A Review of The Algebraic Approaches to Quantum Mechanics. Some Appraisals of Their Theoretical Importance

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Abstract: The main algebraic foundations of quantum mechanics are quickly reviewed. They have been suggested since the birth of this theory till up to last years. They are the following ones: Heisenberg-Born-Jordan’s (1925), Weyl’s (1928), Dirac’s (1930), von Neumann’s (1936), Segal’s (1947), T.F. Jordan’s (1986), Morchio and Strocchi’s (2009) and Buchholz and Fregenhagen’s (2019). Four cases are stressed: 1) the misinterpretation of Dirac’s algebraic foundation; 2) von Neumann’s ‘conversion’ from the analytic approach of Hilbert space to the algebraic approach of the rings of operators; 3) Morchio and Strocchi’s improving Dirac’s analogy between commutators and Poisson Brackets into an exact equivalence; 4) the recent foundation of quantum mechanics upon the algebra of perturbations. Some considerations on alternating theoretical importance of the algebraic approach in the history of QM are offered. The level of formalism has increased from the mere introduction of matrices to group theory and C*-algebras but has not led to a definition of the foundations of physics; in particular, an algebraic formulation of QM organized as a problem-based theory and an exclusive use of constructive mathematics is still to be discovered.

Keywords: Quantum mechanics, Heisenberg’s foundation, Weyl’s foundation, Dirac’s foundation, von Neumann’s formulation and his conversion, T.F. Jordan’s didactic foundation, Segal’s C*-algebra formulation, Morchio and Strocchi’s algebraic exact quantization, Buchholz and Fredenhagen formulation, Foundational appraisals, Alternative Quantum Mechanics.

1. Introduction

More than the results of Planck’s black-body research, Einstein’s 1905 paper on quanta introduced the discrete into theoretical physics. In the first section (“Introduction”) of his paper he contrasts the discrete and the continuum within classical theoretical physics as a fundamental “dichotomy”. In theoretical physics the discrete became unavoidable and in the mathematical representation of reality assumed an importance equal to that of the continuum. In the year 1925 the birth of quantum mechanics (QM) was brought about by the invention of Matrix mechanics which is essentially based on the discrete and algebra. It was, however, soon followed by the invention of Wave mechanics - essentially based on the continuum and differential equations.

However, in the same year their equivalence was proved and some years later confirmed by both Dirac’s and von Neumann’s theories of QM. It seemed that the discrete/continuum dichotomy had been overcome not only in the mathematical formalism but also at the theoretical level. Yet some scholars continued to recognize a dichotomy, albeit of a philosophical nature, in the approaches to QM, concerning two different theoretical approaches. The most significant one was that of Fred Kronz and Tracy Lupher (2019) who contrasted the “rigor” of von Neumann’s approach with the “pragmatism” of Dirac’s.

I, on the other hand, generalize the ancient discrete/continuum dichotomy into the mathematical dichotomy between the two different historical developments of analysis and algebra. This dichotomy may represent also the above philosophical one, since the calculus was considered to be the main representative of the continuum and algebra was for a long time confined to discrete fields.

The present paper is a rapid review of the main respects in which an algebraic approach was introduced into the physical theory of QM, which, more so than special relativity, led theoretical physicists in a new direction[1].

[1] Notice that the algebraic nature of special relativity was long time ignored. Even Albert Einstein who introduced Lorentz group of space-time transformations of electromagnetism into mechanics did not mention its algebraic name: the word “group” occurred in the generic sense of a gathering.

In QM an algebraic approach was present from the outset in Heisenberg's formulation. Yet, some years later the analytical approach of differential equations and then Hilbert space prevailed and became almost the only approach. However, it is interesting to note that subsequently the algebraic approach slowly but decisively re-emerged and gained importance in founding QM. In the following eight different foundations of QM according to the algebraic approach will be presented.

Even more general than the analysis/algebra dichotomy is the dichotomy regarding the two basic kinds of mathematics, constructive mathematics, which developed in the '60s as the mathematics that requires that all mathematical objects be constructed (Markov 1962, Bishop 1967) and classical mathematics. This dichotomy involves the rejection or acceptance of idealist notions: e.g. Weierstrass' theorem of an accumulation point of an infinite set of points, Heine-Borel's theorem, Dirac's $\delta(x)$, Hilbert space, etc. A rapid review of the basic choices on which each algebraic foundation relies is given. No full complete algebraic formulation of QM relying on the alternative choices to that of von Neumann's formulation was achieved.

In the following I do not consider the algebraic approaches to quantum field theory, nor those concerning QM based on information theory, nor the algebraic approaches to QM based on discrete mathematics.

2. Heisenberg's foundation of QM (1925)

Werner Heisenberg's foundation represents the first attempt to formulate QM. Being aimed at solving the problem of spectra in strictly operative terms through the basic concept of the electromagnetic atom it made use of an algebraic mathematics without making an a priori assumption of discreteness, as did Max Born by translating Heisenberg's calculations into the little known mathematical objects, algebraic matrices; Pascual Jordan subsequently improved this matrix calculus.(Born, Heisenberg, Jordan 1925; in the following: H-B-J) The representation of physical operators by matrices incorporates a dependence on time, whereas that of states-vectors is time independent. The main mathematical problem is the resolution of the eigenvalue problem of a matrix in order to obtain the values of the corresponding observable. This was the first theory able to explain theoretically basic quantum phenomena (one-dimensional harmonic oscillator, rotator, Zeeman effect and the fluctuations of the radiation field). However, one year after it was proven that this theory is equivalent to, and less easy than, the following foundation of quantum theory, wave mechanics, where operators are constant and the states are time dependent. Subsequently H-B-J foundation substantially changed. (Beller 1983, p. 470) At present these two foundations of QM are recalled – without mentioning that they naturally descend from the classical Hamiltonian (Donini 1983; Darrigol 2014, pp. 248-253) - as the two main representations of von Neumann's formulation of QM in the infinite dimensional Hilbert space. No explanation is offered by historians as to why this first algebraic foundation could then be included in a more general formulation cancelling its algebraic approach; it is probable that the effectiveness in solving physical problems (also of a foundational nature) through the use of Hilbert space prevailed to such an extent over the wish to understand the foundations of QM that the two mathematical approaches, algebraic and analytical, whose natures are mutually antagonistic, were made equivalent. Above all Norwood R. Hanson (1963), F.A. Muller (1997) and Enrico A. Giannetto (1997) contested this unification. The latter lucidly stressed that the unification disregarded exactly this point, i.e, the different kinds of relationship between mathematics and physics in the two formulations.(p. 203)

