LAMBDA THEORY : INTRODUCTION OF A CONSTANT FOR “NOTHING” INTO SET THEORY, A MODEL OF CONSISTENCY AND MOST NOTICEABLE CONCLUSIONS.

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Abstract

The purpose of this article is to present several immediate consequences of the introduction of a new constant called Lambda in order to represent the object “nothing” or “void” into a standard set theory. The use of Lambda will appear natural thanks to its role of condition of possibility of sets.
On a conceptual level, the use of Lambda leads to a legitimation of the empty set and to a redefinition of the notion of set. It lets also clearly appear the distinction between the empty set, the nothing and the ur-elements.
On a technical level, we introduce the notion of pre-element and we suggest a formal definition of the nothing distinct of that of the null-class. Among other results, we get a relative resolution of the anomaly of the intersection of a family free of sets and the possibility of building the empty set from “nothing”. The theory is presented with equi-consistency results (model and interpretation).
On both conceptual and technical levels, the introduction of Lambda leads to a resolution of the Russell's puzzle of the null-class.

Keywords: nothing, void, empty set, null-class.

1 Introduction

1.1 Why

Our aim is to clarify the real puzzle of Russell's conception of the null class as developed in the “Principles of Mathematics” [1]: ‘But with the strictly extensional view of classes propounded above, a class which has no terms fails to be anything at all: what is merely and solely a collection of terms cannot subsist when all the terms are removed.’; Russell and Whitehead will formally express this inexistence in “The Principia Mathematica” [2]: ‘$\vdash . \neg \exists ! \Lambda$’[a] . Russell could not accept the existence of the null class and assimilates it to “nothing”, while recognising its technical utility, which is not conceptually satisfying for us. Notice that *in fine* Russell conceives the null class as the standard empty set (symbol: $\emptyset$): ‘By symbolic logicians, who have experienced the utility of the null- class, this will be felt as a reactionary view. But I am not at present discussing what should be done in the logical calculus, where the established practice appears to me the best, but what is the philosophical truth concerning the null-class’ [3].

Other logicians and mathematicians saw ontological difficulties with a class free of elements.

[a] N.B. In Russell, Λ denotes the null-class, which is assimilated to *nothing*.

The first of them, Frege[4], strongly inspired Russell with his analytical philosophy approach in general, and his conception of the null-class in particular: 'When a class is composed of objects, when a set is the collective union of these, then it must disappear, when these objects disappear. If we burn down all the trees of a wood, we thereby burn down the wood'.
The fathers of the standard axiomatic set theory agreed with this view. So, in 1908, Zermelo [5] wrote: 'There exists a *fictitious* set, the null set, 0, that contains no element at all.'. In 1923, Fraenkel [6] added: 'For purely formal reasons, i.e. to be able to express some facts in a more simple and adequate manner, let us introduce here an improper set [*uneigentliche Menge*], the alleged set zero [*Nullmenge*] .../... It is defined by the fact that it does not contain any element; so it is not really a set, but it must be taken as such and be designed by 0'.
In his nominalist approach, Lesniewski[7] denies any kind of existence to classes in general and to the null-class in particular: 'I have always rejected, .../..., the existence of theoretical monsters such as the class of squared circles, being aware that nothing can be constituted of what does not exist'. Lesniewski only concedes the use of a nominal constant for denoting the *nothing*.
These quotations show that the doubts about the conceptual legitimacy of the null-class don't come only from detractors of set theory like Lesniewski, but mainly from several fathers themselves of the set theory!

We want to introduce here a clear distinction between the notion of empty set and the one of "nothing" (or "void"), that we will distinguish from $\emptyset$ via the symbol Λ.[b] The "nothing" must be conceived as the free space in any set (so also in the empty set): this is intuitively linked to the naive image of a set, as a "box" containing "objects" and where this is precisely possible because the box presents a "free space". This condition of possibility is also a condition of possibility in other fields, like the one of numbers and letters, see Pythagoras[8]: 'The void exists... It is the void which keeps the things distinct, being a kind of separation and division of things. This is true first and foremost of numbers; for the void keeps them distinct.' Here we see that the "nothing" clearly plays the role of *cut*.

