On Ignorance and Contradiction Considered as Truth-values*

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Abstract

A critical view of the alleged significance of Belnap four-valued logic for reasoning under inconsistent and incomplete information is provided. The difficulty lies in the confusion between truth-values and information states, when reasoning about Boolean propositions. So our critique is along the lines of previous debates on the relevance of many-valued logics and especially of the extension of the Boolean truth-tables to more than two values as a tool for reasoning about uncertainty. The critique also questions the significance of partial logic.

1 Introduction

There have been many discussions as to the actual significance of many-valued logics. It is not indeed very obvious, at first glance, to see what they can be useful for, what kind of phenomena they may account for (see [45] for a critical view and [31] for a more optimistic discussion). Part of this difficulty lies in the controversial interpretation of partial truth in terms of incomplete knowledge. Hähnle [32] faces this difficulty in the field of formal specification of software systems where modeling non-termination and error values in terms of many-valued logic turns out to be problematic.

From the inception of many-valued logics [37], there have been attempts to attach an epistemic flavor to truth degrees. This has led to a very confusing state of facts, and has probably hampered the development of applications of these logics. If we look at the applications of classical logic, there are three main trends in the XXth century. First, classical logic has been proposed as a foundation for mathematics. This program is known to have partly failed after Goedel's incompleteness theorem. Further on, the advent of computers and digital technology has prompted the need for synthetizing Boolean functions. A great deal of results was produced based on this motivation. More recently the emergence of Artificial Intelligence and Logic programming has re-instated classical logic as a tool for knowledge representation and reasoning, a role logic had had since the Antiquity.

The position of many-valued logics in this landscape looks less strong. Although some may be tempted to found new mathematics on many-valued logics [3], this grand purpose still looks out of reach if not delusive. It sounds like a paradox of its own since we use classical mathematics to formally model many-valued logic notions. What could be named "many-valued mathematics" essentially looks like an elegant way of expressing

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properties of many-valued extensions of Boolean concepts in a Boolean-like syntax. For instance, the transitivity property of similarity relations is valid in Łukasiewicz logic, and, at the syntactic level, exactly looks like the transitivity of equivalence relations, but should be interpreted as the triangular inequality of distances measures. The research program of the "many-valued mathematics" school differs from intuitionism, that was motivated by the search for constructive proofs in mathematics and the rejection of the proof by refutation.

Besides, there have been some works proposing to use many-valued logics so as to extend switching logic to analogical devices (especially 3-valued logics). However it is not clear that the current technology of computers will soon move to that direction. Recent works by Mundici and colleagues nevertheless bridge the gap between Łukasiewicz logic and the concise representation and approximation of real-valued functions [1]. Lastly, it is tempting to assume that many-valued logics would provide a more flexible setting for knowledge representation than classical logic. However it is patent that many-valued logics had a very limited impact in Artificial intelligence so far, despite the significant progress recently made in the mechanization of automated reasoning in basic many-valued logics (especially Łukasiewicz logic) [30]. In contrast, Artificial Intelligence has triggered research in symbolic non-classical *practical* logics (nonmonotonic or modal logics) where truth remains two-valued. Interestingly, notions like belief or knowledge, that pervade these logics are often modelled in an all-or-nothing way (even if they are naturally a matter of degree).

Only fuzzy set and fuzzy logic after the pioneering papers of Zadeh [47] [49] seem to have considered many-valued logics as their natural underpinnings. Fuzzy logic is often attacked because it is truth-functional. There is a whole sequence of such papers, perhaps culminating with Elkan [21]'s best paper prize at the 1993 National U.S. AI conference. Looking at these critiques more closely, it can be seen that the root of the controversy lies in a confusion between degrees of truth and degrees of belief (an instance of this confusion is between membership grades and degrees of probability). Indeed, belief is never truth-functional [17], but fuzzy logic is not specifically concerned with belief representation, only with gradual (not black or white) concepts [33]. However this misunderstanding seems to come a long way, and can be traced back to the words the founders of many-valued logics used when speaking of their invention. For instance, a truth-value strictly between true and false was named "possible" [39], a word which refers to uncertainty modelling and modalities. While founders of many-valued logics built a beautiful mathematical and conceptual object, they seem to have been less clear about its purpose.

This paper recalls some elements pertaining to the history of this flaw, and shows that we still suffer from it, not only through the debates around fuzzy logic, but also in some other unrelated attempts at building logics of incomplete knowledge or inconsistency-tolerant formalisms. More specifically, we restrict our attention to partial logic, and Belnap's allegedly useful four-valued logic. This paper relies on a previous one which tried to clarify the confusion between many-valued logic and possibility theory [19]. Our contention is basically that degrees of truth are a matter of convention, while uncertainty handling is a matter of consequencehood and validity, hence a meta-notion with respect to truth-values, be they non-extreme ones. We claim here that we cannot consistently reason under incomplete or conflicting information about Boolean propositions by augmenting the set of Boolean truth-values *true* and *false* with epistemic notions like "unknown" or "contradictory", modeling them as additional genuine truth-values of their own.

2 The meaning of intermediate truth-values: digging into the history of many-valued logics

In the following, 1 stands for *true* and 0 stands for *false*. It is known that Łukasiewicz [39] suggested an interpretation for his third truth-value $\frac{1}{2}$, added to the set $\{0,1\}$. The third truth-value is supposed to be read as *possible*. The idea comes from a debate about future contingents in the Greek Philosophy[1]. It was noticed that statements about future events cannot be assigned a truth-value 1 or 0, unless we live in a predetermined world. So, the fact that, for a proposition p referring to the future, like for instance *The UK will be a republic in year 3000*, one may have both p and its negation $\neg p$ possible, led Łukasiewicz[2] into rejecting the excluded-middle law and building a truth-table for his three-valued logic, reflecting this rejection. In particular, $t(p)=t(\neg p)=\frac{1}{2}$, for future contingents.