> Matrix mechanics implied a change in the relationship between mathematics and physics: as in the case of relativity which involved changing geometry and assuming a "physical geometry" chosen in agreement of its experimentability, so arithmetic and algebra in the case of matrix mechanics could no longer be given a priori, they had to be chosen in relation to experiments, and this led to a "quantum" arithmetic and algebra or "quantum numbers" (matrices of "q-numbers"). The revolutionary consequence of this is the overturning of the relations between "logos mathematikòs" and " physis", which will then be carried out by the radical approaches of "quantum logic". To analyze still today the constitutive works of the mechanics of matrices then does not only have the flavor of a nostalgia or a historical erudition; instead, it concerns a restoration of the extreme radicalism of the original "quantum revolution", not only for its historical value, but for its physical and philosophical consequences that

> oblige us to completely reconsider our own way to relate us to nature from both an epistemological point of view and a point of view that cannot fail to be ethical too: the change in the idea of nature, implied by the mechanics of matrices, involves the abandonment of the illusory modern "era of the images of the world", in which nature was reduced to an image of man … (Giannetto 1997, p. 203)

It is also interesting that after the birth of QM a discussion led physicists to recognize the problem of the simultaneous measurements of two physical quantities; finally Heisenberg introduced the so-called "uncertainty principle", which states that position Q and momentum P cannot be measured simultaneously with an unlimited accuracy. (Heisenberg 1927). It certainly does not represent an axiom for a deductive system, but rather a methodological principle for solving the basic problem of QM of restoring the measurement of a physical system. Born's idea of representing the variables through matrices proved to be a wise move, because their multiplication is non-commutative and hence this non-classical situation was represented through the algebraic language: $QP-PQ = ih/2\pi$, where P and Q are matrices. It may be called the "strange equation". (Jordan 1986, p. 5) because it was absolutely new for a classical physicist.

3. Weyl's book *Group Theory and Quantum Mechanics* (1928)

George W. Mackey summarizes Weyl's novelties in quantum physics:

> … the contributions of Hermann Weyl… began in 1924 with the work on extending the representation theory of finite groups to a class of infinite continuous groups – the compact Lie groups. They culminated in 1927 with Weyl's work in: (a) Unifying group representation theory with Fourier analysis, (b) *Helping to clarify the structure of the new quantum mechanics* that emerged to replace the old quantum mechanics of 1900-1924 after the fundamental discoveries of Heisenberg and Schroedinger in late 1924 and early 1925; (c) Unifying spectral theory with the theory of group representations while applying both to the new quantum mechanics. (Mackey 1988, p. 138; emphasis added)

In the first edition of the book (1928) Weyl develops Heisenberg-Born-Jordan's formalism of Matrix Mechanics. However, in the second edition Weyl accepts a compromise (§. 2); he declares that de Broglie's and Schroedinger's "approach seems to me less cogent, but it leads more quickly to the fundamental principles of QM". (p. 48).

From the outset, rather than real numbers, he made use of the algebra of specific fields of numbers which are suitable for the new TP ("Preface", p. viii), maybe the field of his "elementary mathematics" (Drago 2000a, Drago 2000b).

Weyl devotes Ch. 1 of his book to the notion of an affine space, viewed as the new mathematical framework for advanced TP. Physical states receive the Hermitian representation; operators are represented by matrices. The book extensively introduces group theory, also permutation group theory, into QM (for more details see Drago 2000b, pp. 401-403). But Weyl naively generalizes through mere analogies finite group properties to those of infinity groups (Weyl 1928, pp. 28, 31, 34, 35; notice the use of Dirac's function on pp. 36-37).

> The heart of the book lies in the chapters II, IV and V. Chapter II contains one of earliest systematic coherent accounts of quantum mechanics as a whole. Perhaps only Dirac had as complete an overall view earlier but his early accounts are less complete and well organized. (Mackey 1988, p 147)

In synthesis, Weyl's presentation of QM includes three basic items, i.e. the representation of physical magnitudes in affine space, the mix of discrete-continuum in group theory, the derivation of the commutation relations. The book's main result was the suggestion of the very important applications of groups to QM, which however was no more than a great effort to found QM (and special relativity) on group theory (cleverly improved). His effort to represent a formulation of QM was unsuccessful, otherwise it would have been the first physical theory based on groups ("Weyl Program") in history:

> While more suggestive than persuasive or logically compelling, [Weyl's results] are of importance as perhaps the first step in the program of deriving fundamental relationships in quantum mechanics from group theoretical symmetry principles in a program which I feel it appropriate to call Weyl

program and to distinguish fairly sharply from the related important program inaugurated and much developed by Wigner.(Mackey 1988, p. 146)

His attempt failed in that it contrasted with Janos von Neumann's subsequent theorems on the limitations of the method of matrices (Bernkopf 1968, pp. 341-346). Apparently Weyl's answer to this obstacle was to try to improve his group theory (Weyl 1939). However, Emmy Noether's contemporary axiomatization of group theory (a work considered by most mathematicians as the culmination of a quest lasting half a century to legitimize algebra as a truly general, symbolic mathematics) meant that his attempt was considered backward. (Weyl 1939, p. vi) Again a very promising algebraic approach to QM was put aside by the developments of contemporary scientific research.

