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UNDECIDABILITY OF $L(F_\infty)$ AND OTHER LATTICES OF R.E. SUBSTRUCTURES

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1. Introduction

In this paper we investigate the lattice $L(\mathfrak{M})$ of r.e. substructures of an effective model $\mathfrak{M}$. This was first suggested by Metakides and Nerode [12] as a basic method of analysing the algorithmic content of mathematics. Their original studies concentrated on fields and vector spaces, but subsequently these investigations have been broadened and deepened by many authors (cf. [17]).

The goal of this paper is to prove the undecidability of the first-order theories of a wide class of such lattices. We do so by establishing a general result applying to a class of effective structures along the lines of those considered by Remmel [21]. The actual theorem is a little technical to state here, but we remark that applications include all (recursive) Steinitz systems, free groups, boolean algebras, ideals, theories, and a wide class of modules over locally computable rings. In particular we subsume most currently known undecidability results and also solve a number of open questions (such as $L(F_\infty)$ the lattice of r.e. subfields). We review the setting and these examples in Section 2. The proof of the main result is in Section 3. This falls into two main parts. First we prove the existence of a certain type of r-maximal subset of ω (namely one with the 'lifting property'). Second we show that this existence theorem yields a method of effectively interpreting the theory of the lattice of r.e. sets in $\mathrm{Th}(L(\mathfrak{M}))$ the theory of $L(\mathfrak{M})$. The result will follow because of the work of Harrington [8] and Hermann [9].

In Section 4 we give some applications of the results and techniques of Section 3. We remark that although these 'transfer' techniques introduced here are not very difficult, they are powerful and have numerous applications quite aside from the undecidability results. To demonstrate this we utilize these methods to deduce several other results. For example, we are able to show $L(F_\infty)$ is not recursively presentable.

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2. Setting

As in Remmel [21], we work within an *effective closure system*, $\mathfrak{M}=(M, \mathrm{cl})$. Recall that this is a recursive set M with an operation $\mathrm{cl}: P(M) \rightarrow P(M)$ satisfying
(i) $A \subseteq \mathrm{cl}(A)$,
(ii) $A \subseteq B$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
(iii) $\mathrm{cl}(\mathrm{cl}(A))=\mathrm{cl}(A)$,
(iv) $x \in \mathrm{cl}(A)$ implies that for some finite $A' \subseteq A$, $x \in \mathrm{cl}(A')$,
and furthermore cl is effective on (indices of) finite sets: viz, given $y, x_1, \ldots x_n \in M$ we can decide whether or not $x \in \mathrm{cl}(y_1, \ldots, y_n)$. Here $\mathrm{cl}(y)$ denotes $\mathrm{cl}(\{y\})$. We let $L(\mathfrak{M})$ denote the lattice of r.e. closed sets where A is closed if $\mathrm{cl}(A)=A$. For our later work, let $\mathrm{Th}(L(\mathfrak{M}))$ denote the theory of $L(\mathfrak{M})$. In $\mathfrak{M}$ there is a natural equivalence relation $=^*$ where $A=^* B$ if there exists a finite set F with $\mathrm{cl}(A \cup F)=\mathrm{cl}(B \cup F)$. The idea is to analyse $L(\mathfrak{M})$ under $=^*$ in the same way as we analyse $L(\omega)$ under $=^*$. Thus we additionally require that $M \neq^* \mathrm{cl}(\emptyset)$.

Some notions we need are from [21]. Let $V \in L(M)$. We say V is *decidable* if given $(y, x_1, \ldots, x_n)$ we can decide whether or not $y \in \mathrm{cl}(V \cup \{x_1, \ldots, x_n\})$. From [21] we say a sequence K of elements of M is *special over* V if V is decidable, K is an infinite recursive set and

(i) $\forall A, B\ (A, B \subset K \rightarrow \mathrm{cl}(V \cup A) \cap \mathrm{cl}(V \cup B)=\mathrm{cl}(V \cup(A \cap B))$,

(ii) $\mathrm{cl}(V \cup K)=M$.