It should be clear that, in the above debate, the epistemic notion of *possible* is not restricted to future events, but also applies to past and present events that are unknown by a reasoning agent. Namely, the fact that the statements under concern refer to an unpredictable future or to an ill-known past is immaterial. Even if, contrary to the future, the past is entirely determined, both may be equally unknown to an agent.

The epistemic understanding of truth-functional many-valued logics has been criticized, for instance by Urquhart [45]. Since the main primitive connective in Łukasiewicz logic is the implication $\rightarrow$ (see table 1), the claim that *possible* is a third truth-value leads to the question of assigning a proper truth-value to sentences of the form $p\rightarrow p$, when the truth-value $t(p)=\frac{1}{2}$. Commonsense suggests that $p\rightarrow p$ should be maintained as a tautology so that in particular $\frac{1}{2}\rightarrow\frac{1}{2}=1$, as proposed by Łukasiewicz, in so far as *modus ponens* is going to be used as an inference rule. It is legitimate to consider likewise that $p\rightarrow\neg p$ should not be true when $t(p)=t(\neg p)=\frac{1}{2}$. But since $\frac{1}{2}\rightarrow\frac{1}{2}=1$, $t(p\rightarrow\neg p)=1$, which sounds debatable in this case, since p is not acknowledged as false. Now, postulating that $\frac{1}{2}\rightarrow\frac{1}{2}=\frac{1}{2}$ (an assumption made in Kleene's 3-valued logic) is not very exciting as, to quote Urquhart: "there would be no 3-valued tautologies".

The conclusion here is that, in order to capture the status of future contingents, and more generally any unknown proposition, the very assumption of truth-functionality (building truth-tables for all connectives) is debatable. Combining two propositions whose truth-values are unknown may result in tautological or contradictory statements, whose truth-value can be asserted from the start, even without believing in a predetermined world. As long as p will eventually be either true or false, even if this truth-value cannot be computed or prescribed as of to-day, the proposition $p\wedge\neg p$ can be unmistakably at any time predicted as being false

Table 1. Łukasiewicz implication

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

[1] between Chrysippus school and Epicureans.
[2] adopting the position of Epicureans against Chrysippus.

and $p \vee \neg p$ as being true[3], while $p \wedge p$ and $p \vee p$ remain contingent. So there is no way of defining a sensible truth-table that accounts for the idea of *possible*. This notion of *possible* is in fact a modality and not what is usually understood as a truth-value. In modal logic the fact that *possibly* p and *possibly* $\neg p$ can be true at the same time is not surprizing nor does it question the excluded-middle law. The conjunction of *possibly* p and *possibly* q is in particular not equivalent to the statement that $p \wedge q$ is possible (the latter is a contradiction if $p = \neg q$). Adopting this equivalence usually trivializes modal systems.

In fact, Łukasiewicz seems to have persisted in his view of truth-functional modalities, as he later proposed a 4-valued modal logic in which the modality *possible* ($\Diamond$) obeys $\Diamond p \wedge \Diamond q = \Diamond(p \wedge q)$. Details about this logic are provided by Font and Hájek [22] who show its counterintuitive behaviour, and conclude that it is very difficult to interpret this system as a modal logic.

The basic issue is that the truth or falsity of unknown propositions (such as future contingents) is a matter of belief[4] held by an agent and pertains to the representation of uncertainty. This is the point made very early by De Finetti [15], as a critique of Łukasiewicz's interpretation of his third truth-value. To wit[5]: "Even if, in itself, a proposition cannot be but true or false, it may occur that a given person does not know the answer, at least at a given moment. Hence for this person, there is a third attitude in front of a proposition. This third attitude does not correspond to a third truth-value distinct from yes or no, but to the doubt between the yes and the no (as people, who, due to incomplete or indecipherable information, appear as of "unknown sex" in a given statistics. They do not constitute a third sex. They only form the group of people whose sex is unknown)".

For De Finetti, believing or not a statement about an event is an epistemic notion that encapsulates the statement. The lack of knowledge about truth-values is to be faced and coped with whether one likes it or not. In contrast, what is called a truth set and the number of allowed truth-values are a matter of representation conventions. For the purpose of representing knowledge, we decide to use entities, we call propositions, that are *assumed* to be only true or false, even if these values may be currently unknown because the corresponding statements pertain to the future, or to any ill-known circumstances, be they past or present. Using Boolean variables or not to represent knowledge is a modeling decision. People working in probability theory do use Boolean propositions, but they attach numerical probabilities to them. These probabilities are not truth-values. They are degrees of belief about the truth or falsity of Boolean propositions, interpreted as lottery ticket prices to be paid to gain one money unit if the corresponding events occur. Not knowing in advance if $p =$ *The UK will be a republic in year 3000*, and not having decisive arguments in favor of this statement nor its contrary, the probability $\frac{1}{2}$ will be assigned to both assertions that p and that $\neg p$ (more explicitly, to each of the statements p *is true* and p *is false*).

Note that some authors like Reichenbach [41], more recently Nilsson [40], seemed to consider probabilities as truth-values. The source of this confusion seems to lie in the fact that the negation of *probably true* is *probably false* (while the negation of *possibly true* is not

[3]The case of intuitionism remains to be further discussed along that line. The point of intuitionism is a critique of some notions of proof in mathematics, namely refutation-style proofs, whereby an existence theorem can be proved without exhibiting an example where this theorem holds. Then the loss of the excluded-middle law seems to be due to the decision not to use this property when developing proofs rather than the will to change the Boolean nature of propositional variables.

