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COMPLETELY MITOTIC R.E. DEGREES*

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1. Introduction

An r.e. set A is called *mitotic* if there exist a pair of disjoint r.e. sets A_1, A_2 with $A_1 \cup A_2 = A$ (in this case we write $A_1 \sqcup A_2 = A$) such that $A_1 \equiv_T A_2 \equiv_T A$. We refer to such a splitting as a *mitotic splitting* of A. Lachlan [10] was the first person to show that not all r.e. sets are mitotic. More extensive investigations into (non)mitoticity were provided by Ladner [11, 12] who constructed various types of nonmitotic r.e. sets. He also proved the following very interesting theorem: An r.e. set A is mitotic iff A is *autoreducible* where A is called autoreducible (Trachtenbrot [20]) if there is a functional Φ such that, for all x, $\Phi(A \cup \{x\}; x) = A(x)$. Following Ladner's investigations, there have been several other results concerning the existence of nonmitotic r.e. sets. One example is Ingrassia's [8] result that the degrees containing nonmitotic r.e. sets are dense in $\mathbf{R}$, the r.e. degrees.

The interest in Ingrassia's result is that nonmitotic r.e. sets do not live in all nonzero r.e. degrees. The most difficult of Ladner's results establishes this. That is, in [12] Ladner constructed a *completely mitotic* nonzero r.e. degree $\mathbf{a}$, where $\mathbf{a}$ is completely mitotic if all of its r.e. elements are mitotic.

Our goal in this paper is to investigate the class of completely mitotic degrees. Save for Ladner's one construction of a low_2–low (as P. Cohen observed in [12]) completely mitotic r.e. degree there are no other existence theorems for these degrees. In particular one of the main open questions here was whether or not there exist (even) *low* nonzero completely mitotic degrees.

Ambos-Spies and Fejer [2] have shown that Ladner's *construction* cannot be used to answer this, since his construction automatically gives a *contiguous* r.e. degree (namely an r.e. degree consisting of a single r.e. wtt-degree). In [2] they showed that if $\mathbf{a} \neq \mathbf{0}$ is low and contiguous then $\mathbf{a}$ contains a nonmitotic r.e. set.

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In Section 2 we show that no low promptly simple degree **a** is completely mitotic. (We remind the reader that — with the usual notation — a co-infinite r.e. set A is called *promptly simple* if there exists a recursive function f such that

$$\forall e\,(|W_e| = \infty \rightarrow \exists^{\infty} s, x\,(x \in W_{e,\text{at}\,s} \,\&\, x \in A_{f(s)})).)$$

By [4] the promptly simple degrees are exactly the noncappable degrees and so no low noncappable degree is completely mitotic. In Section 2 we also show that lowness cannot be removed from the hypothesis of our first result, by constructing a promptly simple completely mitotic r.e. degree.

In Section 3, using a completely different construction (which doesn't blend with promptness), we show that there do however exist low completely mitotic nonzero r.e. degrees. The strategies involved are sufficiently flexible that we can modify them to show that there also exist high completely mitotic degrees. We do not know the exact classification of the jumps of completely mitotic degrees but results of Cooper [5] and Shore [15] would seem to suggest that it does not include all the degrees r.e. in and above $\emptyset'$ (and we conjecture this).

In Section 4 we prove some further limiting results on the distribution of completely mitotic degrees. Our results here are that

(1) there exist low_2–low degrees bounding no nonzero completely mitotic r.e. degree,

(2) if **a** is r.e. with $\mathbf{a} \neq \mathbf{0}$, then there exists a nonzero r.e. predecessor **b** of **a** such that every nonzero r.e. degree below **b** contains a nonmitotic r.e. set, and

(3) finally we give a new proof of Ingrassia's theorem and also show that the low completely mitotic r.e. degrees are nowhere dense in the r.e. degrees. That is, we show that if $\mathbf{a} < \mathbf{b}$ are low r.e. degrees, then there exist r.e. degrees **e**, **f** with $\mathbf{a} < \mathbf{e} < \mathbf{f} < \mathbf{b}$ and such that every r.e. degree in $[\mathbf{e}, \mathbf{f}]$ contains a nonmitotic r.e. set.

We do assume some degree of sophistication of the reader (in view of the material being presented). Many of our arguments are 'tree of strategy' Π_2-guessing ones and we assume the reader completely familiar with this technique. We refer him to Soare [18, 19] for expositions of this technique. The main thrust of our arguments will thus be to discuss the strategies rather than the formal details. Some standard notation will be that σ and τ denote *guesses* (mostly members of $2^{<\omega}$). We write the tree order as $\sigma \leqslant_L \tau$ and mean that (for $2^{<\omega}$) either $\sigma \subset \tau$ or $\exists \gamma\,(\gamma\hat{}0 \subset \sigma$ and $\gamma\hat{}1 \subset \tau)$. At stage s all computations are bounded by $s-1$. All other notation and terminology is completely standard.

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2. Prompt simplicity

As a first step towards the classification of the completely mitotic degrees, in this section we shall analyse their relationship with the promptly simple degrees.

Our interests were aroused by the following partial answer to the question of the existence of low completely mitotic r.e. degrees.

(2.1) **Theorem.** *No low promptly simple r.e. degree is completely mitotic.*

Proof. Let A be low, r.e. and promptly simple with witness function p. That is, we have a recursive enumeration $A = \bigcup_s A_s$ such that

(2.2) $\quad |W_e| = \infty \rightarrow \exists^\infty s, x\ (x \in W_{e,\text{at}\,s} \cap A_{p(s)})$.

We shall build $B \equiv_T A$ with B r.e. and nonautoreducible. By Ladner's [11] result, B will thus be nonmitotic. We must satisfy the requirements

$$R_e\colon\quad \exists x \neg(\Phi_e(B \cup \{x\}; x) = B(x)).$$

The argument we shall give is finite injury, and it will suffice to discuss the strategy for the satisfaction of a single requirement. To make $B \equiv_T A$ we shall build B from *coding markers* $\{\Gamma(y, s)\colon y \in \omega\}$. At each stage s, $\Gamma(y, s)$ rests on a member of $\bar{B}_s$. We shall build a recursive strictly increasing function g and shall ensure that the coding markers satisfy the rules

(i) $\Gamma(x, s) < \Gamma(x+1, s)$,
(ii) let $z = \mu x\ (x \in A_{g(s+1)} - A_{g(s)})$, then
 (a) $\Gamma(\hat{z}, s) = \Gamma(\hat{z}, s+1)$ for $\hat{z} < z$, and $B_{s+1} = B_s \cup \{\Gamma(z, s)\}$,
 (b) $\forall \hat{z} \geqslant z\ (\Gamma(\hat{z}, s+1) > \Gamma(z, s))$.

Clearly, these rules ensure that $B \equiv_T A$; the details are quite standard and are left to the reader.

The fundamental idea for satisfying the R_e is the following. Let $l(e, s) = \max\{z\colon \forall y < z\ (\Phi_{e,s}(B_s \cup \{y\}; y) = B_s(y))\}$. Suppose that we see, at stage s,

(2.3) $\quad y \in A_{g(s+1)} - A_{g(s)}$ with $\Gamma(y, s) < l(e, s)$.

Suppose further that it is lucky enough to be a stage such that also

(2.4) $\quad A_{g(s)}[y-1] = A[y-1]$.

Then we can win the R_e requirement as follows. We set $B_{s+1} = B_s \cup \{\Gamma(y, s)\}$ and then reset

(2.5) $$\Gamma(\hat{y}, s+1) = \begin{cases} \Gamma(\hat{y}, s) & \text{if } \hat{y} < y, \\ b_{\Gamma(\hat{y}, s)+s}, & \text{otherwise,} \end{cases}$$

where $\{b_{i,s}\colon i \in \omega\}$ lists in order the members of $\bar{B}_s$. The crucial point here is that by (2.4) and (2.5) we have ensured that

(2.6) $\quad B_{s+1}[u] = B_s[u]$ where $u = u(\Phi_{e,s}(B_s \cup \{\Gamma(y, s); \Gamma(y, s)))$.

Therefore by (2.3) and definition of $l(e, s)$, we see that

$$\Phi_e(B \cup \{\Gamma(y, s)\}; \Gamma(y, s)) = B_s(\Gamma(y, s)) = 0 \neq 1 = B(\Gamma(y, s)).$$

To complete the proof, it thus suffices to describe how we shall achieve (roughly) (2.3) and (2.4). To achieve (2.3) we build an auxiliary r.e. set V_e (whose index is given by the recursion theorem) and use the prompt simplicity of A to ensure that (2.3) occurs infinitely often. Specifically, at any stage we see $l(e, s) > \Gamma(y, s)$ and also $y \notin A_{g(s)}$ and $y \notin V_{e,s}$, we enumerate y into $V_{e,s+1} - V_{e,s}$.

Now we appeal to the Slowdown Lemma ([4, Lemma 1.5] or [19, XIII, Lemma 1.5]) to see if A 'promptly permits' on y. That is, using the recursion theorem we have a recursive function q defined by

$$W_{q(e)} = \{x: (\forall s)[x \in V_{e,s} - W_{q(e),s}]\}.$$

The Slowdown Lemma says that $W_{q(e)}$ so construed, has the property that $V_e = W_{q(e)}$ and an element that occurs in V_e occurs strictly later in $W_{q(e)}$.

Thus we compute the least stage $t > s$ such that $y \in W_{q(e),t}$ and then see if $y \in A_{p(t)}$. If $y \notin A_{p(t)}$ do nothing else except continue. If $y \in A_{p(t)}$ it must be that $p(t) > g(s)$ and we can set $g(s+1) \geqslant p(t)$ to ensure that (2.3) holds. The prompt simplicity condition (2.2) and the fact that $\bar{A}$ is infinite ensure that "$y \in A_{p(t)}$" must occur for infinitely many y (if $l(e, s) \to \infty$, say).

Thus, we have reduced our problem to showing that at least once within the stages where "$y \in A_{p(t)}$" option occurs, we can also arrange that (2.4) occurs. Actually, we need only ensure that (2.4) occurs for the *least* $y \in A_{g(s+1)} - A_{g(s)}$. To do this we use the lowness of A. As with (2.3) we shall construct auxiliary r.e. sets $W_{h(e)}$ with h recursive. This time $W_{h(e)}$ will be a set of canonical indices of finite sets. By Soare [17], as A is low we have $C \leqslant_T \emptyset'$ where

$$C = \{e: (\exists u \in W_{h(e)})[D_u \subset \bar{A}]\}.$$

By the limit lemma there is a recursive function $k(e, s)$ such that $\forall e\,(k(e) = \lim_s k(e, s))$. By the recursion theorem we can use k in the construction. Now we wait till we get a "$y_1 \in A_{p(t)}$" case from (2.3). At such a time we test if $A_{g(s)}[y_1 - 1] = \underline{A[y_1} - 1]$ by enumerating u into our test set $W_{h(e),s+1}$, where D_{u_1} is the set of $\hat{y} \in \bar{A}_{g(s)}$ with $\hat{y} \leqslant y_1$. We search for a stage $t_1 \geqslant g(s)$, $p(t)$ such that either $D_{u_1} \cap A_{t_1} \neq \emptyset$ or $k(e, t_1) = 1$. If $D_{u_1} \cap A_{t_1} \neq \emptyset$, then let y_2 be the least number in $A_{t_1} - A_{g(s)}$. Notice that $y_2 < y_1$ as $D_{u_1} \cap A_{t_1} \neq \emptyset$. In this case let u_2 denote the set of $\hat{y} \in \bar{A}_t$ with $\hat{y} \leqslant y_2$. Find a stage $t_2 > t_1$ such that either $k(e, t_2) = 1$ or $D_{u_2} \cap A_{t_2} \neq \emptyset$. Continue as above until a $y_n \leqslant y_1$, an index u_n and a stage t_n are found with $k(e, t_n) = 1$. Let $y = y_n$. In this case we set $g(s+1) = t_n$.

We must note that y is the least number to occur in $A_{t_n} - A_{g(s)}$ and (2.3) holds for y. What of (2.4)? We do not know that (2.4) holds, but since $k(e, t) = 1$ it appears that (2.4) holds (remember $k(e, t) = 1$ means it looks like $A_{g(s)}[y - 1] = A[y - 1]$ since $k(e) = 1$ means $(\exists u \in W_{h(e)})[D_u \subset \bar{A}]$.). Thus we believe that we have met R_e unless we see that $D_{u_n} \cap A_{\hat{t}} \neq \emptyset$ for some $\hat{t} > t_n$. In this case we search for a new y_1 from (2.3). The crucial observation is that we can wait for some stage $r > t$ with $k(e, r) = 0$ and begin attacking R_e anew. Note that R_e cannot receive attention infinitely often in this way since then $\lim_t k(e, t)$ would then not exist

($k(e, t)$ would change from 0 to 1 with each attack). The remaining details are completely standard finite injury argument obtained from the above strategies and are so left to the reader. □

It seems natural to ask if either hypothesis may be removed from the above. As we shall see in Section 3 there are low completely mitotic r.e. degrees. As our last result for this section we shall show that there are also promptly simple completely mitotic r.e. degrees so both hypotheses are essential. The reader should note that the strategies of Section 3 do not seem to combine with promptness. Our construction here is more along the lines of Ladner's original one of [12].

(2.7) **Theorem.** *There exists a promptly simple completely mitotic r.e. degree.*

Proof. We shall build $A = \bigcup_s A_s$ together with auxiliary sets $C_e = \bigcup_s C_{e,s}$ and $D_e = \bigcup_s D_{e,s}$ to satisfy the requirements

$$P_e:\quad |W_e| = \infty \to \exists s,x\ (x \in W_{e,\text{at}\,s} \to x \in A_{s+1}),$$

$$N_e:\quad \Phi_e(A) = V_e \,\&\, \Gamma_e(V_e) = A \quad \text{implies}$$

$$C_e \sqcup D_e = V_e,\ A \leqslant_T C_e \quad\text{and}\quad A \leqslant_T D_e.$$

Here (Φ_e, V_e, Γ_e) denotes a standard enumeration of triples consisting of 2 reductions and an r.e. set. (Actually we shall be using a tree of strategies argument and the builidng C_σ and D_σ for certain $\sigma \in 2^{<\omega}$ with $\text{lh}(\sigma) = e+1$. We discuss this further later.)

Of course, we must also make $|\bar{A}| = \infty$, but this causes no problem and can be ensured in any of the usual ways. For definiteness we shall only ever choose numbers larger than the e-th member of $\bar{A}_s$ for the sake of P_e. We won't explicitly mention this further but assume it done implicitly.