4. Dirac's foundations (1930)

Dirac's celebrated book[2] locates QM within a vector space (and its dual space) in order to represent both states as vectors and observables as matrices. Its basic mathematical technique is that of the "transformations" (actually, group theory) which the reader later recognizes as canonical transformation. After a first chapter of introduction to the principle of superposition, considered by the author to be the main distinction in the description of quantum and classical systems, five chapters follow dealing with the mathematical preliminaries concerning algebraic structures and contact and canonical transformations. Dirac developed his algebraic framework to such a high degree of clarity and effectiveness that he qualified it as a "symbolic method" (in the 3rd edition it was improved by the introduction of bra-ket formalism) and it greatly impressed its readers. Yet, he added an idealist function, the well-known Dirac's $\delta(x)$, to algebraic mathematics so that his foundation is not rigorously algebraic.

Moreover, the extensive mathematical apparatus of the theory cannot circumvent or justify the basic analogy, which, although incomplete, allowed him to 1) suggest an important connection of QM with more advanced classical mechanics (Hamiltonian formulation): the invariants to the transformations, i.e. Poisson brackets are equated with the commutators. 2) construct most of previous QM (the indeterminacy principle, Matrix mechanics, Schroedinger representation and its equivalence with the previous one, electron dynamics (but not interactions), etc.); 3) add important theoretical improvements.

The following editions of the book had, however, to take in account a number of new developments/ novelties: von Neumann's theorems which barred the way to a rigorous treatment of continuous variables in his matrix formalism, the criticism of a non-rigorous treatment of the subject, and above all von Neumann's impressive formulation in 1932 of QM through the axiomatic method. Dirac introduced some connections to the new advances, but above all he declared the axiomatizing of his theory (4th edition, pp. 14-15), although in the first edition he had said of this method that it was "somewhat artificial" and declared his rejection of it (pp. 16-17).

This non-linear evolution of his theory has justified the now usual appraisal of an initial non-rigorous introduction to QM, then correctly founded by von Neumann in Hilbert space. This appraisal started with von Neumann's influential remark:

> In the preface to his book on the foundations of quantum mechanics, von Neumann says of Dirac's own formulation of quantum theory that it is "scarcely to be surpassed in brevity and elegance," but that it "in no way satisfies the requirements of mathematical rigour." […] Since [von Neumann's] work was published, little has changed to affect the validity of these remarks [….] the Dirac formalism remains far from rigorous, and the formulation in terms of Hilbert space is still the only adequate framework for quantum theory. The very elegance and success of the Dirac formalism have ensured its survival. Most of the current generation of books on quantum theory prefer to take it as their guide, rather than give more than a passing reference to the niceties of Hilbert space. (Roberts 1966, p. 1097)

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[2] I will refer to the first edition of the book, which is online: https://archive.org/details/in.ernet.dli.2015.177580/page/n27/mode/2up. The 2nd edition is the same plus the table of content.

But the same Roberts stresses that making the former book equivalent to the latter one is not correct:

> The most unsatisfactory feature of the present situation is that the gulf between the Dirac formalism and Hilbert space [perceived by those taking Dirac's book as a guide] is quite substantial, so that a lot of rethinking is necessary [to these scholars] before grasping the "correct" way of expressing things in Hilbert space. (*ibidem*)

5. von Neumann's analytical formulation (1932) and his subsequent 'conversion' to an algebraic approach

The theoretical approach of the celebrated book (von Neumann 1932) is radically different from that of Dirac because it was based on a separable infinite dimensional Hilbert space; essentially, analytical mathematics.

Von Neumann criticized Dirac's introduction of mathematical fictions, which was what he considered to be both $\delta(x)$ function and the assumption that any self-adjoint operator can be put in a diagonal form. He wanted a "just clear and unified [treatment of QM as Dirac's was] but without mathematical objections".

The great success of this book seemed to represent a decisive victory of the latter mathematical point of view, a confirmation of the typical approach of analytical mathematical physics of the previous century, which is how current undergraduate teaching of physics presents QM: the only formulation of QM whose mathematics is formally valid.

Not many years after this book, however, von Neumann often expressed his dissatisfaction with Hilbert space. The best known and most remarkable example of his declarations is a 1935 letter to Birkhoff:

> I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of 'conserving the validity of all formal rules' [...]. Now we begin to believe that it is not the vectors which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical states, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the linear closed subspaces.(von Neumann 2005)
>
> The reasons why von Neumann criticized and abandoned Hilbert space QM can be summarized as follows:
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> - Interpretational inconsistencies in Hilbert space QM
>
> *No von Mises type relative frequency interpretation of quantum probability can be given*
>
> - Presence of operationally meaningless mathematical operations and entities in Hilbert space QM
>
> *Usual (composition) product of observables, phase in state vector*
>
> - Mathematical pathologies in Hilbert space QM
>
> *All self-adjoint unbounded operators do not form an algebra*
>
> - New mathematical discoveries
>
> *Existence of finite, continuous von Neumann algebras*
>
> - Perceived greater conceptual coherence of operator algebraic QM
>
> *Unique a priori quantum probability determined by quantum logic*
>
> Again it is remarkable that von Neumann's move beyond Hilbert space QM was *not* motivated by any of the following:
>
> l. Empirical inadequacy of Hilbert space QM
>
> 2. Any new empirical/physical discovery
>
> 3. Mathematical imprecision/nonsense in Hilbert space QM. (Rédei 2002, pp. 240-241)

One scholar commented on the above 'confession' as follows:

> the above words are often quoted as evidence that by 1935 von Neumann favoured the so-called 'lattice theoretical' (also called 'quantum logic') approach, in which the lattice P(H) of projections of infinite dimensional Hilbert space H, the 'logic' of quantum system, is considered as the basic object of the theory. (Rédei 1997, pp. 493-494)

