On ω-categorical, generically stable groups and rings

Jan Dobrowolski and Krzysztof Krupiński*

Abstract

We prove that every ω-categorical, generically stable group is nilpotent-by-finite and that every ω-categorical, generically stable ring is nilpotent-by-finite.

0 Introduction

A general motivation is to understand the structure of ω-categorical groups and rings satisfying various natural model-theoretic assumptions.

There is a long history of results of this kind. The fundamental theorem proved by Baur, Cherlin and Macintyre in [3] and by Felgner in [8] says that ω-categorical, stable groups are nilpotent-by-finite. A long-standing conjecture states that they are even abelian-by-finite, which is known to be true in the superstable case. As to the ω-categorical, stable rings, they are nilpotent-by-finite [2], and it is conjectured that they are null-by-finite. As for groups, this conjecture is known to be true in the superstable case. There are many generalizations and variants of these results. For example, ω-categorical groups with NSOP (the negation of the strict order property) are nilpotent-by-finite [15], and ω-categorical rings with NSOP are nilpotent-by-finite [13], too. It is also known that ω-categorical, supersimple groups are finite-by-abelian-by-finite [7], and ω-categorical, supersimple rings are finite-by-null-by-finite [14].

More recently, in [12], an analysis of ω-categorical groups and rings in the NIP environment has been undertaken. It was proved there that ω-categorical rings with NIP are nilpotent-by-finite, and it was conjectured that ω-categorical groups with NIP are nilpotent-by-finite, too. The conjecture was confirmed, but under the additional assumption of fsg (finitely satisfiable generics). It turns out that ω-categorical groups with NIP and fsg are generically stable in the sense of [10, Definition 6.3]. This notion fits very well with the recent trend in model theory of studying structures some of whose 'pieces' are similar to stable ones. So, one can ask

*Research supported by the Polish Government grant N N201 545938

[0]2010 Mathematics Subject Classification: 03C45, 03C35, 03C60, 20A15, 16B70.

[0]Key words and phrases: ω-categorical group, ω-categorical ring, generic stability

about the structure of ω-categorical, generically stable groups and rings, without the NIP assumption. The following theorem was proved in [5].

Fact 0.1 *Every ω-categorical, generically stable group is solvable-by-finite.*

In this paper, using the above theorem, we obtain the following two results (to be more precise, in the proof of Theorem 2, we will use an appropriate variant of Fact 0.1).

Theorem 1 *Every ω-categorical, generically stable group is nilpotent-by-finite.*

Theorem 2 *Every ω-categorical, generically stable ring is nilpotent-by-finite.*

Theorem 1 proves [5, Conjecture 3.5], and Theorem 2 answers [5, Question 3.6] in the affirmative.

1 Preliminaries

Recall that a first order structure M in a countable language is said to be ω-categorical if, up to isomorphism, $\mathrm{Th}(M)$ has at most one model of cardinality $\aleph_0$. By Ryll-Nardzewski's theorem, this is equivalent to the condition that for every natural number n there are only finitely many n-types over $\emptyset$. Assume M is ω-categorical. If M is countable or a monster model (i.e. a model which is κ-saturated and strongly κ-homogeneous for a big cardinal κ), two finite tuples have the same type over $\emptyset$ iff they lie in the same orbit under the action of the automorphism group of M, and hence for each natural number n the automorphism group of M has only finitely many orbits on n-tuples (which implies that M is locally finite). Moreover, for any finite subset A of such an M, a subset D of M is A-invariant iff D is A-definable.

Let T be a first order theory. We work in a monster model $\mathfrak{C}$ of T.

Let $p \in S(\mathfrak{C})$ be invariant over $A \subset \mathfrak{C}$. We say that $(a_i)_{i\in\omega}$ is a Morley sequence in p over A if $a_i \models p|Aa_{<i}$ for all i. Morley sequences in p over A are indiscernible over A and they have the same order type over A. If $\mathfrak{C}' \succ \mathfrak{C}$ is a bigger monster model, then the generalized defining scheme of p determines a unique A-invariant extension $\widetilde{p} \in S(\mathfrak{C}')$ of p (by the generalized defining scheme of p we mean a family of sets $\{p_i^\varphi : i \in I_\varphi\}$ (with $\varphi(x, y)$ ranging over all formulas without parameters) of complete types over A such that $\varphi(x, c) \in p$ iff $c \in \bigcup_{i\in I_\varphi} p_i^\varphi(\mathfrak{C})$). By a Morley sequence in p we mean a Morley sequence in $\widetilde{p}$ over $\mathfrak{C}$. Finally, $p^{(k)}$ (where $k \in \omega\cup\{\omega\}$) denotes the type over $\mathfrak{C}$ of a Morley sequence in p of length k.

Definition 1.1 *For a small $A \subset \mathfrak{C}$, a global type $p \in S(\mathfrak{C})$ is said to be generically stable over A if it is A-invariant and for each formula $\varphi(x; y)$ there is a natural number m such that for any Morley sequence $(a_i : i < \omega)$ in p over A and any b from $\mathfrak{C}$ either less than m a_i's satisfy $\varphi(b; y)$ or less than m a_i's satisfy $\neg\varphi(b; y)$. A global type $p \in S(\mathfrak{C})$ is said to be generically stable if it is generically stable over some small $A \subset \mathfrak{C}$.*

Let p be a global type invariant over A. Since the type over A of a Morley sequence in p over A does not depend on the choice of this Morley sequence, one can easily check that a generically stable type is in fact generically stable over any small set of parameters over which it is invariant.

Very recently, Adler, Casanovas and Pillay have found an example of an ω-categorical theory and a generically stable type p for which $p^{(2)}$ is not generically stable (see [1, Example 1.7]). This shows that we cannot omit certain extra arguments used in this paper which deal with the case when some powers of a generically stable type are not generically stable.

Recall [16, Proposition 2.1].

Fact 1.2 *If p is generically stable over A, then any Morley sequence in p over A is an indiscernible set over A. In particular, a Morley sequence in p (over $\mathfrak{C}$) is an indiscernible set over $\mathfrak{C}$.*

The following observation was made in [5, Proposition 1.2].

Fact 1.3 *Let $p = tp(a/\mathfrak{C})$ be a type generically stable over A, and assume that $b \in dcl(A, a)$. Then $tp(b/\mathfrak{C})$ is also generically stable over A.*

The next lemma is a variant of a similar result for NIP groups (see [17, Theorem 1.0.5]), and its proof is very similar to the proof of [17, Theorem 1.0.5]. This is also a slight modification of [5, Lemma 2.1(i)]; in fact, it easily implies [5, Lemma 2.1(i)].

Lemma 1.4 *Let G be a group which is $\emptyset$-definable in $\mathfrak{C}$ by a formula $G(x)$. Assume that $p \in S(\mathfrak{C})$ is generically stable over A. Let $H(x, \bar{z}; y)$ be a formula defining a family of groups $H(G, c; g)$, $g \in (p|A)(\mathfrak{C})$, $c \in D$ (D is a definable set). Then there is $N < \omega$ such that for any $c \in D$, $n \in \omega$ and $(g_1, \dots, g_n) \models p^{(n)}|A$ there are $i_1, \dots, i_N \in \{1, \dots, n\}$ for which*

$$\bigcap_{i=1}^{n} H(G, c; g_i) = \bigcap_{j=1}^{N} H(G, c; g_{i_j}).$$

Proof. Let $m > 0$ be such as in the definition of generic stability for p and $H(x, \bar{z}; y)$. We will show that $N := 2m$ satisfies our requirements. Suppose it is not the case. Then there is $n > N$ (even $n = N + 1$ works) such that for some $c \in D$ and $(g_1, \dots, g_n) \models p^{(n)}|A$ the intersection $\bigcap_{i=1}^{n} H(G, c; g_i)$ is not an intersection of at most $n - 1$ groups among $H(G, c; g_i)$, $i = 1, \dots, n$. Hence, for each $j \in \{1, \dots, n\}$ there exists

$$a_j \in \bigcap_{i \neq j} H(G, c; g_i) \setminus \bigcap_{i=1}^{n} H(G, c; g_i).$$

Put $b = \prod_{j=1}^{m} a_j$. We see that

$$b \in H(G, c; g_i) \iff i \in \{m + 1, \dots, n\},$$

which contradicts the choice of m. ■

Recall that a subset of a group G is said to be left generic if finitely many left translates of this set cover G. Assuming that the group G is definable in a model M, a formula $\varphi(x)$ is called left generic if the set $\varphi(G)$ is left generic, and a type is said to be left generic (of G or in G) if every formula in it is left generic. The definitions of right generic sets, formulas and types are analogous. Finally, a subset, formula or type is called generic if it is left generic and right generic.

