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INTERVALS AND SUBLATTICES OF THE R.E. WEAK TRUTH TABLE DEGREES, PART II: NONBOUNDING

R.G. DOWNEY*

Department of Mathematics, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand

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1. Introduction

In this paper we continue our investigations into the structure of $\mathbf{W}$, the r.e. weak truth table (W-) degrees. We use lower case boldface letters $(\mathbf{a}, \mathbf{b}, \ldots)$ to denote r.e. W-degrees. In [4] Fischer showed there exist initial segments of $\mathbf{W}$ that form lattices. In the notation of [2], this would be written as: there exist $\mathbf{a} \neq \mathbf{0}$ such that $\mathbf{W}[\mathbf{0}, \mathbf{a}]$ forms a lattice. In [2] we improved this result to show that all incomplete r.e. degrees are bottoms of lattices, and lattices are dense in $\mathbf{W}$. That is

(1.1) $\quad \forall \mathbf{a} \neq \mathbf{0}'\ \exists \mathbf{b} > \mathbf{a}\ (\mathbf{W}[\mathbf{a}, \mathbf{b}]$ a lattice$)$.

(1.2) $\quad \forall \mathbf{a}, \mathbf{b}\ (\mathbf{a} < \mathbf{b} \rightarrow \exists \mathbf{c}, \mathbf{d}\ (\mathbf{a} \leqslant \mathbf{c} < \mathbf{d} \leqslant \mathbf{b}\ \&\ \mathbf{W}[\mathbf{c}, \mathbf{d}]$ a lattice$))$.

The goal of this paper is to show that (1.1) and (1.2) cannot be combined even for $\mathbf{a} = \mathbf{0}$. Specifically, we show

(1.3) **Theorem.** $\exists \mathbf{a} \neq \mathbf{0}\ \forall \mathbf{b} \leqslant \mathbf{a}\ (\mathbf{W}[\mathbf{0}, \mathbf{b}]$ *is a lattice implies* $\mathbf{b} = \mathbf{0})$.

Actually we do a little better than (1.3). We show

(1.4) **Theorem.** *There exists an r.e. set A of high T-degree such that if W and V are r.e. nonrecursive sets with $W, V \leqslant_{\mathrm{W}} A$, then there exist r.e. sets C and D with $C \leqslant_{\mathrm{W}} W$, $D \leqslant_{\mathrm{W}} V$ and* W-deg$(C) \cap$ W-deg(D) *doesn't exist.*

We remark that (1.4) should be compared with Cooper's [1] result that every r.e. high T-degree T-bounds a minimal pair, since (1.4) also gives the existence of an r.e. set of high degree that W-bounds no (T-) minimal pairs.

Notation and terminology are fairly standard and follow [2, 3]. We use upper case Greek letters $(\Phi, \Gamma, \ldots)$ for functionals, and such letters with 'hats'

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$(\hat{\Phi}, \hat{\Gamma}, \ldots)$ to denote W-functionals. The relevant use functions will be the corresponding lower case Greek letters $(\phi, \gamma, \ldots)$. We always assume such use functions are increasing where defined. This saves on notation. We let $\langle\, , \rangle$ denote a standard pairing function monotone in both variables, and $\omega^{(e)} = \{\langle e, x\rangle : x \in \omega\}$. All computations are bounded by s at stage s.

We have decided to present our argument using the elegant methods of Soare [8, Ch. XIV], and thus assume familiarity with this. Although we describe the basic module and construction in detail, we refer the reader to [7, 8] for motivational comments regarding the intuition behind this type of construction, and furthermore the devices (such as Slaman's 'linking') we use. For the purposes of this paper references such as [8, Lemma 3.7] refer to [8, Ch. XIV].

2. The proof of (1.4)

Let $H = \bigcup_s H_s$ be an r.e. set such that for all e,

$$\text{(2.1)} \qquad \begin{cases} e \in \emptyset'' & \text{implies} \quad H^{(e)} \text{ finite}, \\ e \notin \emptyset'' & \text{implies} \quad H^{(e)} = \omega^{(e)}. \end{cases}$$

For example, define $H^{(e)} = \bigcup_s H_s^{(e)}$ in stages. Define $H_s^{(e)} = \emptyset$ if $W_{e,s} = \emptyset$ and $H_s^{(e)} = \{\langle 0, j\rangle : j \leq h(e, s)\}$ where $h(e, s) = \max\{y : \forall x < y\ (x \in W_{e,s})\}$. To make A of high degree it suffices to make A a *thick subset* of H. That is we build $A \subset H$ and satisfy the requirements

$$P_e: \quad A^{(e)} =^* H^{(e)}.$$

To meet the P_e we basically add as much of H_s to A_s as possible at any stage. The complexity of our argument is derived from the non-bounding requirements. We build auxiliary r.e. sets C_e and D_e and $Q_{e,i}$ to satisfy

$$R_{e,i,j}: \quad \text{if } \hat{\Phi}_e(A) = W_e \;\&\; \hat{\Gamma}_e(A) = V_e \text{ then}$$

$$C_e \leq_{\mathrm{W}} W_e \text{ and } D_e \leq_{\mathrm{W}} V_e \text{ and}$$

$$\text{either } W_e \text{ is recursive, or } V_e \text{ is recursive,} \quad \text{or}$$

$$\text{(2.2)} \qquad \begin{cases} \text{if } \hat{\Phi}_i(C_e) = \hat{\Phi}_i(D_e) = W_i \\ \text{then } Q_{e,i} \leq_{\mathrm{W}} C_e, D_e \text{ and } \hat{\Phi}_j(W_i) \neq Q_{e,i}. \end{cases}$$

We use a gap–cogap argument wherein we attempt to satisfy the part of the $R_{e,i,j}$ given by (2.2) by the Jockusch [5] non-infinum strategy.

(2.3) **The basic module.** To discuss the basic module, we need several auxiliary functions

$$l(e, s) = \max\{x : \forall y < x\ (\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y) \;\&\; \hat{\Gamma}_{e,s}(A_s; y) = V_{e,s}(y))\},$$

$$l(e, i, s) = \max\{x : \forall y < x\ (\hat{\Phi}_{i,s}(C_{e,s}; y) = \hat{\Phi}_{i,s}(D_{e,s}; y) = W_{i,s}(y)\}, \quad \text{and}$$

$$l(e, i, j, s) = \max\{x : \forall y < x\ (\hat{\Phi}_{j,s}(W_{i,s}; y) = Q_{e,i,s}(y) \;\&\; \phi_j(y) < l(e, i, s))\}.$$

It is again important to remember the monotonicity of the use functions here. We shall need 3 restraints associated with the above $r_1(e, i, j, s)$ and $r_2(e, i, j, s)$ that restrain A and $q(e, i, j, s)$ which restrains C_e and D_e. Now assume that $l(e, i, s) \to \infty$. For the next discussion it is convenient to drop some of the subscripts, etc., and thus use $r_1(s)$ for $r(e, i, j, s)$ etc. The basic module for this construction consists of the following steps.

Step 1. For some candidate $x \notin Q_s$ wait till $l(e, i, j, s) > x$. This candidate is targeted for C_e, D_e and $Q_{e,i}$. At stage $s+1$ *open a* W_e*-gap* by resetting $r_1(s) = 0$ and $q(s) = \phi_i(\phi_j(x))$.

Step 2. Wait for the least stage $t > s$ with $l(e, t) > ml(e, t)$ where $ml(e, t) = \max\{l(e, t') : t' < t\}$. At this stage we *close the* W_e*-gap*. We adopt the appropriate case below.

Case 2a (*Successful closure*). If $W_{e,s}[x] \neq W_{e,t}[x]$, then enumerate x into $C_{e,t}$. Now declare an *(e, i)-squeeze to be open*. Go to Step 3.

Case 2b (*Unsuccessful closure*). If $W_{e,s}[x] = W_{e,t}[x]$, then set $r_1(t+1) = \phi_e(x)$, $q(t+1) = 0$, reset x and go to Step 1.

Step 3. Wait for the least stage $m > t$ such that $l(e, i, j, m) > x$. Declare the (e, i)-squeeze to close. *Open a* V_e*-gap* by setting $r_2(m+1) = 0$. (Thus now $q(m+1) = \phi_i(\phi_j(x))$, $r_1(m+1) = r_2(m+1) = 0$.) Go to Step 4.

Step 4. Wait for the least stage $n > m$ such that $l(e, n) > ml(e, n)$. Declare the *V_e-gap to be closed* and adopt the appropriate case below.

Case 4a (*Successful closure*): $V_{e,n}[x] \neq V_{e,m}[x]$. Enumerate x into D_e and x into $Q_{e,i}$, keeping $q(n+1) = \phi_i(\phi_j(x))$ to preserve the disagreement $\hat{\Phi}_{j,n}(W_{i,n}; x) = 0 \neq 1 = Q_{e,i,n+1}(x)$.

Case 4b (*Unsuccessful closure*): $V_{e,n}[x] = V_{e,m}[x]$. Define $r_2(n+1) = r_1(n+1) = \max\{\phi_e(x), \gamma_e(x)\}$, reset x, set $q(n+1) = 0$ and go to Step 1.

Analysis of outcomes (for one requirement). The easy outcomes are that some W_e-gap or V_e-gap or (e, i)-squeeze is opened but not closed. This means that respectively $\hat{\Phi}_e(A) \neq W_e$ or $\hat{\Phi}_e(A) \neq V_e$, or $(\hat{\Phi}_e(C_e) = \hat{\Phi}_i(D_e) = W_i)$. Assuming that each attack is openable and closable the infinitary outcomes are that if there are infinitely many W_e-gaps, but only finitely many V_e-gaps then $W_e \equiv_T \emptyset$, just as in Lachlan's nonbounding theorem of [6, 7, 8].