It is only in the last century that physicists have reflected on the mathematical nature of the set of measurement instrumentations at their disposal. First the paper by (von Murray von Neumann 1936) showed that such a set can be considered an algebra of operators provided that their values are bounded; it was baptized "ring of operators", now called "von Neumann algebra". In the following years von Neumann's mathematical research tried to supersede Hilbert space by uniting probability and logic - since both are based on the concept of orthogonality - by means of continuous geometries with a transition probability. Physically, this choice is tantamount to considering quantum theory with infinitely many particles as more fundamental than simple quantum mechanics. (Stoeltzner 2001, p. 53) His 'type 1 infinite' algebra (a particular ring of operators) corresponds to the infinitely dimensional separable Hilbert space that provides the rigorous framework for wave and matrix mechanics. In addition he investigated what today are called 'type II and type III von Neumann algebras'. However, his research was unsuccessful because the continuous geometries did not directly generalize Hilbert space, as hoped for by von Neumann; their class is too broad for the purposes of axiomatizing QM. (Rédei 1997, p. 509)[3]

6. Segal's formulation of QM through a C*-algebra

In the year 1947 Irvin Segal obtained "spectacular results" (Jammer 1974, p. 381; he made a brief, intelligent summary of Segal's papers; a critical illustration of Segal's results is given by Emch 1984, chp. 9, sect. 1). By essentially adding a norm (which essentially is a bound; more precisely, it is a function which assigns a strictly positive length or size to each vector in a vector space) to von Neumann's ring of operators, Segal (1947) introduced a C*-algebra (a Banach algebra together with an involution satisfying the properties of the adjoint). A C*-algebra of operators substantiates the relationship of QM with mathematics in a more general way than Heisenberg's.

He constructed a new axiomatic of QM on the basis of a set of algebraic and metric axioms which define a more general structure than a C*-algebra (because the former does not require defining adjoint operators; however, most followers dealing with QM exploited a simple C*-algebra). Segal's mathematics may seem less powerful than that of Hilbert space, which makes use of the most general (square summable) mathematical functions. However, more general functions may be obtained, so that a C*-algebra can be linked to Hilbert space. In other works on the representations of algebras, Segal showed that a C*-algebra can be represented within Hilbert space (Gelfand-Naimark-Segal's theorem; and *viceversa* the Stone-von Neumann theorem assures the equivalence of Hilbert space with C*-algebra in the finite case). Hence, here algebra comes first, then the geometry of the representation in Hilbert space follows. As Segal himself remarks:

> it is interesting to note that if our algebraic postulates are strengthened sufficiently, then it can be shown that the collection of observables is isomorphic, (algebraically and metrically) with all self-adjoint operators in an algebra of bounded operators on Hilbert space (the norm corresponding to the operator bound). (Segal 1947, p. 931)

It was the first time that a complete formulation of QM was formulated algebraically.

7. Jordan's didactic version of QM based on matrices (1986)

H-B-J's algebraic approach to QM – i.e. representing the measurable quantities of a quantum system through matrices – was developed by a scholar, Thomas F. Jordan, in order to suggest an interesting didactic formulation of QM (Jordan 1986).

Since in general two matrices do not commute, as the physical variables of QM do, Jordan represents all physical magnitudes by means of finite matrices. He chooses as basic the system of the most simple matrices, Pauli's. His further choice of bounding the theory to finite or discrete

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[3] A side product of this algebraic program of research was the discovery of quantum logic by him and Birkhoff (1936)- This logic is based on the modularity law (which replaces the distributivity law, failing in QM). This von Neumann research orientation also failed to give the hoped for result of recognizing the true quantum logic among those indicated by (Birkhohh and von Neumann 1936). After more than eighty years this research (although corrected by replacing modularity with orthomodularity) is still ineffective and was called "the quantum logic labyrinth". (van Frassen 1974)

variables dramatically simplifies the mathematical problems to be addressed so that “there is no calculus” in his proposal. Measurement is not a projection of an amplitude of probability for obtaining a probability result, but is directly a probability as the mean of the values of the matrix representing the physical variable.

He starts by postulating that the main problem of QM is to explain “The Strange Equation”, i.e. the relation of non-commutation of the two conjugate variables, position and momentum. After a list of preliminaries, it is in chp. 14 that he easily obtains the states of a system of two non-commuting magnitudes, which represent two particles with spin 1/2 through merely on the basis of Pauli's matrices. In chp. 17 Jordan obtains Heisenberg’s uncertainty principle from the properties of matrix product: if A and B are matrices that have real eigenvalues, then the mathematical properties of matrices give the following equality:

$$\sqrt{<(A-\langle A\rangle)^2>}\sqrt{(B-<B>)^2)} = ½ |AB-BA|$$

where the symbol < > represents the average value and | | the module.

The consequences of this equation are then derived. In chp.s 18-22 he describes using of infinite matrices: the harmonic oscillator, Bohr’s model, the quantized levels of angular momentum, rotational energy and and finally the hydrogen atom.

The final chapters of Jordan's book deal with symmetries by treating only small continuous transformations. The expansion in series is considered and the generators are obtained; symmetry matrices are both physical variables and transformation generators. Of course, these generators also undergo the corresponding commutation relations. Then the commutation properties of all the physical quantities arise from the commutation properties of all the different kinds of transformations. In chp.s 23-27, spin rotation, small rotations, and, provided that the continuum is bounded, changes in location, time and velocity; thus covering all successful cases of Heisenberg’s theory (Beller 1989, p. 487).

Yet, the mathematical formalism is not sufficient for treating QM in its entirety, mainly because the continuous variables are bounded and the behavior of bosons (electromagnetic field) is ignored. Moreover he illustrates the dynamics by merely assuming the Hamiltonian as the equation of motion, and then he suggests its solutions to be merely verified as exact solutions.

In sum, his strictly algebraic theory offers a very simplified treatment of a substantial part of QM.

8. Strocchi’s formulation and his new quantization

In 2012 Strocchi revisited “Dirac-von Neumann axioms of QM” and suggested a new formulation of QM based on a C*-algebra inasmuch as he offered a new version of the last axiom:

> *Axiom A*. The observables generate a [polynomial] C*-algebra A, with identity; the states which by eq. (2.1) define positive linear functionals on the Algebras A_A ε A, for any observable A, separate such algebras in the sense of eq. (2.6) and extend to positive linear functional on A.(Strocchi 2012, p. 6).