The naive acceptance of the idea of "set" is then somehow validated in the case of the empty set: the empty set is a collection of "nothing".
Furthermore, this will allow the symbolic representation of the "empty space" that is intuitively present in any set, particularly in the traditional pictures of sets.

It would be natural to use the terminology of "inclusion" for the fact that the "empty space" Λ is "in any set". Nevertheless we show that the same symbol $\in$ can be used safely to express the fact of "belonging" to a set, for an object that is not Λ (and such an object is then called an "element" or a "set"), as well as the fact to be "the space Λ, present in a set" ("space" called "pre-element").

More precisely:

[b] In the use we make of it, Λ is not a class as in Russell; it is a new object that will be defined as a pre-element and a condition of possibility of sets, among others of the empty set.

"$x \in y$" will express that x is an "element" of y only when $x \neq \Lambda$ (corresponding to the usual way of "belonging").
"$\Lambda \in y$" will express that Λ is "present in y"; and we use then the word "pre-element" instead of "element" to avoid any confusion.

Also, when more complex objects are constructed (via "terms", see section 2.2), the same kind of careful distinctions will be taken into account, as several interpretations are available. For example the usual singleton "$\{ a \}$" is simply "standard" in the universe "without Λ", while in the "completed universe" it will appear as something like "$\{ a , \Lambda\}$". This is further discussed in section 2.2).

However, even if the same $\in$-symbol is used in our theory, the roles of the elements/sets and of the unique pre-element are never confused; this immediately comes from the characteristic properties:

x is an "element" $\Longleftrightarrow \exists y \, y \in x$

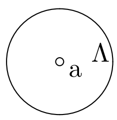
x is a "pre-element" $\Longleftrightarrow [(\nexists y \, y \in x) \,\&\, (\forall z \neq x \; x \in z)]$

and these properties are guaranteed by the axioms (see section 1.2).

In addition, if the nothing-void is conceived as a potential, the Lambda theory is the first step forward to a theory where the notion of "potential membership" ("potentially belongs to") can be conceived. In that way, we can hope to handle the strange "ubiquity property" of the empty set (Theory in development).

Finally, we want to reduce significantly the ontological commitment of set theory. The classical axiom of existence becomes useless: there is no need to postulate the existence of a set any more (should it be the empty set) as Lambda (the "Void", the "Nothing") can be now seen as a generator of a hierarchy of standard sets.

Picture of a set and representation of Lambda
Lambda denotes the free zone around the element "a". The set pictured here is $\{a\}$ in the universe V of a standard set theory $\sum$. In the universe V_Λ of the Λ-theory $\sum_\Lambda$, the set pictured here is $\{a, \Lambda\}$.



1.2 How

Let's simply use the usual symbol $\in$ to express that Λ is "in $\emptyset$", in the same way as Λ is "in any set x". Starting from some set theory $\sum$ (in which the extensionality axiom holds and where $\emptyset$ exists), in the current first-order language $\mathcal{L} = (\in, =)$, we define a new theory $\sum_\Lambda$ in the expanded language $\mathcal{L}_\Lambda = (\in, =, \Lambda)$ (where Λ is a new constant symbol). This allows to give several distinct interpretations to the terms conceived in a classical way. Some of these new distinct interpretations produce interesting results, like: $\{\Lambda\} = \emptyset$, and relative "solutions" to the well-known "anomaly" of the usual phenomenon: "the intersection of an empty family is the universal class". If we call "sets" (in $\sum_\Lambda$) all the objects distinct from Λ, we expect that their behaviour is fundamentally the one described by $\sum$.

The behaviour of Λ will be governed (in $\sum_\Lambda$) by the two following axioms:

(1) Axiom of the *Pre-Element*: $\forall x\,(x \neq \Lambda \Rightarrow \Lambda \in x)$

(2) Axiom of the *Nothing-Void*: $\forall x\,(\neg\,(x \in \Lambda\,))$.

Notice that there can only be one "object" such as Λ, as axiom (1) is in contradiction with: $\exists y \neq \Lambda \forall x \neg\,(x \in y)$.