For $E \subset M$ we write $\mathrm{cl}_V(E)$ for $\mathrm{cl}(V \cup E)$. Now, given K special over V we can define the *support of* x (with respect to K over V) as the unique smallest subset K' of K with $x \in \mathrm{cl}_V(K')$. We denote this by $\mathrm{supp}(x)$. We say K has *local exchange property* (LEP) over V if, given any y and $x \in \mathrm{supp}(y)$, we have

$$x \in \mathrm{cl}_V(\mathrm{supp}(y)-\{x\}) \cup \{y\}).$$

Most of our results will require an effective closure system $\mathfrak{M}$ with a special sequence K with LEP over V. We conclude this section by giving some examples.

(1) (ω, cl), the r.e. sets. Here $\mathrm{cl}(A)=A$.

(2) $(V_\infty, *)$, the r.e. subspaces. Here $(W)^*$ denotes the subspace generated by W in V_∞. See [14], for example.

(3) (F_∞, cl). Here F_∞ denotes a 'fully effective' algebraically closed field, that is one with a recursive infinite transcendence base over a recursive ground field, and cl is an effective algebraic closure operator. See, for example [15] or [16].

(4) *Steinitz systems*. More generally, any recursive Steinitz system $\mathfrak{M}=(M, \mathrm{cl})$. Here $\mathfrak{M}$ is an effective closure system such that $\mathfrak{M}$ also satisfies the *global exchange property* below

$$y \in \mathrm{cl}(A \cup\{x\})-\mathrm{cl}(A) \text{ implies } x \in \mathrm{cl}(A \cup\{y\}).$$

See [16] for further details. In each of examples (2)–(4) we take $V=\mathrm{cl}(\emptyset)$ and any recursive basis as a special sequence (cf. [15], [21]).

(5) *Boolean subalgebras* (Remmel [20, 21]). Given any effective boolean algebra $\mathscr{A}$, within its isomorphism type is a boolean algebra of the form $\tilde{N} \times \tilde{D}$, $\tilde{Q} \times \tilde{D}$ or $\tilde{C}$, where $\tilde{D}$ is an effective boolean algebra and $\tilde{N}$ is an effective presentation of the boolean algebra of finite and cofinite subsets of ω, $\tilde{Q}$ is the atomless countable boolean algebra, and $\tilde{C}$ is the boolean algebra $B_{1+\eta\cdot\omega}$, (the subalgebra generated by the left closed right open subintervals of Q together with $\{\{q\}:q \in Q\}$). In each of $\tilde{N} \times \tilde{D}$, $\tilde{C}$ and $\tilde{Q} \times \tilde{D}$ we can find special sequences with LEP over appropriate V. For example, in $\tilde{N} \times \tilde{D}$ let $K = \{\langle a, 0_{\tilde{D}}\rangle : a$ is an atom of $\tilde{N}\}$ and $V = \{\langle 0_{\tilde{N}}; d\rangle; \langle 1_{\tilde{N}}, d\rangle : d \in \tilde{D}\}$. The others are similar. We refer the reader to [20] for further details.

(6) The above examples can be extended to 'appropriately effective' modules over locally computable rings, and some subrings. Examples include $\bigoplus_{i\in\omega} Z$ and $Z[x_i : i \in \omega]$, and subgroups of certain commutative groups.

(7) *Orderings.* For example $(Q^n, \leqslant)$ with $\leqslant$ the product ordering. Let $V = \{\langle 0, y_2, \ldots, y_n\rangle : y_i \in Q\}$ and $K = \{\langle x, 0, \ldots, 0\rangle : x \in Q\}$.

(8) *Modified examples.* (8a) Sometimes, we need to restrict our domain. For example in $\mathrm{LI}(\tilde{Q})$ the lattice of r.e. ideals of $\tilde{Q}$, of course for all $M \in \mathrm{LI}(\tilde{Q})$, $M =^* \tilde{Q}$, since $\mathrm{cl}(\{1\} \cup M) = \tilde{Q}$. The solution is to pick certain definable substructures and restrict attention there. For example in $\tilde{Q}$, we analyse the lattice of r.e. subideals of a fixed recursive maximal ideal P. Now (P, cl) forms the appropriate closure system, and, as in [21], we can find the desired 'ideal' closure systems in $\tilde{N} \times \tilde{D}$, $\tilde{Q} \times \tilde{D}$ or $\tilde{C}$ by this method.

(8b) Another example of this phenomenon is the lattice of r.e. free subgroups of $G = \langle X, Y \mid -\rangle$. Here we pick a recursive infinitely generated fixed free subgroup G and look at its closure system. Classically, we find bases with LEP for G and the effectivity conditions are guaranteed by Neilson-Schreier theory (cf. Magnus, Karass and Solitar [12]).