I illustrated this novelty - as well as previous case of Segal’s formulation - in detail in the papers (Drago 2018, sections 4-7).

Through algebraic methods Strocchi obtained an even more important result: an accurate definition of Dirac quantization; the original analogy was superseded by an exact set of algebraic rules (Strocchi 2018, chp. 7; Drago 2020); of which Dirac’s identity

$[A, C]\ \{B, D\} = \{A, C\}\ [B, D]$; $\forall\ A, B, C, D \in \mathcal{A}$;

is stressed and the crucial two

5) *Z relates the commutator to the Lie product, in the sense that* $\forall\ E, F, G, H \in \mathcal{A}$

$[E, F] = Z\{E, F\}$, $[Z, \{G, H\}]= 0 = [Z, [G, H]]$.

6) *Z commutes with all the elements of A, i.e. it is a central variable:*

$[Z, A] = 0$ $\forall\ A \in \mathcal{A}$.

are justified. This result provided a solution to an old problem on which scholars have worked hard for over eighty years.(Ali & Englis 2005) One astonishing consequence is that the two cases of classical (Hamiltonian) mechanics and QM correspond to two values (respectively 0 and $ih/2\pi$) of a parameter Λ (I already illustrated this result in Drago 2020) This purely algebraic result frees the foundation of QM from all past tentative analogies and from the rough limit process $h \to 0$, which actually is a singular limit (Rohrlich 1990), hence insufficient for providing a large part of the mathematics of QM.

9. Buchholz and Fredenhagen's formulation of QM

The paper (Buchholz and Fredenhagen 2020a) presents an approach to QM which is entirely based on concepts and facts taken from the classical world - i.e. classical mechanics whose dynamics is described by a Lagrangian - without imposing from the outset any quantization condition for observables. Quantum effects are inherited from the macroscopic arrow of time, determining the causal structure of operations; specific properties, such as commutation relations, then follow from the given dynamics.

The authors start from the configuration space of a finite number of N particles (all of the same mass = 1); it is defined in the classical way, as R^{sN}, so that the state of the system is a vector $\boldsymbol{x}$. Particle motions are orbits c; they are described by functions (depending on time) of $\boldsymbol{x}$ and forming a set C. In space R^{sN} the basic mathematical structure will be defined as an algebra. This algebra describes our interventions in the quantum world. Such perturbations of the system typically arise from measurement arrangements, where forces can be manipulated systematically by human interventions in laboratories; but what surrounds the system also produces perturbations. They are described by functionals F involving arbitrary potentials and some information as to when and for how long these perturbations act.

$F[\boldsymbol{x}] = \int dt F(x(t))$, where $F((\boldsymbol{x}(t)) = \boldsymbol{f}_o(t)\boldsymbol{x}(t) + S g_k(t) V_k(\boldsymbol{x}(t))$,

where $\boldsymbol{f}_o(\boldsymbol{x})$ is a fixed loop (a path starting and ending at 0); $g_k(t)$ is a test function, i.e. a smooth (that is differentiable an infinite number of times) function with compact support; a compact support for a function on the real line means that the function is zero outside of some finite interval; and $V_k(x(t))$ is a continuous bounded function describing a perturbation. Moreover, these functionals can be shifted by loops $\boldsymbol{x}_o \,\varepsilon\, C_o$, a shift is given by $F^{xo}[\boldsymbol{x}] = F[\boldsymbol{x} + \boldsymbol{x}_o]$. Notice that the functionals F describe the envisaged perturbations of the quantum system in the "common language" of the macroscopic world.

These functionals F are non-linear but enjoy the property which is manifest when dividing the full time axis in three disjoint pieces, representing the support of $\boldsymbol{x}_1$, the support of $\boldsymbol{x}_2$ and their common complement $\boldsymbol{x}_3$.

$F[\boldsymbol{x}_1+\boldsymbol{x}_2+\boldsymbol{x}_3] = F[\boldsymbol{x}_1+\boldsymbol{x}_3] - F[\boldsymbol{x}_3] + F[\boldsymbol{x}_2+\boldsymbol{x}_3]$.

The dynamics of the motions is governed by a *classical Lagrangian L* and the corresponding action, as is familiar in classical mechanics.

The second ingredient taken from the macroscopic world is the arrow of time. Time enters into the quantum world since we can firmly state whether some operation was performed after or before some other operation. Moreover, it is impossible to make up for missed operations in the past, because time is has a direction. Since operations can be performed time and again, it is meaningful to assume that they form a semigroup. As a matter of fact, dealing with experimentally accessible systems, it is also plausible that the total effect of two successive perturbations, described by the sum of the underlying functionals, is equal to that of the product of the two individual perturbations. In addition, the effect of some operation can be cancelled out by another one, in accordance with the fact that experiments can be repeated and that their inverses represent the idea that in finite systems it is possible to remove the effects of a perturbation by other suitable perturbations. The operations, which can be performed on quantum systems, therefore form *a group*, whose elements $S(F)$ are labelled by symbols on the functionals F, describing the perturbations. $S(F_1)\, S(F_2)$

The elements $S(F)$ of the group then satisfy two relations which describes *the dynamical evolution* of the system; the former describes the action on the system described by the Lagrangian:

1) $S(F) = S(F(\boldsymbol{x}_o) + \delta L(\boldsymbol{x}_o))$, for all $\boldsymbol{x}_o \,\varepsilon\, C_o$, $F \,\varepsilon\, \mathcal{F}$;

and the second one translates the previous temporal property of Functionals in algebraic terms:

2) $S(F_1+F_2+F_3) = S(F_1+F_2)S(F_3)^{-1} S(F_2 +F_3)$, for arbitrary functional F_3, provided F_1 lies in the future of F_2.