The P_j and the N_e for $e<j$ interact very strongly. For the sake of P_j we put numbers into A causing us to force numbers first into D_e and then into C_e as we describe in the 'basic module' (for the interaction of N_e and P_j) below. Let

$$l(e,s) = \max\{x : \forall y < x\ [\Gamma_{e,s}(V_{e,s}; y) = A_s(y)\ \&$$
$$\forall z\ [z < u(\Gamma_{e,s}(V_{e,s}; y)) \to \Phi_{e,s}(A_s; z) = V_{e,s}(z)]]\}$$

and

$$ml(e,s) = \max\{l(e,t) : t < s\}.$$

The reader should think of $l(e,s)$ above as the 'A-controllable' length of agreement. The basic module for a P_j (for $j>e$) and N_e — with $l(e,s) \to \infty$ — consists of the following steps.

Step 1. Pick a *prefollower* $y = y(e,j) = s_1$ targeted for A. (The notation here is that the "e" refers to N_e and the "j" to P_j.)

Step 2. Wait for the first stage $s_2 > s_1$ to occur with $l(e, s_2) > ml(e, s_2) > y$. Declare y as *e-confirmed* and cancel all followers or prefollowers z targeted for A with $y < z$ $(< s_2)$. (These will be of lower priority.) Now set $x = s_2 = x(e, j)$ as a follower of P_j targeted for A.

Step 3. Wait for the first stage $s_3 > s_2$ with $l(e, s_3) > ml(e, s_3) \geqslant x$. Declare x as e-confirmed and cancel the (lower priority) followers and prefollowers z with $z > x$.

Remark. At the end of Step 3 we have the situation described in Diagram 1 below.

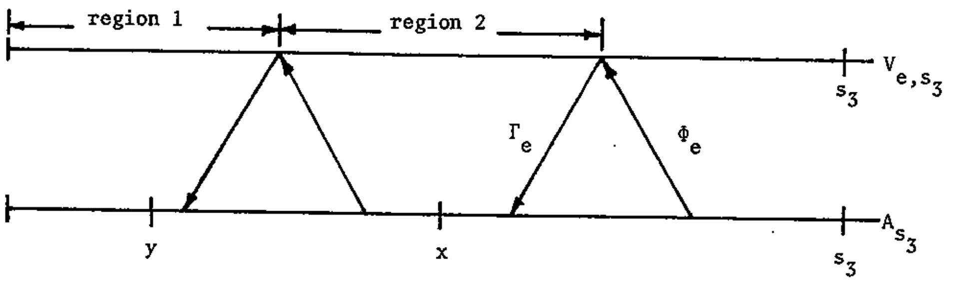

Diagram 1

Notice that the only number alive (i.e. follower or prefollower) between y and s_3 is x. Since we always assign followers or prefollowers to be the stage number, we shall see that after stage s_3 the regions 1 and 2 are fixed unless $A_{s_3}[x] \neq A[x]$ and region 1 is fixed unless $A_{s_3}[y] \neq A[y]$.

Step 4. If at some stage $s_4 > s_3$ we see that $W_{j,s_3} \cap A_{s_3} = \emptyset$ and there exists $z \in W_{j,\text{at}\, s_4}$ with $z > x$ (by convention $z < s_4$), then we set $A_{s_4+1} = A_{s_4} \cup \{\hat{x} \mid x \leqslant \hat{x} \leqslant s_4\}$. (An alternative here is to enumerate z and x into $A_{s_4+1} - A_{s_4}$ and cancel all lower priority prefollowers ($\geqslant x$). The extra dumping achieves a similar effect and simplifies exposition. This idea is also used in [6].) Declare y as *activated.*

Step 5. Wait for a stage $s_5 > s_4$ with $l(e, s_5) > ml(e, s_5)$. The crucial observation is that if we have held $A_{s_5}[y] = A_{s_2}[y]$, that is $A_{s_5}[s_2 - 1] = A_{s_2}[s_2 - 1]$, then $V_{e,s_5} - V_{e,s_4}$ must differ from V_{e,s_2} on region 2 of Diagram 1. We enumerate all such changes into $D_{e,s_5+1} - D_{e,s_5}$. (The other possibility here is that perhaps some P_k i.e. $x(e, k)$ for $k < j$ (of higher priority) has acted.) Now set $A_{s_5+1} = A_{s_5} \cup \{y\}$.

Step 6. At the next stage $s_6 > s_5$ with $l(e, s_6) > ml(e, s_6)$—if no $x(e, k)$ of higher priority has acted—we will have seen a change in $V_{e,s_6} - V_{e,s_5}$ in region 1. Such changes are enumerated into $C_{e,s_6+1} - C_{e,s_6}$. That is, we set $C_{e,s_6+1} = C_{e,s_5} \cup (V_{e,s_6} - (C_{e,s_5+1} \cup D_{e,s_5+1})) = C_{e,s_5} \cup (V_{e,s_6} - V_{e,s_5})$. (If some $x(e, k)$ has acted, perhaps we are at Step 5 again.)

This process satisfies N_e as follows: We wish to decide if $z \in A$ from either C_e or D_e. Now we observe that $z \in A_{u+1} - A_u$ only if z is a follower, prefollower or some follower $\hat{z} \leqslant z$ enters A at the same stage as z. It therefore suffices to argue that both C_e and D_e can recognize entry of followers and prefollowers.

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Suppose therefore, that z is a follower or a prefollower. Find a stage $s = s(z)$ by which z is cancelled, enumerated into A, or z is e-confirmed. First suppose z is a prefollower. Let $u = u(\Gamma_{e,s}(V_{e,s}; z))$. By our remark after Step 3 of the construction, z cannot enter $A - A_s$ unless $V_e[u]$ changes and so unless $A_s[s-1]$ changes. (In fact, $A_s[\hat{u}]$ changes where $\hat{u} = \max\{u(\Phi_{e,s}(A_s; y)) : y \leqslant u\}$.) We claim that this cannot happen unless $C_{e,s}[u]$ changes. But this is not too hard to see, since the last change in any sequence of changes is always a C_e change. The intuition here is that z can enter because we enumerate it into A since its follower goes in (at an earlier stage), or some follower $\hat{z} < z$ enters A causing z's entry into A (by dumping). In the latter case $\hat{z}$ will then activate its prefollower $\hat{y}$ which will try to cause a C_e change. Eventually we must cause a C_e change, because this can only delay matters a finite amount of time. Thus to decide if $z \in A$ or not find the least stage s with $C_{e,s}[u] = C_e[u]$ then $z \in A$ iff $z \in A_s$.

The D_e case is similar. As z is a prefollower we can go to a stage $t > s$ such that either z gets cancelled, enumerated or z gets a follower $\hat{z} > z$. Let $\hat{s}$ be the stage where $\hat{z}$ is confirmed or cancelled or enumerated. By essentially the same argument we see that $\hat{z} \in A$ iff $D_{e,\hat{s}}[\bar{u}] \neq D_e[\bar{u}]$ where $\bar{u} = u(\Gamma_{e,\hat{s}}(V_{e,\hat{s}}; \hat{z}))$.

By Step 6 (i.e. delayed permitting) we can D_e-compute if $z \in A$ as follows. Find the least stage $t_1 > s$ with $l(e, t_1) > ml(e, t_1)$, and $D_{e,t_1}[\bar{u}] = D_e[\bar{u}]$. Now find the least stage t_2 with $t_2 > t_1$ and $l(e, t_2) > ml(e, t_2)$. Then $z \in A$ iff $z \in A_{t_2}$.

The case where z is a follower is entirely similar and left to the reader.

Cooperation

There are several problems concerning the cooperation of the various N_e. In a perfectly standard way we must arrange the N_e with the usual Π_2-guessing strategy on a tree. That is, we don't know which $l(e, s) \to \infty$, but equip followers with guesses as to whether or not $l(e, s) \to \infty$ (as in a minimal pair construction), so that the leftmost path gives the correct outcome. Thus "$l(e, s) \to \infty$" is identified with those $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = e + 1$ and $\sigma = \tau\hat{\ }0$. This changes our sets C_e and D_e to C_σ and D_σ and if σ is on the left-most = true path we ensure that $C_\sigma \cup D_\sigma =^* V_e$ and have the desired properties. In place of our notation $x(e, j)$, if x is a number it will have a *guess* σ and an *association* τ so that we write $x(\sigma, \tau)$. The intention is that $\mathrm{lh}(\tau) = j$, $\mathrm{lh}(\sigma) = e$ and $\sigma \subset \tau$. (It will be the case that $\sigma = \gamma\hat{\ }0$ unless $\sigma = \tau$.)

Now the usual way we would implement this guessing idea for a single P_e cooperating with a single N_j ($j \leqslant e$) is to have two versions of P_e. One guesses $l(j, s) \not\to \infty$ the other, of higher priority guesses $l(j, s) \to \infty$. Of course we only appoint followers to P_e guessing $l(j, s) \to \infty$ at stages when it appears correct. Since there are eventually infinitely many 'j-stages' if $l(j, s) \to \infty$ eventually we pursue the correct strategy.

However, for more than one N_e the problems are more subtle. Let us suppose that $\tau = 1\hat{}0\hat{}1\hat{}0\hat{}1\hat{}0\hat{}1$, say and put $\sigma_1 = 1\hat{}0$, $\sigma_2 = 1\hat{}0\hat{}1\hat{}0$ and $\sigma_3 = 1\hat{}0\hat{}1\hat{}0\hat{}1\hat{}0$. Suppose we are working with a version of P_6 of guess τ. The obvious first approximation to the 'α-strategy' is to simply appoint a follower and a prefollower to P_6 with guess τ at τ-stages. That is at the first stage where τ looks correct we appoint to P_6 a prefollower y with guess τ. Suppose P_6 requires attention and we put x into A at stage s. The intuitive content of the basic module is that we wait until the next *τ-stage* (so that all the σ_1, σ_2, σ_3-computations have all recovered) and then add the prefollower y.

There is a very big problem with this. That is, suppose there is never again another τ-stage. Perhaps σ_2 and σ_3 are strictly left of the true path and there are never again (even) any σ_2-stages. In that case our actions don't matter to σ_3 and σ_2 but really *do matter to* σ_1. After all perhaps σ_1 really is on the true path and since the length of agreement corresponding to σ_1 ($l(\sigma_1, s)$) tends to infinity, σ_1 expects us to build a mitotic splitting $V_1 = C_{\sigma_1} \sqcup D_{\sigma_1}$. Now σ_1 expects the strategy to be: add a follower to A, wait till the next *σ_1-(expansionary) stage* and then add a prefollower to *A for the sake of σ_1*. The crucial point is that this is the *next σ_1-stage* not the *next τ-stage.*

More generally as $\sigma_1 \subset \sigma_2 \subset \sigma_3 \subset \tau$ implicitly we have made certain commitments to C_{σ_i} and D_{σ_i} for $i = 1, 2, 3$. Namely, somehow we have promised to first change D_{σ_i} through some 'confirmed region 2' and then *wait till the next 'σ_i-(expansionary-) stage'* (i.e. when "$l(\sigma_i, s) > ml(\sigma_i, s)$") occurs to then force a C_{σ_i} change. Thus implicitly our action for the sake of τ has committed us to much higher priority activity (namely, e.g., σ_1-activity). The point is from σ_1's point of view, at the next stage $\bar{s}$ with $l(\sigma_1, \bar{s}) > ml(\sigma_1, \bar{s})$, σ_1 expects us to attend to its pending prefollower commitment. On the other hand σ_3, say, wants us to wait till the next stage $\hat{s}$ where, not only does $l(\sigma_1, \hat{s}) > ml(\sigma_1, \hat{s})$ but $l(\sigma_3, \hat{s}) > ml(\sigma_3, \hat{s})$. The point is that there may never be such a stage $\hat{s}$, but there may be a stage $\bar{s}$; perhaps σ_3 is left of the true path but σ_1 is on the true path.

Our solution is to abandon the single prefollower/follower arrangement and give a τ-follower an *entourage of prefollowers* each reflecting its own commitments.

Thus the τ-follower $x = x(\tau, \tau)$ will need a σ_1-prefollower z to fulfil x's pending σ_1-commitment at the next σ_1-stage. We denote this by $z = z(\sigma_1, \tau)$. That is, z is a number devoted to fulfilling a σ_1-commitment initiated by τ. Note that once we add z at the σ_1-expansionary stage we need do nothing more for the sake of σ_1 unless there occurs a σ_2-expansionary stage. Of course we shall need a prefollower $q = q(\sigma_2, \tau)$ which will need be enumerated into A to cause a C_{σ_2} change. This in turn will create a new σ_1-commitment since q will take the role of a follower as σ_1 is concerned and so q must have a σ_1-prefollower $r(\sigma_1, \tau)$. These events are orchestrated as discrete events: namely σ_2 'doesn't believe' computations until all pending σ_1-prefollower activities are completed. Thus the events must happen at $s_1 < s_2 < s_3 < s_4$ with s_4 a τ-stage, s_2 a σ_1-stage that is not a

σ_2-stage, s_3 a σ_2-stage (that must be a σ_1-stage too), and s_4 a σ_1-stage (that precedes the next σ_2-stage after s_3). Of course, in our example τ will also need a σ_3-prefollower which needs a σ_1-prefollower; etc.

The picture we eventually get, in this example, is

$$x_1(\sigma_1, \tau) < x_2(\sigma_2, \tau) < x_3(\sigma_1, \tau) < x_4(\sigma_3, \tau) < x_5(\sigma_1, \tau)$$
$$< x_6(\sigma_2, \tau) < x_7(\sigma_1, \tau) < x_8(\tau, \tau).$$

Diagram 2 below is helpful to see the pattern.

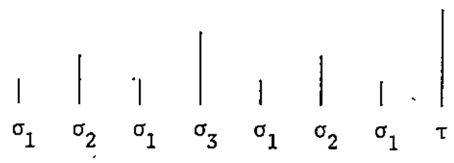

Diagram 2

In summary, the idea is that once we put something into A for the sake of τ we fulfill our σ_1-commitment at the next stage with $l(\sigma_1, s) > ml(\sigma_1, s)$. We then fulfill our σ_2-commitments when $l(\sigma_2, \hat{s})$. This $\hat{s}$ activity creates new σ_1-commitments which we then fulfill, etc. (The reader should note that this means at stages where we are putting numbers into C_{σ_1} we may be putting numbers into D_{σ_2}.) The above idea is the key to the coherence of the strategies, and thus of the whole construction. We now give some formal details, but expect that the reader may prefer to supply them himself.