(9) *Intersection subsystems.* One final way to make such closure systems is as follows. In $\mathfrak{M}$ let K be special with LEP over V. Now let M' be a recursive *set* containing V and K. We can define a new closure system $(M', \mathrm{cl}_{M'})$ via for $B \subset M'$, $\mathrm{cl}_{M'}(B) = \mathrm{cl}(B) \cap M'$. This is a natural way to generate closure systems with many pathological properties. We refer the reader to [16] for some applications of this method.

Now some final pieces of notation and terminology. Recall from, say, [23] that an r.e. set A with A coinfinite is called *maximal* if given any r.e. set $W \supset A$, either $W =^* A$ or $W =^* \omega$. A is called *r-maximal* if given any r.e. sets W, V with $W \cup V = \omega$ either $W \cup A =^* \omega$ or $V \cup A =^* \omega$. A is called *atomless* if A is contained in no maximal r.e. set. An r.e. sequence of disjoint canonical finite sets $\{D_x\}_{x\in W}$ is called a *strong array*. An r.e. sequence of disjoint finite sets $\{W_{f(x)}\}_{x\in\omega}$ is called a *weak array*. If, given any weak array $\{W_{f(x)}\}_{x\in\omega}$ we have $W_{f(x)} \cap \bar{A} = \emptyset$ for some x, we say A is *hyperhyper-simple* (*hh-simple*). We remark that if A is r-maximal and non-maximal, then A is not hh-simple (cf. [23]). We let $\{W_e\}_{e\in\omega}$ denote an effective listing of all r.e. sets. If K is a special sequence over

V we will say "let A have property P in K"; this is meant to be interpreted in $L(K)$, the lattice of r.e. subsets of K. For example, if we say "let A be a maximal subset of K" we shall mean that A is maximal in the lattice of r.e. subsets of K.

For any unexplained terminology and notation we refer the reader to [17] and [23].

3. The main result

We state the main theorem.

3.1. Theorem. *Suppose $\mathfrak{M}$ possesses a special sequence K with LEP over V. Then* $\mathrm{Th}(L(\mathfrak{M}))$ *is undecidable.*

Our first lemma produces an r-maximal subset of ω with a certain property. This property is defined via:

Definition. We say an r.e. set A has the *lifting property* if A is coinfinite, and given any strong array $\{D_x\}_{x\in W}$, $\mathrm{card}(D_x - A) \leq 1$ for almost all x.

We remark that this notion was inspired by one ('co-1-1') from Madan and Robinson [15]. We have

3.2. Theorem. *There exists an atomless r-maximal set A with the lifting property.*

Proof. For our construction of an atomless r-maximal r.e. set with the lifting property, we mainly indicate the modifications necessary to Soare [23, X Theorem 5.4]. Following this, we have markers $\Gamma_{\langle i,j\rangle}$ arranged in a square array where d_n^s denotes the element associated with Γ_n at stage s, and $\omega - A_s = \{d_n^s : n \in \omega\}$. We satisfy

$$N_e:\quad d_e = \lim_s d_e^s \text{ exists}$$

$$P_e:\quad W_e \subseteq^* H_e \text{ or } \omega - A \subseteq^* W_e$$

$$R_e:\quad H_e \text{ is r.e.}$$

where $H_e = A \cup \{d_{\langle i,j\rangle} : i \leq e \text{ and } j \in \omega\}$.

Additionally, we must arrange that A has the lifting property. Let T_n denote the n-th strong array. We must satisfy

$$Q_e:\quad \text{For almost all } D_x \in T_e,\ \mathrm{card}(D_x - A) \leq 1.$$

Now let $V_{e,s} = \{x : \exists t \leq s\ (x = d^t_{\langle i,j\rangle}\ \&\ x \in W_{e,t}\ \&\ e < i)\}$ and $\sigma(e, x, s) = \{i : i \leq e\ \&\ x \in V_{i,s}\}$, the e-state with respect to the V_e array measured at stage s.