Remark. An important, but slightly more subtle point is that we don't use stage numbers for resetting restraints (as in [7, 8]) but use (e.g.) $\phi_e(x)$. Strictly speaking this is unimportant from the basic module's point of view, but crucial from the point of view of the 'α-strategy' since we shall need to argue that at the end of a gap 'α-correct computations' (for x) are still 'α-correct', and so our restraints will be obeyed. The point here is that we will need to guess which $H^{(k)}$ of higher priority are infinite and only believe computations when they are correct according to these guesses.

As in Soare [8], we view the basic module as an automaton that has state $F(s)$

at the end of stage s. The possible outcomes (we list for F) are

$$N = \{s, g_2, g_1, w\},$$

ordered as $s \lneqq_N g_2 \lneqq_N g_1 \lneqq_N w$. The intended meaning is that

w = wait for $l(e, s) > x$ forever for some x,

g_1 = infinitely many W_e-gaps and finitely many V_e gaps,

g_2 = infinitely many V_e-gaps, and

s = successful C_e-gap closure.

The reader should note that g_2 and g_1 both incorporate the fact that every (e, i)-squeeze opened is closed. Like the outcomes 'W_e-opened but not closed' and 'V_e-opened but not closed' the outcome '(e, i)-squeeze opened but not closed' is tested by (essentially) one node above the node coding $R_{e,i,j}$ and doesn't specifically appear in the listed outcomes. (This aspect does make our construction a little more complicated since it necessitates our use of multiple linking.)

(2.4) **The priority tree.** Let $\Lambda = \{s, g_2, g_1, w, 0, 1\}$. Define the priority tree via (for $n \geqslant 1$)

$$T = \{\alpha \in \Lambda^{<\omega}: \alpha(3n) \in \{0, 1\} \;\&\; \alpha(3n+1) \in \{0, 1\}$$
$$\&\; \alpha(3n+2) \in \{0, 1\} \;\&\; \alpha(3n+2) \in \{s, g_2, g_1, w\}\}.$$

For $\sigma \in T$ define $\sigma \leqslant_L \tau$ via $\sigma \leqslant_L \tau$ iff $\sigma \subset \tau$ or

$$\exists \gamma, i, j\, (\gamma\hat{\ }i \subset \sigma \;\&\; \gamma\hat{\ }j \subset \tau \;\&\; [(i = 0 \;\&\; j = 1) \vee (i, j \in N \;\&\; i \lneqq_N j)]).$$

(2.5) **The lists and priority assignment.** To each node α on the tree we wish to assign requirements. For nodes α with $\text{lh}(\alpha) = 3n+2$, we assign a thickness requirement P_n. We can preset this in advance, whereas any other assignments are done inductively. Thus if $\sigma \in T$ and $\text{lh}(\sigma) = 3e+2$, then $\sigma\hat{\ }0$ indicates an outcome (guess) that $H^{(e)} = \omega^{(e)}$ and $\sigma\hat{\ }1$ indicates that $H^{(e)}$ is finite. For such σ define $e(\sigma) = e$, where $\text{lh}(\sigma) = 3e+2$.

For the other nodes we assign requirements via lists L_0 and L_1. We have auxiliary partial functions e, j and i which map $T \to \omega$. For nodes $\alpha \in T$ with $\text{lh}(\alpha) = 3n + j$ for $j \neq 2$, $e(\alpha)$ will be defined. If $\text{lh}(\alpha) = 3n$, then only $e(\alpha)$ is guaranteed defined but perhaps $e(\alpha)$ and $i(\alpha)$ are both defined, perhaps not. This depends on the *type of* α (to be later defined). Finally, if $\text{lh}(\alpha) = 3n+1$, then all of $e(\alpha)$, $i(\alpha)$ and $j(\alpha)$ are defined. The intention here is that the primary task of node α is to test the hypotheses: does $l(e(\alpha), s) \leftarrow \infty$ for $\text{lh}(\alpha) = 3n$ and α of type A; does $l(e(\alpha), i(\alpha), s) \to \infty$ for $\text{lh}(\alpha) = 3n$ and α of type B, and what is the outcome of $R_{e(\alpha), i(\alpha), k(\alpha)}$ assuming that $l(e(\sigma), s) \to \infty$ and $l(e(\alpha), i(\alpha), s) \to \infty$ for $\text{lh}(\alpha) = 3h+1$.

(2.6) **Convention.** It is convenient to regard our pairing functions to be enumerated in such a way that $i \leq \langle i, j\rangle$ for all j, and $\langle i, j\rangle \leq \langle i, j, k\rangle$ for all k.

(2.7) **Lists.** We define $e(\alpha)$, $i(\alpha)$, $j(\alpha)$ and the *lists* by induction on $\mathrm{lh}(\alpha) = n$ as follows.

There will be two lists L_0 and L_1. In the list L_0 the even numbers $2e$ will code requirements of type R_e and the odd numbers $2n+1$ will code requirements of type $R_{e,i}$ where $\langle e, i\rangle = n$. By our convention, please note that $\langle e, i\rangle$ occurs after e.

$n = 0$. Define $e(\emptyset) = 0$ and $L_0(\emptyset) = L_1(\emptyset) = \omega$.

$n > 0$. Let $\alpha = \sigma\hat{\ }a$ for $a \in \Lambda$. Assume that $L_0(\sigma)$ and $L_1(\sigma)$ are defined. Adopt the apropriate case below.

Case 1: $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$. Define $L_0(\alpha) = L_0(\sigma)$ and $L_1(\alpha) = L_1(\sigma)$. Define $m(\alpha) = \mu x\ (x \in L_0(\alpha))$. If $m(\alpha)$ is even, define $e(\alpha) = e$ where $2e = m(\alpha)$ (and have $e(\alpha)$ and $j(\alpha)$ undefined). If $m(\alpha)$ is odd, define $e(\alpha) = e$ and $i(\alpha) = i$ where $m(\alpha) = 2\langle e, i\rangle + 1$ (and $j(\alpha)$ is undefined). If $m(\alpha)$ is even, declare α to be of type A and declare α to be of type B otherwise.

Case 2: $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$.

Step 1. Adopt the appropriate subcase below.

Subcase 1: σ is of type A and $a = 1$. Then $e(\sigma)$ is defined. Set

$$L_0(\alpha) = (L_0(\sigma) - (\{2e(\sigma)\} \cup \{2\langle e(\sigma), k\rangle + 1 : k \in \omega\}))$$
$$\cup \{m : m > 2e(\sigma) \,\&\, \neg\exists k\ (m = 2\langle e(\sigma), k\rangle) + 1\}, \quad \text{and}$$
$$L_1(\alpha) = (L_1(\sigma) - \{\langle e(\sigma), i, j\rangle : i, j \in \omega\}) \cup \{\langle \hat{e}, i, j\rangle : i, j \in \omega \,\&\, \hat{e} > e\}.$$

Subcase 2: σ is of type A and $a = 0$. Then $e(\sigma)$ is defined. Set

$$L_0(\alpha) = L_0(\sigma) - \{2e(\sigma)\}, \quad \text{and} \quad L_1(\alpha) = L_1(\sigma).$$

Subcase 3: σ is of type B and $a = 1$. Then $e(\sigma)$ and $i(\sigma)$ are defined. Set

$$L_0(\alpha) = (L_0(\sigma) - \{2\langle e(\sigma), i(\sigma)\rangle + 1\})$$
$$\cup \{m : m > 2\langle e(\sigma), i(\sigma)\rangle + 1\}, \quad \text{and}$$
$$L_1(\alpha) = (L_1(\sigma) - \{\langle e(\sigma), i(\sigma), k\rangle : k \in \sigma\})$$
$$\cup \{\langle \hat{e}, \hat{\imath}, j\rangle : \langle \hat{e}, \hat{\imath}\rangle > \langle e(\sigma), i(\sigma)\rangle \,\&\, j \in \omega\}.$$

Subcase 4: σ is of type B and $a = 0$. Then $e(\sigma)$ and $i(\sigma)$ are defined. Set

$$L_0(\alpha) = (L_0(\sigma) - \{2\langle e(\sigma), i(\sigma)\rangle + 1\}), \quad \text{and} \quad L_1(\alpha) = L_1(\sigma).$$

Case 2, *Step* 2. Set $e(\alpha) = e$, $i(\alpha) = i$ and $j(\sigma) = j$ where $\langle e, i, j\rangle = \mu x\ (x \in L_2(\alpha))$.

Case 3: $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$. Then $e(\sigma)$, $i(\sigma)$ and $j(\sigma)$ are defined. Adopt the appropriate subcase below.

Subcase 1: $a \in \{g_1, g_2\}$. Define $L_0(\alpha)$ and $L_1(\alpha)$ as in Case 2 (Step 1) Subcase 1. Go to Step 2.

Subcase 2: $a \in \{s, w\}$. Define

$$L_0(\alpha) = L_0(\sigma) \quad \text{and} \quad L_1(\alpha) = L_1(\sigma) - \{\langle e(\alpha), i(\alpha), j(\alpha)\rangle\}.$$

This concludes the description of the priority assignment.

(2.8) **Remarks.** We remark that the intention here is that for $\alpha \in T$ (as we shall see in the construction) the nodes have (roughly) the following meanings.

For $\text{lh}(\alpha) \equiv 2 \pmod 3$:

$\alpha\hat{\ }0$ means "$H^{(e(\alpha))} = \omega^{(e(\alpha))}$",

$\alpha\hat{\ }1$ means "$H^{(e(\alpha))}$ is finite".

For $\text{lh}(\alpha) \equiv 1 \pmod 3$:

$\alpha\hat{\ }g_1$ means "infinitely many $W_{e(\alpha)}$-gaps, infinitely many $(e(\alpha), i(\alpha))$-squeezes, & finitely many $V_{e(\alpha)}$-gaps",

$\alpha\hat{\ }g_2$ means "infinitely many $V_{e(\alpha)}$-gaps",

$\alpha\hat{\ }s$ means "we get a disagreement",

$\alpha\hat{\ }w$ means "some candidate is not realized".

For $\text{lh}(\alpha) \equiv 0 \pmod 3$ *and* α *of type* A:

$\alpha\hat{\ }0$ means "$l(e(\alpha), s) \to \infty$".