Trace entourage

Let $\sigma \in 2^{<\omega}$. Define the rank, rk(σ), of guess σ as $|\{i : \sigma(i) = 0\}|$. (Thus, e.g., $\text{rk}(1\hat{}0\hat{}1\hat{}0\hat{}1\hat{}0\hat{}1) = 3$. We define the *sequence of* σ, seq(σ) by induction on lh(σ) as follows.

(i) $\text{rk}(\sigma) = 0$. Let $\text{seq}(\sigma) = (\sigma)$.

(ii) $\sigma = \tau\hat{}1$. It will be the case that $\text{seq}(\tau) = (\eta_1, \ldots, \eta_k, \tau)$ for some (possibly empty) sequence $(\eta_1, \ldots, \eta_k)$. Define $\text{seq}(\sigma) = (\eta_1, \ldots, \eta_k, \sigma)$. Note $\text{lh}(\text{seq}(\tau)) = \text{lh}(\text{seq}(\sigma))$.

(iii) $\sigma = \tau\hat{}0$. If $\text{rk}(\sigma) = 1$, then set $\text{seq}(\sigma) = (\tau\hat{}0, \tau\hat{}0) = (\sigma, \sigma)$. If $\text{rk}(\sigma) > 1$, then we have already defined $\text{seq}(\tau) = (\gamma_1, \ldots, \gamma_n, \tau)$. We define $\text{seq}(\sigma) = (\gamma_1, \ldots, \gamma_n, \tau\hat{}0, \gamma_1, \ldots, \gamma_n, \tau\hat{}0)$ (in this order).

The idea is to simply capture the situation of Diagram 2. The reader may like to convince himself that it does. Thus for example, seq(σ) has $m(\sigma) = 2^{\text{rk}(\sigma)}$ many elements all except the last of which are of the form $\gamma_i\hat{}0$ (various i).

Now to satisfy P_e at guess τ we will wait until the entourage is complete and confirmed. That is, we have a prefollower/follower for each member of seq(τ). In the hope that it does not confuse, we shall denote the current i-th member of the entourage at stage s by $x_i^s(\mu, \tau)$ where here the i-th member of seq(τ) is μ. Finally we say $x_i^s(\mu, \sigma)$ *connects with* $x_j^s(\alpha, \sigma)$ *at stage s* if for all k with $i \leq k \leq j$, it is the case that for some γ, $x_k^s(\gamma, \sigma)$ is defined, and $i < j$.

Remark 1. Regarding the notation $x_i^s(\sigma, \tau)$, note that the association (τ) reflects only what the entourage pertains to. The true guess of x_i^s (reflecting its priority *should it become active*) is σ. As we see if $x_i^s(\sigma, \tau)$ is alive, then $x_i^s(\sigma, \tau)$ reflects a pending σ-commitment to any active member of $x(\tau, \tau)$'s entourage with which $x_i^s(\sigma, \tau)$ connects.

Remark 2. The point of 'connection' is this. Suppose we have an $x_k^s(\gamma, \sigma)$ and a stage s where the guess γ appears wrong. Then we shall cancel $x_k^s(\gamma, \sigma)$. However $x_i^s(\mu, \sigma)$ may not be cancelled by this process (perhaps $\mu \neq \gamma$). In this case we will see that $x_j^s(\mu, \sigma)$ is no longer connected with any x_j^s for $j > k$. This will signal us to also cancel x_i^s.

Definition. We define the notions *σ-stage, σ-correct length of agreement* and *σ-injurious number* by induction on lh(σ) at stage s:

(i) Every stage s is a $\emptyset$-stage.
(ii) If s is a τ-stage, define a number y to be $\tau\hat{}0$*-injurious* if
 (a) $y = x_i^s(\mu, \gamma)$ for some i, μ, γ,
 (b) y is connected with some active $x_j^s(\alpha, \gamma)$ such that all intervening numbers between y and x_j^s have not lower priority than τ.

Remarks. (1) Note that $x_i^s = y$ and x_j^s are members of the same entourage (i.e. of γ). We can write (b) as

$$(\forall k)\ (i \leq k \leq j, \text{ the guess of } x_k^s(\rho, \tau) \text{ is } \subset \tau)\ (\text{that is } \rho \subset \tau).$$

(2) The idea here is that each τ-injurious number represents a higher priority commitment which is as yet unfulfilled. Returning to the example described earlier (in Diagram 2), suppose $\eta = 1\hat{}0\hat{}1\hat{}0\hat{}1$. Suppose that $x_8(\tau, \tau)$ enters A at some stage s. At this stage we declare $x_7(\sigma_1, \tau)$ as active (as we will see). Now $\eta\hat{}0 = \sigma_3$ 'knows' that the numbers $x_5(\sigma_1, \tau)$, $x_6(\sigma_2, \tau)$ and $x_7(\sigma_1, \tau)$ all correspond to higher priority commitments which must be completed before we want to believe σ_3-computations. Thus we regard x_5, x_6 and x_7 as $\eta\hat{}0$ injurious according to the above.

A slightly more instructive example would be if $\gamma = 1\hat{}0\hat{}1\hat{}0\hat{}0$, in the above example. In this case however γ will cancel σ_3 ($\gamma <_L \sigma_3$) (and so force x_1, x_2 and x_3 to be no longer connected with x_8, and therefore cancel them too). Nevertheless if x_7 were active we would regard x_5, x_6 and x_7 as $\gamma\hat{}0$-injurious.

Now define the $\tau\hat{}0$-*correct length of agreement* as the largest number $z<l(e,s)$ such that for all $\hat{z}\leq z$ there is no $\tau\hat{}0$-injurious number $\leq u(\hat{z},e,s)$ where

$$u(\hat{z},e,s)=\max\{u(\Phi_{e,s}(A_s;y)): y\leq u(\Gamma_{e,s}(V_{e,s};g))\ \&\ g\leq\hat{z}\}.$$

Let $l(\tau\hat{}0,s)$ denote the $\tau\hat{}0$-correct length of agreement. Then if $l(\tau\hat{}0,s)>\max\{l(\tau\hat{}0,t): t$ is a τ-stage and $t<s\}$ we say that s is a $\tau\hat{}0$-stage. Otherwise we say s is a $\tau\hat{}1$-stage.

Remark. The driving force behind the idea of $\tau\hat{}0$-correctness is that we 'don't believe' a computation until we see all injurious 'pending commitments' (to add elements to A) of higher priority are complete.

Definition. Let σ_s denote the unique path of length s such that s is a σ_s-stage.

Definition. We say that P_e *requires attention* at stage $s+1$ if $W_{e,s}\cap A_s=\emptyset$ and one of the following options holds.

(2.8) None of P_e's entourage at guess $\sigma\subset\sigma_s$ with $\mathrm{lh}(\sigma)=e+1$ is defined. That is, $x_1^s(\gamma,\sigma)$ is not defined for all γ.

(2.9) P_e has an incomplete entourage at guess σ and if $x_i^s(\ ,\sigma)$ denotes the current largest (defined) member of this entourage then x_i^s is σ-confirmed.

(2.10) P_e has a τ-confirmed follower x with guess $\tau\leq_{\mathrm{L}}\sigma_s$ (note that it is not necessary for $\tau\subset\sigma_s$) and there exists $y\in W_{e,\mathrm{at}\,s}$ with $y>x$. (Note: having a follower means that the entourage of x is complete.)

Construction, stage $s+1$

Step 1. Cancel all numbers $x_i^s(\alpha,\sigma)$ with $\alpha\not\leq_{\mathrm{L}}\sigma_s$. For such α set $C_{\alpha,s+1}=\emptyset$ and $D_{\alpha,s+1}=V_{e,s}$. We say that such α are *initialized.*

Step 2. Cancel all numbers $x_i^s(\alpha,\sigma)$ with $\sigma\not\leq_{\mathrm{L}}\sigma_s$ and $x_i^s(\alpha,\sigma)$ not connected to an active $x_j^s(\gamma,\sigma)$ still alive after Step 1.

Step 3. Find the least number $x_i^s(\rho,\tau)$ not yet $\alpha\hat{}0$-confirmed for some $\alpha\hat{}0\subset\tau$. (Note the τ here rather than ρ), such that

(i) x_i^s is $\gamma\hat{}0$-confirmed for all $\gamma\hat{}0\subsetneq\alpha\hat{}0$, and

(ii) $l(\alpha\hat{}0,s)>x_i^s(\rho,\tau)$.

Declare x_i^s as $\alpha\hat{}0$-confirmed. Cancel all followers and prefollowers $>x_i^s$. (These will be of lower priority.) Note $\alpha\hat{}0\subset\sigma_s$ here.

Step 4. Now for any active $x_j^s(\gamma,\sigma)$ with $\gamma\subset\sigma_s$, enumerate $x_j^s(\gamma,\sigma)$ into $A_{s+1}-A_s$. If $x_{j-1}^s(\hat{\gamma},\sigma)$ is defined for some $\hat{\gamma}$, declare $x_{j-1}^s(\hat{\gamma},\sigma)$ as active.

Step 5. Find the least e, if any, such that P_e requires attention. Adopt the appropriate case below, choosing Case 3 if more than one pertains.

Case 1: (2.8) *holds*. If $\mathrm{rk}(\sigma)=0$ appoint $x_1^s(\sigma,\sigma)=s$ as a follower of P_e with guess σ. Otherwise let $\alpha\hat{}0 \subset \sigma$ be the unique α with $\text{rk}(\alpha\hat{}0) = 1$. Appoint $x_1^s(\alpha\hat{}0, \sigma) = s$ as the first member of P_e's σ-entourage. Note that $\alpha\hat{}0$ is the first term of $\text{seq}(\sigma)$.

Case 2: (2.9) *holds*. Let $m = 2^{\text{rk}(\sigma)}$. If $i = m$ declare the entourage as complete and declare $x_m^s(\sigma, \sigma)$ as the current follower of P_e. Otherwise, set $x_{i+1}^s(\gamma_{i+1}, \sigma) = s$ as the $(i+1)$-st member of P_e's σ-entourage, where γ_{i+1} is the $(i+1)$-st term of $\text{seq}(\sigma)$.

Case 3: (2.10) *holds*. Set $A_{s+1} = A_s \cup \{\hat{z} : \hat{z} \geq x \;\&\; \hat{z} \leq s\}$. Notice that this meets P_e since $z < s$ by convention. If $x = x_1^s(\tau, \tau)$ (and so $\text{rk}(\tau) = 0$) do nothing else. Otherwise find $j = 2^{\text{rk}(\tau)} - 1$ hence the j-th term γ_j of $\text{seq}(\tau)$. Declare $x_j^s(\gamma_j, \tau)$ as active.

Step 6 (*Recovery*). For each $\tau\hat{}0 \subset \sigma_s$, where indicated, we ensure that $C_{\tau\hat{}0} \sqcup D_{\tau\hat{}0}$ is a mitotic splitting of V_e (where $e = \text{lh}(\tau)$). Let $\hat{s}$ denote the last $\tau\hat{}0$-stage $< s$. Adopt the first case below.

Case 1. For no $\sigma, \gamma \supset \tau\hat{}0$ was $x_i^{\hat{s}}(\gamma, \sigma)$ active at any stage with $\hat{s} \leq t < s$, or $C_{\tau\hat{}0}$ and $D_{\tau\hat{}0}$ have been initialised at some stage t with $\hat{s} \leq t < s$.

Action. In this case we set

$$D_{\tau\hat{}0,s+1} = (V_{e,s} - (C_{\tau\hat{}0,s} \cup D_{\tau\hat{}0,s})) \cup D_{\tau\hat{}0,s} \quad \text{and}$$
$$C_{\tau\hat{}0,s+1} = C_{\tau\hat{}0,s}.$$

Case 2. There exists a $\leq_L$-least $\sigma \supset \tau\hat{}0$ such that at some stage t with $\hat{s} \leq t < s$ for some i, $x_i^t(\sigma, \sigma)$ was enumerated into $A_{t+1} - A_t$.

Action. Proceed as in Case 1.

Case 3. Cases 2 and 1 did not pertain. In this case there existed some active $x_i^{\hat{s}}(\gamma, \sigma)$ at stage $\hat{s}$ with $\tau\hat{}0 \subset \gamma \subset \sigma$. We must determine which of $C_{\tau\hat{}0}$ or $D_{\tau\hat{}0}$ is appropriate for enumeration. This is done by a simple counting argument (see Diagram 2).

Subcase (i): $(i/2^n) \equiv 0 \pmod 2$.

Action. Set

$$C_{\tau\hat{}0,s+1} = C_{\tau\hat{}0,s} \cup (V_{e,s} - (C_{\tau\hat{}0,s} \cup D_{\tau\hat{}0,s})) \quad \text{and}$$
$$D_{\tau\hat{}0,s+1} = D_{\tau\hat{}0,s}.$$

Subcase (ii): Otherwise.

Action. Proceed as in Case 1. □ (End of Construction)

Verification. It remains to verify that the construction indeed does what we ask. To do this, we really only need formalize the intuitive remarks preceding the construction. Thus, in some instances we only sketch the details.

Let β denote the leftmost path. That is, we define β by induction: $\emptyset \subset \beta$. Also if $\tau \subset \beta$, then exactly one of $\tau\hat{}0$ or $\tau\hat{}1 \subset \beta$. It is the case that $\tau\hat{}0 \subset \beta$ iff $\exists^\infty s\, (\tau\hat{}0 \subset \sigma_s)$.

To verify the P_e, let $\alpha \subset \beta$ be such that $\text{lh}(\alpha) = e + 1$. We show that P_e receives attention at most finitely often at α-stages, and is met. But this is quite easy to see. Let s_0 be an α-stage such that for all $s > s_0$

(i) $\alpha \leq_L \alpha_s$,

(ii) whenever $\gamma \leq_L \alpha$ and $\gamma \not\subset \alpha$ we have for all i, δ, $x_i^s(\gamma, \delta) \in A$ iff $x_i^s(\gamma, \delta) \in A_s$,

(iii) for all $\hat{\sigma} \subset \sigma$ with $\hat{\sigma} \neq \sigma$ and for all γ, i we have $x_i^s \in A$ iff $x_i(\gamma, \hat{\sigma}) \in A_s$,

(iv) P_j for $j < e$ do not receive attention at α-stages, and

(v) for all $j < e$ if P_j has a (pre-) follower $x_i^{s_0}(\gamma, \delta)$ with $\gamma \leq_L \alpha$, then this follower never receives any further confirmation after stage s_0. (There are only finitely many, after all.)