At stage $s+1$ of the construction, we have two steps. In step 1, we follow Soare [23]. Find the least e such that for some i, $e < i \leq s$ and $\sigma(e, d_i^s, s) >$

$\sigma(e, d_e^s, s)$. Choose $i > e$ with $\sigma(e, d_i^s, s)$ as large as possible and set $d_e(s+1) = d_i^s$. In this case we say "Γ_e pulls d_i^s". Enumerate d_k^s for $e \leqslant k \leqslant s$, $k \neq i$ into A ('dumping'). Set $d_j(s+1) = d_j^s$ for $j < e$ and $d_{e+k}(s+1) = d_{s+k}^s$ for $k > 0$. If no action is taken, set $d_j(s+1) = d_j^s$ for all j. All of this follows [23].

In step 2, perform the following substages for $0 \leqslant j \leqslant s$.

Substage j (Attacking Q_j). Find the least i and the greatest k if any such that $j < i < k$ and $d_i(s+1)$, $d_j(s+1) \in D_x$ and $D_x \in T_{j,s}$. Dump $d_{i+1}(s+1), \ldots, d_{i+s+1}(s+1)$ into A. Go to substage $j+1$ if $j < s$. If $j = s$, we redefine all d_i^{s+1} to list in order the remaining $d_i(s+1)$, and go to stage $s+2$. (Notice we don't dump the least member of a D_x.)

Verification. The usual arguments show that $\lim_s d_e^s = d_e$ exists. Briefly, once $d_j^s = d_j$ for $j < e$ and all the $D_x \in T_k$ for $k < e$ containing these d_j are already in $T_{k,s-2}$, it is easy to see that d_e^s can only move to higher e-states. Similarly the argument given in [23, Theorem 5.4 (Lemma 2)] will show that all the P_e are met. Step 2 automatically ensures that if $D_x \in T_e$ and $D_x^* \cap \{d_1, \ldots, d_e\} = \emptyset$, then $\text{card}(D_x - A) \leqslant 1$. Hence all the Q_e are met: there are at most finitely many such $D_x \in T_e$. Finally we need to show that H_e is r.e. The proof is essentially the same as that of Soare. Fix e. Let $\sigma = \{i : i \leqslant e \ \& \ \omega - A \subseteq^* V_i\}$. Choose n such that $\sigma(e, d_m) = \sigma$ for all $m \geqslant n$, and s with $d_m^s - d_m$ all $m \leqslant n$. Define

$$\hat{H}_e = \{d_k^t : t+1 > s \ \& \ n \leqslant k \leqslant t \ \& \ \exists i, j \ (k = \langle i, j\rangle \ \& \ i \leqslant e) \ \& \\ \forall m \quad (n \leqslant m \leqslant t \rightarrow \sigma(e, d_m^t, t) = \sigma) \quad \& \quad \forall m \leqslant k \quad (d_m^t = d_m^{t+1})\}.$$

The claim is that $d_k^t \in \hat{H}_e$ at stage $t+1$ iff $d_k^t = d_k$ or $d_k^t \in A$. If $d_k^t \neq d_k$, then either d_k^t is dumped through the action of some Q_j for $j < k$, or Γ_m (for some $m < k$) pulls some element z at stage $t' > t+1$. By definition of $\hat{H}_e$, Γ_m doesn't pull d_k^t at stage t'. Hence, in this case also, d_k^t is dumped into A. Thus if $d_k^{t+1} \neq d_k$, then $d_k^t \in A$. Hence $\hat{H}_e \cup A =^* H_e$ and so H_e is r.e. The result now follows. □

3.3. Remark. For some purposes (in Section 4) we need only an (apparently) weaker property. We say A is *weakly co*-1-1 if for all strong arrays $\{D_x\}$ with $\bigcup_x D_x = \omega$, $\text{card}(D_x - A) \leqslant 1$. We can, for example, show that if A is a major subset of a maximal set, then A is weakly co-1-1. It is unclear if this is also true for the lifting property. Our next lemma is a technical one which shows why we call this property the lifting property.

3.4. Lemma. *Suppose A is an r.e. subset of K with the lifting property, where K is a special sequence with LEP over V. Let $W \in L(\mathfrak{M})$, and suppose $W \supset A \cup V$. Then there exists an r.e. subset D of K with* $\text{cl}_V(D) =^* W$.