$\alpha\hat{\ }1$ means "$l(e(\alpha), s) \not\to \infty$".

For $\text{lh}(\alpha) \equiv 0 \pmod 3$ *and* α *of type* B:

$\alpha\hat{\ }0$ means "$l(e(\alpha), i(\alpha), s) \to \infty$",

$\alpha\hat{\ }1$ means "$l(e(\alpha), i(\alpha), s) \not\to \infty$".

(2.9) **The regions.** Fix $\alpha \in T$ and $e, i \in \omega$.

(i) Define $\tau(\alpha, e)$ via

$$\begin{aligned}\tau(\alpha, e) = (\mu\sigma \subset \alpha)[&\text{lh}(\sigma) \equiv 0 \pmod 3 \;\&\; \sigma \text{ is of type A} \\ &\& \; e(\sigma) = e \;\&\; \neg(\exists\rho)[\sigma \subset \rho \subset \alpha \;\&\; [[\text{lh}(\rho) \equiv 0 \pmod 3) \\ &\& \; \alpha(\text{lh}(\rho)) = 1 \;\&\; ((\rho \text{ is of type A} \;\&\; e(\rho) < e) \\ &\vee (\rho \text{ is of type B} \;\&\; 2\langle e(\rho), i(\rho)\rangle + 1 < 2e))] \\ &\vee [\text{lh}(\rho) \equiv 1 \pmod 3) \;\&\; e(\rho) < e \;\&\; \alpha(\text{lh}(\rho)) \in \{g_1, g_2\}]]].\end{aligned}$$

If none exists, then $\tau(\alpha, e)$ is undefined. Now, if $\tau(\alpha, e)$ is defined, define $E(\alpha, e)$, the *e-region containing* α, to be

$$E(\alpha, e) = \{\sigma: \sigma \in T \;\&\; \tau(\alpha, e) \subset \sigma \;\&\; \tau(\alpha, e) = \tau(\sigma, e)\}.$$

We call $\tau(\alpha, e)$ *the top of* $E(\alpha, e)$. We shall say that α is an *e-boundary* if $\alpha = \sigma\hat{}a$, $e = e(\sigma)$, $a \in \{g_1, g_2, 1\}$ and either σ is of type A and $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$ or $\mathrm{lh}(\sigma) \equiv 1 \pmod 3$.

(ii) Define $\tau(\alpha, e, i)$ via

$$\begin{aligned}\tau(\alpha, e, i) = (\mu\sigma \subset \alpha)[&\mathrm{lh}(\sigma) \equiv 0 \pmod 3 \;\&\; \sigma \text{ is of type B} \;\&\; e(\sigma) = e \\ &\&\; i(\sigma) = i \;\&\; \neg(\exists\rho)[\sigma \subset \rho \subset \alpha \;\&\; [\mathrm{lh}(\rho) \equiv 0 \pmod 3 \\ &\&\; \alpha(\mathrm{lh}(\rho)) = 1 \;\&\; ((\rho \text{ is of type A} \;\&\; 2e(\rho) < \langle e, i\rangle + 1) \\ &\vee (\rho \text{ is of type B} \;\&\; \langle e(\rho), i(\rho)\rangle < \langle e, i\rangle)) \\ &\vee [\mathrm{lh}(\rho) \equiv 1 \pmod 3 \;\&\; e(\rho) < e \;\&\; \alpha(\mathrm{lh}(\rho)) \in \{g_1, g_2\}]]]].\end{aligned}$$

If none exists, then $\tau(\alpha, e, i)$ is undefined. Now define the *(e, i)-region containing* α to be

$$E(\alpha, e, i) = \{\sigma : \sigma \in T \;\&\; \tau(\alpha, e, i) \subset \sigma \;\&\; \tau(\alpha, e) = \tau(\alpha, e, i).$$

We call $\tau(\alpha, e, i)$ *the top of* $E(\alpha, e, i)$. We say that α is an *(e, i)-boundary* if either $\alpha = \sigma\hat{}a$ for $a \in \{g_1, g_2\}$ with $e = e(\alpha)$ and $\mathrm{lh}(\sigma) \equiv 1 \pmod 3$, or $\alpha = \sigma\hat{}1$, $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$ and either (a) or (b) below holds:

(a) σ is type A and $e = e(\sigma)$.

(b) σ is type B, $e = e(\sigma)$ and $i = i(\sigma)$.

(2.10) **Lemma.** (i) *Let E be an e-region with top τ and $e = e(\tau)$. Then*

(a) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}a \;\&\; e(\alpha) < e \;\&\; \mathrm{lh}(\alpha) \equiv 1 \pmod 3 \;\&\; a \in \{g_1, g_2\}]$,

(b) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}1 \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* A $\&\; e(\alpha) < e]$,

(c) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}1 \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* B $\&\; 2\langle e(\alpha), i(\alpha)\rangle + 1 < 2e]$, *and*

(d) $\neg(\exists\sigma \in E)[\tau \subset \sigma \;\&\; \tau \neq \sigma \;\&\; \mathrm{lh}(\sigma) \equiv 0 \pmod 3 \;\&\; \sigma$ *of type* A $\&\; e(\sigma) = e)$.

(ii) *Let E be an (e, i)-region with top τ and with $e = e(\tau)$ and $i = i(\tau)$. Then*

(a) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}a \;\&\; \mathrm{lh}(\alpha) \equiv 1 \pmod 3 \;\&\; e(\alpha) < e \;\&\; a \in \{g_1, g_2\}]$,

(b) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}1 \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* A $\&\; 2e(\alpha) < \langle e, i\rangle + 1]$,

(c) $\neg(\exists\sigma \in E)[\sigma = \alpha\hat{}1 \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* B $\&\; \langle e(\alpha), i(\alpha)\rangle < \langle e, i\rangle]$,

(d) $\neg(\exists\sigma \in E)[\tau \subset \sigma \;\&\; \tau \neq \sigma \;\&\; \mathrm{lh}(\sigma) \equiv 0 \pmod 3 \;\&\; \sigma$ *of type* A $\&\; e(\sigma) = e]$, *and*

(e) $\neg(\exists\sigma \in E)[\tau \subset \sigma \;\&\; \tau \neq \sigma \;\&\; \mathrm{lh}(\sigma) \equiv 0 \pmod 3 \;\&\; \sigma$ *of type* B $\&\; e(\sigma) = e \;\&\; i(\sigma) = i]$.

Proof. Straightforward induction (cf. [8, Lemma 4.6]). □

Our next lemma is our analogue of Soare's 'finite injury along any path' lemma [8, Lemma 4.7].

(2.11) **Lemma.** *For any infinite path β of T, and for any $e, i \in \omega$,*

(a) $(\exists^{<\infty}\alpha \subset \beta)[e(\alpha) = e \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* A],

(b) $(\exists^{<\infty}\alpha \subset \beta)[e(\alpha) = e \;\&\; i(\alpha) = i \;\&\; \mathrm{lh}(\alpha) \equiv 0 \pmod 3 \;\&\; \alpha$ *of type* B], *and*

(c) $(\exists^{<\infty}\alpha \subset \beta)[e(\alpha) = e \;\&\; \mathrm{lh}(\alpha) \equiv 1 \pmod 3 \;\&\; \alpha\hat{\ }a \subset \beta \;\&\; a \in \{g_1, g_2\}]$.

Proof. Fix β, e, and i and assume the lemma holds for all f, h, i and $f < e$ and $2\langle h, i\rangle + 1 < 2e$. Let $\gamma \subset \beta$ be least such that for all σ, if $\gamma \subset \sigma$, then

(i) $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$ and σ of type A implies $e(\sigma) \geqslant e$,

(ii) $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$ and σ of type B implies $2\langle e(\sigma), i(\sigma)\rangle + 1 \geqslant 2e$, and

(iii) $\mathrm{lh}(\sigma) \equiv 1 \pmod 3$ and $e(\sigma) < e$ implies $\sigma\hat{\ }g_1 \not\subset \beta$ and $\sigma\hat{\ }g_2 \not\subset \beta$.

It is not too difficult to see that the next α with $\gamma \subset \alpha \subset \beta$ and $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ will be of type A and have $e(\alpha) = e$. Now $2e$ is deleted from the list $L_0(\alpha^+)$ for $\alpha^+ = \beta(\mathrm{lh}(\alpha))$. Also $2e$ can only be added back to this list at some $\rho^+ \in T$ if

(a) $e(\rho) < e \;\&\; \mathrm{lh}(\rho) \equiv 1 \pmod 3 \;\&\; \rho^+ = \rho\hat{\ }g_1 \;\&\; \rho^+ = \rho\hat{\ }g_2$,

(b) $e(\rho) < e \;\&\; \mathrm{lh}(\rho) \equiv 0 \pmod 3 \;\&\; \rho^+ = \rho\hat{\ }1 \;\&\; \rho$ is of type A, or

(c) $2\langle e(\rho), i(\rho)\rangle + 1 < 2e \;\&\; \mathrm{lh}(\rho) \equiv 0 \pmod 3 \;\&\; \rho^+ = \rho\hat{\ }1 \;\&\; \rho$ is of type B.

By our assumptions (i), (ii) and (iii) above, no such ρ^+ can exist with $\gamma \subset \rho^+ \subset \beta$. Therefore (a) holds since for all $\sigma \supset \alpha$, if σ is type A and $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$ then $e(\sigma) > e$. Parts (b) and (c) are entirely similar and are left to the reader. □

Take $e, i \in \omega$ and any path β through T. Let α be the $\subset$-maximal node with $\alpha \subset \beta$, $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$, and α of type A and $e(\alpha) = e$. In this case we say $E(\alpha, e)$ is the *final e-region of* β and write $E(\beta, e) = E(\alpha, e)$. Similarly define $\tau(\beta, e) = \tau(\alpha, e)$. Let γ be the $\subset$-maximal node with $\gamma \subset \beta$ and $\mathrm{lh}(\gamma) \equiv 0 \pmod 3$, $e(\gamma) = e$, and $i(\gamma) = i$ with γ of type B. Define $E(\beta, e, i) = E(\gamma, e, i)$, $\tau(\beta, e, i) = \tau(\gamma, e, i)$ and refer to $E(\beta, e, i)$ as the *final (e, i)-region of* β. We note that

(2.12) $\forall\gamma\,[(\tau(\beta, e) \subset \gamma \subset \beta \;\&\; e(\gamma) = e) \rightarrow \gamma \in E(\beta, e)]$.