It is easy to construct (in a metatheory like Zermelo-Fraenkel) a model M_Λ for $\sum_\Lambda$, starting from a model M for $\sum$: we just artificially add a new element ("Λ") to the universe of M and extend adequately the $\in$-relation of M. The axiom of extensionality will still be applicable in M_Λ. It is easy, modulo some minimal conditions on $\sum$, to improve this result, namely to give an interpretation of $\sum_\Lambda$ in $\sum$ (instead of a *stricto sensu* "model" as just described), and to clarify the possibility of using Λ as parameter in several comprehension axioms: *inter alia* the example of separation, which is valid in M_Λ even for $\mathcal{L}_\Lambda$-formulas, once it is valid in M (for $\mathcal{L}$-formulas).

2 The theory

We start with a set theory $\sum$, expressed in $\mathcal{L} = (\in, =)$, and assume "$\top$", "$\bot$" (respectively "true", "false") as primitive symbols in our (classical) logic.

We expect $\sum$ to satisfy at least the 3 following conditions:

- $\sum \vdash$ EXT,
where EXT is the Extensionality axiom: $(\forall x \forall y \forall t (t \in x \Longleftrightarrow t \in y)) \Longrightarrow x = y$.

- $\sum \vdash \exists a \forall x\,(x \notin a\,)$; so "$\exists\,\emptyset$".

- $\sum \vdash \forall a \forall b \exists c \forall x\,(x \in c \Longleftrightarrow (x = a \vee x = b\,)\,)$;
(the classical "Pairing axiom").

Our theory $\sum_\Lambda$, in the language $\mathcal{L}_\Lambda = (\in, =, \Lambda)$ initially assumes the axioms described hereunder (2.1), but can surely be enriched based on the observation of the model M_Λ obtained by modification of M (see section 3). For convenient purposes, we introduce the following abbreviations:

- "$\forall^* x$" for "$\forall x \neq \Lambda$".

- "$\exists^* x$" for "$\exists x \neq \Lambda$".

Λ will be called "the Nothing" or "the Void"; and the "sets" are the objects x such that $x \neq \Lambda$.

For φ a formula in $\mathcal{L}_\Lambda$ (with "$\top, \bot$", allowed), φ^* will be obtained from φ by replacing in φ each $\forall$ by $\forall^*$ and each $\exists$ by $\exists^*$.

If Γ is a theory (list of axioms), Γ^* will denote the list of σ^*, with σ in Γ.

2.1 Axioms of $\sum_\Lambda$

(1) $\forall^* x(\Lambda \in x)$.

(2) $\forall x(x \notin \Lambda)$.

(3) σ^* for any axiom σ of $\sum$ (so $\sum_\Lambda$ "contains" $\sum$).

Remarks:

One can easily check that:

- $\sum_\Lambda \vdash$ EXT, i.e. EXT is applicable in the "full" universe (sets + Λ).

- $\sum_\Lambda \vdash \forall x(x \in \emptyset \Longleftrightarrow x = \Lambda)$, i.e. $\emptyset$ is the "singleton" of Λ (cf. hereunder our discussion about "terms").

2.2 Interpretations for terms

Usually, the term $\tau = \{x \mid \varphi\}$ is the name of the (unique via EXT) set b such that $\forall x(x \in b \Longleftrightarrow \varphi)$. In the theory $\sum_\Lambda$ however, we can now distinguish different interpretations for a term $\tau = \{x \mid \varphi\}$ based on a formula φ (in $\mathcal{L}_\Lambda$):

Definitions:

1) $\tau^* = \{x \mid \varphi\}^*$ is the unique set (if it exists) b (so b $\neq \Lambda$) such that : $\forall^* x(x \in b \Longleftrightarrow \varphi^*)$, or equivalently: $(\forall x(x \in b \Longleftrightarrow \varphi))^*$.

2) $\tau_\Lambda = \{x \mid \varphi\}_\Lambda$ is the unique set (if it exists) b (so b $\neq \Lambda$) such that : $\forall^* x(x \in b \Longleftrightarrow \varphi)$, or equivalently (in $\sum_\Lambda$): $(\forall x(x \in b \Longleftrightarrow (\varphi \vee x = \Lambda))$.