Proof. Let $W = \{a_0, a_1, \ldots\}$ with $a_0 \notin V$. Let $D_0 = \text{supp}(a_0) \neq \emptyset$. $R_0 = D_0$ and $b_0 = a_0$. At stage $s+1$, let b_{s+1} be the least a_i with $\text{supp}(a_i) \nsubseteq R_s$. Set $R_{s+1} = R_s \cup \text{supp}(a_i)$ and $D_{s+1} = \text{supp}(a_i) - R_s$. Clearly $\{D_x\}_{x\in\omega}$ is a strong array. It follows that for almost all x, $\text{card}(D_x - A) \leqslant 1$. Let $t = \max\{x : \text{card}(D_x - A) \geqslant$

2}. Let $B = A \cup R_{t+1}$. Then $B =^* A$. We see that $\forall x$ (card$(D_x - B) \leq 1$). We claim that for all x, $D_x \subset \mathrm{cl}_V(W \cup R_{t+1})$. We prove this by induction. Certainly, $D_0 \subset B$ by definition. Now suppose $\forall y < x$, $D_y \subset B$. Consider D_x. Now $D_x = \mathrm{supp}(a_j) - \bigcup_{y<x} D_x$, for some j; and we may clearly suppose $x > t+1$. Hence $D_x = \{b\} \cup D'_x$ where $D'_x \subset B$ and $b \in K$ with (we suppose) $b \notin B$. It follows that $\mathrm{supp}(a_j) = \{b\} \cup B'$ some $B' \subseteq B$. By LEP, $b \in \mathrm{cl}_V(B' \cup \{a_j\})$. But $\mathrm{cl}_V(B' \cup \{a_j\}) \subseteq \mathrm{cl}_V(W \cup R_{t+1})$. Hence $b \in \mathrm{cl}_V(W \cup R_{t+1})$ as required. It follows that $W =^* \mathrm{cl}_V(W \cup R_{t+1}) = \mathrm{cl}_V(R)$, and $R \subseteq K$, giving the result. □

Now, let $V_1, V_2 \in L(\mathfrak{M})$, with $V_1 \subseteq V_2$. We define the *interval lattice* $L(V_1, V_2)$ to be the lattice generated by $\{W : W \in L(\mathfrak{M})$ and $V_1 \subseteq W \subseteq V_2\}$. For any such lattice we can naturally associate a partial ordering $L^*(V_1, V_2)$ by factoring out with $=^*$.

3.5. Lemma. *Let K be special with LEP over V. Let A be an r.e. atomless r-maximal subset of K. Then there exists $W \in L(\mathfrak{M})$ such that $W \supset \mathrm{cl}_V(A)$ and $L^*(\mathrm{cl}_V(A), W)$ is recursively isomorphic to $L^*(\omega)$, the lattice of r.e. sets modulo finite sets.*

Proof. Let A, K, and V satisfy the hypotheses of 3.5. Now A is not hyperhyper-simple in K. Hence there is a weak array $\{F_0, F_1, \ldots\}$ of disjoint finite subsets of K such that $\forall x$ $(F_x \cap \bar{A} \neq \emptyset)$. In fact, it is clear that by slowing down the enumeration of each F_x we can ask that card$(F_x - A) = 1$. Let $W = \mathrm{cl}_V(A \cup \bigcup_x F_x)$. Define a function f via $f : x \to A \cup F_x$. We may extend this to a mapping from $L(\omega)$ via $f(W_e) = A \cup \{F_x : x \in W_e\}$. Finally set $g(W_e) = \mathrm{cl}_V(f(W_e))$. Now g is clearly a recursive 1–1 function. It remains to show that g induces an isomorphism from $L^*(\omega)$ tp $L^*(\mathrm{cl}_V(A), W)$. Let $Q \in L^*(\mathrm{cl}_V(A), W)$. By 3.4 and the fact that $A \cup \bigcup_x F_x \subset K$, there exists an r.e. set B such that $B \supset A$ and $Q =^* \mathrm{cl}_V(B)$. Now find the unique r.e. collection $\{F_x : x \in W_e\}$ with $A \cup \bigcup_x \{F_x : x \in W_e\} = B$. Evidently $g : W_e \to \mathrm{cl}(B)$. Thus g is onto, it takes the $=^*$ equivalence class of W_e to that of $\mathrm{cl}(B)$. It is obviously join preserving. We now also require that it be meet preserving. Let C and D be r.e. It clearly suffices to show that for all Q, $R \in L(\mathfrak{M})$, $Q =^* \mathrm{cl}_V(A \cup \{F_x : x \in C\})$ and $R =^* \mathrm{cl}_V(B \cup \{F_x : x \in D\})$ implies that for all $W \in L(\mathfrak{M})$ if $W \leq^* Q$, R then $W \leq^* \mathrm{cl}_V(A \cup \{F_x : x \in C \cap D\})$. But this is obvious from the proof of Lemma 3.4. □