(2.13) $\forall\gamma\,[(\tau(\beta, e, i) \subset \gamma \subset \beta \;\&\; e(\gamma) = e \;\&\; i(\gamma) = i) \rightarrow \gamma \in E(\beta, e, i)]$.

(2.14) $\forall i\,(\tau(\beta, e, i) \supset \tau(\beta, e))$.

(2.15) **Parameters and notation.** To state the construction we need a little more notation. For $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ we have the 'α-parameters' for the α-module as in the basic module.

$x(\alpha, s) =$ the current candidate for the α-module,

$r_1(\alpha, s) =$ the current restraint imposed on A to preserve $W_{e(\alpha)}[x(\alpha, s)]$,

$r_2(\alpha, s) =$ the current restraint imposed on A to preserve $V_{e(\alpha)}[x[\alpha, s)]$,

$r(\alpha, s) = \max\{r_1(\alpha, s), r_2(\alpha, s)\}$,

$q(\alpha, s) =$ the current restraint imposed on $D_{e(\alpha)}$ to preserve $\hat{\Phi}_{i(\alpha)}(D_{e(\alpha)}; z)$ for $z \leq \phi_{j(\alpha)}(x(\alpha, s))$ or on $C_{e(\alpha)}$ to preserve $\hat{\Phi}_{i(\alpha)}(C_{e(\alpha)}; z)$ for $z \leq \phi_{j(\alpha)}(x(\alpha, s))$, and

$F(\alpha, s) =$ the current state $a \in \{s, g_2, g_1, \omega\}$ of the α-module.

Remark. We point out that strictly speaking the restraint $q(\alpha, s)$ will be unnecessary because of *cancellation.* It is merely convenient for our presentation.

To *reset a candidate* $x(\alpha, s-1)$ at stage s means to cancel $x(\alpha, s-1)$ and find a large fresh number y (exceeding all previously mentioned ones) and set $x(\alpha, s) = y$. To *initialize node α at stage s* means: rest $x(\alpha, s)$, set $F(\alpha, s) = w$, set $r_i(\alpha, s) = 0$ for $i = 1, 2$; cancel any $W_{e(\alpha)}$- or $V_{e(\alpha)}$-*links* with top or bottom equal to α. (Links are defined later.)

For α with $\mathrm{lh}(\alpha) \not\equiv 1 \pmod 3$ we always have $r_i(\alpha, s) = r(\alpha, s) = 0$ for all α.

At the end of the construction, we will have the true path β through T. We approximate β at each stage s by a string $\sigma_s \in T$. This is defined in substages $t \leq s$ during which we define a string $\sigma(t, s)$ with $\sigma(t, s) \subset \sigma(t+1, s)$. We let σ_s denote the last $\sigma(t, s)$ defined for $t \leq s$.

(2.16) **α-correct computation.** Because we are guessing whether or not $H^{(e)} = \omega^{(e)}$ we need a notion of *α-correct computation.* We say a computation "$\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y)$" is α-correct where $e = e(\alpha)$ if for all $\tau\hat{\ }0 \subset \alpha$ with $\mathrm{lh}(\tau) \equiv 2 \pmod 3$, (2.17) below holds.

(2.17) If $\max\{r(\rho, s) : \rho \leq_{\mathrm{L}} \alpha\} < z \leq \phi_{e,s}(y)$ & $z \in \omega^{(e(\tau))}$ then $z \in A_s$.

We can similarly define: "$\hat{\Gamma}_{e,s}(A_s; y) = V_{e,s}(y)$" to be α-correct as above, but with $\gamma_{e,s}$ in place of $\phi_{e,s}$ in (2.17).

Now for $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ or $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ we define the *α-length of agreement* via

$$l(\alpha, s) = \max\{z : \forall y < z\ (\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y)\ \&\ \hat{\Gamma}_{e,s}(A_s; y) = V_{e,s}(y)$$
$$\text{and these computations are } \alpha\text{-correct}\}.$$

For $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α of type B or $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$, define "$\hat{\Phi}_{i(\alpha),s}(C_{\tau,s}; x) = \hat{\Phi}_{i(\alpha),s}(D_{\tau,s}; x) = W_{i(\alpha),s}(x)$" for $\tau = \tau(\alpha, e(\alpha))$ to be *α-correct* if

(i) $x < \max\{z: \forall y < z\ (\hat{\Phi}_{i(\alpha),s}(C_{\tau,s}; y) = \hat{\Phi}_{i(\alpha),s}(D_{\tau,s}; y) = W_{i,s}(y))\}$, and

(ii) both $\hat{\Phi}_{e,s}(A_s; z) = W_{e,s}(z)$ and $\hat{\Gamma}_{e,s}(A_s; z) = V_{e,s}(z)$ are α-correct for all $z \leqslant \phi_i(y)$.

Now for $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α of type B or $\mathrm{lh}(\alpha) \equiv \pmod 3$, let

$$l(\alpha, i, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{i,s}(C_{\tau,s}; y) = \hat{\Phi}_{i,s}(D_{\tau,s}; y) = W_{i,s}(y) \text{ for } i = i(\alpha) \text{ and } \tau = \tau(\alpha, e(\alpha)) \text{ and the computations are } \alpha\text{-correct}\}.$$

Finally, we say a computation $\hat{\Phi}_{j,s}(W_{i,s}; y) = Q_{\tau,s}(y)$ is α-correct (where $j = j(\alpha)$, $i = i(\alpha)$ and $\tau = \tau(\alpha, e(\alpha), i(\alpha)))$ if $\max\{x: \forall y < x\ (\hat{\Phi}_{j,s}(W_{i,s}; y) = Q_{\tau,s}(y))\} > \phi_j(y)\ \&\ l(\alpha, i, s) > \phi_j(y)$. In this way we define (of course, for $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$)

$$l(\alpha, i, j, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{j,s}(W_{i,s}; y) = Q_{\tau,s}(y) \text{ via } \alpha\text{-correct computations}\}$$

$$(\text{where } \tau = \tau(\alpha, e(\alpha), i(\alpha)).$$

(2.18) **Definition.** Let $\alpha \in T$.

(i) We say $s+1$ is an α-stage if $\alpha \subset \sigma_{s+1}$. In addition 0 is an α-stage.

(ii) We say $s+1$ is a *genuine α-stage* if $\sigma(t, s+1) = \alpha$ for some substage t of stage $s+1$. Let G denote the collection of genuine α-stages.

(iii) Suppose $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α has type A. We say that u is an *α-expansionary* stage if $u = 0$ or $u = s+1$ where

(a) $s+1$ is a genuine α-stage, and

(b) $l(\alpha, s) > \max\{l(\alpha, \hat{u})$: $\hat{u}$ is an α-expansionary stage and $\hat{u} < u\}$.

Let $ls(\alpha, s)$ denote the last α-expansionary stage $<s$.

(iv) Suppose $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α has type B. We say that u is an *(α, i)-expansionary stage for $i = i(\alpha)$* if $u = 0$ or $u = s+1$ and

(a) $s+1$ is a genuine α-stage, and

(b) $l(\alpha, i(\alpha), u) > \max\{l(\alpha, i(\alpha), \hat{u})$: $\hat{u}$ is an (α, i)-expansionary stage with $\hat{u} < u\}$.

Similarly define $ls(\alpha, i, s)$.

(v) Finally, if $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$, we say that u is an (α, i, j)-expansionary stage for $i = i(\alpha)$ and $j = j(\alpha)$ if $u = 0$ or $u = s+1$ and

(a) $s+1$ is a genuine α-stage, and

(b) $l(\alpha, i(\alpha), j(\alpha), u) > \max\{l(\alpha, i(\alpha), j(\alpha), \hat{u})$: $\hat{u}$ is an (α, i, j)-expansionary stage with $\hat{u} < u\}$.

Similarly we define $ls(\alpha, i, j, s)$.

(2.19) **Linking.** At stage s, if a node α (necessarily, with $\text{lh}(\alpha)=1\ (\text{mod}\ 3)$) opens a $W_{e(\alpha)}$-gap, then at the same time it opens an $(e(\alpha), i(\alpha))$-squeeze. We indicate this by constructing a pair of 'short-circuits' or *links* (τ_1, α) and (τ_2, α) where

$$\tau_1 = \tau(\alpha, e(\alpha)), \quad \text{and} \quad \tau_2 = \tau(\alpha, e(\alpha), i(\alpha)).$$

The link (τ_1, α) remains in force until the next τ_1-expansionary stage when we *travel the link* (τ_1, α). Namely, there will be a substage $\sigma(t, s+1)$ with $\sigma(t, s+1)=\tau_1$ and $\sigma(t+1, s+1)=\alpha$. The link (τ_1, α) is then cancelled. The link (τ_2, α) remains in force until the next τ_2-expansionary stage when we travel (τ_2, α) and then cancel it. The idea of linking is an important idea used to simplify $\mathbf{0}'''$-arguments and is due to Slaman. The reader should either consider them as short circuits or promises at τ that we are allowed to hop like a kangaroo directly from τ_1 to α ignoring what happens in between. The reader should note that when the link (e.g.) (τ_1, α) is travelled, the nodes γ with $\tau_1 \subsetneqq \gamma \subsetneqq \alpha$ are γ-stages but not *genuine* γ-stages since they are not accessible to receive attention.

Finally we only have one type of link $(\tau(\alpha, e(\alpha)), \alpha)$ for $V_{e(\alpha)}$-gaps.