Definition 1.5 *Let G be a group definable in $\mathfrak{C}$ by a formula $G(x)$. G has finitely satisfiable generics (fsg) if there is a global type p containing $G(x)$ and a model $M \prec \mathfrak{C}$, of cardinality less than the degree of saturation of $\mathfrak{C}$, such that for all $g \in G$, gp is finitely satisfiable in M.*

Let G be a group $\emptyset$-definable in $\mathfrak{C}$. By G^{00} we will denote the smallest type-definable subgroup of bounded index (if it exists). We do not know whether G^{00} always exists when T is ω-categorical. Notice, however, that if it exists, then, being $\emptyset$-invariant, it must be $\emptyset$-definable and of finite index in G.

The following fact was proved in [11, Section 4].

Fact 1.6 *Suppose G has fsg, witnessed by p. Then:*
(i) a formula is left generic iff it is right generic (iff it is generic),
(ii) p is generic,
(iii) the family of non-generic sets forms an ideal, so any partial generic type can be extended to a global generic type,
(iv) G^{00} exists, it is type-definable over $\emptyset$, and it is the stabilizer of any global generic type of G.

Recall [6, Proposition 0.26].

Fact 1.7 *Suppose G has fsg and G^{00} is definable. Then G^{00} has a unique global generic type.*

The next definition was introduced in [10, Section 6].

Definition 1.8 *(i) Let G be a group definable in $\mathfrak{C}$. We say that G is generically stable if it has fsg and some global generic type of G is generically stable.*
(ii) Let R be a ring definable in $\mathfrak{C}$. We say that R is generically stable if its additive group is generically stable.

We say that a group [or ring] definable in a non-saturated model is generically stable if the group [or ring] defined by the same formula in a monster model is such.

When we are talking about an ω-categorical, generically stable group [or ring], we mean a generically stable group G [or ring R] definable in a monster model $\mathfrak{C}$ of an ω-categorical theory. Replacing $\mathfrak{C}$ by G [or by R] equipped with the structure induced from $\mathfrak{C}$, neither ω-categoricity nor generic stability is lost. So, whenever we want to prove some algebraic properties of G [or R], we can assume that $\mathfrak{C} = G$ [or $\mathfrak{C} = R$] (possibly with some extra structure).

We say that G is connected if it does not have a proper, definable subgroup of finite index, and we will say that G is absolutely connected if it does not have a proper, type-definable subgroup of bounded index (i.e. $G = G^{00}$).

Recall some basic notions from ring theory. In this paper, rings are associative, but they are not assumed to contain 1 or to be commutative. An element r of a ring R is nilpotent of nilexponent n if $r^n = 0$ and n is the smallest number with this property. The ring is nil [of nilexponent n] if every element is nilpotent [of nilexponent $\leq n$ and there is an element of nilexponent n]. The ring is nilpotent of class n if $r_1 \cdots r_n = 0$ for all $r_1, \ldots, r_n \in R$ and n is the smallest number with this property. An element r is null if $rR = Rr = \{0\}$. The ring is null if all its elements are.

The Jacobson radical of a ring R, denoted by $J(R)$, is the collection of all elements of R satisfying the formula $\phi(x) = \forall y \exists z (yx + z + zyx = 0)$ (that is, it is the set of all elements which generate quasi-regular left ideals.). Equivalently, $J(R)$ is the intersection of all the maximal regular left [or right] ideals, where a left ideal I is said to be regular if there is $a \in R$ such that $x - xa \in I$ for all $x \in R$ (notice that for rings with 1 all ideals are regular). For any ring R, $J(R)$ is a two-sided ideal. We say that R is semisimple if $J(R) = \{0\}$. $R/J(R)$ is always a semisimple ring. For details on Jacobson radical see [9, Chapter 1].

Recall that a ring R is a subdirect product of rings $R_i, i \in I$, if there is a monomorphism of R into $\prod_{i \in I} R_i$ whose image projects onto each R_i. The following fact is [2, Corollary 1].

Fact 1.9 *If R is a semisimple, ω-categorical ring, then R is a subdirect product of complete matrix rings over finite fields. Moreover, only finitely many different matrix rings occur as subdirect factors.*

By [2, Lemma 1.3] and [4] we have:

Fact 1.10 *If R is an ω-categorical ring, then $J(R)$ is nilpotent.*

2 ω-categorical, generically stable rings

This section is devoted to the proof of Theorem 2 from the introduction. After a reduction to the situation when there is a unique global generic type, our proof splits into two cases depending on whether the generic type has non-nilpotent or nilpotent realizations. If they are non-nilpotent, the proof is a slight elaboration of the proof of [12, Theorem 2.1], which is based on Facts 1.9 and 1.10. For the reader's convenience, we include most of the details. The argument in the nilpotent case is completely different; in particular, it uses a variant of Fact 0.1 and some ideas from the proof of [13, Theorem 2.1(i)]. It will be noted in the course of the proof that if the ring in question is commutative, then it is enough to consider the non-nilpotent case.

Since [1, Example 1.7] shows that the generic stability of a type p does not imply the generic stability of all its powers $p^{(n)}$, $n \geq 1$, we will see that Fact 0.1 is too weak to complete the proof of Theorem 2 in the nilpotent case. Actually, we have to use a certain variant (in fact, strengthening) of Fact 0.1. Literally, it will be a strengthening of [5, Theorem 2.3], obtained by the same proof as in [5] modulo obvious modifications and applications of an appropriate strengthening of [5, Lemma 2.2] which is described below.

Lemma 2.1 *Let G be a $\emptyset$-definable group in a monster model $\mathfrak{C}$ of an ω-categorical theory. Assume that $G_1 \trianglelefteq G$ is infinite, $\emptyset$-definable, and characteristically simple in $(G, \mathfrak{C})$, i.e. it has no non-trivial, proper subgroup which is invariant under conjugations by the elements of G and invariant under $Aut(\mathfrak{C})$. Let $p \in S(\mathfrak{C})$ be a type generically stable over $\emptyset$. Suppose that for some $\emptyset$-definable function f and Morley sequence $(g_i)_{i<\omega}$ in p over $\emptyset$, $f(g_0, \dots, g_{k-1}) \in G_1 \setminus \{e\}$. Assume additionally that whenever $(h_i)_{i<k}$ is a Morley sequence in p over some $g \in G$, then the conjugate $f(h_0, \dots, h_{k-1})^g$ equals $f(h'_0, \dots, h'_{k-1})$ for some Morley sequence $(h'_i)_{i<k}$ in p over g.*

Then G_1 is abelian.

