ldots, z_{N-1}, a_1)$ is consistent for any $N < \omega$. Because $p_N(z_0, \ldots, z_{N-1}, a_0) \vdash (\bigwedge_{i\neq j} x_i \neq x_j) \wedge \bigwedge_{i<N} p(z_i, a_0)$, we get that $p(z, a_0) \cup p(z, a_1)$ has a realization c such that the coordinate b of c corresponding to x is not in $acl(A)$. Note that c is an enumeration of $acl(b)$. Since b, a_0, a_1 is an independent triple, we get that $acl(ba_0) \cap acl(a_0a_1) = acl(a_0)$ and $acl(ba_1) \cap acl(a_0a_1) = acl(a_1)$, so by the definition of T_P we can arbitrarily choose which of the elements of $acl(ba_0)\backslash acl(a_0)$ and $acl(ba_1)\backslash acl(a_1)$ are in P. Since $acl(ba_0) \cap acl(ba_1) = acl(b) = c$ and $tp_L(ca_0) = tp_L(ca_1)$, we can do this in such a way that $acl(a_0b_0) \cong acl(a_0b) \cong acl(a_1b)$ as L_P-structures. Then, by Fact 4.1(3), we get that $tp_{L_P}(a_0b_0) = tp_{L_P}(a_0b) = tp_{L_P}(a_1b)$. In particular, $\models \phi(b, a_0) \wedge \phi(b, a_1)$, a contradiction. □

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