Construction

Stage 0. Initialize all $\alpha \in T$. Define $\sigma_0 = \emptyset$.

Stage $s+1$

Step 1. We refer to substage t of stage $s+1$ as *stage* $(t, s+1)$. The value of a parameter p with $p \neq \sigma$ at the end of substage t is denoted by p_t.

Substage $t=0$. Define $\sigma(0, s+1)=\emptyset$.

Substage $t+1$ $(t \leqslant s)$. We are given $\sigma(t, s+1)$ and for all $\alpha \in T$ with $\text{lh}(\alpha)\equiv 2\ (\text{mod}\ 3)$, $F_t(\alpha)$. We recall that $F_t(\alpha)=F_t(\alpha, s+1) \in \{s, g_1, g_2, w\}$ and is the current stage of the α-module. First define $\sigma(t+1, s+1)$ as follows.

Case 1: $\text{lh}(\sigma(t, s+1) \equiv 1\ (\text{mod}\ 3)$. Define $\sigma(t+1, s+1) = \sigma(t, s+1)\hat{}\ F_t(\sigma(t, s+1))$.

Case 2: $\text{lh}(\sigma(t, s+1) \equiv 0\ (\text{mod}\ 3)$ and $\sigma(t, s+1)$ has type A. Adopt the appropriate subcase below.

Subcase 1: Stage $s+1$ is not $\sigma(t, s+1)$-expansionary. Define $\sigma(t+1, s+1)=\sigma(t, s+1)\hat{}1$.

Subcase 2: Stage $s+1$ is $\sigma(t, s+1)$-expansionary and there is no link with top $\sigma(t, s+1)$. Define $\sigma(t+1, s+1)=\sigma(t, s+1)\hat{}0$.

Subcase 3: Stage $s+1$ is $\sigma(t, s+1)$-expansionary and there is a link (σ, ρ) with top $\sigma=\sigma(t, s+1)$. Define $\sigma(t+1, s+1)=\rho$. (This will set $\sigma_{s+1}=\sigma(t+1, s+1)$, as we shall see later.)

Case 3: $\text{lh}(\sigma(t, s+1) \equiv 0\ (\text{mod}\ 3)$ and $\sigma(t, s+1)$ has type B. Adopt the appropriate subcase below (for $\sigma=\sigma(t, s+1)$).

Subcase 1: Stage $s+1$ is not (σ, i)-expansionary. Define $\sigma(t+1, s+1)= \sigma(t, s+1)\hat{}1$.

Subcase 2: Stage $s+1$ is (σ, i)-expansionary and there is no link with top α. Define $\sigma(t+1, s+1) = \sigma(t, s+1)\hat{}0$.

Subcase 3: Stage $s+1$ is (σ, i)-expansionary and there is a link (σ, ρ). Define $\sigma(t+1, s+1) = \rho$. (This will set (as we see later) $\sigma_s = \sigma(t+1, s+1) = \rho$ or we will create a new link $(\hat{\sigma}, \rho)$ with $\hat{\sigma} \subset \sigma$.)

Case 4: $\mathrm{lh}(\sigma(t, s+1)) \equiv 2 \pmod 3$. Let $\mathrm{lh}(\sigma(t, s+1)) = 3e+2$. If $\mathrm{card}(H^{(e)}_{s+1} - H^{(e)}_g) > 0$ where g is the last genuine $\sigma(t, s+1)$-stage $\leqslant s$, set $\sigma(t+1, s+1) = \sigma(t, s+1)\hat{}0$. Otherwise set $\sigma(t+1, s+1) = \sigma(t, s+1)\hat{}1$. Now let $\alpha = \sigma(t+1, s+1)$. We say that α requires attention at stage $s+1$ if one of the following conditions holds (as defined precisely later):

$$(2.20)\quad \begin{cases} \text{ready to open a } W_{e(\alpha)}\text{-gap (at } \alpha), \\ \text{ready to close a } W_{e(\alpha)}\text{-gap (at } \alpha), \\ \text{ready to close an } (e(\alpha), i(\alpha))\text{-squeeze (at } \alpha), \text{ or} \\ \text{ready to close a } V_{e(\alpha)}\text{-gap (at } \alpha). \end{cases}$$

If α does not require attention and $t<s$ to to substage $t+2$. If α requires attention choose the first clause (2.21–2.24) below to pertain. If $t<s$ and α opens a $W_{e(\alpha)}$-gap or α closes an $(e(\alpha), i(\alpha))$-squeeze and opens a $V_{e(\alpha)}$-gap, then go to substage $t+2$. Otherwise go to step 2 setting $\sigma_s = \sigma(t+1, s+1)$.

(2.21) Ready to open a $W_{e(\alpha)}$-gap:

(a) $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$,
(b) $\tau(\alpha, e(\alpha))$ is defined,
(c) $\tau(\alpha, e(\alpha), i(\alpha))$ is defined,
(d) $F_t(\alpha) = w$, and
(e) $l(\sigma, e, i, j) > x(\alpha)$.

Action. Open a $W_{e(\alpha)}$-gap by defining $F(\alpha, s+1) = g_1$, $r_1(\alpha, s+1) = 0$. Initialize all γ such that $\alpha\hat{}w \leqslant_L \gamma$. Create links (τ_1, α), (τ_2, α) with tops $\tau_1 = \tau(\alpha, e(\alpha))$ and $\tau_2 = \tau(\alpha, e(\alpha), i(\alpha))$ and bottom α. Set $q(\alpha, s+1) = \phi_i(\phi_j(x(\alpha)))$ where $i = i(\alpha)$ and $j = j(\alpha)$.

(2.22) Ready to close a $W_{e(\alpha)}$-gap.

(a) $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$,
(b) $F_t(\alpha) = g_1$, and
(c) $s+1$ is τ-expansionary for $\tau = \tau(\alpha, e(\alpha))$.

Action. Let $u+1 < s+1$ be the stage when the current $W_{e(\alpha)}$-gap opened. Let $x = x(\alpha, s)$. Close the $W_{e(\alpha)}$-gap by adopting the appropriate case below.

Case (a) (*Successful closure*): $W_{e(\alpha),s}[x] \neq W_{e(\alpha),u}[x]$. Keep $F(\alpha) = g_1$ and $r_2(\alpha, s+1) = r_2(\alpha, s)$, $r_1(\alpha, s+1) = 0$, and $q(\alpha, s+1) = q(\alpha, s)$. Declare an $(e(\alpha), i(\alpha))$-squeeze to be open. Enumerate x into $C_{\tau, s+1}$ where $\tau(\alpha, e(\alpha)) = \tau$. Remove the link with top $\tau(\alpha, e(\alpha))$ but keep the link $(\tau(\alpha, e(\alpha), i(\alpha)), \alpha)$.

Case (b) (*Unsuccessful closure*): $W_{e(\alpha),a}[x] = W_{e(\alpha),u}[x]$. Set $r_1(\alpha, s+1) = \phi_{e(\alpha)}(x)$. Reset $x(\alpha, s+1)$ and initialize all γ such that $\alpha\hat{}1 \leqslant_L \gamma$. Remove both links $(\tau(\alpha, e(\alpha)), \alpha)$ and $(\tau(\alpha, e(\alpha), i(\alpha)), \alpha)$. Set $q(\alpha, s+1) = 0$.

(2.23) Ready to close an $(e(\alpha), i(\alpha))$-squeeze.

(a) $\text{lh}(\alpha) \equiv 1 \pmod 3$,

(b) $F_t(\alpha) = g_1$, and

(c) $s+1$ is τ-expansionary for $\tau = \tau(\alpha, e(\alpha), i(\alpha))$.

Action. Close the $(e(\alpha), i(\alpha))$-squeeze by removing the link $(\tau(\alpha, e(\alpha), i(\alpha)), \alpha)$. Open a $V_{e(\alpha)}$-gap by setting $r_2(\alpha, s+1) = r_1(\alpha, s+1) = 0$. Set $q(\alpha, s+1) = q(\alpha, s)$. Create a link $(\hat{\tau}, \alpha)$ where $\hat{\tau} = \tau(\alpha, e(\alpha))$. Define $F(\alpha) = g_2$ and initialize all γ with $\alpha\hat{}g_1 \leqslant \gamma$.

(2.24) Ready to close a $V_{e(\alpha)}$-gap.

(a) $\text{lh}(\alpha) \equiv 1 \pmod 3$,

(b) $F_t(\alpha) = g_2$, and

(c) $s+1$ is τ-expansionary where $\tau = \tau(\alpha, e(\alpha))$.

Action. Let $u+1 < s+1$ be the stage where the current $V_{e(\alpha)}$-gap opened and let $x = x(\alpha, s)$. Remove the link (τ, α) and close the $V_{e(\alpha)}$-gap by adopting the appropriate case below.

Case 1 (*Successful closure*). If $V_{e(\alpha),s}[x] \neq V_{e(\alpha),u}[x]$, then enumerate x into $Q_{\tau(\alpha,e(\alpha),i(\alpha)),s+1}$ and into $D_{\tau(\alpha,e(\alpha)),s+1}$. Keep $q(\alpha, s+1) = q(\alpha, s)$. Initialize all η for $\alpha\hat{}g_2 \leqslant_L \eta$. Set $F(\alpha) = s$.

Case 2 (*Unsuccessful closure*). If $V_{e(\alpha),s}[x] = V_{e(\alpha),u}[x]$, set $r_1(\alpha, s+1) = \phi_{e(\alpha)}(x(\alpha, s))$, $r_2(\alpha, s+1) = \gamma_{e(\alpha)}(x(\alpha, s))$. Reset $x(\alpha, s+1)$ and initialize all η for $\alpha\hat{}g_1 \leqslant_L \eta$. Set $q(\alpha, s+1) = 0$. Set $F(\alpha) = w$.

This completes the description of Step 1 of the construction. The reader should note that (in Step 1) if α receives attention, then $\text{lh}(\alpha) \equiv 1 \pmod 3$.