Now after stage s_0 any (pre-) follower appointed to P_e is uncancellable. Since $\alpha \subset \beta$ each becomes eventually α-confirmed and so P_e eventually gets a complete entourage. It is quite easy then to see that if P_e fails to receive attention, then $|W_e| < \infty$ or $W_e \cap A \neq \emptyset$ promptly.

Finally, we turn to the N_e with α and s_0 as above. An induction easily shows that for all $s_1 > s_2 > s_0$ we have $C_{\alpha,s_1} \supset C_{\alpha,s_2}$ and $D_{\alpha,s_2} \supset D_{\alpha,s_1}$. Moreover, if $\Phi_e(A) = W_e$ and $\Gamma_e(V_e) = A$, then it must be that $\alpha = \tau\hat{}0$ for some τ with $\mathrm{lh}(\tau) = e$. Therefore there are infinitely many stages at which Step 6 pertains to $\tau\hat{}0$, and so $C_\alpha \sqcup D_\alpha = V_e$, with C_α and D_α r.e.

Now, for example, to compute if $z \in A$ or not from D_α proceed as follows. Let $\hat{z} = \max\{z, s_0\}$. Compute the least α-stage s_1 exceeding $\hat{z}$. If $z \notin A_{s_1}$, then z can enter A after s_1 only at the same time as some follower $\hat{z} \leq z$ (if z is not a prefollower), or z is a prefollower. In both cases $\hat{z}$ or z must already be present at stage s_1 and thus, as in the intuitive discussion, it suffices to determine whether or not a (pre-) follower will enter A.

For example, suppose z is a pre-follower. Then $z = x_i(\gamma, \delta)$ for some γ, δ.

Now at stage s_1 if z is not yet cancelled and yet still can enter A it must be that either z is α-confirmed and $\alpha \subset \delta$, or z is connected with some active $x_i(\hat{\gamma}, \delta)$ with $\hat{\gamma} \subset \sigma_s$.

In the first case, in exactly the same way as in the intuitive discussion, we see that $z \in A$ iff $z \in A_{s_2}$ where s_2 is the least α-stage $> s_1$ with $D_{\alpha,s_2}[s_1] = D_\alpha[s_1]$.

In the case that z is connected, we have that either $\hat{\gamma} \subset \alpha$ in which case $z \in A$, or $\alpha \subset \hat{\gamma}$ in which case we proceed as above since in that case z is α-confirmed.

All of the other cases are essentially similar and are left to the reader. □

We conclude this section with a brief technical discussion as to the nature of the proof of (2.7). The reader may proceed directly to Section 3 with no loss of continuity.

From a slightly different (higher?) point of view, what the above construction achieves is this: in using our confirmation/tracing procedure to satisfy the N_e we actually build a wtt-reduction Δ_e (of course Δ_σ with $\mathrm{lh}(\sigma) = e + 1$) from V_e to A. For the sake of this discussion, let us drop all subscripts. The reduction procedure Δ is a simple permitting one. To decide if $x \in A$ or not find the least σ-stage $s_1 > x$

with all followers and prefollowers $<x$, σ-confirmed (or $x \in A_{s_1}$). Compute the least σ-stage $s_2 > s_1$ with $A_{s_1}[s_1] = A[s_1]$. Then in the same way as for C_σ we see that $x \in A$ iff $x \in A_{s_2}$. Now let us speed up the enumeration of A so that we have $A = \bigcup_s \tilde{A}_s$ where $\tilde{A}_s = A_{t_s}$ where t_s is the s-th σ-stage, and $V = \bigcup_s \tilde{V}_s$ similarly. We can thus regard $\Delta(V) = A$ such that $l(s) > l(s+1)$ where $l(s) = \max\{x: \forall y < x\ (\Delta_s(\tilde{V}_s; y) = \tilde{A}_s(y))\}$. (All of this says: look at V and A only at σ-stages.)

Now define a new set $E = \bigcup_s E_s$ as follows: at *stage* $s+1$ let

$$E_{s+1} = E_s \cup \{\mu_z\ (z \in \tilde{A}_{s+1} - \tilde{A}_s)\}.$$

Thus z is the least number to enter A between the s-th and the $(s+1)$-st σ-stages.

We invite the reader to verify:

(2.12) **Corollary** (to the construction of (2.7)). *E is mitotic.*

The point of this discussion is that another proof of (2.7) can be obtained from the above observations and the following lemma of Ambos-Spies.

(2.13) **Theorem** (Ambos-Spies [1, §2, Lemma 2]). *Let A and B be r.e. with recursive enumerations $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ for which there exists a wtt-reduction procedure $\Gamma(B) = A$ with $l(s+1) > l(s)$ for all s, where $l(s) =$* $\max\{x: \forall y < x\ (\Gamma_s(B_s; y) = A_s(y))\}$. *Define an r.e. set $E = \bigcup_s E_s$ via $E_{s+1} = E_s \cup$ $\{\mu z\ (z \in A_{s+1} - A_s)\}$.*

Then, if $E = E^1 \sqcup E^2$ is any r.e. splitting of C, there exists an r.e. splitting $A^1 \sqcup A^2 = A$ of A with $E^i \leqslant A^i$ for $i = 1, 2$.

Summarizing, an alternative view of the previous construction is that it consists of two steps. First it builds a wtt-reduction from V to A. Second it ensures that the set E is given by the Ambos-Spies theorem above (whose splittings are 'covered' by those of V) is mitotic and hence so too is V (by (2.13)).

The results of [2, Corollary 3.8], namely that no low contiguous degree is completely mitotic, show that the set A we constructed cannot be low, since by the usual arguments it is seen to be contiguous. (And hence, as Cohen-Ladner observed, there are low$_2$–low contiguous degrees.) (Also see Section 4.) Furthermore any construction which builds a mitotic least r.e. wtt-degree as part of its strategy of ensuring complete mitoticity can't construct a high r.e. degree by the results of [3]: Ambos-Spies, Cooper, and Jockusch showed that no high r.e. degree contains at least r.e. wtt-degree.

3. A low and a high completely mitotic degree

The difficulty of combining techniques along the lines of those of Section 2 with any form of jump control led Ladner, Cooper and others to suggest that there were — in particular — no low non-zero completely mitotic degrees. The main result of this section is the construction of a low nonrecursive completely mitotic r.e. degree. The strategy used is sufficiently flexible to combine with a coding strategy to build a high completely mitotic degree.

(3.1) **Theorem.** *There exists a low nonzero completely mitotic r.e. degree.*

Proof. We shall build $A=\bigcup_s A_s$, $C_e=\bigcup_s C_{e,s}$ and $D_e=\bigcup_s D_{e,s}$ to satisfy the requirements below:

P_e: $\bar{A}\neq W_e$.

R_e: $\Phi_e(A)=V_e\ \&\ \Gamma_e(V_e)=A$ implies

$C_e\sqcup D_e=V_e$ and $A\leqslant_{\mathrm{T}} C_e, D_e$.

N_e: $\exists^\infty s\,(\Xi_{e,s}(A_s;e)\downarrow)\rightarrow \Xi_e(A;e)\downarrow$.

Here $\langle \Phi_e, \Gamma_e, V_e\rangle_{e\in\omega}$ is a standard enumeration of all triples consisting of two functionals (Φ_e, Γ_e) and an r.e. set (V_e), and $\langle \Xi_e\rangle_{e\in\omega}$ is an enumeration of all functionals.

The principal difficulty in satisfying N_j in the presence of R_e (or vice-versa) using the strategy of Section 2 is this: suppose some P_k for $k>j>e$ receives attention. This action will probably initiate a sequence of codings that we have to fulfil for the sake of R_e of higher priority than P_k. In particular, if $l(e,s)\rightarrow\infty$, for the sake of e we will need to enumerate some $x_i^s(\sigma\hat{\ }0,)$ with $\mathrm{lh}(\alpha)=e$ into A. Now suppose this requested coding occurs at exactly the stage when we see $\Xi_{j,s}(A_s;j)\downarrow$. We would then like to not enumerate x_i^s into A but preserve the j-computation. We cannot do so since $e<j$. However, this process can occur infinitely often. Namely we can see some P_k initiating an e-action injuring N_j infinitely often. Because of this N_j may never be met.

Our solution, therefore, is to find a strategy that allows us to halt any sequence of numbers being put into A — for the sake of P_k's cooperation with R_e — should N_j request it. In this way we can meet the N_j and the P_k too since N_j will only request this finitely often. To make life simpler, we shall adopt the convention that N_j simply cancels all potentially injurious numbers of lower priority. Thus, if x is a follower or trace associated with P_k for $k>j$ and $x<u(\Xi_{j,s}(A_s;j))$, then we cancel x. If each P_k only causes finitely many numbers to enter A, then this meets N_j in a completely familiar and standard way.

The way P_k lives with this and cooperates with the R_e is as follows. Let $l(e,s)$ denote the length of agreement as given by the previous construction. As in (2.7), we monitor V_e at e-expansionary stages and ensure that $C_e\sqcup D_e=V_e$. We shall build reduction procedures $\Delta_e(D_e)=A$ and $\Lambda_e(C_e)=A$ to meet R_e.

The basic idea

The reduction procedures Δ $(=\Delta_e)$ and Λ $(=\Lambda_e)$ have uses $\delta(x,e,s)$ and $\lambda(x,e,s)$ respectively. Before we give exact rules governing Δ and Λ, we feel that it will be helpful to discuss a specific case first.

For a single P_k, N_j, R_e with $e<j<k$ we shall proceed as follows. For the sake of P_k we shall appoint a follower $x=x_0$. At the first e-expansionary stage t with $l(e,t)>x$ we appoint to x_0 a 'postfollower' or 'trace' $x_1>t$. Note that we can use the usual tree machinery to get x_0 e-confirmed (i.e. cancel all numbers between x_0 and x_1). in general this gives a little more than we need (see Step 3 below) but it is helpful for this construction to consider it done (and we shall).

Again we wait until a stage $s>t$ when $l(e,s)>x_1$. We define $\delta(x_0,e,s)$ and $\lambda(x_0,e,s)$ to exceed $u(x_0,e,s)$ and pick a new trace $x_2>s$. Here, as in Section 2, $u(x_0,e,s)$ denotes the use of the total e-computation involving x_0. (The reader should note that we now have a way of causing—roughly speaking—two changes in V_e. First enumerate x_1 into A, wait till an e-expansionary stage and then enumerate x_0 into A. This causes two changes in V_e below *both* $\delta(x_0,e,s)$ *and* $\lambda(x_0,e,s)$.)

Now we wait until the least e-expansionary stage $\hat{s}$ where $l(e,\hat{s})>x_2$ and then define $\delta(x_1,e,\hat{s})$ and $\lambda(x_1,e,\hat{s})$ so that they exceed $u(x_2,e,\hat{s})$ and $x_3>\hat{s}$. We continue this process for the sake of P_k cooperating with R_e until x_0 is cancelled (by N_j) or x_0 occurs in $W_{k,s}$. A typical situation for x_4 occurs in Diagram 3 below.

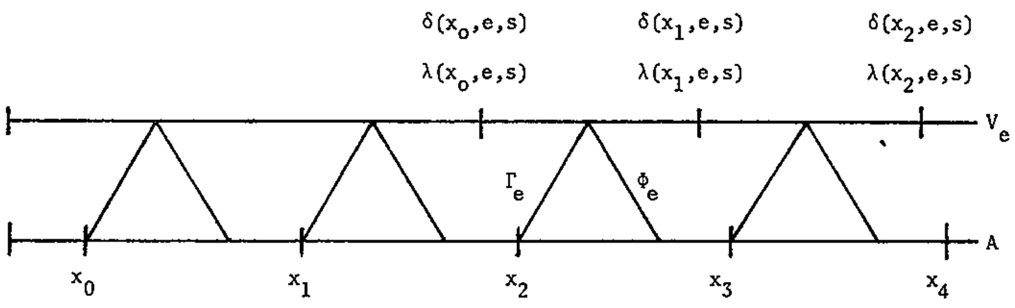

Diagram 3

The typical rules regarding uses are in force: x_i can enter A provided both D and C change below $\delta(x_i,e,s)$ and $\lambda(x_i,e,s)$ respectively. The fundamental idea is that if we see $x=x_0\in W_{k,s}$ then we try to enumerate the 'entourage' $x_n, x_{n-1},\ldots, x_0$ into A at e-expansionary stages in reverse order.

Thus suppose $x_0\in W_{k,s}$ in the situation of the diagram. When this occurs (at an e-expansionary stage s) we first would enumerate both x_3 and x_4 into A. The reader should note that no axioms involving x_3 or x_4 have been enumerated into Δ or Λ. The key point is that x_3's entry into A causes a change in V_e below $u(x_3,e,s)$ and so below both $\delta(x_2,e,s)$ and $\lambda(x_2,e,s)$.

Now at the next e-expansionary stage s_1 there are two possibilities. Either x_0, x_1 and x_2 have been cancelled for the sake of N_j or they haven't (in which case we would wish to enumerate x_2 into A.

In the first case, we shall simply enumerate all V_e-changes-since s-arbitrarily into C. Although this allows us to reset $\lambda(x_2,e,s_1)$ it doesn't matter since x_2 has been cancelled. The point is from D's point of view the original $\Delta(D;x_2)$ computation saying $x_2 \notin A$ was correct.

In the case that x_0 is still alive we shall enumerate all V_e-changes since stage s into D and enumerate x_2 into A. The reader should note that Δ can now be corrected on x_2 as we have caused a change in D below $\delta(x_2, e, s)$. However, Λ is incorrect on x_2 *so we have a pending commitment to C.* The reader should note that our enumeration of x_2 at stage s_1 will cause a change in $V_{e,s_2} - V_{s,s_1}$ below $\delta(x_1, e, s)$ and $\lambda(x_1, e, s)$ where s_2 denotes the next e-expansionary stage.