3) $\tau = \{x \mid \varphi\}$ is the unique object (if it exists) b (it could be Λ) such that : $\forall x(x \in b \Longleftrightarrow \varphi)$.

We will also use these indices "*" and "Λ" for the notations that abbreviate several classical terms, like:

$\{a\} := \{x \mid x = a\}$ (singleton)

$\{a, b\} := \{x \mid x = a \vee x = b\}$ (pair)

$\wp a := \{x \mid x \subseteq a\}$ (power set)

$\bigcup a := \{x \mid \exists y \in a, x \in y\}$ (general union)

$a \cup b := \{x \mid x \in a \vee x \in b\}$ (binary union)

$\bigcap a := \{x \mid \forall y \in a, x \in y\}$ (general intersection)

$a \cap b := \{x \mid x \in a \wedge x \in b\}$ (binary intersection)

With these clarifications, one can easily check that, in $\sum_\Lambda$:

- $\{\Lambda\} = \{\Lambda\}_\Lambda = \{\Lambda\}^* = \emptyset$: the empty set is the singleton of Λ.

- $\wp\Lambda = \wp_\Lambda\Lambda = \wp^*\Lambda = \emptyset$: the empty set is the Power set of Λ.

- $\bigcap_\Lambda \emptyset = \emptyset$: this constitutes a *relative* solution (Indeed, as we will see in section 4.4, it is the case that $\bigcap_\Lambda \Lambda = V$) to the well known classical "anomaly" of $\bigcap \emptyset = V$, that is in dissymetry with $\bigcup \emptyset = \emptyset$. In the Lambda theory, $\bigcap_\Lambda \emptyset = \bigcup_\Lambda \emptyset = \emptyset$.

- $\bigcap \emptyset = \Lambda$. In the same way, $\bigcup \emptyset = \Lambda$. So, once again we have a symetry between union and intersection of an empty family.

- Notice that $\bigcap^* \emptyset = V$, as in the "classical" situation.

3 Modelisation

3.1 The Idea

Working in Zermelo-Fraenkel as meta-theory, we can start with a model (in the *stricto sensu* sense, as in [9]) for $\sum$:

$M = (U_M, \in_M)$, where U_M is a set and $\in_M$ is a binary relation on M.

The desired model for $\sum_\Lambda$ is simply $M_\Lambda = (U_\Lambda, \in_\Lambda)$, where $U_\Lambda = U_M \cup \{\Lambda\}$ and $\in_\Lambda$ is the obvious extension of $\in_M$ such that: $\forall x \in U_M (\Lambda \in_\Lambda x)$ and $\forall x \in U_\Lambda \neg(x \in_\Lambda \Lambda)$, where Λ is some chosen element, not in U_M.

One can easily check that M_Λ models $\sum_\Lambda$.

The initial set theory $\sum$ should only satisfy the basic conditions described in section 2. When stronger theories $\sum$ are considered, new interesting properties appear in M_Λ, for example when $\sum$ satisfies the Power set Axiom, or other specific forms of comprehension. For further details, please refer to section 3.3. Examples: one can take (for $\sum$) ZF, or NF (Quine's New Foundations), or a "positive set theory" [10]. Furthermore, we can verify that for these "agreeable theories", there are corresponding comprehension axioms still applicable in M_Λ, even when the involved formula φ is in $\mathcal{L}_\Lambda$ (instead of in $\mathcal{L}$). As a consequence, Λ may appear as a parameter.
For example: the set $\{x \in a \mid \varphi\}^*$ exists in M_Λ when M is a model of ZF, even when φ is in $\mathcal{L}_\Lambda$; similarly, $\{x \mid \varphi\}^*$ exists in M_Λ when M is a model of NF (and φ is stratified): the reason is that by replacing in φ any atomic formula $x \in \Lambda, \Lambda \in x, x = \Lambda$, etc. by (the "ad hoc") $\perp$ or $\top$, one gets an equivalent formula in $\mathcal{L}$, stratified if φ was.