Sketch of proof. Define

$$H = \bigcap_{i_1 < \cdots < i_k} C_{G_1}(f(g_{i_1}, \dots, g_{i_k})).$$

It was shown in the course of the proof of [5, Lemma 2.2] that H is invariant under $Aut(\mathfrak{C})$ (this follows from the generic stability of p over $\emptyset$).

Now, we will show that H is normal in G. Consider any $g \in G$. Choose a Morley sequence $(h_i)_{i<\omega}$ in p over g. By the invariance of H under $Aut(\mathfrak{C})$ and the uniqueness of a Morley sequence in p over $\emptyset$ up to the type, we have that

$$H = \bigcap_{i_1 < \cdots < i_k} C_{G_1}(f(h_{i_1}, \dots, h_{i_k})).$$

Thus,

$$H^g = \bigcap_{i_1 < \cdots < i_k} C_{G_1}(f(h_{i_1}, \dots, h_{i_k})^g).$$

By assumption, for every $i_1 < \cdots < i_k$ there are $h'_{i_1}, \dots, h'_{i_k}$ forming a Morley sequence in p over g for which $f(h_{i_1}, \dots, h_{i_k})^g = f(h'_{i_1}, \dots, h'_{i_k})$. Once again by the invariance of H under $Aut(\mathfrak{C})$, we get that $H \leq C_{G_1}(f(h'_{i_1}, \dots, h'_{i_k}))$. Therefore, $H \leq H^g$. This shows that H is normal in G.

Knowing that H is invariant under $Aut(\mathfrak{C})$ and normal in G, the rest of the proof of [5, Lemma 2.2] works in our context, yielding the desired conclusion. ■

Having Lemma 2.1, in order to get the next theorem (which strengthens [5, Theorem 2.3]), one should apply the proof of [5, Theorem 2.3] with several obvious modifications.

Theorem 2.2 *Let G be a group $\emptyset$-definable in a monster model $\mathfrak{C}$ of an ω-categorical theory. Let $p \in S(\mathfrak{C})$ be a type generically stable over $\emptyset$. Suppose that for some $\emptyset$-definable function f and Morley sequence $(g_i)_{i<\omega}$ in p over $\emptyset$, $tp(f(g_0, \dots, g_{k-1})/\emptyset)$ is a generic type of G. Assume additionally that whenever $(h_i)_{i<k}$ is a Morley sequence in p over A, g (for some $A \subseteq \mathfrak{C}$ and $g \in G$), then the conjugate $f(h_0, \dots, h_{k-1})^g$ equals $f(h'_0, \dots, h'_{k-1})$ for some Morley sequence $(h'_i)_{i<k}$ in p over A, g. Then G is solvable-by-finite.*

We would like to remark that while repeating the proof of [5, Theorem 2.3] to get Theorem 2.2, the function f is replaced by some other functions, but always defined by means of f in a purely group-theoretic way (e.g. by taking quotients or iterated commutators) and that is why the property of f described in the penultimate sentence of Theorem 2.2 is never lost (and we can use Lemma 2.1).

Remark 2.3 *Suppose R is a ring definable in a monster model and that R^{00} in the additive sense exists. Then R^{00} is an ideal of R.*

Proof. Consider any $r \in R$. We will show that $rR^{00} \subseteq R^{00}$; the inclusion $R^{00}r \subseteq R^{00}$ can be proved analogously.

Let $f : R \to R$ be an additive homomorphism defined by $f(x) = rx$. Then $f[R^{00}]$ is a type-definable subgroup of R.

We need to show that $f[R^{00}] \subseteq R^{00}$. Suppose this is not the case. Then $A := R^{00} \cap f[R^{00}]$ is a proper, type-definable subgroup of $f[R^{00}]$ of bounded index. Hence, $R^{00} \cap f^{-1}[A]$ is a proper, type-definable, bounded index subgroup of R^{00}, which is not possible. ■

Proof of Theorem 2. Let R be a generically stable ring which is definable in a monster model of an ω-categorical theory. By Fact 1.6(iv), R^{00} (in the additive sense) exists, and since it is $\emptyset$-invariant, it follows by ω-categoricity that it is $\emptyset$-definable. So, it has finite index in R, and, by Remark 2.3, we can assume that $R = R^{00}$ (because, being an additive translate of a generically stable generic type, the generic type of R^{00} is also generically stable by Fact 1.3). Fact 1.10 tells us that $J(R)$ is nilpotent, so we can assume that R is semisimple (replacing R by $R/J(R)$ and using Fact 1.3). If R is finite, we are done, so we can assume that R is infinite. By Fact 1.7, R has a unique (global) generic type $p \in S_1(R)$. Thus, p is generically stable over $\emptyset$. So, without loss of generality, the monster model is just the ring R (possibly with some extra structure).

As it was mentioned at the beginning of this section, the proof splits into two cases depending on whether the realizations of p are non-nilpotent or nilpotent. Notice, however, that in the special case of commutative rings, all non-zero elements of R are non-nilpotent (since, being semisimple, R does not have non-trivial nil ideals). So, for commutative rings it is enough to consider only the first case.

Case 1. Realizations of p are not nilpotent.

By Fact 1.9, we can assume that R is a subring of $\prod_{i \in I} R_i$, where each R_i is finite, and $|\{R_i : i \in I\}| < \omega$. Let π_i be the projection onto the i-th coordinate. For $i_0, \dots, i_n \in I$ and $r_0 \in R_{i_0}, \dots, r_n \in R_{i_n}$, we define

$$R^{r_0,\dots,r_n}_{i_0,\dots,i_n} = \left\{ r \in R : \bigwedge_{j=0}^{n} \pi_{i_j}(r) = r_j \right\}.$$

Claim 1 *There are $i_0, i_1, \dots \in I$, non-nilpotent elements $r_j \in R_{i_j}$ and a Morley sequence $(\eta_i)_{i<\omega}$ in p over $\emptyset$ such that $\eta_n \in R^{0,\dots,0,r_n}_{i_0,\dots,i_{n-1},i_n}$ for every $n < \omega$.*

Proof of Claim 1. Assume that we have already constructed $i_0, \dots, i_n$, $r_0, \dots, r_n$ and $(\eta_i)_{i\leq n}$. Let $p_n = p|(\eta_i)_{i\leq n} \in S_1((\eta_i)_{i\leq n})$. If $R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R) = \emptyset$, then $R \setminus p_n(R)$ is generic in R (since $R^{0,\dots,0}_{i_0,\dots,i_n}$ has finite index in R), so, by compactness, there is $\phi \in p_n$ such that $\neg\phi$ is generic. From Fact 1.6(iii), we get that $\{\neg\phi\}$ extends to a global generic type, which contradicts the uniqueness of the generic type in R. So, $R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R) \neq \emptyset$. Take $\eta_{n+1} \in R^{0,\dots,0}_{i_0,\dots,i_n} \cap p_n(R)$. By the assumption of Case 1, η_{n+1} is non-nilpotent. Since $|\{R_i : i \in I\}| < \omega$, there is $i_{n+1} \in I$ such that $\pi_{i_{n+1}}(\eta_{n+1})$ is non-nilpotent. We put $r_{n+1} := \pi_{i_{n+1}}(\eta_{n+1})$. So, we have found i_{n+1}, r_{n+1} and η_{n+1} with the desired properties. □

By ω-categoricity, the two-sided ideals RrR, $r \in R$, are uniformly definable (because ω-categoricity implies that there is K such that every element of any RrR is the sum of at most K elements of the form $r_1 r r_2$ for $r_1, r_2 \in R \cup \{1\}$). Thus, there is a formula $H(x, z; y)$ expressing that $x \in R(z-y)R$. Let N be as in Lemma 1.4 for the type p, formula $H(x, z; y)$ and $D := R$.