Again at the next e-expansionary stage s_2 we must decide where to enumerate this change. There are two possibilities, again depending on whether or not N_j has cancelled x_0 and x_1. If N_j has cancelled x_0 and x_1, then we simply attend the pending C-commitment and enumerate $V_{e,s_2} - V_{e,s_1}$ into C causing Λ to be correct on x_2. Note that both Λ and Δ correctly tell us that $x_1, x_0 \notin A$ in this case. On the other hand, if N_j has not cancelled x_0 and x_1 we again enumerate all changes $V_{e,s_2} - V_{e,s_1}$ into D and also enumerate x_1 into A causing V_e to change below $u(x_1, e, s_2) = u(x_1, e, s)$ and so causing another change below $\lambda(x_1, e, s)$ (and a change below both $\delta(x_0, e, s)$ and $\lambda(x_0, e, s)$). The reader should note that this still delays our pending commitment to Λ regarding x_2 and makes another involving x_1. In the same way, at the next e-expansionary stage s_3 either N_j has acted and we have cancelled x_0, in which case we enumerate $V_{e,s_3} - V_{e,s_2}$ into C, or we enumerate all changes into D as well as x_0 into A. In the latter case at the e-expansionary s_4 stage following s_3 we enumerate $V_{e,s_4} - V_{e,s_3}$ into C fulfilling all commitments.

The reader should note that at each e-expansionary stage we have the option of enumerating all changes (since the last e-expansionary stage) into either C or D. The relevant sequence thus 'looks like' $D, D, D, \ldots, D, C$ whereas in (2.7) it was D, C, D, C, D, C.

The basic module

More generally, dropping the subscript 'e' our procedures satisfy the following rules. (We shall always drop the e-subscript from Δ or Λ when things are clear from context.)

1. To ensure that $\Delta(D) = A$, we only allow a number x to enter A at stage s if $\Delta(D, x)$ is currently undefined. (Of course, we can make $\Delta(D, x)$ undefined by enumerating new axioms pertaining to x into Δ at e-expansionary stages. That is, we ensure that new elements enter D below the use of $\Delta(D, x)$ via A-changes, at e-expansionary stages.)

2. To ensure that $\Lambda(C) = A$ we only allow x to enter A at a stage when one of the following holds.

(i) There is no current $\Lambda(C, x)$ computation.

(ii) The current $\Lambda(C, x)$ computation has current use $\lambda(x, e, s)$ and $u(x, e, s) < \lambda(x, e, s)$; where as in Section 2, $u(x, e, s)$ denotes the use of the R_e-computations pertaining to x.

Moreover, if there is a number y with $\Lambda(C; y)=0$ but $y \in A_s$, then one of the following holds.

(iia) Some number must enter C below $\lambda(x, e, s)$.

(iib) Some number x less than y must enter A.

3. To ensure that Λ and Δ are total we must enumerate new axioms into Λ and Δ at e-expansionary stages. That is for any $x < l(e, s)$ we need relevant axioms pertaining to x in Λ and Δ. Note that if a strategy of higher priority requests it, the enumeration of new axioms can be delayed for finitely many expansionary stages, and the use of new axioms can be increased. We also enumerate the axioms so their use is increasing (as a function of argument and stage).

The reader should note that if we additionally ensure that $\Phi_e(A)=V_e$ and $\Gamma_e(V_e)=A$ implies that $\forall x$ ($\delta(x, e, s)$ & $\lambda(x, e, s)$ are defined and are reset finitely often), then $\Delta(D_e)=A$ and $\Lambda(C_e)=A$. In the first case when $D_{e,s}[\delta(x, e, s)]=D_e[\delta(x, e, s)]$, then $x \in A$ iff $x \in A_s$ (by rule 1). In the Λ case we see that whenever y enters A at stage s_1, either $\Lambda(C_{e,s}; y)$ is not defined, or $u(x, e, s) < \lambda(x, e, s)$. In the first case, there is no C-computation wrong about y. In the second case, V_e must change below $u(e, x, s)$. At this point either C_e is changed below $\lambda(x, e, s)$ by enumerating the new elements of V_e into C_e, or these numbers go into D_e but also a number less than y enters A. By induction, there is a least such $\hat{y}$ to enter $A_{\hat{s}}$ and for this $\hat{y}$ C must change below $\lambda(\hat{y}, e, \hat{s})$.

In general, for a single P_k, N_j and R_e with $e<j<k$, the way we meet the P_k and respect the e-strategy is given by the following steps.

Step 1. Pick a follower x and a trace $x_1 > x_0 = x$. Assume these numbers are fresh, and we have ensured that $x_1 > u(x, e, s)$ at an e-expansionary stage s.

Step 2. In general, we can assume we have two numbers x_{n-1} and x_n, P_k is not yet satisfied, and no axioms for x_{n-1} or x_n are yet enumerated. We wait till $l(e, s) > x_n$ and now declare that any axioms enumerated by R_e for x_{n-1} should have use $>u(x_n, e, s)$. We also pick a new trace $x_{n+1} > s$.

Step 3. So that the construction respects Δ and Λ we ask that for all i if y enters A below $u(x_i, e, s)$ then $\delta(x_i, e, s)$ is undefined and either $\lambda(x_i, e, s)$ is also undefined or $u(y, e, s) < \lambda(x_i, e, s)$.

Step 4. We also ask that if V_e changes below either $\delta(x_i, e, s)$ or $\lambda(x_i, e, s)$, then every x_j with $j > i$ is cancelled.

In general, we keep performing the above steps until we see $x = x_0 \in W_{k,s}$ at some e-expansionary stage s_1. At this stage we have an 'entourage' $x_0, \ldots, x_n$, say and as in the example, we try to enumerate the x_i into A at e-expansionary stages in reverse order. Thus if Step 4 did not pertain and we see $x \in W_{k,s}$ with s e-expansionary we enumerate x_n into A_s. In general, we can assume that at the last e-expansionary stage $\hat{s}$ we enumerated x_i into A for some i. We then adopt the appropriate case below.

Case 1: $i=0$, or x_{i-1} has been cancelled (in particular, because of the action of an N_j for $j > k$). Enumerate all new elements of V_e since $\hat{s}$ into C_e. This causes all of C_e's computations to be corrected. If $i=0$, then P_k is met and also has no further effect on the construction.

Case 2: Otherwise. Then $i>0$ and x_{i-1} is yet defined. In this case we enumerate all new elements of V_e into D_e causing $\Delta(D_e, x_{i-1})$ to become undefined. (See Steps 2 and 3.) We then enumerate x_{i-1} into A cancelling x_{i-1}.

The key idea of the above strategy is this. The enumeration of x_{n+1} is organised in such a way that it is possible to arrange that at the next e-expansionary stage we can enumerate V_e-changes into *either of C_e or D_e and end the effect of P_k* (or R_e depending on your point of view). Thus we have no other pending e-commitments than eventually putting something small (i.e. caused by x_j for $j \leq n+1$) into C_e. Remember, we have fulfilled our D_e commitments at the last e-expansionary stage when we created an even smaller C_e-commitment. Thus we never have *D_e-commitments* (although we may decide to enumerate changes into D_e delaying our C_e-commitment). Consequently N_j may interrupt this sequence and R_e may still remain satisfied. Thus N_j's action doesn't injure R_e any more, only P_k. (Of course, N_j's injuries to P_k are finite.)

It is quite easy to see that P_k can be satisfied using the above strategy (at least for one P_k, R_e). To see this go to a stage s_0 where all the $N_{\hat{j}}$ for all $\hat{j} \leq k$ cease acting. After s_0, any number appointed to P_k is uncancellable. Now, if P_k fails, then eventually at some stage $s_1 > s_0$ we have $x_0 \in W_{k,s}$. At this stage we have a '*k-entourage*' $x_0, \ldots, x_n$ say. It is really quite easy to see that after n further e-expansionary stages, we have put x_0 into A meeting P_k forever.

Coherence

It remains, therefore, to give a technique to combine the above with a Π_2 guessing procedure and thus give a nesting which makes the above strategies cohere. Let us consider two mitotic requirements R_e and R_f interacting with some P_k. We assume $e<f<k$ and assume that we have $\Phi_i(A)=V_i$ and $\Gamma_i(V_i)=A$ for $i=e$ or f. In the nested version of R_f on the tree we will have f-expansionary stages occurring within e-expansionary stages (at least along the true path). The important point is how to build R_f's reduction procedures cooperatively with e.

The crucial observation is this. For a single requirement e, our Friedberg strategy gives us a method of (eventually) enumerating a number into A should a certain Σ_1-event occur in the construction. For the P_k case, this Σ_1-event is $\forall s\,(x_0 \in W_{k,s} \,\&\, W_{k,s} \cap A_s = \emptyset)$, however any Σ_1-event will do. This provides the key to the strategy below:

Specifically for the sake of P_k we initially choose x_0 as before. Now the problem is that there may be infinitely many e-expansionary stages but only finitely many f-expansionary ones. Therefore we cannot afford to wait until the next f-expansionary stage to define our e-axioms and still give R_e an environment in which it can survive. Thus at the next e-expansionary stage we fulfil our commitments to R_e by defining an e-sequence as if e were the only requirement around. We denote these numbers by y_i. Thus, by the next f-expansionary stage

s_1 (which is also e-expansionary) we may have the scenario

$x_0,\quad y_1, y_2, \ldots, y_{n_1}$ where here

y_i takes the role of x_{i+1} in the construction for e alone as previously discussed.

(That is, for example y_1 is chosen when we see $l(e, s) > x_0$ as are $\delta(x_0, e, s_1)$ when $l(e, s_1) > y_1$.) Now at this stage we simply select x_1 $(= y_{n_1+1})$.

When we define $x_2 = y_{n_2+1}$ we simultaneously define $\lambda(x_0, t, s_2)$ and $\delta(x_0, t, s_2)$ as we do $\delta(y_{n_2-1}, e, s_2)$ and $\lambda(y_{n_2-1}, e, s_2)$. The following Diagram 4 might be helpful.

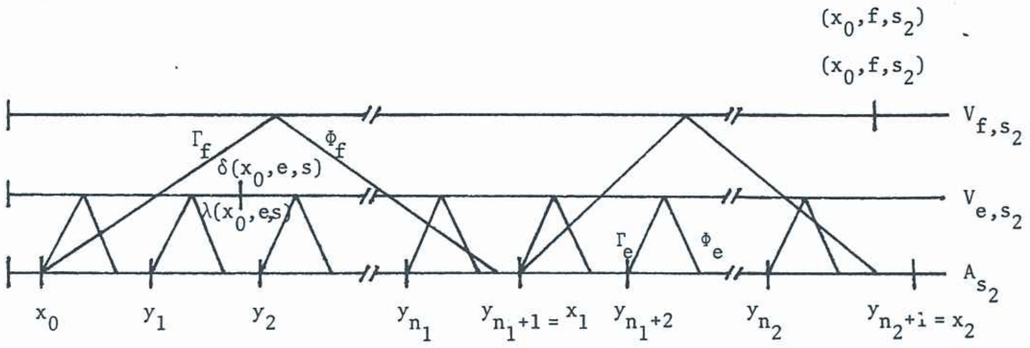

Diagram 4

Thus at any stage if there is no cancellation we will have a sequence of numbers

$$x_0, y_1, \ldots, y_{n_1}, y_{n_1+1} = x_1, y_{n_1+2}, \ldots, y_{n_2}, y_{n_2+1} = x_2, \ldots$$

The above sequence thus satisfies both R_e and R_f in a perfectly standard way except that we have the following problem. Suppose $x_0 \in W_{k,s}$ is seen to occur at some stage. Then in reverse order we must put the numbers above into A at the appropriate expansionary stages. This is fine from R_e's point of view but it creates problems from R_f's point of view due to *timing problems*.

For example suppose we have a sequence

(3.2) $\quad x_0, y_1, y_2 = x_1, y_3, y_4, y_5 = x_2, y_6, y_7 = x_3, y_8, y_9$

and $x_0 \in W_{k,s}$. Now we begin to put the y_i into A in reverse order. Since f can only operate in stages also good for e, f can afford to wait (for $x_1 = y_2$) until y_i for $i > 2$ enter A before we believe an f-computation. In this way we can keep our commitments to e for the y_i for $i > 1$. However, there is a real problem for $y_2 = x_1$ caused by conflict between e and f. The problem is that when we get a stage s for which we wish to enumerate $y_2 = x_1$ into A we cannot do so unless we have seen, in particular $l(f, s) > y_2$. On the other hand an e-expansionary stage may not be f-expansionary so that although e is asking us to put x_3 into A, f is asking us to wait until an f-expansionary stage (or at least until $l(f, s) > y_2$) (which from e's point of view might not occur). The crucial point here is that until we have seen an f-expansionary stage, we haven't seen the V_f-change below $u(x_2, f, s_1)$ (and so below $\delta(x_1, f, s_1) = \lambda(x_1, f, s_1)$) caused by $y_5 = x_2$'s entry into A. (Remember x_3 is defined so these are set.)

The way we resolve this conflict is to notice that "there exists another f-expansionary stage" is also a Σ_1-event, and so e can solve the problem induced by f's 'slowness' by beginning (say at stage t) a new Friedberg strategy for y_2 with the Σ_1-event being "there exists another f-expansionary stage" which we denote by $\exists s.(P(f, s))$. However this Friedberg strategy must only respect e rather than both e and f since we only create it to keep e happy whilst we are waiting for f to reveal his actions.

Explicitly, at stage t we begin a new e-sequence $z_0 = y_2$, z_1 and continue to build it at e-expansionary stages until we see, for $s > t$, $P(f, s)$ holds. At such a stage s our sequence will appear as, say,

$$x_0, y_1, y_2 = x_1 = z_0, z_1, z_2, \ldots, z_n.$$

Now we would like to fulfil our f commitments by enumerating x_1 into A. Now, however we can't do so because of the new e-commitments we have created. Thus we must put the z_i into A in reverse order first. On the surface the same problem may occur by stage $\hat{s}$ when we get back to $y_2 = x_3 = z_0$. But now we have salvation, because we know that we have seen $l(f, s) > y_2$ and so *we know that we can enumerate axioms below $\delta(f, y_2, s)$ since V_f must have changed there* (when $l(f, s) > y_2$). And so can enumerate y_2 into A at the next e-expansionary stage after z_1 enters A (rather than the next f-expansionary stage, which is the whole point of the procedure).

Now this idea in turn creates a new problem regarding C and D we solve by delay. For example consider the sequence devoted to D_j given by

(3.3) $\quad x_0, y_1, y_2 = x_1, y_3, y_4, y_5 = x_2.$

Now suppose we pursue the above strategy but we get stuck at $y_2 = x_1$ say at stage s. Now at this stage we enumerate the last changes into C and begin a new e-sequence $z_0, z_1, z_2, \ldots$ devoted to solving the $l(f, s) > y_2$ problem. The reader should note that only $\lambda(y_2, e, s)$ has changed. $D[\delta(y_2, e, s)]$ is the same as it was at the stage t when we first began enumerating y_i's into A (or, indeed when it was first defined). Thus we might get Diagram 5 below.