Step 2. Now for any $x \in H^{(e)}_{s+1}$ with $x \notin A_s$, enumerate x into A_{s+1} if $x > \max\{r(\rho, s+1) : \rho \leqslant_L \alpha \text{ for } \text{lh}(\alpha) = 3e+2\}$. □ End of Construction

We turn to the verification. This is accomplished by a series of lemmas based on Soare's [8] scheme.

The following lemma summarizes the elementary properties of links that we shall implicitly use in the later lemmas. Most of its proof is left to the reader since although tedious it is essentially straightforward, and mainly depends on the construction of the priority tree.

(2.25) **Lemma** ('The link lemma'). (i) *If* (τ, α) *is a link, then* $\alpha \supset \tau\hat{}0$. *Furthermore* $\text{lh}(\alpha) \equiv 1 \pmod 3$ *and either*

(a) $\text{lh}(\tau) \equiv 0 \pmod 3$, $\tau = \tau(\gamma, e(\alpha))$, *$\tau$ has type* A, *and* $e(\alpha) = e(\tau)$, *or*

(b) $\text{lh}(\tau) \equiv 0 \pmod 3$, $\tau = \tau(\gamma, e(\alpha), i(\alpha))$, *$\tau$ has type* B, $e(\alpha) = e(\tau)$ *and* $i(\alpha) = i(\tau)$.

(ii) *Any link* (τ, α) *once created may be travelled at most once before it is removed.*

(iii) *There are at most two links with bottom* α *at any stage. Furthermore, if* (τ_1, α) *and* (τ_0, α) *are two links existing at the end of stage s, then for some i,* $\tau_i = \tau(\alpha, e(\alpha))$ *and* $\tau_{1-i} = \tau(\alpha, e(\alpha), i(\alpha))$, *without loss,* $\tau_0 = \tau(\alpha, e(\alpha))$. *Then* $\tau_0 \subset \tau_1$ *and* $\tau_0 \neq \tau_1$. *(Hence the links are nested with* (τ_1, α) *inside* (τ_0, α).)

(iv) *Suppose* (τ_1, α_1) *is a link created at stage s and* (τ_2, α_2) *is a link with* $\tau_1\hat{}0 \subset \alpha_2$ *which is created at a later substage of stage s, or* (τ_2, α_2) *is created at a later stage, and suppose* $\alpha_1 \neq \alpha_2$. *Then* $\alpha_1 \subset \alpha_2$ *and*

(a) *if* τ_2 *has type* A, *and* $\tau_2 \subset \tau_1$ *then*
  (i) *if* τ_1 *has type* A *then* $e(\tau_2) < e(\tau_1)$,
  (ii) *if* τ_1 *has type* B *then* $2e(\tau_2) < 2\langle e(\tau_1), i(\tau_1)\rangle + 1$,
  (iii) *there is no* $\hat{e}$*-boundary* ρ *with* $\alpha_1 \subset \rho \subset \alpha_2$ *and* $\hat{e} < e(\tau_2)$,
  (iv) *there is no* e', i'*-boundary* ρ' *with* $\alpha_1 \subset \rho' \subset \alpha_2$ *and* $2\langle e', i'\rangle + 1 \leq 2e(\tau_2)$, *and*
  (v) *furthermore, if* τ_1 *has type B then there is a unique link* (τ_3, α_1) *with* $\tau_3 \neq \tau_1$. *Then* τ_3 *has type* A *and* $\tau_2 \subset \tau_3 \subset \tau_1$, *and*

(b) *if* τ_2 *has type* B *and* $\tau_2 \subset \tau_1$ *then*
  (i) *if* τ_1 *has type* A *then* $2\langle e(\tau_2), i(\tau_2)\rangle + 1 < 2e(\tau_1)$,
  (ii) *if* τ_1 *has type* B *then* $\langle e(\tau_2), i(\tau_2)\rangle < \langle e(\tau_1), i(\tau_1)\rangle$,
  (iii) *there is no* $\hat{e}$*-boundary* ρ *with* $\alpha_1 \subset \rho \subset \alpha_2$ *and* $2\hat{e} < 2\langle e(\tau_2), i(\tau_2)\rangle + 1$ *or* $\hat{e} < e(\tau_2)$.
  (iv) *there is no* e', i'*-boundary* ρ' *with* $\alpha_1 \subset \rho' \subset \alpha_2$ *and* $\langle e', i'\rangle < \langle e(\tau_2), i(\tau_2)\rangle$, *and*
  (v) *furthermore, if* τ_1 *has type* B *then there is a (unique) link* (τ_3, α) *with* $\tau_3 \neq \tau_1$. *Then* τ_3 *has type* A *and* $\tau_2 \subset \tau_3 \subset \tau_1$, *and*

(c) *if* τ_2 *has either type* A *or* B, *but* $\tau_1 \subset \tau_2$ *then* $\alpha_1 \subset \tau_2$.

Remark. The gist of the lemma is that links may be nested but never crossed. Also if (τ_1, α_1) is nested within (τ_2, α_2), so is any link (τ, α_1), and furthermore either $\alpha_1 = \alpha_2$ or (τ_2, α_2) is created after (τ_1, α_1) and has higher priority than (τ_1, α_1) (and will be removed first).

Proof. Each of the above is proved by straightforward induction and/or analysis of the priority tree. We simply sketch the details.

To see (i) links are only created when α requires attention and then only when (2.23) or (2.21) holds. In both cases, $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ and one of (a) or (b) must hold. Finally, suppose $\tau\hat{}1 \subset \alpha$. First suppose τ has type A. Then by construction of the priority tree for all $j, i \in \omega$ all $\langle e(\tau), i, j\rangle$ are deleted from $L(\tau\hat{}1)$ and all $2\langle e(\tau), i\rangle + 1$ are deleted from $L_0(\tau\hat{}1)$. Now as $e(\alpha) = e(\tau)$ and τ is the top of α's $E(\alpha, e(\alpha))$ region, (2.10) says that there is no higher priority boundary between τ and α. But then it is impossible for $e(\alpha) = e(\tau)$ since e cannot be re-added to the $L_0(\rho)$-list for any $\tau\hat{\ }1 \subset \rho \subset \alpha$. If τ has type B, the argument is similar.

(ii) is very easy and is left to the reader.

For (iii) we need that $e < \langle e, i\rangle$ and whenever e is put back onto the L_0-list, so is $\langle e, i\rangle$ and so e is always removed first. Given these facts, it is quite easy to see that (iii) holds.

(iv) (a) Suppose (τ_1, α_1) is created at stage s and (τ_2, α_2) is later created. τ_2 has type A and $\tau_2 \subset \tau_1$. By (i) above since (τ_1, α_1) is a link $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ and $\alpha_1\hat{\ }g_i \subset \alpha_2$ for $i = 1, 2$. Now, if $e(\tau_1) < e(\tau_2)$, then $e(\tau_2)$ would be re-added to the L_0-list and by definition of $\tau(\alpha_2, e(\alpha_2))$ (as α_2 cannot cross the $e(\tau_1)$-boundary, τ_1) $\tau_2 = \tau(\alpha_2, e(\alpha_2))$ must occur below τ, and so $\tau_1 \subset \tau_2$. Similarly $e(\tau_1) = e(\tau_2)$ is only possible if $e(\tau_1)$ has somehow been re-added to the list. Only higher priority boundaries do this and this would contradict the position of τ_2 and τ_1. τ_2 of type B is similar.

We remark that all of the remaining observations of the lemma may be proved by entirely similar methods and are left to the reader. □

(2.26) **Definition.** The *true path* β of T is defined by induction on n as follows. Let $\alpha \subset \beta$ with $\mathrm{lh}(\alpha) = n$. Define $\alpha\hat{\ }i \subset \beta$ via:

Case 1: $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α of type A. Then $\alpha\hat{\ }0 \subset \beta$ if $\exists^\infty s$ (s is α-expansionary); and $\alpha\hat{\ }1 \subset \beta$ otherwise.

Case 2: $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α of type B. Then $\alpha\hat{\ }0 \subset \beta$ if $\exists^\infty s$ (s is $(\alpha, i(\alpha))$-expansionary); and $\alpha\hat{\ }1 \subset \beta$ otherwise.

Case 3: $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$. Then $\alpha\hat{\ }a \subset \beta$ if $\lim_s F(\alpha, s)$ exists and $a \in \{s, w\}$, $\alpha\hat{\ }g_2 \subset \beta$ if α opens infinitely many $V_{e(\alpha)}$-gaps, and $\alpha\hat{\ }g_1 \subset \beta$ otherwise.

Case 4: $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$. Then $\alpha\hat{\ }0 \subset \beta$ if there are infinitely many $\alpha\hat{\ }0$-stages and $\alpha\hat{\ }1 \subset \beta$ otherwise.

(2.27) **Lemma** ('The leftmost path lemma'). *Fix n and let $\alpha \subset \beta$ with $\mathrm{lh}(\alpha) = n$. Then*

(i) $\exists^{<\infty} s\ (\sigma_s \leqslant_L \alpha \;\&\; \sigma_s \not\subset \alpha)$, *and*

(ii) $|G^\alpha| = \infty$ (*where G^α denotes the set of genuine α-stages*).