The second property is a *"causal" relation* imposing the direction of time because it describes the ordering effects of time on the basis of the fact that any functional comprises information as to when the corresponding perturbation takes place. This allows us to incorporate the arrow of time into the group by relying on the temporal order of perturbations. At this point the direction of time enters also at the microscopic level, because one can firmly state that some operation has occurred earlier, respectively later, than another one. Owing to the directionality of time this group is non-commutative. One could therefore argue that it is this discrete arrow of time which is at the origin of the "quantization" of the classical theory. The specific form of commutation relations then follows from the underlying classical dynamics.(Buchholz and Fredenhagen 2020b)

In sum, the two basic ingredients, Lagrangian and time causality, determine the structure of the *dynamical group* of a Langrangian L, G_L, whose element is designated by the generating symbol $S(F)$ and whose aim is to describe the dynamical effects of perturbations on the underlying system. Hence, without stipulating from the out-set their "quantization", their concrete implementation in the quantum world emerges from the inherent structure of the algebra.composed by Lagrangian and time causality.

A standard procedure then leads from the group G_L to a C*-algebra A_L. This algebra is by definition the complex linear span of the elements $S \in G_L$. By easily equipping this algebra with a norm one obtains Segal's C*-algebraic setting.

The resulting C*-algebraic setting covers the full set of operations and resultant observables of the system. The C*-algebra contains in the non-interacting case unitary exponentials of the position and momentum operators (Weyl operators), satisfying the Heisenberg relations for position and velocity measurements. It is shown that Hilbert space representations of the algebra lead to the conventional formalism of quantum mechanics, where operations on states are described by time-ordered exponentials of interaction potentials. It is then proven that the dynamical algebras are irreducibly and regularly represented in the Schrödinger representation by time-ordered exponentials of functions of the position and momentum operators, described in terms of the classical theory. Interacting systems can be described within the algebraic setting by a rigorous version of the interaction picture. Thus, a "classical approach" reproduces the structure of QM in every asspect.

10. Appraisals of the above foundations of QM according to the two basic dichotomies

In a previous paper I suggested that the foundations of a scientific theory are constituted by the choices regarding two basic dichotomies, one regarding the kind of infinity (either potential (PI) or actual (AI) – to which the formal dichotomy corresponds: either constructive mathematics or classical mathematics) - and another regarding the kind of organization (either aimed at solving a problem (PO) or based on principles-axioms (AO) - to which the formal dichotomy corresponds: making use of either intuitionist or classical logic).(Drago 2012)

Notice that the dichotomy algebra/analysis is specific to each kind of mathematics, hence it is included in the first dichotomy regarding the kind of mathematics, either constructive or classical mathematics. The addition of one more dichotomy, that of the kind of organization of a theory[4], suggests moreover a more general framework in which to examine the previous foundations of QM. Such an examination would involve recognizing which choices regarding the two dichotomies underlie each of the previous foundations.

Heisenberg-Born-Jordan's formulation makes use of finite matrices which manifestly belongs to a **PI** mathematics (apart from the problem of the multiplicity of the eigenvalues of a

4 One might add that the algebraic approach was traditionally linked to a problem-based approach to a theory, whereas the analytical approach was currently associated with a deductive, possibly axiomatic approach to a theory.

matrix). Moreover, it is a **PO** theory inasmuch as it is aimed at solving the problem of extending Bohr's atom theory. (Drago 2016)

Weyl's book. Its organization is not clearly definable. It starts by merely describing the theoretical results of QM; then it applies group theory only to quantum general problems; hence, there is no manifest choice regarding the kind of organization. As for its mathematics, the book starts by dealing with finite groups (hence PI mathematics) without support, but then it generalizes the results to infinite groups. Hence also regarding the kind of infinity there is no clear choice. (Drago 2000a)

Dirac's book relies on the algebra of vector space and group of transformations; but this algebraic approach of PI kind is later contradicted by the introduction of Dirac's function $\delta(x)$), clearly appealing to **AI**. The extensive mathematical apparatus of the theory cannot cancel the crucial step in the development of the theory, which is an analogy aimed at overcoming the radical difference between classical mechanics and QM. In the first edition of the book Dirac's choice of the kind of organization is clearly **PO** (as a matter of fact, the theory is also based on the characteristic features of a PO theory; doubly negated propositions, *ad absurdum* proofs and the principle of sufficient reason). (Drago 2019, App.). However, in the subsequent editions he aligned himself with the axiomatic approach (see pp. 14-15 of the fourth edition).

Von Neumann's search for a new algebraic approach is manifestly based on the same choices of his book: **AI** and **AO**. (Drago 1991)

Segal's formulation is based on the choices **AI** (its two following results have no counterparts in constructive mathematics: a norm and Gelfand-Naimark-Segal's theorem (GNS) providing a representation of a C*-algebra in Hilbert space) and **AO** (it suggests a set of axioms for QM). (Drago 2018)

Jordan's didactic formulation is manifestly based on the choice **PI** and may be presented in an even clearer **PO**. (Drago 2016, sect. 3)

Morchio and Strocchi's "operative" formulation of QM in C*-algebra relies on the choices **AI** and **AO** (Rather On the other hand, Strocchi's rigorous translation of Dirac's quantization may be represented as a **PO** theory; Drago 2020).

Buchholz and Fredenhagen apply sophisticated mathematical notions as topological loops, test functions, compact support, which apparently appeal to **AI**. Owing to the a priori hypothesis of a space as composed by loops, the kind of organization is **AO.**

9. A table summarizing the eight algebraic approaches spanning over a century

	Basic math. notion	***Mathem. technique***	***Basic physical notion***	***Basic physical principle***	***Result***	***Kind of infinity***	***Kind of organiz ation.***	***Problem s within CoM***[5]
Heisenberg-Born-Jordan (1925)	Matrices	Algebraic calcolosa	Observab les	Operativis m, Non-commuta-tion	Hydrogen atom	PI	PO	No (multiple eigenval ues?)
Weyl (1928)	Affine space	Transf. groups	System's state	Schroeding er Equation	Heisenberg Relations, Groups in QM	AI?	PO?	H
Dirac (1930)	Vector space	Transf. groups	System's state	Hamilton equations	Analogy with Hamiltonia n	AI	PO	Imp ($\delta(x)$)
(Murray &)	Ring	Continuou s geometry	Linear closed	Non-com-mutation	Type II and III	AI	AO	Imp?