Claim 2 *There are natural numbers $n(0) < n(1) < \cdots < n(N)$ and a Morley sequence $(a_i)_{i\leq N}$ (of length $N+1$) in p over $\emptyset$ such that*

$$a_0 \in R^{r_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, a_1 \in R^{0,r_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a_N \in R^{0,\dots,0,r_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}}. \qquad (*)$$

Proof of Claim 2. First, we will find natural numbers

$$n(0) < n'(0) < n(1) < n'(1) < \cdots < n(N-1) < n'(N-1) < n(N)$$

such that for $a_k := \eta_{n(k)} - \eta_{n'(k)}$, $k = 0, \dots, N-1$, and $a_N := \eta_{n(N)}$ the condition $(*)$ is satisfied. This follows exactly as in the proof of Claim 2 in [12, Theorem 2.1], but we give some details for completeness.

Let $c = \max_{i\in I} |R_i|$. Define numbers $L_N, \dots, L_1, L_0$ recursively by:

$$\begin{array}{lcl} L_N & = & c+1, \\ L_{N-k} & = & c^{L_N+\cdots+L_{N-k+1}+1}+1 \ \text{ for } \ k = 1, \dots, N-1, \\ L_0 & = & 0. \end{array}$$

Put $I_N = \{L_0 + \cdots + L_N\}$, and define intervals $I_0, \dots, I_{N-1}$ as

$$I_k = [L_0 + \cdots + L_k, L_0 + \cdots + L_{k+1} - 1].$$

For each $k \in \{0, \dots, N-1\}$, by the pigeonhole principle, we can find two natural numbers $n(k) < n'(k)$ in I_k such that $\pi_{i_j}(\eta_{n'(k)}) = \pi_{i_j}(\eta_{n(k)})$ for every $j \in I_{k+1} \cup \dots \cup I_N$. Put additionally $n(N) = L_0 + \dots + L_N$. Now, it is easy to check that $(*)$ is satisfied for $a_k := \eta_{n(k)} - \eta_{n'(k)}$, $k = 0, \dots, N-1$, and $a_N := \eta_{n(N)}$.

It remains to show that $(a_i)_{i \leq N}$ is a Morley sequence in p over $\emptyset$. Fix any $k < N$. We have that $\eta_{n'(k)} \models p|(\eta_{n'(i)})_{i<k}(\eta_{n(i)})_{i\leq k}$. By the uniqueness of the generic type in R, we get that $\eta_{n(k)} - \eta_{n'(k)} \models p|(\eta_{n'(i)})_{i<k}(\eta_{n(i)})_{i<k}$, so $a_k \models p|(a_i)_{i<k}$. It is also clear that $a_N \models p|(a_i)_{i<N}$. This shows that $(a_i)_{i\leq N}$ is a Morley sequence in p over $\emptyset$. $\square$

Let $c = \sum_{i \leq N} a_i$ and $b_j = \sum_{i \neq j} a_i = c - a_j$ for $j = 0, \dots, N$. Using Claim 2 and the choice of N, we reach a final contradiction in the same way as in the proof of [12, Theorem 2.1]. Namely,

$$\pi_{i_{n(j)}}[Rb_0R \cap \dots \cap Rb_NR] = \{0\} \text{ for } j = 0, \dots, N. \tag{$**$}$$

On the other hand, $\prod_{k \neq j} b_k \in \bigcap_{k \neq j} Rb_kR$ for $j = 0, \dots, N$. We also have that $\pi_{i_{n(j)}}[\prod_{k\neq j} b_k] = r_{n(j)}^N \neq 0$ as $r_{n(j)}$ is non-nilpotent. So,

$$\pi_{i_{n(j)}}\left[\bigcap_{k \neq j} Rb_kR\right] \neq \{0\} \text{ for } j = 0, \dots, N. \tag{$***$}$$

By $(**)$ and $(***)$, $Rb_0R \cap \dots \cap Rb_NR \neq \bigcap_{k \neq j} Rb_kR$ for all $j = 0, \dots, N$. This is a contradiction to the choice of N, because $Rb_iR = R(c - a_i)R = H(R, c; a_i)$ and $(a_i)_{i \leq N}$ is a Morley sequence in p over $\emptyset$.

Case 2. Realizations of p are nilpotent.

By ω-categoricity, R has a finite characteristic c. Put $R_1 = R \times \mathbb{Z}_c$, and define $+$ and $\cdot$ on R_1 by $(a,k) + (b,l) = (a+b, k +_c l)$ and $(a,k) \cdot (b,l) = (ab + l \times a + k \times b, k \cdot_c l)$, where $+_c$ and $\cdot_c$ are addition and multiplication modulo c, and $l \times a := a + \dots + a$ (l-many times). Then R_1 is a ring with 1 interpretable in R, and R is a two-sided ideal of finite index in R_1. Let G be the subgroup of $\mathrm{GL}_3(R_1)$ generated by $\{t_{ij}(\alpha) : \alpha \in R,\ i,j \in \{1,2,3\},\ i \neq j\} \cup \{t_j(\beta) : \beta \in (1+R) \cap R_1^*,\ j \in \{1,2,3\}\}$, where $t_{ij}(\alpha)$ is the matrix with 1's on the diagonal, α on the (i,j)-th position and 0's elsewhere, and $t_j(\beta)$ is the matrix with β on the (j,j)-th position, 1's on the rest of the diagonal and 0's elsewhere. Since G is invariant over finitely many parameters (those over which R_1 is defined), it follows by ω-categoricity that it is definable. Let $(a_{ij})_{1 \leq i,j \leq 3} \models p^{(9)}$ (in a bigger monster model $\mathfrak{C} \succ R$); note that the order of a_{ij}'s is irrelevant, because p is generically stable and we have Fact 1.2. Define

$$A = \begin{pmatrix} 1 + a_{11} & a_{12} & a_{13} \\ a_{21} & 1 + a_{22} & a_{23} \\ a_{31} & a_{32} & 1 + a_{33} \end{pmatrix}.$$

Claim 3 *$A \in G(\mathfrak{C})$, where $G(\mathfrak{C})$ is the interpretation of G in $\mathfrak{C}$.*

Proof of Claim 3. The idea is to show that we can transform A to the identity matrix by a Gaussian elimination process in which all elementary matrices belong to $G(\mathfrak{C})$ (because then $BA = I$ for some $B \in G(\mathfrak{C})$, so $A = B^{-1} \in G(\mathfrak{C})$).

The following well-known remark is fundamental for our process: if $r \in R$ satisfies $r^n = 0$ for some n, then $(1 + r)(1 - r + r^2 - \cdots \pm r^{n-1}) = 1$, so $(1 + r)^{-1} \in (1 + R \cap dcl(r)) \cap R_1^*$.