[4]or knowledge; here we are not concerned by the fact that the beliefs of the agent are or not in agreement with the real world. So, we shall use the words "belief" and "knowledge" indifferently in this paper.

[5]Our translation from the French, already cited in [19].

TABLE 2. Kleene strong disjunction

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

possibly false, but *certainly false*). So the truth set $\{0,1\}$ looks superseded by the unit interval of probabilities. As pointed out by Carnap [12], what is superseded by the probability scale are the notions of *verified* and *falsified*, not the truth-values per se.

Adopting Boolean variables to describe past, present or future states of facts, we are forced to take all tautologies and contradictions of classical logic for granted. It does not presuppose any philosophical attitude regarding the issue of determinism, nor on the nature of truth. It is just a mathematical consequence of our modelling convention. Other conventions may be adopted (as acknowledged by De Finetti himself) like defining a proposition as a 3-valued entity, where the third truth-value $\frac{1}{2}$ genuinely means *half-true*. For instance, I can see a bottle on a table and declare that the statement *the bottle is full* has truth-value $\frac{1}{2}$. By this claim, I precisely mean the bottle is half-full. Note that in this example, there is no uncertainty involved (I can see the bottle and measure the amount of liquid in it). Moreover it is clear that an infinite truth-set looks natural since the bottle may be in many more states than full, empty, and half-full. Here the use of a Boolean variable is not natural because the negation of *full* is not *empty* (the latter is the antonym of *full*).

Three-valued logics other than Łukasiewicz's propose different truth tables and different views of the third truth-value. Kleene's three-valued logic [35] considers truth-values as the result of a computation by a machine. The third truth-value is used when the machine fails to provide an answer. Even if the reference to a computer makes the third truth-value more down-to-earth than the philosophical standpoint of Łukasiewicz, the point of view on propositions after Kleene remains the same: if the machine cannot compute the truth-value of a proposition couched in terms of Boolean variables, the agent who runs the computation ends up not knowing this truth-value. The truth-table for strong disjunction in Kleene's logic (see table 2) stipulates that $\frac{1}{2}\vee\frac{1}{2}=\frac{1}{2}$. So, this logic implicitly presupposes that if the machine fails to compute the truth-value of p and the one of $\neg p$ then it cannot compute the truth-value of the syntactic expression $p\vee\neg p$. But the latter being a well-known tautology inside a Boolean modeling framework, there is no need to use the machine to compute its truth-value. It does not question the fact that determining if a proposition with complex syntactic format is a tautology or not can be difficult in practice, nor that it may be useful to handle complex syntactic entities in a distinguished way. It only suggests that for any proposition that possesses the syntactic form $p\vee\neg p$, the decision problem is straightforward, even if the truth-value of p and the one of $\neg p$ are not known. Unfortunately, truth-tables do not seem to be the right tool to make the difference between Boolean tautologies expressed as complex well-formed formulas (wffs) that can hardly be proved tautological in practice, and obviously tautological wffs. It means that Kleene truth-tables cannot straightforwardly apply to this problem (see also [32]). So the interpretation of the third truth-value in Kleene's logic seems to suffer from the same defects as in Łukasiewicz logic.

In Bochvar system [11], the third truth-value means *meaningless*. A formula is assigned this truth-value if one of its component is considered meaningless. This seemed to be motivated by mathematical issues concerning Russell's paradox that are not relevant here. However the issue of meaningless statements may be of interest in the scope of database research as one possible view of null-values. This aspect is not considered further in this paper.

3 Truth vs. belief in propositional logic

In a previous paper [19] we pointed out that while classical (propositional) logic is always presented as the logic of the true and the false, this description neglects the epistemic aspects of this logic. Namely, if a set B of well-formed Boolean formulae is understood as a set of propositions believed by an intelligent agent (a *belief base*) then the underlying uncertainty theory is ternary and not binary: it is conceivable that some proposition is neither believed nor is disbelieved by a particular agent. Moreover, belief is not compositional even in propositional logic.

Building a belief base can be achieved by, for instance, querying an agent about what he believes to be true. Note that, in this respect, asking the agent whether a proposition p is true is misleading (even if this is the way such questions are formulated usually). The right question it stands for is : "does the agent BELIEVE p to be true?"[6] Hence, responses to this question do not directly provide access to truth, only to the agent's beliefs about the truth or falsity of propositions. So, in order to lay bare this epistemic ingredient at the syntactic level, propositions in B should be prefixed by a modality like knowledge or belief, attached to the agent. This is done in modal epistemic logic [34]. However, in the works of many authors, like Gärdenfors [26], for instance, who defines a *belief set* as a deductively closed set of formulas, such modalities are omitted.

Insofar as the agent supplied all of his knowledge in B, and up to the logical omniscience assumption, the logical consequences of B can be considered as a faithful description of all that is known by the agent. As an agent usually has but limited knowledge, the epistemic status of each proposition in the language is not Boolean, but three-valued:

- p is believed (or known), which is the case if B implies p
- its negation is believed (or known), which is the case if B implies $\neg p$
- neither p nor $\neg p$ is believed, which is the case if B implies neither $\neg p$ nor p

It is clear that belief representation refers to the notion of validity of p in the face of B and is a matter of consequencehood, not truth-values. In fact, one can represent belief by means of *subsets of possible truth-values* enabled for p by taking propositions in B for granted. Full belief in p corresponds to the singleton $\{1\}$ (only the truth-value "true" is possible); full disbelief in p corresponds to the singleton $\{0\}$ (only the truth-value "false" is possible); the situation of total uncertainty relative to p for the agent corresponds to the set $\{0,1\}$. This set is to be understood disjunctively (both truth-values for p remain possible due to incompleteness, but only one is correct). Under such conventions, the characteristic function of $\{0,1\}$ is viewed as a possibility distribution π (Zadeh [48]). Namely, $\pi(0)=\pi(1)=1$ means that both 0 and 1 are possible. It contrasts with other uses of subsets of truth-values, interpreted conjunctively, whereby $\{0,1\}$ is understood as the simultaneous attachment of "true" and "false" to p (expressing a contradiction, see Dunn [20] and the section on Belnap's logic in this paper). This is another convention based on necessity degrees

[6] or equivalently: "Is the agent sure that p is true?"