3.2 Interpretation of $\sum_\Lambda$ in $\sum$

The interpretation of $\sum_\Lambda$ in $\sum$ here developed guarantees the equi-consistency of $\sum$ and $\sum_\Lambda$; the converse interpretation (of $\sum$ in $\sum_\Lambda$) is obviously given by the initial universe of $\sum$. The construction described in 3.1 is the classical model-theoretic one. However if equi-consistency only is considered, this construction can be improved and we can give a direct interpretation of $\sum_\Lambda$ in $\sum$.

Just take, in the universe U of $\sum$, a copy U' of that universe, such that $U' \neq U$; this allows to choose an object in $U \setminus U'$, and we call this object "Λ".
The usual technical trick to get such a U' and Λ (consider f.ex. $U' := U \times \{\emptyset\}$, and $\Lambda := (\emptyset, \{\emptyset\})$ is perfectly available here (modulo our conditions on $\sum$; cf. section2).

Of course we transfer isomorphically the $\in$-relation on the universe U to the universe U', so that $(U', \in')$ satisfies $\sum$. As universe for our interpretation of $\sum_\Lambda$ in $\sum$, we take then the class $U_\Lambda := U' \cup \{\Lambda\}$, and apply on it the obvious class-relation $\in_\Lambda$ defined by:
$x \in_\Lambda y$ iff $[\, (x \in U' \wedge y \in U' \wedge x \in' y) \vee (x = \Lambda \wedge y \in U')\,]$.
The conclusion is now similar to the one of 3.1: $(U_\Lambda , \in_\Lambda)$ interprets $\sum_\Lambda$ (in $\sum$).

3.3 Enriched Theories

We already mentioned in 3.1 that M_Λ (model in 3.1 or interpretation in 3.2) presents new interesting properties, when stronger theories $\sum$ are considered. This is particularly the case for comprehension schemes, and can be explained very simply modulo the following technical remarks.

From the construction, it is obvious that, in M_Λ, any atomic formula containing the symbol Λ ($x \in \Lambda$, $\Lambda \in \Lambda$, $\Lambda \in x$, $x = \Lambda$, etc) is equivalent to $\perp$ or $\top$, if the variables are supposed to represent objects distinct from Λ.

As a consequence, for any sentence σ (sentence: formula without free variable) in the language $\mathcal{L}_\Lambda$, the sentence σ^* is equivalent, in M_Λ, to a sentence $(\tilde{\sigma})^*$, where $\tilde{\sigma}$ is obtained from σ by replacing each atomic formula containing the symbol "Λ" (adequately) by "$\perp$" or "$\top$", the choice being determined by the axioms (concerning Λ) of $\sum$ (see 2.1).

Examples:

One will replace "$x \in \Lambda$", "$\Lambda \in \Lambda$", "$x = \Lambda$" by "$\perp$"; and "$\Lambda \in x$" by "$\top$".

This elementary fact proves the following technical lemma:

If $\sum \vdash (\tilde{\sigma})^*$, then $M_\Lambda \models \sigma^*$.

This has interesting consequences on several so-called "comprehension schemes"; three examples are described below:

1) stratified comprehension, i.e. the scheme of axioms:

σ (the universal closure of): $\exists a \forall x (x \in a \iff \psi)$, for each stratified formula ψ.
Let's consider here even a stratified ψ in $\mathcal{L}_\Lambda$ (ψ "stratified" for $\mathcal{L}_\Lambda$ is obtained from a stratified formula in $\mathcal{L}$, where one or more free variables have been replaced by "Λ"). Then it is clear that $\tilde{\varphi}$ is again stratified (in $\mathcal{L}$ this time).

So, if $\sum$ is the system NF (cf.[9]), then $M_\Lambda \models \sigma^*$, even when σ is a stratified comprehension axiom with ψ in $\mathcal{L}_\Lambda$.

2) Separation and Replacement (as in ZF):

for σ an instance of one of these classical schemes, with Λ now admitted as parameter in the involved formulas ψ, we obviously have that $\tilde{\sigma}$ is again an instance of the same scheme (in $\mathcal{L}$ this time).

Let us briefly detail this for separation (the case of replacement is analogous):

Let's consider σ (the universal closure of): $\forall b \exists a \forall x (x \in a \iff (x \in b \wedge \varphi))$, with φ in $\mathcal{L}_\Lambda$ (our formula ψ here is: $x \in b \wedge \varphi$). Then $\tilde{\sigma}$ is (the universal closure of): $\forall b \exists a \forall x (x \in a \iff (x \in b \wedge \tilde{\varphi}))$, which is again an axiom of separation (in $\mathcal{L}$).