Now, we describe the first step of the process. We have

$$t_{21}\left(-a_{21}(1+a_{11})^{-1}\right)A = \\ \begin{pmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+a_{22}-a_{21}(1+a_{11})^{-1}a_{12} & a_{23}-a_{21}(1+a_{11})^{-1}a_{13} \\ a_{31} & a_{32} & 1+a_{33} \end{pmatrix}.$$

But $a_{21}(1 + a_{11})^{-1}a_{12} \in \mathfrak{C} \cap dcl((a_{ij})_{(i,j)\neq(2,2)})$, so, by the uniqueness of the generic type,

$$b_{22} := a_{22} - a_{21}(1 + a_{11})^{-1}a_{12} \models p|R, (a_{ij})_{(i,j)\neq(2,2)}.$$

Similarly,

$$b_{23} := a_{23} - a_{21}(1 + a_{11})^{-1}a_{13} \models p|R, (a_{ij})_{(i,j)\neq(2,3)}.$$

Therefore,

$$t_{21}\left(-a_{21}(1+a_{11})^{-1}\right)A = \begin{pmatrix} 1+a_{11} & a_{12} & a_{13} \\ 0 & 1+b_{22} & b_{23} \\ a_{31} & a_{32} & 1+a_{33} \end{pmatrix},$$

where $((a_{ij})_{(i,j)\neq(2,2),(2,3)}, b_{22}, b_{23}) \models p^{(8)}$. Continuing Gaussian elimination in this way, we obtain a matrix $C \in G(\mathfrak{C})$ such that

$$CA = \begin{pmatrix} 1+b_1 & 0 & 0 \\ 0 & 1+b_2 & 0 \\ 0 & 0 & 1+b_3 \end{pmatrix} = t_1(1+b_1)t_2(1+b_2)t_3(1+b_3)$$

for some $(b_1, b_2, b_3) \models p^{(3)}$. But $t_1(1+b_1), t_2(1+b_2), t_3(1+b_3) \in G(\mathfrak{C})$, so we conclude that $A \in G(\mathfrak{C})$. $\square$

Claim 4 *All translates of the type $q := tp(A/R)$ by the elements of G are finitely satisfiable in some small model. Thus, G has fsg and q is a global generic type of G.*

Proof of Claim 4. We will show that every translate of A by an element of G belongs to the set

$$Z := \{(k_{ij} + b_{ij})_{i,j\in\{1,2,3\}} : k_{ij} \in \mathbb{Z}_c,\ (b_{ij})_{i,j\in\{1,2,3\}} \models p^{(9)}\}.$$

This will complete the proof of Claim 4, as every element of Z is in the definable closure of $\mathbb{Z}_c$ and some realization of the type $p^{(9)}$, and $p^{(9)}$ is finitely satisfiable in some small model. So, it suffices to show that Z is invariant under multiplication by the elements of the set $\{t_{ij}(\alpha) : \alpha \in R,\ i,j \in \{1,2,3\},\ i \neq j\} \cup \{t_j(\beta) : \beta \in (1+R) \cap R_1^*,\ j \in \{1,2,3\}\}$ (notice that this set is closed under the group inversion). Choose any $B = (k_{ij} + b_{ij})_{i,j \in \{1,2,3\}} \in Z$.

First, consider any element of Z of the form $t_j(\beta)$, where $\beta = 1 + r \in R_1^*$ for some $r \in R$. Denote the entries of the matrix $t_j(\beta)B$ by d_{im} ($i,m \in \{1,2,3\}$). Then $d_{im} = k_{im} + b_{im}$ for all m and $i \neq j$. Take any $m \in \{1,2,3\}$. Then $d_{jm} = \beta(k_{jm} + b_{jm}) = (1+r)(k_{jm} + b_{jm}) = k_{jm} + k_{jm} \times r + (1+r)b_{jm}$. Since multiplication by $(1+r)$ is a definable automorphism of $(R,+)$, by the uniqueness of the generic type, we get that $(1+r)b_{jm} \models p|R, (b_{il})_{(i,l)\neq(j,m)}$, and hence $k_{jm} \times r + (1+r)b_{jm} \models p|R, (b_{il})_{(i,l)\neq(j,m)}$. This easily implies that $t_j(\beta)B \in Z$.

Now, consider any $t_{ij}(\alpha)$, where $i \neq j$ and $\alpha \in R$. Denote the entries of $t_{ij}(\alpha)B$ by f_{ij} ($i,j \in \{1,2,3\}$). Choose any $m \in \{1,2,3\}$. For all $l \neq i$ we have $f_{lm} = k_{lm} + b_{lm}$. Moreover, $f_{im} = k_{im} + b_{im} + \alpha(k_{jm} + b_{jm})$. But $\alpha(k_{jm} + b_{jm}) \in dcl(R, b_{jm})$, so $b_{im} + \alpha(k_{jm} + b_{jm}) \models p|R, (b_{pq})_{(p,q)\neq(i,m)}$. Hence, $t_{ij}(\alpha)B \in Z$. This completes the proof of Claim 4. □

If we knew that $p^{(9)}$ were generically stable (recall that we know that p is generically stable), then Claim 4 would imply that G is generically stable, so, by Fact 0.1, we would conclude that G is solvable-by-finite and we could turn to the last paragraph of the proof. Since, in general, we cannot conclude that $p^{(9)}$ is generically stable, we will prove Claim 5 below and then apply Theorem 2.2 in order to get that G is solvable-by-finite.

Adding to the language the appropriate parameters, we can assume that everything is definable over $\emptyset$.

Claim 5 *Let a $\emptyset$-definable function $f : M_{3\times 3}(R) \to M_{3\times 3}(R_1)$ be defined by*

$$f((x_{ij})_{1\leq i,j\leq 3}) = \begin{pmatrix} 1+x_{11} & x_{12} & x_{13} \\ x_{21} & 1+x_{22} & x_{23} \\ x_{31} & x_{32} & 1+x_{33} \end{pmatrix}.$$

Then, whenever $(h_{ij})_{1\leq i,j\leq 3}$ is a Morley sequence in p over A, g (for some $A \subseteq R$ and $g \in G$), then $f((h_{ij})_{1\leq i,j\leq 3}) \in G$ and $f((h_{ij})_{1\leq i,j\leq 3})^g = f((h'_{ij})_{1\leq i,j\leq 3})$ for some Morley sequence $(h'_{ij})_{1\leq i,j\leq 3}$ in p over A, g.

Proof of Claim 5. Let $(h_{ij})_{1\leq i,j\leq 3}$ be a Morley sequence in p over A, g. The fact that $f((h_{ij})_{1\leq i,j\leq 3}) \in G$ follows from Claim 3 and the uniqueness of a Morley sequence in p over $\emptyset$ up to the type.

For the second part, first notice that it is enough to prove the statement for g of the form $t_{ij}(\alpha)$ (for $\alpha \in R$ and distinct $i,j \in \{1,2,3\}$) and $t_j(\beta)$ (for $\beta \in (1+R) \cap R_1^*$ and $j \in \{1,2,3\}$), which follows easily from the fact that G is generated by these elements and by the independence of the choice of a Morley sequence $(h_{ij})_{1\leq i,j\leq 3}$ in p over A, g. Next, apply a similar argument to the proof of Claim 4 in order to get that the conjugates by the elements of the form $t_{ij}(\alpha)$ or $t_j(\beta)$ have the desired property. □

From Claims 4 and 5 and Theorem 2.2, we conclude that G is solvable-by-finite.

The rest of the proof follows exactly as in [13, Theorem 2.1(i)], but we will repeat the argument for the reader's convenience. Let H be a normal subgroup of G of finite index, which is solvable. We have the following well-known formulas:

$$t_{ij}(\alpha)t_{ij}(\beta) = t_{ij}(\alpha + \beta) \text{ and } [t_{ik}(\alpha), t_{kj}(\beta)] = t_{ij}(\alpha\beta) \tag{†}$$

for pairwise distinct i, j, k. Define $I = \{r \in R : (\forall i \neq j)t_{ij}(r) \in H\}$. Using the normality of H in G and (†), we see that $I \lhd R$. If $|R/I| \geq \omega$, then, by Ramsey's theorem, for some distinct $i, j \in \{1, 2, 3\}$ there are r_k, $k < \omega$, such that $t_{ij}(r_n - r_m) \notin H$ for every $n < m < \omega$. But, by (†), $t_{ij}(r_n - r_m) = t_{ij}(r_n)t_{ij}(r_m)^{-1}$, which contradicts the finiteness of $[G : H]$. So, $|R/I| < \omega$. Since H is solvable, there exists n for which the n-th derived subgroup $H^{(n)}$ is trivial. Then (†) implies that for every $r_1, \dots, r_{2^n} \in I$ and distinct $i, j \in \{1, 2, 3\}$ we have $t_{ij}(r_1 \dots r_{2^n}) \in H^{(n)} = \{e\}$, so $r_1 \dots r_{2^n} = 0$. This shows that I is a nilpotent ideal of R of finite index. ■

3 ω-categorical, generically stable groups

The goal of this section is to prove Theorem 1 from the introduction. In the final part of the proof, we will apply Fact 0.1 and the argument from page 490 of [15]. However, in order to do that, first we need to prove a certain non-trivial lemma (a variant of [15, Corollary 3.5]), which uses some ideas from the proof of Theorem 2 in Case 1 and from the final part of the proof of [13, Corollary 3.17].