$N(0)=1-\pi(1); N(1)=1-\pi(0)$. Then clearly, $N(0)=1=N(1)$ indicates a strong contradiction. However we shall not use the latter convention[7], and $\{0,1\}$ attached to p stands for lack of knowledge about p.

It must be emphasized that $\{0\},\{1\}$, and $\{0,1\}$ are not truth-values of propositions in B. They express what can be called *epistemic values* whereby the agent believes p, believes $\neg p$, or is ignorant about p respectively. They are like modalities. Attaching the epistemic annotation $\{1\}$ to p is like asserting $\Box p$ using a necessity modality interpreted as belief or knowledge. Clearly, the negation of the statement *p is believed* (inferred from B) is not the statement *¬p is believed* , it is *p is not believed*. However, the statement *p is not believed* cannot be written in B because the syntax of classical logic does not allow for expressing ignorance in the object language. The latter requires a modal logic, since in classical logic $\neg p$ means $\Box\neg p$, not $\neg\Box p$ (that cannot be expressed).

The fact that epistemic modalities $\{0\},\{1\}$, and $\{0,1\}$ do not behave compositionally like genuine truth-values is simply examplified by the fact that $\Box(p\vee q)$ is generally not equivalent to $\Box p\vee\Box q$ in modal logic. So, $p\vee q$ is believed does not mean that either p is believed or q is believed. Attaching $\{1\}$ to $p\vee q$ is actually weaker than attaching $\{1\}$ to one of p or q. In the case of ignorance about p, $\{0,1\}$ is attached to p and to $\neg p$, yet, only $\{1\}$ can be attached to their disjunction (since it is a tautology). This fact only reminds us that the union of deductively closed belief sets need not be deductively closed. Assuming compositionality of epistemic annotations by means of truth tables of connectives in a three-valued logic provides only an imprecise approximation of the actual Boolean truth-value of complex formulas (using Kleene logic, see De Cooman, [14]).

Interestingly the representation of beliefs in classical logic can be cast inside the usual modal epistemic logic KD45, as shown by the following result

Proposition 3.1 *(Dubois, Hájek Prade [16]) : In the KD45 system, if B is a propositional belief base and $\Box B=\{\Box p, p\in B\}$, where $\Box$ is the belief modality then B implies p classically if and only if $\Box B$ implies $\Box p$.*[8]

Note that the above embedding of classical logic inside a modal logic is not the usual one: Usually, modal logics contain propositional logic as a fragment without modalities. The above theorem concerns another fragment, namely all wffs made of classical propositions prefixed by $\Box$.

Embedding epistemic notions in a modal object language brings us back to a (higher-order) truth-functional setting for reasoning about belief-prefixed propositions. Indeed, the truth-value of $\Box p$ tells whether p is believed or not: *$\Box p$ is true* precisely means that $\{1\}$ is the subset of truth-values left to p, i.e. *it is true that p is believed (to be true)*. So, what belief internally means may be captured by a kind of external truth-set, say $\{\mathbf{0},\mathbf{1}\}$. Mind that the value $\mathbf{1}$ in $t(\Box p)=\mathbf{1}$ and the value 1 in $t(p)=1$ refer to different truth-sets (and different propositions). This trick can be used for probability theory and other non-compositional uncertainty theories (see Godo Hájek et al. [29], [28]) and leads to legitimating many-valued logic approaches to uncertainty management, where the lack of compositionality of belief

[7]The convention based on necessity degrees hardly extends to truth-sets T with more than two truth-values: when more than two truth-values are allowed, the degrees of necessity of singletons $N(\{t\})=1-\Pi(T\setminus\{t\})$ are all zero, generally, as soon as more that one truth-value is possible.

[8]In fact not all axioms of KD45 are needed to get this result. Any modal logic where the K axiom $\Box(p\to q)\to(\Box p\to\Box q)$ holds verifies this embedding property.

is captured in the object language. For instance, the degree of probability $Prob(p)$ can be modeled as the truth-value of the proposition "Probable(p)" (which expresses the statement that p is probably true), where *Probable* is a many-valued predicate, but it is not the allegedly multivalued truth-value of the (Boolean) proposition p.

4 Partial Logic vs. Supervaluations

Partial logic starts from the claim that truth-values of propositions can be left open, and that such undefinedness may stem from a lack of information. This program is clearly in the scope of our investigation, as theories of uncertainty and partial belief were precisely introduced as well to cope with limited knowledge. Other interpretations of partiality exist, that are not considered here. From a historical perspective, the formalism of partial logic is not so old, but has its root in Kleene [35]'s three-valued logic, where the third truth-value expresses the impossibility to decide if a proposition is true or false. The reader is referred to the dissertation of Thijsse [42] and a survey paper by Blamey [10].