[5] This column represents an evaluation of the difficulties met in attempting to translate each foundation into constructive mathematics (CoM): "No" means no problem; "H" a hard taget; "Imp" a hopeless goal.

von Neumann (1936)			subspace		algebras			
Segal (1947)	Physical operator	Non-commutation	C*-algebra	C*-algebra of operators, GNS	Axiomatic	AI	AO	H (GNS, norm)
T.F. Jordan (1986)	Pauli matrices	Algebra of finite matrices	Spin	Hamiltonian	Introduction to part of QM	PI	PO	No
Strocchi (2012)	C*-algebra of operators	GNS representation in Hilbert space	Physical operators	Non-commutation	"Opera-tive" axiomatic	AI	AO	H (GNS, norm)
Buchholz and Freden-hagen (2020a)	Space of configu-rations, Loops	Group and C*-algebra of perturbations	Perturbations	Lagrangian	Algebraic foundation of QM	AI	AO	Imp?

Let us read the table by columns, disregarding the fourth and sixth rows since they concern incomplete foundations). The mathematical viewpoint of each algebraic foundation of MQ is considered in columns 1 and 2; their two sequences show a progress in formulating QM through progressively improving algebraic structures that range from Heisenberg's elementary technique, vector space, group theory and C*-algebra, to Buchholz and Fredenhagen's algebraic framework.

Let us then consider the physical features of these foundations. Regard to the basic physical notion, the sequence of column 3 shows the use of every aspect of a quantum system: observables, system's state, physical operators, perturbations. The sequence of column 4, regarding the basic principle of a foundation, manifests a tendency to abandon the specific principle of QM, i.e. non-commutativity, in order to restore the classical framework of theoretical physics – respectively, the Hamiltonian by both Dirac and Strocchi, the Lagrangian by Buchholz and Fredenhagen -, in order to promote it as a universal theoretical framework for both kinds of mechanics. This sequence may be interpreted as an effort to discover a common foundation for all the theories of theoretical physics. Such a foundation was recognized in a basic physical theory - rather than in a basic notion, whether physical (eg space, possibly composed by loops) or mathematical (eg Hilbert space) or philosophical (eg . causality). However, only two attempts were made to go beyond the traditional organization of a deductive theory: by Heisenberg in a rough way and by Dirac. Although without historical consequences, they show that this kind of organization, PO, is not extraneous to QM.

The above foundations were all formulated on the assumption that classical mathematics is the only kind of mathematics. Only the mathematical apparatuses of two foundations are apparently of a constructive kind: the H-B-J foundations and Jordan's. This shows that constructive mathematics has not been extraneous to QM since its inception. Column 9 presents evaluations of the difficulties met in translating the mathematical apparatus of each of the other foundations within Constructive mathematics. These problems are certainly difficult, but perhaps not impossible, to solve [6].

Let us consider columns 7 and 8 concerning the basic choices regarding the two dichotomies, actually ignored by all theoretical physicists. Four out of eight foundations rely on the dominant choices AI&AO; at most two foundations (Dirac's and maybe Weyl's) rely on the mixed choices AI&PO. Only one foundation relies on the alternative choices, PI and PO, to the dominant ones; within the historical development of QM it was a very important one because it was the first

[6] More details are presented in my papers (Drago 2016, §. 2) for H-B-J foundation, (Drago 2018, §§. 9-10) for Morchio and Strocchi's formulation.

foundation of QM (B-H-J); yet, it is algebraically incomplete and was not followed by any other foundation based on the same choices.

In sum, still today the progressive invention of new algebraic foundations of the QM has not yet exhausted all the possibilities provided for by the four choices regarding the two dichotomies; not only because foundations based on the couple of choices PO&AO do not exist, but above all because a complete formulation of QM based on the choices PI and PO has still to be achieved.

11. Conclusions

A previous review showed that, after a long period of being undervalued, algebra finally plays a crucial role in suggesting foundations of QM, to the extent that it now has the same status as, or even a higher status than, the analytical approach.

Current university teaching of physics ignores the algebraic approach to QM because it incorrectly interprets Dirac's as an AO, that is, with the same choice of the dominant formulation, that of von Neumann, about which yet von Neumann himself became sceptical. The time has come for a radical reform of the teaching of QM in universities.

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Bibliography

Ali S.T., Englis M. (2005). "Quantization Methods: A Guide for Physicists and Analysts". *Review of Mathematical Physics*, 17, pp. 391-490.

Beller M. (1989), "Matrix Theory Before Schroedinger. Philosophy, Problems, Consequences." *ISIS*, 74, pp. 469-491.

Bernkopf M. (1968), "A History of Infinite Matrices", *Arch. Hist. Ex. Sci.*, 4, 308-358.

Birkhoff G., von Neumann J. (1936). "The Logic of Quantum Mechanics", *Annals of Mathematics*, 37, pp. 823-843.

Bishop E. (1967), *Foundations of Constructive Analysis*, New York: AcademicP..

Born M., Heisenberg W., and Jordan P. (1925), "Zur Quantenmechanik II", *Zeitschrift für Physik*, **35**, 557-615, 1925 (B. L. van der Waerden (ed.) (1968), *Sources of Quantum Mechanics*, New York: Dover, pp. 321-386).

Buchholz D. and Fredenhagen K. (2020a), "Classical Dynamics, Arrow of Time, and Genesis af Heisenberg Commutation Relations", *Expositiones Mathematicae*, 38, pp. 150-167.

Buchholz D. and Fredenhagen K. (2020b), "From Path Integrals to Dynamic Algebras: A Macroscopic View of Quantum Physics", *Foundations of Physics*, 50 (7), pp. 727-734.

Darrigol O. (2014), *Physics and Necessity*, Oxford: Oxford U.P.

Dirac P.A.M. (1930), *Principles of Quantum Mechanics*, Oxford: Oxford U.P..