It is worth recalling that [13, Theorem 3.15] says that whenever each ring interpretable in a given ω-categorical structure (in which all definable groups have connected components) is nilpotent-by-finite, then each solvable group definable in this structure is also nilpotent-by-finite. This was used in [12] to conclude that ω-categorical groups with NIP and fsg are nilpotent-by-finite (using the fact that they are solvable-by-finite and that each ω-categorical ring with NIP is nilpotent-by-finite). The reason why, having Theorem 2, we cannot apply Fact 0.1 and [13, Theorem 3.15] in order to get Theorem 1 is that rings interpretable in a given ω-categorical, generically stable group need not to be generically stable. In the proof of Lemma 3.3 below, we undertake a detailed analysis of the relevant interpretable rings.

Recall a few basic definitions. Let H be a group. The commutator of $h_0, h_1 \in H$ is defined as $[h_0, h_1] = h_0^{-1}h_1^{-1}h_0h_1$. The iterated commutators γ_n on H are defined inductively as follows:

$$\gamma_1(h_0) = h_0 \text{ and } \gamma_{n+1}(h_0, \dots, h_n) = [\gamma_n(h_0, \dots, h_{n-1}), h_n].$$

For two subsets A and B of H, $[A, B]$ denotes the subgroup of H generated by all commutators $[a, b]$, where $a \in A$ and $b \in B$. The lower central series $H = \Gamma_1(H) \geq \Gamma_2(H) \geq \dots$ of H is defined by:

$$\Gamma_1(H) = H \text{ and } \Gamma_{n+1}(H) = [\Gamma_n(H), H].$$

The group H is nilpotent of class n if $\Gamma_{n+1}(H) = \{e\}$ and n is the smallest number with this property. The following formulas for commutators will be very useful:

$$[x, zy] = [x, y][x, z]^y \text{ and } [xz, y] = [x, y]^z[z, y]. \tag{1}$$

Recall that, in this paper, a group H definable in a monster model is said to be absolutely connected if $H^{00} = H$.

Remark 3.1 *If H is an absolutely connected, nilpotent group definable in a monster model of an ω-categorical theory, then each group $\Gamma_k(H)$ is also definable and absolutely connected.*

Proof. The definability of $\Gamma_k(H)$ is a standard consequence of ω-categoricity: there is l such that each element of $\Gamma_k(H)$ is a product of at most l iterated commutators of length n. By induction on the nilpotency class n of H, we will show that each term of the lower central series is absolutely connected. For $n = 1$ it is clear. Assume H is of class $n + 1$ and the conclusion holds for groups of smaller class.

First, suppose for a contradiction that $\Gamma_{n+1}(H)$ is not absolutely connected, i.e., it has a proper, type-definable subgroup C of bounded index. Since $\Gamma_{n+1}(H)$ is central in H, C is a normal subgroup of H. For any $h/C \in \Gamma_n(H)/C$, $[h/C, H/C] \leq \Gamma_{n+1}(H)/C$ is finite, so $C_H(h/C)$ is of finite index in H, and hence, by the absolute connectedness of H, it is equal to H. Thus, $\Gamma_{n+1}(H) = [\Gamma_n(H), H] \leq C$, which implies $\Gamma_{n+1}(H) = C$, a contradiction to the properness of C. So, we have proved that $\Gamma_{n+1}(H)$ is absolutely connected. Since $H/\Gamma_{n+1}(H)$ is of nilpotency class n, by the induction hypothesis, each quotient $\Gamma_k(H)/\Gamma_{n+1}(H)$ is absolutely connected. These two observations imply that each term $\Gamma_k(H)$ is also absolutely connected. ■

Lemma 3.2 *Let H be an absolutely connected group with fsg (H is definable over A in a monster model). Then for every $n \in \omega \setminus \{0\}$ each element of $\Gamma_n(H)$ is a product of conjugates of elements from the set $\{\gamma_n(g_0, \dots, g_{n-1}) : (g_0, \dots, g_{n-1}) \models p^{(n)}|A\}$, where p is the unique global generic type in H.*

Proof. Using (1), we easily get by induction that for every $x_0, \dots, x_k, y$ the commutator $[x_0 \dots x_k, y]$ is a product of conjugates of commutators $[x_i, y], i = 0, \dots, k$. To prove the lemma, we proceed by induction on n. The induction starts, since every element of H is a product of two realizations of $p|A$ (which follows from the uniqueness of the generic type). Suppose that the conclusion of the lemma is satisfied for n. Take any $g \in \Gamma_{n+1}(H)$. Then $g = [a, b]$ for some $a \in \Gamma_n(H)$ and $b \in H$. By the inductive hypothesis, $a = \prod_{i<l} \gamma_n(\overline{g_i})^{c_i}$ for some $\overline{g_i} \models p^{(n)}|A$ and $c_i \in H$ (for $i = 0, \dots, l-1$). So, by (1), $[a, b] = \prod_{i<l}[\gamma_n(\overline{g_i})^{c_i}, b]^{d_i}$ for some $d_i \in H$, $i = 0, \dots, l-1$. Choose $b_1, b_2 \models p|A, (\overline{g_i}, c_i)_{i<l}$ such that $b_1 b_2 = b$. Then, by (1), for every $i < l$ we have

$$\begin{aligned}
[\gamma_n(\overline{g_i})^{c_i}, b]^{d_i} &= [\gamma_n(\overline{g_i})^{c_i}, b_2]^{d_i}[\gamma_n(\overline{g_i})^{c_i}, b_1]^{b_2 d_i} \\
&= [\gamma_n(\overline{g_i}), b_2^{c_i^{-1}}]^{c_i d_i}[\gamma_n(\overline{g_i}), b_1^{c_i^{-1}}]^{c_i b_2 d_i} \\
&= \gamma_{n+1}(\overline{g_i}, b_2^{c_i^{-1}})^{c_i d_i}\gamma_{n+1}(\overline{g_i}, b_1^{c_i^{-1}})^{c_i b_2 d_i}.
\end{aligned}$$

By the uniqueness of the generic type, $b_1^{c_i^{-1}}, b_2^{c_i^{-1}} \models p|A, \overline{g_i}$. So, $(\overline{g_i}, b_1^{c_i^{-1}}), (\overline{g_i}, b_2^{c_i^{-1}}) \models p^{(n+1)}|A$. This completes the proof of the lemma. ■

Lemma 3.3 *We work in a monster model $\mathfrak{C}$ of an ω-categorical theory. Let H be a nilpotent, absolutely connected, generically stable group definable in $\mathfrak{C}$ acting definably and by automorphisms on a definable vector space V over $F := GF(q^a)$ (q is a prime number), and assume that H has no elements of order q. Then $\mathrm{Stab}_H(V) = H$.*

Proof. First, we will prove the following claim.