At the semantic level, the main idea of partial logic is to change interpretations into partial interpretations (also called *coherent situations*) obtained by assigning a Boolean truth-value to some (but not all) of the propositional variables forming a set $Prop = \{a, b, c, \ldots\}$. A coherent situation can be represented as any conjunction of literals pertaining to distinct propositional variables. Denote by s a situation, S the set of situations, and $V(a,s)$ the partial function from $Prop \times S$ to $\{0,1\}$ such that $V(a,s) = 1$ if a is true in s, 0 if a is false in s, and is undefined otherwise. Then, two relations are defined for the semantics of connectives, namely *satisfies* ($\models_T$) and *falsifies* ($\models_F$):

- $s \models_T a$ if and only if $V(s,a) = 1$;
- $s \models_F a$ if and only if $V(s,a) = 0$;
- $s \models_T \neg p$ if and only if $s \models_F p$;
- $s \models_F \neg p$ if and only if $s \models_T p$;
- $s \models_T p \wedge q$ if and only if $s \models_T p$ and $s \models_T q$;
- $s \models_F p \wedge q$ if and only if $s \models_F p$ or $s \models_F q$;
- $s \models_T p \vee q$ if and only if $s \models_T p$ or $s \models_T q$;
- $s \models_F p \vee q$ if and only if s $\models_F p$ and $s \models_F q$.

In partial logic a coherent situation can be encoded as a truth-assignment t_s mapping each propositional variable to the set $\{0, \frac{1}{2}, 1\}$, understood as a partial Boolean truth-assignment in $\{0,1\}$. Let $t_s(a) = 1$ if a appears in s, 0 if $\neg a$ appears in s, and $t_s(a) = \frac{1}{2}$ if a is absent from s. Thus, t_s encodes a partial interpretation where only part of the propositional variables are assigned truth-values 0 and 1. The basic partial logic can thus be described by means of a three-valued logic. The truth set is $\{0, \frac{1}{2}, 1\}$, where $\frac{1}{2}$ (again) means *unknown*. The connectives can be expressed by means of the following functions: $1 - x$ for the negation, max for disjunction, min for the conjunction, and $\max(1-x, y)$ for the implication. Note that if $t_s(p) = t_s(q) = \frac{1}{2}$, then also $t_s(p \vee q) = t_s(p \wedge q) = t_s(p \rightarrow q) = \frac{1}{2}$ in this approach.

Since these definitions express truth-functionality in a three-valued logic, a situation may fail to satisfy classical tautologies (Thijsse, 1992). But this anomaly stems from the same difficulty again, that is, no three-element set can be endowed with Boolean algebra structure! A coherent situation s can be interpreted as a special set $A(s)$ of standard Boolean interpretations, and can be viewed as a disjunction thereof. Namely, s is of the form

$\wedge\{a_i \in Prop^+(s)\} \bigwedge \wedge\{\neg a_i \in Prop^-(s)\}$ where $Prop^+(s)$ (resp. $Prop^-(s)$) is the set of propositional variables assigned to 1 (resp. 0) in situation s. Then $A(s)$ contains all models of this formula, that can be built just completing s by all possible assignments of 0 or 1 to the variables not in $Prop^+(s) \cup Prop^-(s)$. A coherent situation is thus an attempt to model an epistemic state reflecting a lack of information.

If this view is correct, the equivalence $s \models_T p \vee q$ if and only if $s \models_T p$ or $s \models_T q$ cannot hold under classical model semantics. Indeed $s \models_T p$ supposedly means $A(s) \subseteq [p]$ and $s \models_F p$ supposedly means $A(s) \subseteq [\neg p]$, where $[p]$ is the set of interpretations where p is true. But while $A(s) \subseteq [p] \cup [q]$ holds whenever $A(s) \subseteq [p]$ or $A(s) \subseteq [q]$ holds, the converse is invalid, as already seen!

This is the point made by Van Fraassen [46] who first introduced the notion of supervaluation to account for this situation. A supervaluation SV over a coherent situation s is (in our terminology) a function that assigns, to each proposition in the language and each coherent situation s, the *super-truth-value* $SV(p,s)=1$ (0) to propositions that are true (false) for all Boolean completions of s. It is clear that p is "super-true" ($SV(p,s)=1$) if and only if $A(s) \subseteq [p]$, so that supervaluation theory recovers missing classical tautologies by again giving up truth-functionality: $p \vee \neg p$ is always super-true, but $SV(p \vee q, s)$ cannot be computed from $SV(p,s)$ and $SV(q,s)$.

One interesting question is whether we can preserve truth-functionality for partial logic, in special cases, under partial information. Clearly, if $p = a_1$ and $q = a_2$ are independent literals, the property holds because $A(s) \subseteq [p \vee q]$ implies either $s = a_1 \wedge a_2$ or a_1 or a_2. It is not clear if we can go beyond this case without losing classical tautologies.

Lastly, note that interpreting $s \models_T p$ as $A(s) \subseteq [p]$ points out the relationship between partial logic and the belief calculus of propositional logic in the previous section (of which it is a special case). Indeed it is clear there is a classical belief base $B(s)$ such that $s \models_T p$ if and only if $B(s)$ classically implies p. Namely $B(s)$ is semantically equivalent to the conjunction of literals appearing in s. The term "super-true" in the sense of Van Fraassen stands for "certainly true" in the scope of belief management in possibility theory. In partial logic, the set $A(s)$ formed by completions of s is a very special subset of interpretations. It is a conjunction of literals, a Cartesian product within 2^n. Hence partial logic only captures special types of situations of partial knowledge: those where ignorance only pertains to the truth-values of some propositional variables, not linked with any dependence relation. Note that not all subsets of the set of Boolean interpretations, viewed as incomplete epistemic states can be modeled by a three-valued (partial) interpretation. For instance if $Prop = \{a, b\}$, then the set of coherent situations corresponding to proper partial models is $\{\top, a, b, \neg a, \neg b\}$[9]. But an epistemic state captured by the knowledge base $B = \{\neg a \vee \neg b, a \vee b\}$ cannot be expressed by means of a coherent situation in partial logic. In contrast, we claim that the belief calculus at work in propositional logic is doing the job and covers the semantics of partial logic as a special case. It exactly coincides with the semantics of the supervaluation approach.