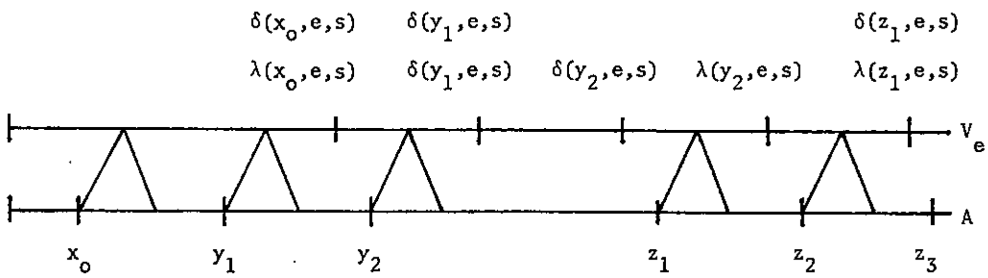

Diagram 5

The problem here is that perhaps z_1's entry into A might not cause a V_e-change below $\delta(y_2, e, s)$. However, we do know that if $\hat{s}$ is the stage when z_1 enters A, then at the next e-expansionary stage q either our sequence will be halted and numbers enter C, which is fine since V_e does change below $\lambda(y_2, e, q)$, or we will then enumerate y_2 into A causing V_e to change below $\delta(y_2, e, s)$. Thus we need only delay our C and D decisions until the least e-expansionary stage after q.

The only remaining problem concerns when to build D_f and C_f. The point is that we may be forced to stop our enumeration of the z_i's because some N_j for $j<e$ cancels them. Therefore at the f-expansionary stage s when we start to enumerate the z_i's into A in reverse order we do not attend to R_f at all but (as we mentioned in (3.1)) *delay* R_f's action until all the z_i's have revealed their eventual behaviour. That is, if no higher priority activity has upset the situation, we put the new axioms into D_f at the e-expansion (not f-expansionary) stage $\hat{s}$ when $z_0 = y_2$ enters A. If we get interrupted (by N_j, say) and can't finish the Friedberg strategy for y_2, then we put them into C_f. This delay is of course fine from R_f's point of view since (this version of) R_f is guessing that there are infinitely many e-expansionary stages, and hence can afford to wait for all pending e-actions to finish.

Summarizing, when we are dealing with P_k cooperating with a version of R_f guessing that R_e acts infinitely often, (and with $e<f<k$), initially we build a sequence for the sake of R_f. However for each element of this sequence we will want to put that number into A if a certain Σ_1-event is seen to occur. Thus, as we have seen above each element of the sequence constitutes a Friedberg requirement that must respect only R_e. Since f-expansionary stages here must also be e-expansionary, we see that the sequences for e and for f are compatible. It remains to observe that when the sequences become active (i.e. we start enumerating numbers into A for the sake of P_k) whenever we get to a situation where we must wait for an f-computation, we can create a new Friedberg strategy assigned to solving this problem. This sequence of numbers is built purely for the sake of e and once we see an f-expansionary stage we activate this new sequence and delaying our f-commitments a finite number of steps until all pending e-commitments are finished.

We call a strategy designed to satisfy a Friedberg requirement (i.e. when a Σ_1-event occurs) and yet still meet a single R_e of higher priority a depth-1 strategy. The depth-2 strategy is as outlined above for R_e and R_f. The general depth-n strategy is defined as above, but replacing our depth-1 strategy (for R_e) by a depth-$(n-1)$ strategy. It is clear that no new problems arise for n requirements and we thus have met the coherence criterion for any collection of R_j.

A more elegant version of the depth-2 strategy is obtained by not using numbers (like $y_2 = x_3$ in (3.2)) for both e- and an f-role. That is we select a number y_2 and x_3 for each. It is easy to see that simultaneously selecting y_2 and x_3 in this way is quite compatible (y_2 and x_3 will enter at the same stage). This then allows us to visualise the y_i, x_i sequence as a tree of sequences with stem the sequence for R_f and branches the sequences for R_e (which are depth-1 strategies). Such a visualisation is more easily extended to depth-n. This representation makes some of the internal logic of the system more visible at the expense of a little more book-keeping.

Finally we should remark about the coherence of the Friedberg strategies (interacting with some R_e). Thus let $e<i$. We analyse the situation for P_i, P_{i+1} and P_{i+2}. It is perhaps easiest to use distinct numbers as traces and followers of distinct P_n's. (It is possible for them to serve dual roles, we refer the reader to the next theorem.) Also should P_i require attention, it is easiest to initialize the $P_{\hat{i}}$ for $\hat{i}>i$ and, in particular P_{i+1} and P_{i+2}.

There are various ways to mesh the P_i, P_{i+1} and P_{i+2} strategies. Although we must modify the technique in (3.5) below, in this construction we can use a very simple meshing which exploits the finite injury nature of the proof. It is convenient to denote traces of P_i by x_n^s, of P_{i+1} by y_n^s and of P_{i+2} by z_n^s. For these three Friedberg requirements the relevant order is

$$x_0, x_1, y_0, x_2, y_1, z_0, x_3, y_2, z_1, x_4, y_3, z_2, x_5, \ldots$$

For this construction the idea is that if P_{i+1} is yet unsatisfied, R_e delays defining $\delta(x_0, e, s)$ and $\lambda(x_0, e, s)$ until we define y_0^s, etc. At stage s the relevant picture would appear as Diagram 6.

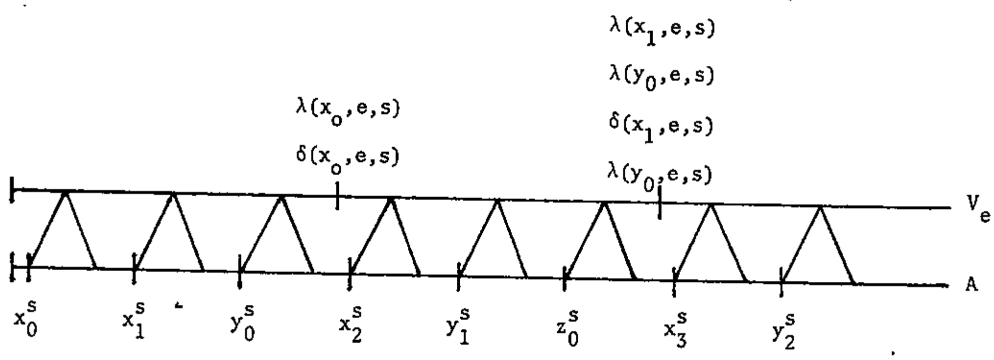

Diagram 6

The only proviso is that if (e.g.) $y_k^s < x_{k+1}^s$ is enumerated, we get to reset x_j^s for $j \geqslant k+1$ and *both* $\lambda(x_{k-1}, e, s)$ and $\delta(x_{k-1}, e, s)$. The fact that if, for example, y_0^s enters A for the sake of P_j we might only reset $\lambda(x_0, e, s)$ is irrelevant since the resulting x_i-sequence would still obey the rules for Δ and Λ. The point is that the above meshing will reset any x_k^s only finitely often, and it is quite easy to see that strategies meshed in this way will achieve the desired results.

As in the previous argument (2.7) the formal details are to arrange the R_e on the usual Π_2-guessing tree. As this is standard and yields no new insight we leave these details to the reader. (We hope that the details are obvious at this stage.)

Remark. We point out that there are other techniques of meshing Friedberg sequences. One such technique is to use traces of some x_0 following P_i as followers of P_j for $j > i$. Thus for a sequence $x_0, x_1^s, x_2^s, \ldots$ we can let any x_j^s for $j \geq k$ follow P_{i+k} as well as tracing x_0. The only problem with this technique is that it necessitates the further use of the delay involved in (3.3). For instance if x_1^s follows P_{i+1} it may be that x_1^s enters A yet x_0 has not yet required attention. Upon resetting at stage $\hat{s} > s$ we would then only have that $\lambda(x_0, e, \hat{s}) > u(x_1^{\hat{s}}, e, s)$— but $\delta(x_0, e, \hat{s}) < u(x_1^{\hat{s}}, e, \hat{s})$—since $\delta(x_0, e, \hat{s}) = \delta(x_0, e, s)$. Indeed we might get a stage t where after $x_2^{\hat{s}}$ entered A for the sake of P_{i+1} yet $x_1^t = x_1^{\hat{s}}$ so that $\delta(x_1^t, e, t) = \delta(x_1^t, e, \hat{s}) < u(x_2^t, e, t)$. Again this all involves a finite delay in the definition of Δ and Λ (and C and C). But this causes no real problems. Actually a mild variation of this version involves much less notation for (3.5) below. Of course the trade-off is the added conceptual difficulty in the delay required in the definition of C and D and the functionals Δ and Λ.

One corollary of course is the existence (by (2.1) of nonzero cappable completely mitotic degrees. However, this is not particularly significant since the mitotic construction of (2.7) is easily seen to blend with a minimal pair construction. It is however clear that there seems no way to combine (3.1) with promptness and so it seems that (3.1) always builds cappable degrees. This is important only in that we do not know which degrees r.e. in and above $\emptyset'$ (REA($\emptyset'$)) are the jumps of completely mitotic degrees. If jump inversion is possible then—by the observations above—it would seem that another construction is needed. This follows since Shore [15] and Cooper [5] have shown that not every REA($\emptyset'$) degree is the jump of a cappable degree.

It is relatively easy to modify the strategy of (3.1) to show the result below whose proof we only sketch due to its similarity with (3.1).

(3.5) **Theorem.** *There exists a high completely mitotic degree.*

Proof (sketch). The proof relies on a modification of the strategy of (3.1), or, rather, checking that the strategy is compatible with the usual 'piecewise thick' highness requirements. Specifically, let H be an r.e. set such that $H^{(e)}$ is either $\omega^{(e)}$ or a finite initial segment of $\omega^{(e)}$, and such that $|H^{(e)}| < \infty$ iff $e \in \emptyset''$. Then it suffices—as usual—to build a thick subset A of H (i.e. $A \subset H$ and $A^{(e)} =^* H^{(e)}$ for all e) to make A high. We reserve row $\omega^{(0)}$ for tracing and so we shall satisfy

$$P_e: \quad A^{(e+1)} =^* H^{(e+1)}$$

and ensure that $A^{(j)} \subset H^{(j)}$ for all $j \geq 1$. We retain the R_e of Section 3.

As usual the grafting of infinitary positive requirements involves guessing which columns of H are infinite or not. R_e must thus guess whether $j \in \emptyset''$ for $j < e$ and the behaviour of the R_j for $j < e$. We assume the reader is familiar with this completely standard process and refer him to (e.g.) [18] for further details.

The important new features of our new construction are to ensure that our uses $\lambda(x, e, s)$ and $\gamma(x, e, s)$ settle down if $\liminf_s l(e, s) \to \infty$ and to deal with meeting the P_k for $k > e$ if $l(e, s) \to \infty$ but $\liminf_s l(e, s) < \infty$.

Notice that each $\langle e, x \rangle$ must be given an entourage (taken from $\omega^{(0)}$) in some fair way. The first major difference in our activities is that we can no longer cancel followers of P_k for $k > j$ if P_j acts putting some member of $\omega^{(j+1)}$ into A. We describe the necessary coherence of two P_j, P_k with one $\hat{R}_e$ for $k > j > e$. For definiteness, we take $k = j + 1$.

Initially there will be a least unrestrained number x_0^s in $\omega^{(j)}$ not yet in A. When the e-correct version of (Φ_e, Γ_e) gives $l(e, s) > x_0^s$ we pick x_1^s and then take as our 'Σ_1-event' that all members z of $\omega^{(j)}$ that satisfy the inequality $x_0^s \leq z \leq x_1^s$ have occurred in $\omega^{(j)}$ at stage $\hat{s}$. Until this Σ_1-event occurs we restrain $A_s[x_1]$. Since we are dealing with e-correct computations this will preserve the appropriate region $u(x_0^s, e, s)$. The basic idea is that if $\omega^{(j)}$ is finite then eventually we'll get stuck on some x_0^s. (The point is that should $x_1^s \in \omega_{\hat{s}}^{(j)}$, then eventually we'll enumerate x_0^s into A_t for some $t > s$. At this time we'll choose a new x_0^{t+1}.)

Obviously for this strategy, when we pick $x_2^{\bar{s}} > x_1^{\bar{s}}$ when $l(e, \bar{s}) > x_1^{\bar{s}}$, we can't similarly restrain $A_{\bar{s}}[x_2]$ since perhaps $x_0^s \notin \omega^{(j)}$ and in this case all of A would eventually be restrained. Thus we must mesh the P_k strategy with the P_j one. In fact we need two versions of P_k to guess whether $|\omega^{(j)}| = \infty$ or not.

The first version of P_k is guessing $|\omega^{(j)}|$ is finite and it is played (roughly) like the strategies of (3.1), but with an intrinsic commitment to believing $x_0^s \notin \omega^{(j)}$. Thus, as in (3.1) a typical situation would be Diagram 7.

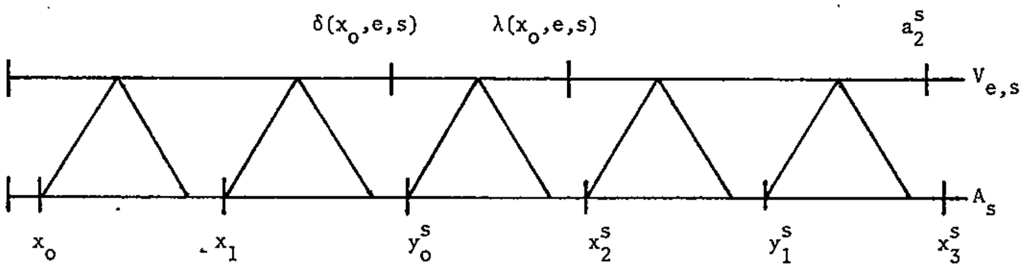

Diagram 7

Here a_2^s denotes $\lambda(y_0^s, e, s)$, $\delta(y_2^s, e, s)$, $\lambda(x_1, e, s)$ and $\delta(x_1, e, s)$. We write x_0 and x_1 to suggest that the reader think of them in their limit positions. The reader should note that $\delta(x_0, e, s)$ may not respect y_0^s as in Diagram 5 (only perhaps $\lambda(x_0, e, s)$). The reason is that y_0^s may not be the same as y_0^t where y_0^t was set at the stage when $\delta(x_0, e, t) = \delta(x_0, e, s)$ was set. Note that since the last in any sequence is a C-change, at best we could have reset $\lambda(x_0, e, s)$, but $\delta(x_0, e, s)$ remains the same. Of course the internal integrity of $\delta(x_0, e, t)$ with x_1 remains, and hence we still remain consistent with the construction.