Claim 1 *If H is a nilpotent group of class n which satisfies the assumptions of the lemma, then $\mathrm{Stab}_{\Gamma_n(H)}(V) = \Gamma_n(H)$.*

Proof of Claim 1. We can assume that everything is $\emptyset$-definable in $\mathfrak{C}$. Then the unique global generic type p of H is generically stable over $\emptyset$. Put $A = \Gamma_n(H)$. As in [15], we define W as the sum of all finite dimensional FA-submodules of V. Then W is a subspace of V which is definable in $\mathfrak{C}$ and invariant under A. By [15, Proposition 3.4] (see also [13, Fact 3.16]) and Remark 3.1, it is enough to show that $W = V$. Suppose for a contradiction that $W \subsetneq V$, and put $\overline{V} = V/W$. Exactly as in in the proof of [15, Corollary 3.5], we get:

$$\text{The } FA\text{-module } \overline{V} \text{ has no non-trivial, finite dimensional } FA\text{-submodules.} \qquad (*)$$

Choose a non-trivial $v \in \overline{V}$, and put $V_0 = \mathrm{Lin}_F(Av)$. By ω-categoricity, V_0 is interpretable. Let R be the ring of endomorphisms of V_0 generated by A. Since A is commutative (as $A = \Gamma_n(H)$ and H is nilpotent of class n), R is a commutative ring interpretable in $\mathfrak{C}$. Adding some parameters to the language, we can assume that R is interpretable in $\mathfrak{C}$ over $\emptyset$.

Let $(g_i)_{i<\omega} \models p^{(\omega)}|\emptyset$. We will show that

$$\gamma_n(g_0, \dots, g_{n-2}, g_n) - \gamma_n(g_0, \dots, g_{n-2}, g_{n-1}) \in J(R). \qquad (**)$$

Suppose for a contradiction that this is not the case. Put

$$a = \gamma_{n-1}(g_0, \dots, g_{n-2}).$$

By Fact 1.9, we can assume that $R/J(R)$ is a subring of $\prod_{i\in I} R_i$, where each R_i is finite and $|\{R_i : i \in I\}| < \omega$. Let $\pi_i : R \to R_i$ be the quotient map $R \to R/J(R)$ composed with the projection onto the i-th coordinate. For $i_0, \dots, i_m \in I$ and $r_j \in R_{i_j}$, we define

$$R^{r_0,\dots,r_m}_{i_0,\dots,i_m} = \left\{ r \in R : \bigwedge_{j=0}^{m} \pi_{i_j}(r) = r_j \right\}.$$

Subclaim 1 *There are $i_0, i_1, \dots \in I$, non-nilpotent elements $r_j \in R_{i_j}$ and a Morley sequence $(\eta_i)_{i<\omega}$ in p over a such that $[a, \eta_{2m+1}] - [a, \eta_{2m}] \in R^{0,\dots,0,r_m}_{i_0,\dots,i_{m-1},i_m}$ for every $m < \omega$.*

Proof of Subclaim 1. Suppose we have already constructed $i_0, \dots, i_{k-1}$, $r_0, \dots, r_{k-1}$ and $(\eta_i)_{i<2k}$. Let $(h_i)_{i<\omega} \models p^{(\omega)}|a, (\eta_i)_{i<2k}$. Choose $j < l < \omega$ such that $\pi_{i_m}([a, h_j]) = \pi_{i_m}([a, h_l])$ for every $m < k$. Put $\eta_{2k} = h_j$ and $\eta_{2k+1} = h_l$. Since $R/J(R)$ is a semisimple, commutative ring, the only nilpotent element of $R/J(R)$ is zero. Hence, by our assumption that $(**)$ does not hold, $([a, \eta_{2k+1}] - [a, \eta_{2k}])/J(R)$ is non-nilpotent. Therefore, since $|\{R_i : i \in I\}| < \omega$, there is $i_k \in I$ such that $\pi_{i_k}([a, \eta_{2k+1}] - [a, \eta_{2k}])$ is non-nilpotent. Putting $r_k = \pi_{i_k}([a, \eta_{2k+1}] - [a, \eta_{2k}])$, the construction is completed. $\square$

Now, using the assumption that H is nilpotent of class n and (1), we get

$$\begin{aligned} ([a, \eta_{2k+1}] - [a, \eta_{2k}])[a, \eta_{2k}^{-1}] &= [a, \eta_{2k+1}][a, \eta_{2k}^{-1}] - [a, \eta_{2k}][a, \eta_{2k}^{-1}] \\ &= [a, \eta_{2k+1}][a, \eta_{2k}^{-1}]^{\eta_{2k+1}} - [a, \eta_{2k}][a, \eta_{2k}^{-1}]^{\eta_{2k}} \\ &= [a, \eta_{2k}^{-1}\eta_{2k+1}] - 1 \end{aligned}$$

for any $k < \omega$. But $[a, \eta_{2k}^{-1}]$ is an invertible element of R, so putting $h_k = \eta_{2k}^{-1}\eta_{2k+1}$ for $k < \omega$, we obtain that $[a, h_k] - 1 \in R^{0,\dots,0,s_k}_{i_0,\dots,i_{k-1},i_k}$ for every $k < \omega$, where $s_k \in R_{i_k}$ are non-nilpotent. Also, by the uniqueness of the generic type, $(h_i)_{i<\omega}$ is a Morley sequence in p over a.

Let $H(x, \overline{z}; y)$ be a formula expressing that $x \in R(z_1 - [z_2, y] + 1)R$, where $\overline{z} = (z_1, z_2)$. Choose N as in Lemma 1.4 for the type p, formula $H(x, \overline{z}; y)$ and $D := R \times \{a\}$.

Now, exactly as in Claim 2 in the proof of Theorem 2, we find

$$n(0) < n'(0) < n(1) < n'(1) < \dots < n(N-1) < n'(N-1) < n(N) < n'(N)$$

such that for $a_k := ([a, h_{n(k)}] - 1) - ([a, h_{n'(k)}] - 1)$, $k = 0, \dots, N$, we have:

$$a_0 \in R^{s_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, a_1 \in R^{0,s_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a_N \in R^{0,\dots,0,s_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}}.$$

By the fact that H is nilpotent of class n and (1), for all $k = 0, \dots, N$ we have

$$a'_k := a_k[a, h^{-1}_{n'(k)}] = [a, h_{n(k)}][a, h^{-1}_{n'(k)}] - [a, h_{n'(k)}][a, h^{-1}_{n'(k)}] = [a, h^{-1}_{n'(k)}h_{n(k)}] - 1.$$

Since each $[a, h^{-1}_{n'(k)}]$ is invertible in R, we obtain

$$a'_0 \in R^{s'_{n(0)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, a'_1 \in R^{0,s'_{n(1)},0,\dots,0}_{i_{n(0)},\dots,i_{n(N)}}, \dots, a'_N \in R^{0,\dots,0,s'_{n(N)}}_{i_{n(0)},\dots,i_{n(N)}},$$

for some non-nilpotent elements $s'_{n(0)} \in R_{i_{n(0)}}, \dots, s'_{n(N)} \in R_{i_{n(N)}}$. Moreover, we see that $(h^{-1}_{n'(k)}h_{n(k)})_{k<\omega}$ is a Morley sequence in p over a. So, by the choice of N, the argument after Claim 2 in the proof of Theorem 2 leads to a contradiction, which completes the proof of $(**)$.