5 A Critical Discussion of Belnap Four-Valued Logic

Two seminal papers of Belnap([4], [5]) proposed an approach to reasoning both with incomplete and with inconsistent information. It relies on a set of truth-values forming a bilattice, further studied by scholars like Ginsberg [27] and Fitting [23]. The aim of this section is

[9]where $\top$ denotes tautology. This set excludes complete models.

to focus on the two papers of Belnap, showing that his approach, considered as a system for reasoning under imperfect information, suffers from the same difficulties as partial logic, and for the same reason. Indeed one may consider this logic as using the three epistemic values already considered in the previous sections (*certainly true, certainly false and unknown*), along with an additional epistemic value. The latter accounts for epistemic conflicts, namely the situation where the agent hears that a proposition p is claimed to be true by one source and false by another source. Belnap proposes truth-tables for this calculus, and they are extensions of Kleene's truth-tables for three-valued logic, adding an extra truth-value standing for the contradictory epistemic value. It turns out that adding contradiction to incompleteness, this truth-functional logic is not better off. A similar critique of Belnap already appears in a short note by Fox [25], who also relies on the Boolean nature of propositions to question the legitimacy of losing classical tautologies in the face of incompleteness and contradiction.

5.1 The contradiction-tolerant setting

The initial idea behind Belnap's logic is very attractive. The author considers an artificial information processor capable of answering queries on propositions of interest. The system is fed from a variety of sources. It is clear that in a multisource environment, inconsistency threatens, all the more so as the information processor is supposed never to subtract information. So the basic assumption is that the computer receives information about atomic propositions in a cumulative way from outside sources, each asserting for each atomic proposition whether it is true, false or being silent about it. The notion of *epistemic set-up* is defined as an assignment of one of four values, denoted $\mathbf{T}, \mathbf{F}, \mathbf{BOTH}, \mathbf{NONE}$, to each atomic proposition $a, b, \ldots$:

- Assigning $\mathbf{T}$ to a means the computer has only been told that a is true.
- Assigning $\mathbf{F}$ to a means the computer has only been told that a is false.
- Assigning $\mathbf{BOTH}$ to a means the computer has been told at least that a is true by one source and false by another.
- Assigning $\mathbf{NONE}$ to a means the computer has been told nothing about a.

The whole approach relies on the approximation lattices proposed by Scott. In an approximation lattice, the ordering relation reflects the idea of approximation of an element by another element : $x \leq y$ means that x approximates y. For instance, if x and y are intervals, $x \leq y$ means interval x is less precise than (= contains) interval y. Of importance are monotonic functions f, that respect the ordering in the lattice. Again, in terms of intervals, one expects that if x contains y then $f(x)$ contains $f(y)$ for well-behaved functions. The set $\mathbf{4} = \{\mathbf{T}, \mathbf{F}, \mathbf{BOTH}, \mathbf{NONE}\}$ is equipped with an ordering $\sqsubset$ that is viewed by Belnap as an approximation lattice. According to this ordering: $\mathbf{NONE} \sqsubset \mathbf{T} \sqsubset \mathbf{BOTH}; \mathbf{NONE} \sqsubset \mathbf{F} \sqsubset \mathbf{BOTH}$. This ordering reflects the inclusion relation of the sets $\emptyset$, $\{0\}, \{1\}$, and $\{0,1\}$ in the sense of the approximation lattice. These sets respectively encode $\mathbf{NONE}$, $\mathbf{F}$, $\mathbf{T}$, $\mathbf{BOTH}$, respectively, under Dunn [20]'s convention (conjunctive sets, opposite the convention in possibility theory [10]). $\mathbf{NONE}$ is at the bottom because (to quote) "it gives no information at all". $\mathbf{BOTH}$ is at the top because (following Belnap) it gives too much information. So the ordering $\sqsubset$

[10] In possibility theory $\mathbf{NONE} = \{0,1\}, \mathbf{BOTH} = \emptyset$, as pointed out in Section 3; these subsets represent mutually exclusive truth-values, one of which is the right one. Dunn's convention uses Boolean necessity degrees.

TABLE 3. Belnap disjunction

$\cup$	**F**	**NONE**	**BOTH**	**T**
F	**F**	**NONE**	**BOTH**	**T**
NONE	**NONE**	**NONE**	**T**	**T**
BOTH	**BOTH**	**T**	**BOTH**	**T**
T	**T**	**T**	**T**	**T**

TABLE 4. Belnap conjunction

$\cap$	**F**	**NONE**	**BOTH**	**T**
F	**F**	**F**	**F**	**F**
NONE	**F**	**NONE**	**F**	**NONE**
BOTH	**F**	**F**	**BOTH**	**BOTH**
T	**F**	**NONE**	**BOTH**	**T**

could be dubbed information *acquisition* ordering since it intends to reflect the amount of (possibly conflicting) data provided by the sources.

In a further step, connectives of negation, conjunction and disjunction are defined truth-functionally on the approximation lattice. Belnap truth-tables are rigorously derived from mathematical principles present in approximation lattices, and the coincidence with classical logic for **F**, **T**. Namely

1. Keep the usual connectives for the restriction of negation $\sim$, conjunction $\cap$ and disjunction $\cup$ to $\{\mathbf{T},\mathbf{F}\}$;
2. Assume monotonic connectives w.r.t. ordering $\sqsubset$;
3. Assume $A \cap B = A$ if and only if $A \cup B = B$;
4. Commutativity, associativity of $\cup, \cap$, De Morgan laws.