The most important point, however is to ensure that $\lim_s \lambda(x_i, e, s)$ exists, and indeed $\lim_s x_i^s$ exists. The point is that x_2^s is constantly reset whenever y_0^s enters A_s

and a new y_0^t is later chosen. In the previous construction, the finite injury nature gave us this for free. In the current construction we explicitly ensure this by waiting until a stage s where

(3.6) For some z set aside for tracing P_j we have
(i) $z > u(x_1, e, s)$ and
(ii) $y_0^s > u(z, e, s)$.

Notice that if truly $\Phi_e(A) = V_e$ and $\Gamma_e(V_e) = A$, then such a stage $\hat{s}$ must occur lest $u(\hat{z}, e, s) \to \infty$, where $\hat{z}$ is the least number set aside for P_j with $\hat{z} > u(x_1, e, s)$. At this stage, we let $x_2 = z$ *and shift everything one position left.* The relevant picture would be Diagram 8.

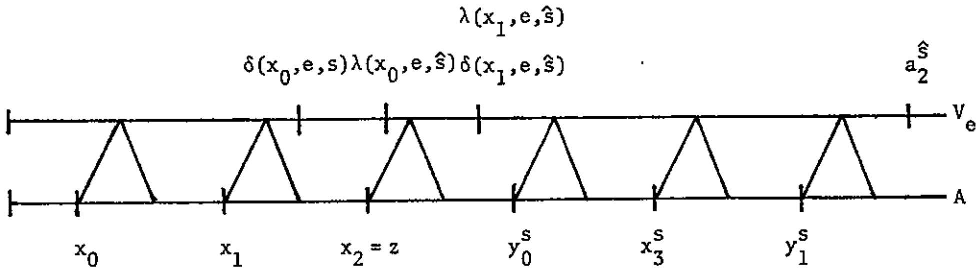

Diagram 8

Here $a_2^{\hat{s}} = \lambda(x_2, e, \hat{s}) = \delta(x_2, e, \hat{s}) = \lambda(y_0^s, e, s) = \delta(y_0^s, e, s)$. Note that now x_2, $\lambda(x_0, e, \hat{s})$ and $\delta(x_0, e, \hat{s})$ have reached their limits as essentially have $\lambda(x_1, e, \hat{s})$ and $\delta(x_1, e, \hat{s})$. The reader should note that in this way eventually all the x_i will reach their limits and yet we are still able to meet P_k.

Thus we have dealt with the case where $|\omega^{(j)}| < \infty$. We must also be able to deal with the other version: namely $|\omega^{(j)}| = \infty$ (i.e. P_k guesses this). Roughly speaking for this version of P_k typically we'll have the situation above reversed, but with the added proviso that y_0^s can't cause x_i^s to enter A (i.e. P_k respects P_j). Now in this case P_k *will wait for the action of P_j to cause x_i^s to enter A.* The important situation is when y_0^s's enter $\omega^{(k)}$ 'slowly', but x_0^s's enter $\omega^{(j)}$ 'quickly'.

Shifting occurs as outlined above, and a typical situation would be Diagram 9.

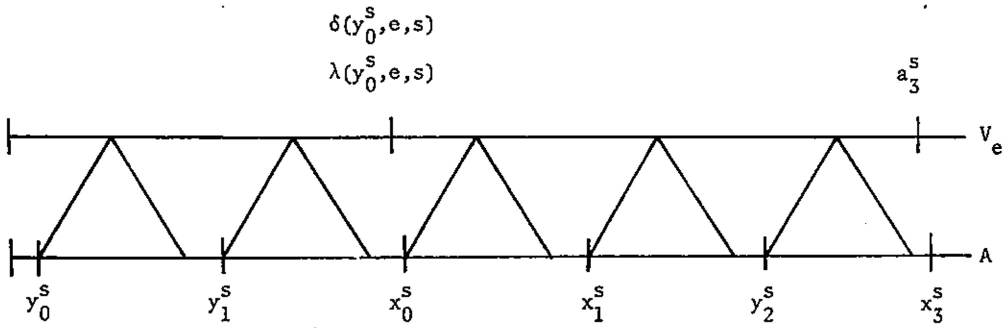

Diagram 9

Here a_3^s would denote $\delta(x_0^s, e, s) = \lambda(x_0^s, e, s) = \delta(y_1^s, e, s) = \lambda(y_1^s, e, s)$. The important point is that this version of P_k 'knows' that eventually x_0^s will enter A. Thus once it sees $y_0^s \in \omega^{(k)}$ it then simply waits until x_0^s enters A. P_j then delays picking new x_0^s until y_1^s and y_0^s get enumerated into A (or stuck by an R_t for $t < j$). Such delay is fine from P_j's point of view since it is guessing $l(e, s) \to \infty$ and so can wait two more expansionary stages since it knows $l(e, s) \to \infty$.

In essence, the only major problem is in ensuring that all the δ's and λ's settle down. The modifications above clearly ensure this for the version of R_e on the true path. The remaining details go through as before and we leave these to the reader. □

Thus we know that completely mitotic r.e. degrees can be high, low, and low_2–low (Cohen–Ladner in [12]). We know nothing else about the jumps of such degrees. In particular we remark that we don't know if there are such degrees in $\mathbf{H}_{n+1} - \mathbf{H}_n$, $\mathbf{L}_{n+1} - \mathbf{L}_n$ and **Int**. We point out that pseudo-jumps can't be used here since $\mathbf{0}'$ is not completely mitotic.

4. Limiting results

In this section, we examine results which limit the existence of completely mitotic r.e. degrees. In Section 2 we saw that no low promptly simple r.e. degree was completely mitotic. An early result of Ladner [11] shows that $\mathbf{0}'$ is not completely mitotic either:

(4.1) **Theorem** (Ladner [11]). *There exists a non-mitotic complete set.*

Proof (sketch). For completeness, we provide a quick sketch proof. Again we make A nonautoreducible. We satisfy the requirements

$$R_e\text{:} \quad \Phi_e(A \cup \{x\}; x) \neq A(x) \quad \text{for some } x.$$

At each stage s we place markers $\Lambda(e, s)$ on members of $\bar{A}_s$. Let $K = f(\omega)$ be a 1–1 enumeration of a creative set. To meet R_e wait till $l(e, s) > \Lambda(e, s)$ where

$$l(e, s) = \max\{y\colon \forall z < \ (\Phi_{e,s}(A_s \cup \{z\}; z) = A_s(z))\}.$$

Assuming that $e \leqslant f(s)$ set $A_{s+1} = A_s \cup \{\Lambda(e, s)\}$, $\Lambda(e+i, s+1) = \Lambda(e+i+s, s)$ for all $i \in \omega$, and $\Lambda(j, s+1) = \Lambda(j, s)$ for $j < e$. If no e receives attention this way, set $A_{s+1} = A_s \cup \{\Lambda(f(s), s)\}$, $\Lambda(f(s)+i, s+1) = \Lambda(f(s)+i+1, s)$ for all i and $\Lambda(j, s+1) = \Lambda(j, s)$ for $j < f(s)$. It is really quite easy to show that $\lim_s \Lambda(e, s) = \Lambda(e)$ exists, $A \equiv_T K$ and that the R_e receive attention at most a finite (bounded) number of times. □

Another limiting result is given by analysing some results from Downey and Jockusch [6].

(4.2) **Theorem.** *There exists a* low_2*–low r.e. degree* $\mathbf{a}$ *such that for all r.e.* $\mathbf{b}$ *with* $\mathbf{0}<\mathbf{b}<\mathbf{a}$, $\mathbf{b}$ *contains a nonmitotic r.e. set.*

Proof. In [6], Downey and Jockusch construct an r.e. set A such that A is incomplete, such that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A, then $\inf\{\deg(A_1), \deg(A_2)\} = \mathbf{0}$, and such that if $B \leqslant_{\mathrm{T}} A$ then $B \leqslant_{\mathrm{m}} A$. They also showed that such r.e. sets are low_2–low.

In [1], Ambos-Spies shows that if D and C are r.e. sets with $C \leqslant_{\mathrm{wtt}} D$, then there exists an r.e. set $\hat{C} \equiv_{\mathrm{wtt}} C$ such that if $C_1 \sqcup C_2 = \hat{C}$ is an r.e. splitting of C, then there exists an r.e. splitting $D_1 \sqcup D_2 = D$ of D with $C_i \leqslant_{\mathrm{wtt}} D_i$ (cf. (2.13)).

Putting these two results together shows that if $B \leqslant_{\mathrm{T}} A$, then $\deg(B)$ contains an r.e. set $\hat{B}$ every splitting of which is a minimal pair and so is certainly not mitotic. □

It seems quite probable that all high r.e. degrees bound nonzero completely mitotic degrees. It would be interesting therefore to know how high the top of a 'nonmitotic cone' (as in (4.2)) can be. Certainly every nonzero r.e. degree has a predecessor with this property:

(4.3) **Theorem.** *Let* $\mathbf{a}$ *be r.e. nonrecursive. Then there exists an r.e. degree* $\mathbf{b}$ *with* $\mathbf{0}<\mathbf{b}\leqslant\mathbf{a}$ *such that for all* $\mathbf{c}\leqslant\mathbf{b}$ *if* $\mathbf{c}$ *is completely mitotic then* $\mathbf{c}=\mathbf{0}$.

Proof. Let A be an r.e. nonrecursive set with $A = \bigcup_s A_s$ a recursive enumeration. We build $B \leqslant_{\mathrm{T}} A$ by simple permitting. We shall satisfy the requirements

$$P_e:\quad \bar{B} \neq W_e,$$

$$N_{e,i}:\quad \Phi(B) = C_e \text{ implies } C_e \equiv_{\mathrm{T}} \hat{C}_e \text{ and}$$
$$\text{either } C_e \text{ is recursive, or } \exists x\, (\Gamma_i(\hat{C}_e \cup \{x\}; x) \neq \hat{C}_e(x)).$$

Here $(\Phi_e, C_e)_{e\in\omega}$ is a standard enumeration of apirs consisting of an r.e. set and a functional, $(\Gamma_i)_{i\in\omega}$ is an enumeration of all functionals, and $\hat{C}_e$ is a set we build (if $\Phi_e(B) = C_e$). As usual, we regard $\Phi_e(B)$ as controlling C_e and hence, if

$$l(e, s) = \max\{z : \forall y < z\, (\Phi_{e,s}(B_s; y) = C_{e,s}(y))\} > x,$$

then don't allow $C_{e,s}(x)$ to change unless $B_{s+1}[u(e, x, s)] \neq B_s[u(x, e, s)]$ where

$$u(x, e, s) = \max\{u(\Phi_{e,s}(B_s; z)) : z \leqslant x\}.$$

Now let

$$l(e, i, s) = \max\{x : \forall y < x\, (\Gamma_{i,s}(\hat{C}_{e,s} \cup \{y\}; y) = \hat{C}_{e,s}(y))$$
$$\&\ \forall z\, (z < u(\Gamma_{i,s}(\hat{C}_{e,s} \cup \{y\}; y)) \to l(e, s) > z)\},$$

the 'B-controllable' length of agreement. To ensure that $\hat{C}_e \equiv_{\mathrm{T}} C_e$ we add to $\hat{C}_e$ the least number to enter C_e between e-expansionary stages (namely when $l(e, s) > \max\{l(e, t) : t < s\}$).

To meet the $N_{e,i}$ we attempt to preserve any perceived disagreement by cancelling lower priority followers at e-expansionary stages. If we fail to meet $N_{e,i}$ in this way, we must argue that C_e is recursive. The resultant argument is a fairly easy finite injury one which we give below.

We say P_e *requires attention* at stage $s+1$ if e is least with $W_{e,s} \cap B_s = \emptyset$ and such that one of the options below hold:

(4.4) P_e has no follower x with $x \notin W_{e,s}$

(4.5) P_e has a follower x with $x \in W_{e,s}$ and
A permits on x (i.e. $A_{s+1}[x] \neq A_s[x]$).

Construction, stage $s+1$

Step 1. For each $e \leq s$ if s is e-expansionary let $z = \mu y\ (y \in C_{e,s} - C_{e,\hat{s}})$ where $\hat{s}$ is the last e-expansionary stage less than s. Define $\hat{C}_{e,s+1} = \hat{C}_{e,s} \cup \{z\}$. Now, if for any $\hat{z}$ and i we have

(i) $\Gamma_{i,s}(\hat{C}_{e,s+1} \cup \{\hat{z}\}; \hat{z}\}\downarrow \neq C_{e,s+1}(\hat{z})$, and

(ii) $l(e, s) > u$ where $u = u(\Gamma_{i,s}(\hat{C}_{e,s+1} \cup \{\hat{z}\}; \hat{z}))$,

then cancel all followers of P_k for $k > \langle e, i\rangle$ and say that $N_{\langle e,i\rangle}$ receives attention at stage s.

Step 2. Now find e such that P_e requires attention. If (4.5) holds, set $B_{s+1} = B_s \cup \{x\}$. Initialize. If (4.4) holds, and (4.5) doesn't pertain assign $x = s$ as a follower of P_e. □(End of Construction)

The following lemma is standard and easy.

(4.6) **Lemma.** (i) $B \leq_T A$.

(ii) *If the $N_{\langle f,i\rangle}$ for $\langle f, i\rangle \leq e$ receives attention only finitely often, then P_e receives attention only finitely often and is met.*

Thus to complete our verification, we check

(4.7) **Lemma.** *Each $N_{\langle e,i\rangle}$ receives attention at most finitely often and is met.*

Proof. Let s_0 be a stage such that all the P_j for $j < \langle e, i\rangle$ cease receiving attention. If $l(e, s) \to \infty$ there are infinitely many e-expansionary stages and so $C_e \equiv_T \hat{C}_e$. First suppose that $N_{\langle e,i\rangle}$ receives attention at some (e-expansionary) stage $s > s_0$. Then at this stage for some (least) $\hat{z}$ we have $\Gamma_{i,s}(\hat{C}_{e,s+1} \cup \{\hat{z}\}; \hat{z}) \neq \hat{C}_{e,s+1}(\hat{z})$. This preserved disagreement is preserved forever since we can cancel all potentially injurious numbers from possible entry into B, ensuring that $C_{e,s}[u] = C_e[u]$ where $u = u(\Gamma_{i,s}(\hat{C}_{e,s+1} \cup \{\hat{z}\}; \hat{z}))$ and so $\hat{C}_e[u] = C_{e,s+1}[u]$. Thus $\Gamma_i(\hat{C} \cup \{\hat{z}\}; \hat{z}) \neq \hat{C}_e(\hat{z})$. Hence $N_{\langle e,i\rangle}$ can receive attention at most once after stage s_0.