Conclusion: if $\sum$ is ZF, then M_Λ satisfies the versions of Separation* and replacement* that admit Λ as parameter. (i.e. involved formulas ψ in $\mathcal{L}_\wedge$)

3) Positive Comprehension:

several such systems have been proposed and studied; a description and references can be found in [9].
The basic idea is to consider comprehension for "positive" formulas, i.e. formulas not allowing negation (nor, of course, implication); notice that "$\perp$" and "$\top$"

are considered as positive formulas. The corresponding scheme is then made of sentences σ (universal closure of): $\exists\, a\, \forall\, x\, (\, x \in a \iff \varphi)$, for any positive φ.

Now, let's allow also positive formulas φ in $\mathcal{L}_\Lambda$. It is obvious that $\tilde{\varphi}$ is again positive (in $\mathcal{L}$), so that $\tilde{\sigma}$ is still in the same scheme.

So, if $\sum$ is a positive set theory (one of the existing variants), then M_Λ satisfies the version of $\sum^*$ that allows Λ as parameter in the comprehension scheme.

Synthetic conclusion:
Our model/interpretation construction (cf. 3.1, 3.2) gives equiconsistency results for several "enriched" theories; more precisely:

for $\sum$ satisfying specific "comprehension schemes" (as described above), we have the equiconsistency between $\sum$ and $\sum_\Lambda^+$, where: $\sum_\Lambda^+$ is $\sum_\Lambda$ enriched with Γ^*, Γ being one of the types of schemes 1), 2), 3), that admits here Λ as parameter in the involved formulas φ.

4 Interest, Nature & Properties of Λ

4.1 Terminology

From an ontological point of view, we insist on the fact that here (in $\sum_\Lambda$), we clearly distinguish two types of objects:

- the "sets", *elements* x characterized (equivalently) by: $x \neq \Lambda$; $\Lambda \in x$.

- the "void" or "nothing" or *pre-element* Λ characterized by our axioms (1), (2) (section 1).

4.2 Internal and External Condition of Possibility

Lambda is a condition of possibility of elements (in the state of affairs, sets) in two ways:
- as an internal condition of possibility, Λ enables a set to contain elements. Lambda is the fundamental constituent of any set. This is expressed by the axiom of the pre-element. Indeed, in order for a set to contain other sets, an available space is necessary. Without the internal condition of possibility Λ, a set would be an atom, an ur-element, because there would be no way to make a distinction between the elements of a set.
- as an external condition of possibility, Λ also allows to have different sets. Indeed, the "Nothing" plays the role of cut, the *physical* separation between sets.

Thanks to this statute of condition of possibility, we think that the use of Lambda in set theory and the construction of $\sum_\Lambda$ are not artificial.

4.3 Lambda and the "contradictory property"

The contradictory property is traditionally sufficient to define $\emptyset$: $\{x : x \neq x\} = \emptyset$. Here in $\sum_\Lambda$, that property offers some more possibilities:

- $\{x : x \neq x\} = \Lambda$.

- $\{x : x \neq x\}_\Lambda = \{x : x \neq x\}^* = \emptyset$.

4.4 Lambda versus the Null-Class (or Empty Set $\emptyset$)

It is fundamentally clear that Λ is not $\emptyset$, as the first is the (unique) pre-element, while the second is an element (or set) (cf. 4.1). This has, of course, many consequences on their respective behaviours; we give here some interesting examples, involving cases where they behave in an analogous manner, as well as cases where they don't. For the notations (terms): cf. 2.2.

Lambda (Λ)	Null Class ($\emptyset = \{\Lambda\}$)
$\bigcap \Lambda = V$	$\bigcap \emptyset = \Lambda$.
$\bigcap_\Lambda \Lambda = V$	$\bigcap_\Lambda \emptyset = \emptyset$.
$\bigcap^* \Lambda = V$	$\bigcap^* \emptyset = V$.
$\Lambda \cap \emptyset = \Lambda$	$\emptyset \cap \emptyset = \emptyset$.
$\Lambda \cap \Lambda = \Lambda$	$\emptyset \cap \emptyset = \emptyset$.
$\{\Lambda\} = \emptyset$	$\{\emptyset\} \not\exists$.