3.6. Remark. Notice in particular, that for (F_∞, cl) we have found V, $W \in L(F_\infty)$ such that for Q, $R \in L(F_\infty)$ with $V \subseteq Q$, $R \subseteq W$, "$[Q] \wedge [R]$" is meaningful in the lattice theoretic sense. (Here $[G]$ denotes the $=^*$ equivalence class of G.) Namely, $[Q] \wedge [R] = [H]$ where H is an r.e. subfield such that $H \leq^* Q$, R and for all W if $W \leq^* Q$, R then $W \leq^* H$. Notice that we do not assert that $L^*(\mathrm{cl}_V(A), W)$ forms a lattice by defining $[A] \wedge [B] = [A \cap B]$, for this is not true. ($=^*$ is not a congruence.) See, for example, [16]. We remark that it is unknown whether or not $L^*(F_\infty)$ is a lattice if we define meet via the lattice-theoretic definition in 3.5.

3.7. Conclusion of proof of 3.1. We may now conclude the proof of the main result as follows. Using parameters v, w we can now effectively interpret $\mathrm{Th}(L^*(\omega))$ in $\mathrm{Th}(L^*(\mathfrak{M}))$. It follows that $\mathrm{Th}(L^*(\mathfrak{M}))$ is undecidable. Moreover in $\mathrm{Th}(L(\mathrm{cl}_V(A), W))$ we can interpret $\mathrm{Th}(L^*(\mathrm{cl}_V(A), W))$ for $\mathrm{cl}_V(A) \subseteq Q \subseteq W$; since $\mathrm{cl}_V(A) =^* Q$ iff for all Q' with $\mathrm{cl}_V(A) \subseteq Q' \subseteq Q$, Q' is complemented in $L(\mathrm{cl}_V(A), W)$. To see this suppose $\mathrm{cl}_V(A) =^* Q$. Let Q' be as above. Now there is a finite set G with $\mathrm{cl}_V(A \cup G) \supseteq Q'$. Find $G' \subset G$ as follows. Let $G = \{b_0, \ldots, b_n\}$. Now find the least j such that $b_j \in \mathrm{cl}_V(Q' \cup \{b_i; i<j\})$. If none exists, set $G' = G$. Otherwise delete b_j from G. Now find the least j' such that

$$b_{j'} \in \mathrm{cl}_V(Q' \cup (\{b_i : i<j'\} - \{b_j\})).$$

Now, this process halts after at most $n+1$ steps to define G'. The claim is that $Q' \cap \mathrm{cl}_V(R) = \mathrm{cl}_V(A)$ where $R = G' \cup (A \cup (\bigcup_x F_x) - G)$. Since $\mathrm{supp}(Q') \subset A \cup G$, it suffices to show that $Q' \cap \mathrm{cl}_V(G' \cup A) = \mathrm{cl}_V(A)$. Let $y \notin \mathrm{cl}_V(A)$ and $y \in Q' \cap \mathrm{cl}_V(G' \cup A)$. Let $b_j = \max\{b_i : b_i \in \mathrm{supp}(y) \cap \hat{G}'\}$. By LEP, if $\mathrm{supp}(y) - A = \{b_{i_0}, \ldots, b_{i_m}, b_i\}$, we see that

$$b_i \in \mathrm{cl}_V(A \cup \{b_{i_0}, \ldots, b_{i_m}\} \cup \{y\}) \subseteq \mathrm{cl}_V(Q' \cup \{b_{i_0}, \ldots, b_{i_m}\}).$$

Hence $b_i \notin G'$ by construction: contradiction. Thus for Q_1, $Q_2 \in L(\mathrm{cl}_V(A), W)$, $Q_1 =^* Q_2$ is definable in $\mathrm{Th}(L(\mathrm{cl}_V(A), W))$ and hence definable with parameters in $\mathrm{Th}(L(\mathfrak{M}))$. Thus we can effectively interpret $\mathrm{Th}(L^*(\omega))$ in $\mathrm{Th}(L(\mathfrak{M}))$. Now by the results of Harrington [8] or Hermann [9] we know $\mathrm{Th}(L^*(\omega))$ is undecidable. Thus $\mathrm{Th}(L(\mathfrak{M}))$ is undecidable. □

Remark. Strictly speaking, we use (e.g.) that Hermann [9] showed $\mathrm{Th}(L^*(\omega))$ by showing all boolean pairs definable with parameters in $L^*(\omega)$.