Donini E. (1983), "Meccanica ondulatoria e meccanica delle matrici: letture divergenti della meccanica classica", *Atti III Congr. Naz. Storia della Fisica*, Palermo, 1982, pp. 498-504.

Drago A. (1991), "Alle origini della meccanica quantistica: le sue opzioni fondamentali", in G. Cattaneo, A. Rossi (eds.): *I fondamenti della meccanica quantistica. Analisi storica e problemi aperti*, Cosenza: Editel, pp. 59-79.

Drago A. (2000a), "Which Kind of Mathematics for Quantum Mechanics? The Relevance of H. Weyl", in C. Garola, A. Rossi (eds.), *Foundations of Quantum Mechanics. Historical Analysis and Open Questions,* Singapore: World Scientific, 167-193.

Drago A. (2000b), "A New Mathematics for the New Theory of Quantum Mechanics. The Fate of H. Weyl's Program of Research", *Atti Fond. Ronchi*, 55 (2000), pp. 387-425.

Drago A. (2012). "Pluralism in Logic. The Square of Opposition, Leibniz's Principle and Markov's Principle". In J.-Y. Beziau, & D. Jacquette (Eds.), *Around and beyond the Square of Opposition* , Basel: Birckhaueser, pp. 175-189

Drago A. (2016), “The three formulations of quantum mechanics founded on the alternative choices”, in Esposito S. (ed.), *Società Italiana degli Storici della Fisica e dell’Astronomia: Atti del XXXV Convegno annuale / Proceedings of the 35th Annual Conference*, Pavia U.P., pp. 251-259.

Drago A. (2018), “Is the C*-Algebraic Approach to Quantum Mechanics an Alternative to the Dominant One?”, *Advances in History of Science*, 7 (2).

Drago A. (2019), “A Mathematical Structure Defining the Incommensurability of Quantum Mechanics and Classical Mechanics”, *Atti della “Fondazione Giorgio Ronchi”*, 74, n. 1, pp. 119-129.

Drago A. (2020), “Dirac’s Quantization Improved by Morchio & Strocchi to an Algebraic Structure Founding Both Classical Mechanics and Quantum Mechanics”, Esposito S. et al. (eds.), *Atti XXXVIII Congresso Nazionale della Società Italiana degli Storici della Fisica e dell’Astronomia Messina*, 2018, Pavia U.P., Pavia, pp. 65-73.

Emch G.G. (1984), *Mathematical and Conceptual Foundations of Quantum of 20th Century Physics*, Amsterdam: North-Holland.

Giannetto E.A. (1997), “Note sulla rivoluzione della meccanica delle matrici di Heisenberg, Born e Jordan e sul problema dell’equivalenza con la meccanica di Schrödinger”, in Tucci P. (ed.), *Atti del XVII Congresso Nazionale di Storia della Fisica e dell’Astronomia*. Milano: Università Milano, pp. 199-208.

Hanson N. (1963), *The Concept of the Positron. A Philosophical Analysis*, Cambridge University Press, Cambridge.

Heisenberg, W. (1927), "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik", *Zeitschrift für Physik*, **43** (3–4): 172–198.

Jordan T.F. (1986), *Quantum Mechanics in Simple Matrix Form*. New York: Wiley & Sons.

Mackey G.W. (1988), “Hermann Weyl and the Application of Group Theory to Quantum Mechanics”, in Doppelt W. et al. (eds.) *Exacts sciences and Their Philosophical Foundations*, Frankfurt: Lang, pp. 131-159.

Markov, A. A. (1962). “On Constructive Mathematics”. *Trudy Mathematicieskie Institut Steklov, 67,* 8-14. [English Translation in (1971) *American Mathematical Society Translations, 98,* 1-9.]

Morchio G., Strocchi F. (2009). “Classical and Quantum Mechanics from the Universal Poisson-Rinehart Algebra of a Manifold”, *Reports of Mathematical Physics*, 64, pp. 33-48.

Muller F. A. (1997), “The Equivalence Myth of Quantum Mechanics”, I & II, *Studies in History and Philosophy of Modern Physics*, 28 , pp. 35-61, and.28, pp. 219-247.

Murray F.J., Von Neumann J. (1936), “On Rings of Operators”, *Ann. of Math.*, 37, pp. 116–229.

Rédei M. (1997), “Why John von Neumann did not Like the Hilbert Space Formulation of Quantum Mechanics (and What he Liked instead)”, *Stud. Hist. Phil. Modern Sci.*, 27, 4, pp. 493-510.

Rédei M. (2002), Mathematical Physics and Philosophy of Physics (with special consideration to von Neumann’s work)”, in Heidelberg M. and Stadler F. (eds.), *History of Philosophy of Science*, Dordrecht: Kluwer, pp. 239-243.

Segal I.E. (1947), “Postulates for General Quantum Mechanics”, *Ann. Math.*, 48, pp. 930-948.

Stoeltzner M.(2001), “Opportunistic Axiomatic – von Neumann and the Methodology of Mathematical Physics”, in Redéi M. and Stoeltzner M. (eds.), *John von Neumann and the Foudations of Physics*, Dordrecht: Kluwer, pp. 35-62.

Strocchi F. (2012), “The Physical Principles of Quantum Mechanics: A Critical Review”. *The European Physical Journal Plus*, 127, 12.

Strocchi F. (2018), *A Primer of Analytical Mechanics*, Berlin: Springer, Chp. 7.

van Frassen B.C. (1974), “The Labyrinth of Quantum Logic”, in Cohen R.S. (ed.), *Logical and Epistemological Studies in Contemporary Physics*, BSPS no. 13, Dordrecht: Reidel, pp. 224-254.

von Neumann J. (1932), *MathematischeGrundlagen der Quantenmechanik*, Springer, Berlin.

von Neumann J. (2005), *Selected Letters*, M. Rédei (ed.), Providence: American Mathematical Society.
Weyl H. (1928), *Group Theory and Quantum Mechanics*, New York: Dover.
Weyl H. (1939), *The Classical Groups*, Princeton: Princeton U.P.

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