By $(**)$ together with the fact that H is nilpotent of class n and (1), we obtain

$$\begin{aligned} J(R) &\ni (\gamma_n(g_0, \dots, g_{n-2}, g_n) - \gamma_n(g_0, \dots, g_{n-2}, g_{n-1}))\gamma_n(g_0, \dots, g_{n-2}, g_{n-1}^{-1}) \\ &= \gamma_n(g_0, \dots, g_{n-2}, g_{n-1}^{-1}g_n) - 1, \end{aligned}$$

so $\gamma_n(h_0,\dots,h_{n-1})-1 \in J(R)$ for every $(h_0,\dots,h_{n-1}) \models p^{(n)}|\emptyset$. But, by Lemma 3.2, every element of A is a product of finitely many elements of the form $\gamma_n(h_0,\dots,h_{n-1})$, where $(h_0,\dots,h_{n-1}) \models p^{(n)}|\emptyset$. Also, every element of R is a sum of a fixed number of elements of the set $A \cup \{-h : h \in A\} \cup \{0\}$. So, we conclude that $J(R)$ is of finite index in R.

The rest of the proof follows the lines of the final part of the proof of [13, Corollary 3.17]. Namely, choose representatives $r_1,\dots,r_m$ of all cosets of $J(R)$ in R. Let $k \geq 1$ be the least number such that $J(R)^k = \{0\}$ (such a number exists by Fact 1.10). Take any non-trivial $i \in J(R)^{k-1}$ (where $J(R)^0 := R$). Then $i(v) \neq 0$ and $Ai(v) \subseteq Ri(v) = \{r_1 i(v),\dots,r_m i(v)\}$. Thus, $\mathrm{Lin}_F(Ai(v))$ is a non-trivial, finite-dimensional (over F) FA-submodule of $\overline{V}$. This is a contradiction to $(*)$, which completes the proof of Claim 1. $\square$

Using Claim 1, the lemma follows easily by induction on the nilpotency class of H. $\blacksquare$

Notice that the only places in the proof of Lemma 3.3 in which the assumption that H has no elements of order q is used are the proof of [15, Proposition 3.4] (an application of Maschke's theorem) and the proof of $(*)$.

Having Fact 0.1 and Lemma 3.3 in hand, the proof from page 490 of [15] goes through under the assumption of Theorem 1 after a slight modification (which is necessary, because the generic stability of a group is not inherited by definable subgroups).

Proof of Theorem 1. Let G be a generically stable group $\emptyset$-definable in a monster model of an ω-categorical theory. By fsg, G^{00} exists, and by ω-categoricity, it is $\emptyset$-definable. Hence, G^{00} has finite index in G, and we can assume that $G = G^{00}$. By Fact 0.1, G is solvable-by-finite, so it has a definable, solvable subgroup of finite index (by ω-categoricity, the group generated by all normal, solvable subgroups of finite index does the job). Thus, G is solvable.

We will argue by induction on the maximal possible length of a chain of distinct, characteristic (in the group-theoretic sense) subgroups of G. Suppose $\{e\} = G_0 < G_1 < \dots < G_t = G$ is a chain of maximal length of distinct, characteristic subgroups of G and that the theorem holds for absolutely connected, generically stable groups with smaller maximal length of such a chain. Notice that all groups G_i are $\emptyset$-invariant and so $\emptyset$-definable. They are also normal in G.

The group G_1 is characteristically simple and solvable, so it is abelian. The group G/G_1 is absolutely connected and generically stable, hence, by the induction hypothesis, it is nilpotent-by-finite and so nilpotent (by absolute connectedness). Put $N := G_1$.

Since any nilpotent, ω-categorical group is a direct product of its Sylow subgroups, we may write

$$G/N = P_1 \times \dots \times P_n, \tag{$\dagger$}$$

where each P_i is a Sylow p_i-subgroup of G/N. By ω-categoricity, G/N has bounded exponent, so every P_i is definable. Hence, using (†), one can easily check that every P_i is a nilpotent, absolutely connected, generically stable group. Thus, we can apply Lemma 3.3 to definable actions of these groups on the appropriate abelian groups. Having this, the rest of the proof from [15] goes through in our context, which we sketch below.

Let Q_i be the preimage of P_i, $i = 1, \dots, n$, under the quotient map $G \to G/N$. Applying Lemma 3.3 to the action of P_1 on the subgroup of N consisting of the elements whose order is co-prime to p_1, we get that the elements of Q_1 of co-prime orders commute, and so Q_1 is locally nilpotent. Since Q_1 is also solvable, ω-categoricity and [18, 4(8)] imply that Q_1 is nilpotent. Since the group Q_2Q_1/Q_1 is definably isomorphic with $P_2/P_2 \cap P_1$, we see that it is a nilpotent, absolutely connected and generically stable p_2-group. Thus, applying Lemma 3.3 to the action of Q_2Q_1/Q_1 on successive quotients of the lower central series of the normal Sylow r-subgroup of the nilpotent group Q_1 (for $r \neq p_2$), we conclude that the elements of co-prime orders in Q_2Q_1 commute. As before, this implies that Q_2Q_1 is nilpotent. Continuing in this way, we get that $G = Q_n \dots Q_1$ is nilpotent. ■

References

[1] H. Adler, E. Casanovas, A. Pillay, *Generic stability and stability*, preprint, 2012.

[2] J. Baldwin, B. Rose, *$\aleph_0$-categoricity and stability of rings*, Journal of Algebra 45, 1-16, 1977.

[3] W. Baur, G. Cherlin, A. Macintyre, *Totally categorical groups and rings*, Journal of Algebra 57, 407-440, 1979.

[4] G. Cherlin, *On $\aleph_0$-categorical nilrings. II*, Journal of Symbolic Logic 45, 291-301, 1980.

[5] J. Dobrowolski, K. Krupiński, *On ω-categorical, generically stable groups*, Journal of Symbolic Logic (77), 1047-1056, 2012.

[6] C. Ealy, K. Krupiński, A. Pillay, *Superrosy dependent groups having finitely satisfiable generics*, Annals of Pure and Applied Logic 151, 1-21, 2008.

[7] D. Evans, F. Wagner, *Supersimple ω-categorical groups and theories*, Journal of Symbolic Logic 65, 767-776, 2000.

[8] U. Felgner, *$\aleph_0$-categorical stable groups*, Mathematische Zeitschrift 160, 27-49, 1978.

[9] I. N. Herstein, *Non-commutative Rings*, Carus Mathematical Monographs, No. 15, Mathematical Association of America, 1968.

[10] E. Hrushovski, A. Pillay, *On NIP and invariant measures*, Journal of the European Mathematical Society 13, 1005-1061, 2011.

[11] E. Hrushovski, A. Pillay, Y. Peterzil, *Groups, measures, and the NIP*, Journal of the American Mathematical Society 21, 563-595, 2008.

[12] K. Krupiński, *On ω-categorical groups and rings with NIP*, Proceedings of the American Mathematical Society, accepted.

[13] K. Krupiński, *On relationships between algebraic properties of groups and rings in some model-theoretic contexts*, Journal of Symbolic Logic 76, 1403-1417, 2011.

[14] K. Krupiński, F. Wagner, *Small profinite groups and rings*, Journal of Algebra 306, 494-506, 2006.

[15] H. D. Macpherson, *Absolutely ubiquitous structures and $\aleph_0$-categorical groups*, Quart. J. Math. Oxford (2) 39, 483-500, 1988.

[16] A. Pillay, P. Tanović, *Generic stability, regularity, and quasi-minimality*, CRM Proceedings and Lecture Notes 53, 189-211, 2011.

[17] F. Wagner, *Stable groups*, London Math. Soc. Lecture Notes Series 240, Cambridge University Press, UK, 1997.

[18] J. Wilson, *The algebraic structure of ω-categorical groups*, in: Groups-St. Andrews, Ed. C. M. Campbell, E. F. Robertson, London Math. Soc. Lecture Notes 71, Cambridge, 345-358, 1981.

Address:
Instytut Matematyczny, Uniwersytet Wrocławski,
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.

E-mail addresses:
Jan Dobrowolski: dobrowol@math.uni.wroc.pl
Krzysztof Krupiński: kkrup@math.uni.wroc.pl