As we have announced in section 2.2 as well, the classical anomaly of the intersection reappears at a deeper level, at Lambda level. Nevertheless we find interesting to see that it does not appear on the level of the sets any more.

4.5 Lambda versus Ur-elements

The use of "nothing" also enables the distinction of the empty set from ur-elements (or "atoms"), which are generally considered as kinds of empty sets: with u for an ur-element and x for a set, the expression $u \in x$ (which can be true or false) is syntactically admitted, while the expression $x \in u$ is not syntactically allowed.
Of course, "no thing" belongs to an ur-element, even not Lambda. This is precisely why an ur-element is a kind of atom. But we can make the distinction between Lambda and an ur-element too. Lambda belongs to the empty set, and more, Lambda belongs to every set. As we have defined it, as condition of possibility of elements, Lambda is a *pre-element*.
Finally, Lambda is different from the empty set by definition, and by its behaviour as we have seen in point 4.4.
Notice that Λ also enables the distinction $\emptyset$ from any ur-element.

4.6 Lambda as Generator of Sets

It is "ontologically interesting" to notice that, while we presented here $\sum_\Lambda$ as constructed on the basis of $\sum$, where "$\emptyset$" is already present, we can easily give an autonomous direct presentation of $\sum_\Lambda$ where $\emptyset$ would be "generated" (for example as $\{\Lambda\}$ or $\wp\Lambda$, as we have seen) if $\sum_\Lambda$ assumes the Pairing axiom or the Power set axiom.

So it is possible to make redundant the classical axiom of existence; there is no need any more to postulate the existence of a set. Moreover, we can build the hierarchy V_ω of sets starting from Lambda (assuming the Pairing Axiom), or even the Von Neumann Hierarchy (of the well-founded sets), if $\sum$ extends ZF:

- Let V_0 be Λ.
- For any ordinal number β, let $V_{\beta+1}$ be the Power set of V_β. So, V_1 is $\wp(\Lambda) = \emptyset$.
- For any limit ordinal λ, let V_λ be the union of all the V-stages so far: $V_\lambda := \bigcup_{\beta \leq \lambda} V_\beta$.

The class V is defined as the union of all the V-stages: $V := \bigcup_\alpha V_\alpha$.

For those who consider the need to postulate some primitive entities to be problematic, we hope that Lambda will appear as a more attractive entity than the *empty set* or any other *primitive set*. Indeed, in this way, the theory is completed by the bottom, in a "minimal way" (Lambda being "nothing"), and is more in adequacy with a possible "mathematical reality" or at least with "formal possibilities".

4.7 Lambda and Simplification of the Axiom of Infinity

Another nice consequence of the use of Λ is the possibility - modulo a slight modification - of simplifying the classical axiom of infinity, as used in ZF.
That axiom starts with an initial set b (often, but not necessarily, $\emptyset$) and postulates the existence of an infinite *set* ("x"):

$\exists x(b \in x \wedge \forall y\,(y \in x \Rightarrow y \cup \{y\} \in x))$.

In $\sum_\Lambda$, the part "$b \in x$" can be removed, as Λ is "omnipresent" as pre-element in any set. So that the axiom of infinity can be reformulated as:

$\exists^* x\, \forall y(y \in x \Rightarrow y \cup \{y\} \in x)$.

4.8 Lambda as solution to the Puzzle of the Null-Class

We have seen that the Puzzle of the Null-Class as found in Russell consists in the dichotomy between the technical legitimacy of the use of the Null-Class and its ontological illegitimacy.
In the "Principia", Russell justifies this ontological illegitimacy in two ways:
- the null-class does not exist because it does not contain anything.
- the null-class does not exist because it cannot belong to any class.[c]

In the "Principles", Russell did not really succeed in giving conceptual legitimacy to the null-class; in the "Principia", he does not even try to do it. However he seems to be satisfied in some way with the conceptual use of the "nothing"

[c]A.N. Whitehead, B. Russell. *Principia Mathematica To *56.* Ćambridge University Press, (1997) p.227.

since he reduces the null-class to it. Indeed, if the "Nothing" had no legitimacy at all, the reduction of the null-class to the "Nothing" would make no sense.