Proof. Clearly, (i) and (ii) hold for $n = 0$. Fix $n \geqslant 0$ and assume (i) and (ii) hold for n. Let $\rho \subset \beta$ with $\mathrm{lh}(\rho) = n$ and let $\alpha = \rho\hat{\ }a$ for $\alpha \subset \beta$. For an induction, let s_1 be a stage such that for all $s \geqslant s_1$

(a) $\sigma_s \leqslant_L \rho$ implies $\sigma_s \subset \rho$,

(b) if $\mathrm{lh}(\rho) \equiv 0 \pmod 3$ and ρ is of type A and $a = 1$, then s is not ρ-expansionary,

(c) if $\mathrm{lh}(\rho) \equiv 0 \pmod 3$ and ρ is of type B and $a = 1$, then s is not (ρ, i)-expansionary,

(d) if $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$, then $F(\rho, s) \nless a$, and

(e) for all e, x with $3e + 2 \leqslant \mathrm{lh}(\rho)$, $H^{(e)}$ is finite and $x \in H^{(e)}$ we have $x \in A$ iff $x \in H^{(e)}$,

(i) Now, suppose $\sigma_s \leqslant_L \alpha$ but $\sigma_s \not\subset \alpha$ for infinitely many s. Choose $b <_A a$ with $\rho\hat{}b \subset \sigma_s$ for infinitely many s. Now by choice of s_1 there is no genuine $\rho\hat{}b$-stage $s \geqslant s_1$. Hence there are infinitely many stages $s \geqslant s_1$ at which there is some link of the form (τ, η) with $\tau \subset \rho$ and $\rho\hat{}b \subset \eta$ such that (τ, η) is travelled at stage s.

By the hypotheses, the collection of genuine ρ-stages is infinite. Let $s_2 + 1 > s_1$ with $s_2 + 1 \in G^\rho$. Let t be such that $\sigma(t, s+1) = \rho$. At stage $t(s_2 + 1)$ there can be at most one link (τ, η) as above, and furthermore $\tau = \eta$. This link must be travelled at stage $(t+1, s_2+1)$ and this link will then be removed.

Now, if $\tau = \tau(\eta, e)$ for some $e = e(\eta)$, by the paragraph following (2.20) the removal of this link finishes stage $s_2 + 1$. In particular at the end of stage $s_2 + 1$ there is no link $(\hat{\tau}, \hat{\eta})$ with $\hat{\tau} \subset \rho$ and $\rho\hat{}b \subset \hat{\eta}$. But now it follows that $\rho\hat{}b \not\subset \sigma_s$ for all $s > s_2 + 1$ and there cannot be any further (τ, η).

Thus the only possibility is that $\tau = \tau(\eta, e, i)$ with $e = e(\eta)$ and $i = i(\eta)$. Now, by the analysis above if infinitely many $(\hat{\tau}, \hat{\eta})$ exist, it must be that at stage $(t+1, s_2+1)$ we successfully close an $(e(\eta), i(\eta))$-squeeze. We thus open a $V_{e(\eta)}$-gap with a link (τ_1, η) for $\tau_1 \subset \tau$ by the link lemma. Also by the link lemma this link is the innermost of any link (τ_2, η_2) created after stage $(t+1, s_2+1)$ but before (τ_1, η) is removed. In particular we note that there is now no link with top $\tau = \rho$. Now since there must be infinitely many (genuine) ρ-stages there must be some stage (t_1, s_3+1) with $s_3 + 1 > s_2 + 1$ and $\sigma(t_1, s_3+1) = \tau_1$. Then at stage (t_1+1, s_3+1), we remove (τ_1, η) and then by the paragraph following (2.20), we finish stage $s_3 + 1$. It again now follows that there are no links $(\hat{\tau}, \hat{\eta})$ with $\hat{\tau} \subset \rho$ alive at stage $s_3 + 2$ and so for all $s > s_3 + 1$, $\rho\hat{}b \not\subset \sigma_s$ and there cannot be any further $(\hat{\tau}, \hat{\eta})$. This clinches (i).

(ii) We must show that G^α is infinite. If $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ and α has type A or α has type B or $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$, then this follows because the above analysis says that there are infinitely many genuine ρ-stages with no links originating at ρ. If, for example, $\alpha = \rho\hat{}0$ and ρ has type A, then no link can originate from ρ until the next genuine ρ-expansionary stage. The other cases are similar. Finally, if $\mathrm{lh}(\rho) \equiv 1 \pmod 3$, then the result follows because if $(t, s+1)$ for $s > s_1$ is a genuine ρ-stage, then it is a genuine $\rho\hat{}i$-stage for $i \in \{s, w, g_1, g_2\}$. □

(2.28) **Lemma** ('Restraint lemma'). *Let* $\alpha \subset \beta$ *and let* $\alpha^+ = \alpha\hat{}\beta(n)$ *where* $n = \mathrm{lh}(\alpha)$. *Then*

$$\lim\{\hat{r}(\alpha, s) : s \in \alpha^+\} \quad \textit{exists}$$

where

$$\alpha^+ \textit{ is the set of } \alpha^+\textit{-stages, and } \hat{r}(\alpha, s) = \max\{r(\rho, s) : \rho \leqslant_L \alpha\}.$$

Furthermore, if $q(\alpha, s)$ *is defined, then* $\hat{q}(\alpha) = \lim\{\hat{q}(\alpha, s) : s \in \omega\}$ *exists, where* $\hat{q}(\alpha, s) = \max\{q(\rho, s) : \rho \leqslant_L \alpha\}$.

Proof. Let α^- be the predecessor (if any) of α and define

$$\hat{r}(\alpha^-) = \lim\{r(\alpha^-, s) : s \in \omega\}.$$

Now, if $\text{lh}(\alpha) \not\equiv 1 \pmod 3$, then $r(\alpha, s) = 0$ for all s. The lemma then holds because — by choice of $\alpha \subset \beta$ — $\lim_s \{r(\rho, s) : \rho \leqslant_L \alpha \;\&\; \rho \not\subset \beta\}$ exists (by the previous lemma, and the fact that $r(\rho, s)$ is reset only at *genuine* ρ-stages).

On the other hand, if $\text{lh}(\alpha) \equiv 1 \pmod 3$, choose s_1 such that for all $s \geqslant s_1$, $\sigma_s \leqslant_L \alpha$ implies $\sigma_d \subset \alpha$ (via (2.27)). Now as in the basic module, $r(\alpha) = \liminf r(\alpha, s)$ exists as does $q(\alpha) = \liminf q(\alpha, s)$. If $\alpha^+ = \alpha\hat{\ }a$ and $a \in \{s, w\}$, the neither $r(\alpha, s)$ nor $q(\alpha, s)$ are reset after the next $\alpha\hat{\ }a$-stage. If $a = g_2$, then $q(\alpha) = 0$ and $r(\alpha) = 0$. Finally, if $a = g_1$, then after the final $\alpha\hat{\ }g_2$-stage s_2 we have $r(\alpha, s) = r(\alpha, s_2)$. □

(2.29) **Lemma.** *All the P_e are met. That is, for all e, $H^{(e)} =^* A^{(e)}$.*

Proof. Let $\alpha \subset \beta$ with $\text{lh}(\alpha) = 3e + 2$. Then by (2.28) $\liminf \bar{r}(\alpha, s) = \bar{r}(\alpha)$ exists. By Step 2 of the construction if $x \in H^{(e)} \;\&\; x > \bar{r}(\sigma)$, then we add x to A at some genuine α-stage, since at such a stage $r(\eta, s) = 0$ for $\eta \not\leqslant_L \alpha$. □

(2.30) **Lemma** ('Truth of outcome lemma'). *Let $\alpha \subset \beta$.*

(i) *If $\text{lh}(\alpha) \equiv 2 \pmod 3$, then $\alpha\hat{\ }0 \subset \beta$ implies $H^{(e(\alpha))} = \omega^{(e(\alpha))}$.*

(ii) *If $\text{lh}(\alpha) \equiv 2 \pmod 3$, then $\alpha\hat{\ }1 \subset \beta$ implies $H^{(e(\alpha))}$ finite.*

(iii) *If $\text{lh}(\alpha) \equiv 0 \pmod 3$ and α has type* A, *then $\alpha\hat{\ }0 \subset \beta$ implies there are infinitely many α-expansionary stages.*

(iv) *If $\text{lh}(\alpha) \equiv 1 \pmod 3$ and α has type* A, *then $\alpha\hat{\ }1 \subset \beta$ implies that either $\hat{\Phi}_{e(\alpha)}(A) \neq W_{e(\alpha)}$ or $\hat{\Gamma}_{e(\alpha)}(A) \neq V_{e(\alpha)}$.*

(v) *If $\text{lh}(\alpha) \equiv 0 \pmod 3$ and α has type* B, *then $\alpha\hat{\ }0 \subset \beta$ implies that there are infinitely many $(\alpha, i(\alpha))$-expansionary stages.*

(vi) *If $\text{lh}(\alpha) \equiv$* (mod 3) and α has type B, *then $\alpha\hat{\ }1 \subset \beta$ implies that*

$$\text{either} \quad \hat{\Phi}_{i(\alpha)}(C_\tau) \neq W_{i(\alpha)}, \quad \text{or} \quad \hat{\Phi}_{i(\alpha)}(D_\tau) \neq W_{i(\alpha)},$$

where $i = i(\alpha)$ and $\tau = \tau(\alpha, e(\alpha), i(\alpha))$.

(vii) *If $\text{lh}(\alpha) \equiv 1 \pmod 3$ and $\alpha\hat{\ }g_1 \subset \beta$, then $W_{e(\alpha)}$ is recursive.*

(viii) *If $\text{lh}(\alpha) \equiv 1 \pmod 3$ and $\alpha\hat{\ }g_2 \subset \beta$, then $V_{e(\alpha)}$ is recursive.*

(ix) *If $\text{lh}(\alpha) \equiv 1 \pmod 3$ and $\alpha\hat{\ }w \subset \beta$ and $\tau = \tau(\alpha, e(\alpha), i(\alpha))$ is defined, then $\hat{\Phi}_{j(\alpha)}(W_{i(\alpha)}) \neq D_\tau$.*

(x) *If $\text{lh}(\alpha) \equiv 1 \pmod 3$ and $\alpha\hat{\ }s \subset \beta$, then $\tau = \tau(\alpha, e(\alpha), i(\alpha))$ is defined and for some $x(\alpha) = x(\alpha, s)$ we have $\hat{\Phi}_{j(\alpha)}(W_{i(\alpha)}; x(\alpha)) \neq Q_\tau(x(\alpha))$.*

Proof. (i) and (ii). There are infinitely many genuine α-stages. At such stages $\alpha\hat{\ }i$ is genuine and we choose $\alpha\hat{\ }0$ only if $H_{s+1}^{(e(\alpha))}$ exceeds $H_u^{(e(\alpha))}$ where u is the last genuine α-stage. Thus $\alpha\hat{\ }0 \subset \beta$ iff $H^{(e(\alpha))} = \infty$ and so $H^{(e(\alpha))} = \omega^{(e(\alpha))}$.