The first requirement enforces $\sim\mathbf{T}=\mathbf{F}$ and $\sim\mathbf{F}=\mathbf{T}$ and then, the monotonicity requirement forces the negation $\sim$ to be such that $\sim\mathbf{BOTH}=\mathbf{BOTH}, \sim\mathbf{NONE}=\mathbf{NONE}$[11]. The tables for $\cap$ and $\cup$ are also uniquely determined (see tables 3 and 4). As announced above, the restrictions of all connectives to the subsets $\{\mathbf{T},\mathbf{F},\mathbf{NONE}\}$ and $\{\mathbf{T},\mathbf{F},\mathbf{BOTH}\}$ coincide with Kleene's three-valued truth-tables, encoding **BOTH** and **NONE** as $\frac{1}{2}$. For instance, **T** is the identity for $\cap$, absorbing for $\cup$. In other words there is a second ordering on $\{\mathbf{T},\mathbf{F},\mathbf{BOTH},\mathbf{NONE}\}$, say $\succ$, that Belnap calls *the logical lattice ordering*, according to which $\mathbf{T} \succ \mathbf{BOTH} \succ \mathbf{F}$ and $\mathbf{T} \succ \mathbf{NONE} \succ \mathbf{F}$ reflecting the truth-set of Kleene's logic. So $\succ$ means "truer than". Note that Belnap's conjunction and disjunction operations $\cup$ and $\cap$ exactly correspond to the lattice meet and joint for the lattice defined by the logical lattice ordering, and not by the approximation lattice defined by $\sqsubset$. In fact, **BOTH** and **NONE** cannot be distinguished by $\succ$ and play symmetric roles in the truth-tables.

The major new point is the result of combining conjunctively and disjunctively **BOTH** and **NONE**. The only possibility left for such combinations is that $\mathbf{BOTH} \cap \mathbf{NONE} = \mathbf{F}$ and $\mathbf{BOTH} \cup \mathbf{NONE} = \mathbf{T}$. This looks surprising[12], but there is no other choice. For instance,

[11] In [5], p. 13, the truth-table for negation is different and such that $\sim\mathbf{BOTH}=\mathbf{NONE}$; this must be a typographical mistake.

[12] Belnap acknowledges it.

monotonicity w.r.t. $\sqsubset$ and the natural assumption that $\mathbf{F}\cap\mathbf{NONE}=\mathbf{F}$ allow the following derivations:

Since $\mathbf{F}\sqsubset\mathbf{BOTH}$, $\mathbf{F}\cap\mathbf{NONE}=\mathbf{F}\sqsubseteq\mathbf{BOTH}\cap\mathbf{NONE}$; but $\mathbf{BOTH}\cap\mathbf{NONE}\sqsubseteq\mathbf{BOTH}\cap\mathbf{F}=\mathbf{F}$ since $\mathbf{NONE}\sqsubset\mathbf{F}$; so, $\mathbf{BOTH}\cap\mathbf{NONE}$ is exactly $\mathbf{F}$ (using antisymmetry)[13].

5.2 This is not really how a computer should think

Belnap's calculus is an extension of partial logic to the truth-functional handling of inconsistency. In his paper, Belnap does warn the reader on the fact that the four values are not ontological truth-values but epistemic ones. But what is exactly meant by this phrase "epistemic truth-values" is left to the reader. In any case, they are qualifications referring to the state of knowledge of the agent (here the computer). But then **T** and **F** are not truth-values in the usual sense: assigning **T** to an atomic proposition expresses the idea that the computer is prone to assert the truth of this proposition because only evidence in its favour has been collected. Of course, **F** is interpreted likewise. **NONE** expresses ignorance about truth and falsity due to lack of any evidence, and **BOTH** describes a confused state of knowledge due to an overflow of information some being evidence for the proposition, some being evidence against it. Since ultimately a proposition is true or false here, part of the information justifying the value **BOTH** is erroneous, even if the computer has no tool for testing it. The information accumulation ordering indicates that each of the three assignments **T**, **F** and **NONE** are provisional, in the sense that they may change with the arrival of new evidence, and the direction of change is indicated by the ordering in the approximation lattice. So, for instance, **T**, **F** may become **BOTH** and thus differ from usual truth-values, which, in a static world, cannot change.

Additionally, the set-representation Dunn [20] suggested for the alleged truth-values in Belnap's logic **4** ($\mathbf{T}=\{1\},\mathbf{F}=\{0\},\mathbf{NONE}=\emptyset,\mathbf{BOTH}=\{0,1\}$), is in agreement with the idea of accumulation of information. This notation (like the dual one in possibility theory) rather comforts the idea that these are not truth-values. For instance $\{1\}$ is a subset of $\{0,1\}$ while 1 is an element thereof. Interpreting Belnap's epistemic truth-values as genuine truth-values comes down to confusing elements of a set and singletons included in it.

Belnaps explicitly claims that the systematic use of the truth-tables of **4** "tells us how the computer should answer questions about complex formulas, based on a set-up representing its epistemic state"([4], p. 41). However, if one admits that $\mathbf{T},\mathbf{F},\mathbf{BOTH},\mathbf{NONE}$ are not genuine truth-values, the use of truth-functional connectives becomes again problematic. Indeed, since the truth-tables of conjunction and disjunction extend the ones of partial logic so as to include the value **BOTH**, Belnap's logic inherits all difficulties of partial logic pointed out in the previous section in the way the computer should handle Boolean formulas that are, regardless of their epistemic status, true or false. Mathematical soundness and depth are no guarantee for the appropriateness of a theory in a practical problem. It is not clear that incompleteness and inconsistency are properly modelled in this logic. Denoting $E(p),E(q)$ the assignment of epistemic values to propositions p,q (previously computed), and letting $\star$ be a connective, can $E(p\star q)$ be determined non-ambiguously by combining epistemic values of atoms $E(p)$ and $E(q)$? In other words, can an epistemic value on p and one on q