The trick of Lambda theory here consists in starting from and exploiting this *conceptual* legitimacy of the "Nothing" in order to give it a *technical* legitimacy as well.

The way to give technical legitimacy to the "Nothing" is the introduction of the axiom of the "pre-element".
While Russell justifies the inexistence of the null-class by denying it the privilege of belonging to another class, the axiom of the "pre-element" says that Lambda denotes the "Nothing" because it belongs to *any* set. In the spirit of the definition of the inclusion of the empty set in any set, we could say that, since Lambda denotes the "Nothing", there is no set to which Lambda cannot belong. So Lambda belongs to the empty set too.
The empty set becomes the set that contains only Lambda.

The fact that our intuitive wishes about an adequate behaviour of Lambda can be formally axiomatized and lead to equiconsistency results gives a technical legitimacy to the notion of "void".

In this way, not only the null-class seems to acquire complete (conceptual *and* technical) legitimacy in set theory, but the "Nothing" does too.

5 Conclusion

"*Nothing*" is added to set theory. This means: the introduction of the constant Lambda denoting the "Nothing" (the Void) in the language of set theory does not imply that some new thing (in the sense of: "new set") is added to the theory. The Nothing is subjacent to the standard set theory as a pre-element. It is a condition of possibility for the elements of the theory. The Nothing has two functions:
- the first one is the function of *internal* condition of possibility. This enables a set to contain elements. Lambda is the fundamental constituent of any set, and this is expressed by the axiom of the pre-element;
- the second one is the function of *external* condition of possibility. Lambda plays the role of the *physical* space, of cut, between sets and allows to have distinct sets; this can be more specifically studied by formal ontology.

We believe that, as condition of possibility of sets, Lambda is more fundamental than sets and should not be reduced to a technical artifice. The Lambda theory is a natural approach of set theory and probably introduces the smallest, the minimal constituent that can be added to a set theory.

The use of Lambda leads to several interesting and/or surprising conceptual and/or technical results.

1) Introducing the Nothing in the language of set theory allows to distinguish

the Nothing from the empty set, solving what we called the puzzle of Lambda in Russell's approach (see section 1.1).

2) The notion of "set" acquires here a real ontological dimension. In the *naive* conception of a set, a set is a collection of elements, and therefore it is not easy to give conceptual legitimacy to the empty set. The Lambda theory legitimates and gives an emblematic status to the empty set, now defined in a positive way: it is the only set that contains only Nothing.

3) More generally, the Lambda theory also allows to redefine the concept of set: a set can contain sets or objects thanks to the free space denoted by Λ. The *naive* acceptance of set is in some way validated in the case of the empty set: the empty set is a collection of "Nothing".

4) The Lambda theory also allows to build the empty set by means of the axiom of pairing and also by means of the axiom of the power set applied to Lambda: $\emptyset = \{\Lambda\} = \wp(\Lambda)$. In classical ZF set theory, the existence of a set is to be postulated; in Lambda theory, the first set is built from "Nothing".

5) In the Lambda theory, the axiom of the existence of the empty set or the construction of the empty set by means of a contradictory property becomes useless. "Something", or rather a *pre-thing* belongs to the empty set: the "Nothing", and remember that the empty set is the only set to which only Lambda belongs.

6) It also allows to distinguish the empty set from ur-elements, which are generally considered as kinds of empty sets. No thing belongs to an ur-element, not even Lambda. This is why an ur-element is a kind of atom. Now, no thing belongs to Lambda either. But we can make the distinction between Lambda and an ur-element: Lambda belongs to the empty set, moreover, Lambda belongs to every set.

7) Finally, the Lambda theory is the first step forward to a theory where the notion of "potential membership" ("potentially belongs to") can be considered and formalized.

So, the use of Lambda in set theory appears natural and could even be seen as necessary.
For the theory, the balance in terms of investment and profits is clearly positive: the investment is *quasi-null*, the gains are numerous.

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