3.8. Corollary. *Under the conditions of* 3.1, $\mathrm{Th}(L^*(\mathfrak{M}))$ *is undecidable.*

4. Associated results

By showing every finite distributive lattice is a filter in $L^*(V_\infty)$, Nerode and Smith [18] originally showed $\mathrm{Th}(L(V_\infty))$ and $\mathrm{Th}(L^*(V_\infty))$ were undecidable. (However this proof doesn't extend to $L(F_\infty)$.) Another result subsumed here is that of Downey [4] who observed that the lattice of r.e. ideals of $\tilde{Q}$ is undecidable. Finally we get an unpublished result of Smith [private communication] that the theory of the lattice of r.e. free subgroups of $\langle x, y \mid - \rangle$ is undecidable.

Caroll [3] has announced the undecidability of the lattice of r.e. subalgebras of *any* recursive infinite boolean algebra. We remark that our results do not seem to apply to give the full power of this result since it is unclear whether or not every recursive infinite boolean algebra has a special sequence of the desired type. Our result merely works for boolean algebras discussed in Section 2.

One associated application here is to complemented members of $L(\mathfrak{M})$. In the r.e. set case, the collection of complemented (= recursive) sets form a lattice with a decidable first-order theory. Let $S(\mathfrak{M})$ denote the partial ordering of complemented members of $L(\mathfrak{M})$. Using the techniques of Ash and Downey [1], we have

4.1. Theorem. (i) *Let $\mathfrak{M}$ be a recursive Steinitz system such that given any infinite independent set I in M and $y \in M$, $\dim(\mathrm{cl}(\{y\} \cup I)) = \infty$. Then* $\mathrm{Th}(S(\mathfrak{M})$ *is undecidable.*

(ii) *Let $\mathfrak{M}$ be the lattice of r.e. ideals of $\bar{Q}$. Then* $\mathrm{Th}(S(\mathfrak{M}))$ *is undecidable.*

Proof. (sketch). (i) Generalize the results from [1] to an arbitrary Steinitz system. Briefly, the assumption on $\mathfrak{M}$ allows us to show that for all $V \in L(\mathfrak{M})$ there exist $D_1, D_2 \in S(\mathfrak{M})$ with $D_1 \oplus D_2 = V$. With this we can effectively interpret $\mathrm{Th}(L(\mathfrak{M}))$ in $\mathrm{Th}(S(\mathfrak{M}))$. See [1] for further details. (ii) is similar and relies on Remmel's result (in [22]) that any r.e. ideal can be split into a pair of complemented ones. □

Now some other applications. One curious one is:

4.2. Theorem. *There is a recursively presented vector space Q such that $L^*(Q)$ is recursively isomorphic to $L^*(\omega)$.*

Proof. In the setting $M = V_\infty$ over an infinite field construct $\mathrm{cl}_V(A) = V'$ and W as in 3.1. Let V'' be a recursive subspace of V' with $\dim(V'/V'') = 1$. Then $Q = W \bmod V''$ has the desired properties. □

The use of the lifting property is of course not restricted to r-maximal sets. The following result can be established by similar modifications to Soare [23, X Theorem 7.2], and we leave this to the reader.

4.3. Theorem. *Let $\mathfrak{B}$ be any Σ_3^0 boolean algebra. There exists a hh-simple $A \in L(\omega)$ whose lattice of supersets (modulo $=^*$) is isomorphic to $\mathfrak{B}$, and A has the lifting property.*

4.4. Corollary. *Let $\mathfrak{M}$ be as in* 3.1. *Then*

(i) *Every Σ_3^0 boolean algebra is a filter in $L^*(\mathfrak{M})$, and*

(ii) *$L(\mathfrak{M})$ is not recursively presentable.*

Proof. (ii) This also requires Feiner's result [7] that there are Σ_3^0 boolean algebras that are not recursively presentable. □

There are several useful applications of the lifting property and that of being 'weakly co-1-1' (cf. Remark 3.3). For example, if A is a weakly co-1-1 r-maximal subset of K, then it is easy to show that $\mathrm{cl}_V(A)$ is r-maximal. It is unknown if, for example, an r-maximal subset of a recursive basis generates an r-maximal subspace. Nevertheless, by using weakly co-1-1 sets we certainly get existence theorems. We remark that an example in [11] shows that there *are* r-maximal sets that are not weakly co-1-1. It would be interesting to know if any of these notions are definable in $L(\omega)$.