Parts (iii)–(vi) are entirely similar and are left to the reader.

(vii) Let $\alpha\hat{\ }g_1 \subset \beta$ as above. We claim $W_{e(\alpha)}$ is recursive. This is where use the fact that we are using W-reductions rather than T-reductions. Choose a genuine α-stage s_1 such that $\forall s > s_1$ $(\sigma_s \leqslant_L \alpha\hat{\ }g_1$ implies $\sigma_s \subset \alpha\hat{\ }g_1)$ (by Lemma (2.27)).

Now we may also assume by the link lemma that there are no links (τ, α) alive at stage s_1. Also, since $\alpha^\wedge g_1 \subset \beta$, it must be that $\tau(\alpha, e(\alpha), i(\alpha))$ and $\tau(\alpha, e(\alpha))$ are defined. Set $e = e(\alpha)$ and $i = i(\alpha)$. Also, by Lemma (2.28) we may assume that $\hat{r}(\sigma, s) = \hat{r}(\sigma)$ at each (genuine) $\alpha^\wedge g_1$ stage for all $\sigma \leqslant_L \alpha$. Let $s_2 \geqslant s_1$ be an $\alpha^\wedge g_1$-stage at which we open a $W_{e(\alpha)}$-gap. This is only possible if $l(\alpha, i, j, s_2) > x(\alpha, s_2)$ *via α-correct computations.* Now by monotonicity we know $l(\alpha, s_2) > x$ via α-correct computations. In particular then by choice of s_1 we know that if $\tau^\wedge 0 \subset \alpha$ with $\text{lh}(\tau) = 3j + 2$, then if $z \in \omega^{(j)}$ and $z \leqslant \phi_{e(\alpha)}(x)$ then

(2.31) $\quad z \in A \quad \text{iff} \quad z \in A_{s_2}$.

Also, if $\tau^\wedge 1 \subset \alpha$ with $\text{lh}(\tau) = 3j + 2$, then if $z \in \omega^{(j)}$ we have

(2.32) $\quad z \in A \quad \text{iff} \quad z \in A_{s_2}$.

Now at the next genuine α-stage $s_3 > s_2$ when we close the $W_{e(\alpha)}$-gap, this closure must be unsuccessful. In particular, $W_{e(\alpha),s_2}[x] = W_{e(\alpha),s_3}[x]$. At this stage we have $l(\alpha, s_3) > x(\alpha, s_2)$. *Now since we are dealing with W-reductions, our use function $\phi_e(x)$ has not changed and hence our computations*

$$\text{“}\hat{\Phi}_{e,s_3}(W_{e(\alpha),s_3}; z) = W_{e(\alpha),s_3}(z) \textit{ for } z \leqslant x\text{”}$$

are (still) α-correct.

Now at stage s_3 we re-impose the $r_1(\alpha, s_3)$-restraint to hold $W_{e(\alpha),s_3}[x]$ and by α-correctness this restraint succeeds in holding $W_{e(\alpha),s_3}[x]$ during the co-gap, that is, until the next genuine $\alpha^\wedge g_1$-stage. By similar reasoning, we see $W_{e(\alpha)}[x] = W_{e(\alpha),s_2}[x]$, for larger and larger x and so $W_{e(\alpha)}$ is recursive.

(viii) As in (vii) but using $\Gamma_{e(\alpha)}$, $V_{e(\alpha)}$ and g_2.

(ix) Then $\lim_s F(\alpha, s) = w$ and $\lim_s x(\alpha, s) = x(\alpha)$ for some x. It is really quite easy to see that — as $\tau(\alpha, e(\alpha))$ and $\tau(\alpha, e(\alpha), i(\alpha))$ are defined — we have $l(\alpha, e(\alpha)) \to \infty$ but $l(\alpha, i(\alpha), j(\alpha)) \not\to \infty$.

(x) Assume $\lim_s F(\alpha, s) = s$ and hence $\lim_s x(\alpha, s) = x(\alpha)$ exists. Choose s_0 so that these values are constant for all $s > s_0$. Then, if we take the (genuine) α-stage $s_1 > s_0$ where the $W_{e(\alpha)}$-gap eventually adds $x(\alpha)$ to Q_τ where $\tau = \tau(\alpha, e(\alpha), i(\alpha))$, we must have $l(\alpha, i(\alpha), j(\alpha), s) > x(\alpha)$. By the $q(\alpha, s_1)$-restraint it is quite easy to see that $D_{\tau_1,s_1}[\phi_i(\phi_j(x(\alpha)))] = D_{\tau_1,s_2}[\phi_i(\phi_j(x(\alpha)))]$ where $\tau_1 = \tau(\alpha, e(\alpha))$ and s_2 is the stage where we open the $V_{e(\alpha)}$-gap for $x(\alpha)$. But at this stage is must be that $l(\alpha, i(\alpha), j(\alpha)) > x(\alpha)$. Hence the $q(\alpha, s_2) = q(\alpha, s_1) = q(\alpha)$-restraint ensures that $C_{\tau_1,s_2}[\phi_i(\phi_j(x(\alpha))] = C_{\tau_1}[\phi_i(\phi_j(x(\alpha)))]$ and hence $l(\alpha, i(\alpha), j(\alpha)) > x(\alpha)$ and we must have $W_{i(\alpha),s_2}[\phi_{j(\alpha)}(x(\alpha))] = W_{i(\alpha)}[\phi_{j(\alpha)}(x(\alpha))]$. This means that

$$\hat{\Phi}_{j(\alpha)}(W_{i(\alpha)}; x(\alpha)) = 0 \neq 1 = Q_{\tau,s_2+1}(x(\alpha)).$$

This gives the desired result. □

Thus it remains to prove that all the reduction procedures we need exist. This is accomplished by the following lemma, which concludes our proof.

(2.34) **Lemma** ('The regions lemma'). *Fix e such that* $\hat{\Phi}_e(A) = W_e$ *and* $\hat{\Gamma}_e(A) = V_e$ *with* W_e *and* V_e *non-recursive. Let* $\tau = \tau(\beta, e)$. *Then* C_τ *and* D_τ *are r.e.,* $C_\tau \leqslant_W W_e$ *and* $D_\tau \leqslant_W V_e$. *Furthermore, if* $\hat{\Phi}_i(C_\tau) = \hat{\Phi}_i(D_\tau) = W_i$, *then* $Q_{\tau_1} \leqslant_W C_\tau, D_\tau$ *and if* W_e *and* V_e *are nonrecursive, then* $Q_{\tau_1} \nleqslant_R W_i$ *where* $\tau_1 = \tau(\beta, e, i)$.

Proof. By the finite injury lemma fix τ. Then C_τ is the collection of $x(\alpha)$ such that $\alpha \in E(\tau, e)$ and $e(\alpha) = e$ and α enumerates $x(\alpha)$ into C_τ at the close of a (successful) $W_{e(\alpha)}$-gap (so the $W_{e(\alpha)}[x]$ changes). Then C_τ is clearly r.e. To see that $C_\tau \leqslant_W W_{e(\alpha)}$ it suffices to show that every $E_{e(\alpha)}$-gap opened by α for $e(\alpha) = e$, $\alpha \in E(\tau, e)$ and $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ is eventually closed. But this is immediate. As $\tau \subset \beta$ any link (τ, α) created at an α-stage is removed at the next genuine τ-stage, and we know (by Lemma (2.27)) that there are infinitely many genuine τ-stages. Thus $C_\tau \leqslant_W W_{e(\alpha)}$. The proof that $D_\tau \leqslant_W V_{e(\alpha)}$ is virtually the same and is left to the reader.

Now suppose that additionally $\hat{\Phi}_i(C_\tau) = \hat{\Phi}_i(D_\tau) = W_i$. Now as above for $\alpha \in E(\tau_1, e, i)$ any (τ_1, α) link (and of course any (τ, α) link) created is later removed and so every $(e(\alpha), i(\alpha))$-squeeze once opened is later closed, and moreover, opens a $V_{e(\alpha)}$-gap if successful. We only add x to Q_{τ_1} at the end of a successful $V_{e(\alpha)}$-gap. These facts together mean $Q_{\alpha_1} \leqslant_W D_\tau$ by simple permitting. Furthermore, $Q_{\tau_1} \leqslant_W C_\tau$ by delayed permitting; that is, if x enters Q_{τ_1} at the close of the $V_{e(\alpha)}$-gap that is opened at the time x enters C_τ. Hence $Q_{\tau_1} \leqslant_W C_\tau, D_\tau$.

Finally, we need to show that $Q_{\tau_1} \nleqslant_W W_i$. Suppose for a contradiction that $Q_{\tau_1} \leqslant_W W_i$. Find $\alpha \in E(\beta, e, i)$ with $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ and $\hat{\Phi}_{j(\alpha)}(W_i) = Q_{\tau_1}$. By the truth of outcomes as W_e and V_e are nonrecursive, $\alpha^\wedge g_i \not\subset \beta$ for $i = 1, 2$. But now this means $\alpha^\wedge s$ or $\alpha^\wedge w \subset \beta$ and again by truth of outcome this means $\hat{\Phi}_{j(\alpha)}(W_i) \neq Q_{\tau_1}$ giving the desired result. □

We believe that the above techniques can be extended to show

(2.35) $\forall \mathbf{a} \neq \mathbf{0}'\ \mathbf{b} > \mathbf{a}\ \forall \mathbf{c}\ ((\mathbf{a} \leqslant \mathbf{c} < \mathbf{b}\ \&\ W[\mathbf{a}, \mathbf{c}]\ \textit{a lattice}) \rightarrow \mathbf{a} = \mathbf{c}))$.

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