Now we must argue that if $N_{\langle e,i\rangle}$ does not receive attention after stage s_0 and $l(e, i, s)\to\infty$, then $\hat{C}_e$ is recursive. To compute $\hat{C}_e(z)$ find a stage $s > s_0$ such that $l(e, i, s) > z$. Since no number enters $\hat{C}_e$ between stage s and the next e-expansionary stage $s_1 > s$, it must be that $l(e, i, s_1) > z$ also. It is clear that for all $\hat{z} \leqslant z$, it must be that $\Gamma_{i,s_1}(\hat{C}_{e,s_1} \cup \{\hat{z}\}; \hat{z}) = \hat{C}_{e,s_1}(\hat{z}) = C_{e,s_1}(\hat{z}) = C_{e,s_1+1}(\hat{z})$. Otherwise in Step 1 we would kill $N_{\langle e,i\rangle}$ at some such z. These observations hold for any s with $l(e, i, s) > z$ and constitute a proof that $\hat{C}_{e,s}(z) = \hat{C}_e(z)$. □

Finally, we turn to density properties. First we give a new proof of Ingrassia's theorem on the density of degrees containing non-mitotic r.e. sets. His proof is quite complex and involves the use of 'p-generic' and 'intro-reducible' r.e. sets. Ours is direct, simpler and rather more amenable to modification.

(4.8) **Theorem** (Ingrassia [8]). *The degrees containing nonmitotic r.e. sets are dense in* $\mathbf{R}$.

Proof. We are given r.e. sets $E <_T F$. By Sacks density theorem [14] it suffices to construct an r.e. set A with $A \oplus E \leqslant_T F$ satisfying

$$P_e\colon\ \exists x\,(\Phi_e(A \oplus E \cup \{x\}; x) \neq (A \oplus E)(x)).$$

Let $l(e, s) = \max\{x\colon \forall y < x\,(\Phi_{e,s}(A_s \oplus E_s \cup \{y\}; y) = (A_s \oplus E_s)(y))\}$. The basic strategy for satisfying P_e remains the same: pick a follower x, wait till $l(e, s) > 2x$ and enumerate x into A_{s+1} setting $r(e, s+1) = u(2x, e, s)$ where

(4.9) $\quad u(x, e, s) = u(\Phi_{e,s}(A \oplus E \cup \{2x\}; 2x).$

In itself F-permitting causes no real problems. As in (4.3) this really only involves an infinite collection of followers. However E-coding causes two rather serious problems. The first problem is that E-coding can injure the computations of (4.9). The second problem is a coherence one, which we delay discussing until later.

The solution to the first problem involves arranging matters so that if, for all x, $\Phi_e(A \oplus E \cup \{x\}; x) = (A \oplus E)(x)$, then E can compute F, giving a contradiction. We implement this as follows. At each stage s we have a collection $x_{1,s} < \cdots < x_{n,s}$ of followers following P_e. We refer to i as the permitting number of $x_{i,s}$. These followers satisfy the three rules below:

(4.10) (*Cancellation*). If $x_{i,s}$ is currently active (that is, $x_{i+1,s}$ is currently defined or $x_{i,s} \in A_s$) but we discover that $u(2x_{i,s}, e, s)$ is really E-incorrect, we choose the least such i and cancel $x_{j,s}$ for $j > i$. If $x_{i,s} \in A_s$, we also cancel $x_{i,s}$. Declare $x_{i,s}$ as inactive.

(4.11) (*Appointment*). If $x_{i,s}$ is currently defined and $x_{i+1,s}$ is not, then if $l(e, s) > 2x_{i,s}$ declare $x_{i,s}$ as active and set $x_{i+1,s} = s$. Set $r(r, s+1) = \max\{u(x_{k,s}, e, s)\colon k \leqslant i\}$.

(4.12) (*Permission*). If $x_{i,s}$ is active and $i \in F_{\text{at } s}$, then enumerate $x_{i,s}$ into A_{s+1}. Cancel $x_{j,s}$ for $j > i$. We regard $x_{i,s}$ as still active.

The reader should note that the rules allow followers of P_e to still 'receive attention' whilst P_e appears satisfied, provided that our new attack is more likely to succeed. The important point is that P_e *cannot get new followers whilst it appears satisfied.*

The above rules are sufficient to satisfy a single P_e (overcoming the first problem), as we see below.

(4.13) **Lemma.** *Suppose that* $\forall x\,(\Phi_e(A \oplus E \cup \{x\}; x)\downarrow)$. *Then* $\exists y\,(\Phi_e(A \oplus E \cup \{y\}; y) \neq (A \oplus E)(y))$ *and* P_e *acts finitely often.*

Proof. Suppose otherwise. We show $F \leqslant_T E$. We show by induction that

(4.14) (i) All the $x_{i,s}$ eventually become permanently defined, that is $\lim_s x_{i,s} = x_i$ exists with $x_i \notin A$.

(ii) Once x_{k+1} is defined at stage t, $\forall s > t\,(u(2x_k, e, t) = u(2x_k, e, s) = u(e, 2x_k))$.

(iii) $\forall s\,(x_{i+1} > \max\{u(e, 2x_k): k \leqslant i\}$.

(iv) *E can recognise when* (i) *occurs.*

Once we have (4.14) we E-compute F as follows. Let $q \in \omega$. E-recursively compute a stage s where x_{q+1} is defined. Then x_q is active, $x_q \notin A_s$ and since x_{q+1} is final, for all $j \leqslant q$, the $u(2x_j, e, s)$ computations are E-correct. By restraints (unless P_e acts) it must be that the $u(2x_j, e, s)$ computations are final. Hence $q \in F$ iff $q \in F_s$ since otherwise q's entry into F would meet P_e.

It remains, therefore, to verify (4.14). Suppose that we have E-recursively computed $x_1, \ldots, x_k$ and a stage $\hat{s}$ where $\forall s > \hat{s}\,(x_{k,s} = x_{k,\hat{s}} = x_k)$. By hypothesis also (i), (ii) and (iii) hold for x_j for $j < k$. Now $x_k \notin A_s$ otherwise the $\Phi_{e,s}(A_s \oplus E_s \cup \{2x_k\}; 2x_k)$ computations must be E-incorrect (since we have that $\forall x\,(\Phi_e(A \oplus E \cup \{x\}; x) = (A \oplus E)(x))$ and so x_k would be reset (by (4.10)). Now E-recursively find a stage $s > \hat{s}$ with $l(e, s) > 2x_k$ via E-correct computations. Then $x_{k+1,s+1} = x_{k+1}$. □

Thus, by (4.13) we now have a way of making $\Phi_e(A \oplus E \cup \{x\}; x) \neq (A \oplus E)(x)$ for some x: either we will meet P_e by divergence for some x, or the strategy outlined above meets P_e with finite effect.

The second problem we mentioned earlier is caused by our solution to the first. It occurs due to a combination of (4.10) and (4.11) causing disaster for the P_j for $j > e$ although P_e is met by divergence. Specifically the case we must worry about is that for some (least) x_k we have $u(2x_k, e, s) \to \infty$. Now we can see that $x_1, \ldots, x_{k-1}$ don't matter, but infinitely our x_i-list is cut back to x_k (i.e. we cancel $x_{j,s}$ for $j > k$). At the next stage $t > s$ when $l(e, t) > 2x_k$ we reset $r(e, t)$ to

$u(2x_k, e, t)$. Thus although P_e is met by divergence, the unbounded use of $2x_k$ may cause us to not meet P_j. The reader should note that this problem occurs even if we allow $r(e, t)$ to drop back at nondeficiency stages due to the fact that we also need F-permitting to enumerate x_i into A for (4.12). For example, P_j might wish to add some follower y_n to A since it sees $n \in F_{\text{at } s}$. However, it may be that $r(e, s) > 2y_n$, although $r(e, s)$ is really E-incorrect. Having lost our chance for y_n, E now declares $r(e, s)$ incorrect and now lets it drop back. But we no longer have a chance to put y_n into A under our current permission rules.

The key observation that allows coherence of the requirements is that E knows if a current $r(e, s)$ is E-correct or not (remember, in essence, $r(e, s)$ drops back only due to E-incorrect computations); *and whatever E knows F knows since* $E \leqslant_T F$. Thus our solution is to use *delayed permitting* for y_n. That is, when we see $n \in F_{\text{at } s}$ if $r(e, s) > 2y_n$, then we *declare* $2y_n$ *as F-permitted.* Now, should we ever see (with $2y_n$ still alive) $r(e, s)$ drop back because of E-incorrectness, we then allow y_n to enter A. The whole point is that A remains $\leqslant_T F$ since F can decide (via E) if an F-permitted follower will ever enter A.

In general to satisfy P_1 in P_0's environment we have — as usual on a tree, say — two versions of P_1. One is guessing that P_0 has finite effect, and the other is guessing that P_0 has infinitely many cutback stages. The first version of P_1 just treats P_0 as it would in a finite injury argument. The second version 'knows' that $\liminf r(e, s) < \infty$ and uses delayed permission. (More thematically the second version (to the left of the first, of course) could 'not believe' an F-permission until the next stage it is accessible.) Notice that, although F can't determine which is the true version (a Π_2-question), F can decide for any particular follower from either version whether or not it will succeed in entering A, keeping $A \leqslant_T F$.

There are clearly no further problems with the coherence of $n > 2$ requirements than there are with 2 and we leave any further formal details to the reader. □

Using the ideas of Ambos–Spies and Fejer [2], we can strengthen (4.8) if we only consider low degrees: we call a class $\mathbf{C}$ of degrees *nowhere dense* if given any interval $[\mathbf{a}, \mathbf{b}]$ in $\mathbf{R}$ (i.e. $\mathbf{a} < \mathbf{b}$) there exists a nontrivial subinterval $[\mathbf{e}, \mathbf{f}]$ with $\mathbf{a} < \mathbf{e} < \mathbf{f} < \mathbf{b}$ and $\forall \mathbf{g}\, (\mathbf{g} \in [\mathbf{e}, \mathbf{f}] \rightarrow \mathbf{g} \notin \mathbf{C})$.

(4.15) **Theorem.** *The low completely mitotic r.e. degrees are nowhere dense.*

Proof. This proof involves not much more than combining the ideas of (4.3), (4.8) and using low oracles. Hence we only give a sketch, referring the reader to [2] for further information.

Again, let $E <_T F$ be low r.e. sets. We build r.e. A, B with $E \oplus A \oplus B \stackrel{\text{def}}{=} \hat{B} \leqslant_T F$ satisfying

$$P_e{:}\quad \Phi_e(E \oplus A) \neq \hat{B},$$

$$R_{e,i}{:}\quad \Gamma_e(\hat{B}) = W_e \oplus A \text{ implies } V_e \equiv_T W_e \oplus A$$
$$\text{and } \exists x\, (\Phi_i(V_e \cup \{x\}; x) \neq V_e(x)).$$

Here we work with pairs $(\Gamma_e, W_e)_{e\in\omega}$ functionals $(\Phi_e)_{e\in\omega}$ and we build the auxiliary sets $(V_e)_{e\in\omega}$. We need the following auxiliary functions. Let

$$l(e, s) = \max\{x\colon \forall y < x\ (\Gamma_{e,s}(\hat{B}_s; y) = (W_{e,s} \oplus A_s)(y))\},$$

$$ml(e, s) = \max\{l(e, t)\colon t < s\}, \quad \text{and}$$

$$l(e, i, s) = \max\{y\colon \forall y < x\ [\Phi_{i,s}(V_{e,s} \cup \{y\}; y) = V_{e,s}(y)$$
$$\& \ \forall z(z \leq u(\Phi_{i,s} \cup \{y\}; y)) \rightarrow l(e, s) > z\}.$$

Now, as in (4.3) when s is e-expansionary (i.e. $l(e, s) > ml(e, s)$), we enumerate the least number to have entered $W_e \oplus A$ since the last e-expansionary stage into $V_{e,s+1}$. This clearly ensures that if $l(e, s) \rightarrow \infty$ then $V_e \equiv_T W_e \oplus A$.

Without the E-coding, we meet the $R_{e,i}$ by

(4.16) Pick a follower x_j, wait till $l(e, i, s+1) > 2x_j + 1$ with s-expansionary, set $r(e, i, s)$ to preserve the "$l(e, i, s) > 2x_j + 1$" computation and declare x_j as active.

(4.17) When j is F-permitted, enumerate x_j into A. Now at the next e-expansionary stage s x's entry into A has caused a change in $W_e \oplus A$ below $2x_j + 2$. Let y be the last number to have entered $W_e \oplus A$ since the last e-expansionary stage. Then our V_e action will ensure $\Phi_{i,s}(V_{e,s+1} \cup \{y\}; y) \neq V_{e,s+1}(y)$.

It really only remains to show how the strategy outlined above can survive E-coding. This is where lowness comes to the rescue. As in [2] or in our construction (2.1) we can ask if the "$l(e, i, s) > 2x_j + 1$" computations are E-correct. If our oracle answers us "Yes" we proceed as above, whereas if we are told they are not correct, then we don't let x receive attention. As usual we can be lied to with a "Yes" answer only finitely often. Thus, eventually we get a truly E-correct follower x_j. (Note that we don't need delayed permission here, the argument is now finite injury.) The remaining details are to show that if we fail, then $F \leq_T E$ as in (4.13). The P_e requirements similarly cause no problem using the lowness oracle to test correctness. We refer the reader to [2] for further details. □

We do not know if we can extend (4.15) to all r.e. degrees. It seems feasible that a variation of the above strategy might work for low_2 r.e. degrees, using the oracle methods of Bickford and Mills. The general question of nowhere density of all completely mitotic degrees would seem to require new technology. We remark that techniques sufficiently powerful to answer this question would probably be sufficient to answer similar questions for contiguous degrees, degrees containing sets with the universal splitting property (cf. Lerman and Remmel [13]) and several other related degree classes.

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