As a final result, we shall prove the undecidability of another lattice of r.e. substructures which is not directly covered by our earlier results. Let V_∞ be over a recursively ordered recursive field. Define a closure system $(V_\infty, \langle\,,\,\rangle)$ as in [6] or [10] by $x \in \langle y_1, \ldots, y_n \rangle$ iff $x = \sum \lambda_i y_i$ with $0 \leqslant \lambda_i \leqslant 1$ and $\sum \lambda_i = 1$, and also ask that $\langle\,,\,\rangle$ is effective. Then the lattice $K(V_\infty)$ of closed sets is called the lattice of *r.e. convex sets*. This example allows another (indirect) application of our results. There are no special sequences with LEP in $(V_\infty, \langle\,,\,\rangle)$. Nevertheless we can prove:

4.5. Theorem. $\mathrm{Th}(K(V_\infty))$ *is undecidable.*

Proof. We need the abbreviations

$$Z(x) = \text{“}x \text{ is empty”} = \forall y\ (y \leqslant x \rightarrow y = x)$$

$$P(x) = \text{“}x \text{ is a point”} = \neg Z(x)\ \&\ \forall y\ (y \leqslant x \rightarrow (y = x \vee Z(y))$$

$$C(x, y, z) = \text{“}x = \langle y, z\rangle\text{”} = y \leqslant x\ \&\ z \leqslant x\ \&\ \forall q\ ((y \leqslant q\ \&\ z \leqslant q) \rightarrow x \leqslant q)$$

$$S(x) = \text{“}x \text{ is a segment”} = \neg p(x)\ \&\ \exists y\ \exists z (p(y)\ \&\ p(z)\ \&\ C(x, y, z)).$$

Thus we define $l(x) =$ “x is a line” via

$$\begin{aligned} l(x) = {} & \neg Z(x)\ \&\ \neg P(x)\ \&\ \forall y\ (S(y) \rightarrow \neg(x \leqslant y))\ \& \\ & \forall y\ \forall z\ \forall q\ [[y \leqslant x\ \&\ z \leqslant x\ \&\ q \leqslant x\ \&(y)\ \&\ p(z)\ \&\ p(q)\ \& \\ & \forall h\ \forall k\ \forall m\ ((c(h, y, z)\ \&\ c(k, z, q)\ \&\ c(m, y, q)) \rightarrow S(h)\ \&\ S(k)\ \&\ S(m))] \rightarrow \\ & [\forall h\ \forall k\ \forall m\ (c(h, y, z)\ \&\ c(k z, q)\ \&\ c(m, y, q)) \rightarrow (q \leqslant h\ \&\ y \leqslant k\ \&\ z \leqslant m)]. \end{aligned}$$

(That is, any three points on x are co-linear, and x is not $\leqslant$ a segment.)

With this we can define

$$\begin{aligned} A(x) &= \text{“}x \text{ is an affine subspace”} \\ &= Z(x) \vee P(x) \vee \forall y\ ((y \leqslant x\ \&\ S(y)) \rightarrow \exists z\ (z \leqslant x\ \&\ y \leqslant z\ \&\ l(z)). \end{aligned}$$

Now here $V \subseteq V_\infty$ is an *affine subspace* if given $y_1, \ldots, y_n \in V$, for all λ_i with $\sum \lambda_i = 1$, $\sum \lambda_i y_i \in V$. The collection of affine subspaces of V form a recursive Steinitz system and hence by 3.1, have an undecidable first-order theory. By the above, we can effectively interpret this theory in $\mathrm{Th}(K(V_\infty))$. Thus $\mathrm{Th}(K(V_\infty))$ is undecidable. □

We remark that we do not know if Th($K^*(V_\infty)$) is undecidable. It does form a lattice. There is a fair amount of unpublished material concerning $K(V_\infty)$ due to Downey and Kalantari. These results however will appear elsewhere.

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