[13]The non-modal fragment of the four-valued modal logic Ł of Łukasiewicz studied by Font and Hájek [22] is defined via truth-tables for negation and implication $\to$. The conjunction is $p\wedge q=\neg(p\to\neg q)$. The truth-table for negation in Ł differs from the one in Belnap logic. In fact, the truth-set of Ł is $\{0,1\}\times\{0,1\}$ where operations act coordinatewise, hence not a biliattice.

characterize a single epistemic value for $p\star q$? The discussion in the previous sections has warned us that this is impossible unless a high price is paid, that is, the loss of classical tautologies, which induces paradoxes for reasoning based on Boolean variables. For instance $\mathbf{NONE}\cap\mathbf{NONE}=\mathbf{NONE}$, $\mathbf{NONE}\cup\mathbf{NONE}=\mathbf{NONE}$. So if asked about the status of formulas such as $p\wedge\neg p$ and $p\vee\neg p$, where p is **NONE**, a computer following Belnap logic is bound to say that $p\wedge\neg p$ and $p\vee\neg p$ are both **NONE**. But this is not what is expected since, even if no information has been received by the computer regarding the truth or falsity of atom p, the former should be false and the latter should be true, if the domain of p is two-valued. Belnap truth-tables should not be used to compute the epistemic status of formulas involving logically dependent components.

Some may object that the computer is not aware of the strong link between p and $\neg p$ and that applying the truth-table (i.e., returning **NONE** for $p\wedge\neg p$ and $p\vee\neg p$) merely expresses that the computer did not receive any information about this dependence. However, it means the computer has too little information about what a proposition is: even if the computer never received information about atomic facts, it sounds reasonable to assume that the most elementary rules of classical logic (like being aware that formulas of the form $p\wedge\neg p$ are ever false, etc.) are part of its background knowledge.

The same anomaly will be observed with the epistemic value **BOTH** because it plays the same role as **NONE** for the logical lattice ordering. Claiming that $\mathbf{BOTH}\cap\mathbf{BOTH}=\mathbf{BOTH}$, $\mathbf{BOTH}\cup\mathbf{BOTH}=\mathbf{BOTH}$ is hardly acceptable when applied to propositions of the form p and $\neg p$. It is strange for a tautology like $p\vee\neg p$ to receive epistemic value **BOTH**.

Another issue is how to interpret the results $\mathbf{BOTH}\cap\mathbf{NONE}=\mathbf{F}$ and $\mathbf{BOTH}\cup\mathbf{NONE}=\mathbf{T}$. These results sound quite counterintuitive[14]. One may rely on bipolar reasoning and argumentation to defend that when p is **BOTH** and q is **NONE**, $p\wedge q$ should be $\mathbf{BOTH}\cap\mathbf{NONE}=\mathbf{F}$. Suppose there are two sources providing information, say S_1 and S_2. Assume S_1 says p is true and S_2 says it is false. This is why p is **BOTH**. Both sources say nothing about q, so q is **NONE**. So one may consider that S_1 would have nothing to say about $p\wedge q$, but one may legitimately assert that S_2 would say $p\wedge q$ is false. In other words, $p\wedge q$ is **F**: one may say that there is one reason to have $p\wedge q$ false, and no reason to have it true.

However, this part of the truth-table also leads to debatable epistemic value assignments. Suppose two atomic propositions a and b with $E(a)=\mathbf{BOTH}$ and $E(b)=\mathbf{NONE}$. Then $E(a\wedge b)=\mathbf{F}$. But since Belnap negation is such that $E(\neg a)=\mathbf{BOTH}$ and $E(\neg b)=\mathbf{NONE}$, we also get $E(\neg a\wedge b)=E(a\wedge\neg b)=E(\neg a\wedge\neg b)=\mathbf{F}$. Hence

$$E((a\wedge b)\vee(\neg a\wedge b)\vee(a\wedge\neg b)\vee(\neg a\wedge\neg b))=\mathbf{F}$$

using table 3, that is $E(\top)=\mathbf{F}$ which is hardly acceptable again.

One might thus argue that the combination of epistemic values has no unique result. So one might think of using set-valued connectives on the bilattice, a technique recently suggested by Avron [2], which is a genuine way of capturing the partial lack of information pervading the agent's knowledge at the level of truth-tables. Namely, it "saves" truth-functionality while admitting that the combination of two truth-values may result in a set of possible truth-values. However one may suspect that the price paid by this form of non-deterministic

[14] Even to Belnap, as it seems, considering his insistance to present them as unavoidable consequences of his formal setting. This is an example of a theory supposed to supersede commonsense reasoning, as when the use of Bayesian priors in statistical reasoning delivers conclusions that challenge our intuition.

truth-functionality is a loss of precision (still not recovering tautologies of classical logic for instance).

5.3 Does the bilattice structure capture belief ?

It may be tempting to interpret the epistemic truth-values of Belnap as describing the state of knowledge of an intelligent agent, viewing the idea of information accumulation in terms of belief. Suppose we take Belnap's word "epistemic" for granted and go ahead interpreting **T** as "(at least provisionally) surely true", **F** as "surely false", and **NONE** as "unknown" for the agent. Then according to the information acquisition ordering $\sqsubset$, **BOTH** appears as more surely true than **T** and more surely false than **F**, which may not be Belnap's intention. Indeed, moving upward according to $\sqsubset$ in the lattice may misleadingly suggest an improvement in the epistemic state of the agent.

However while inconsistency does result from an excess of information, the computer's epistemic position about the truth or the falsity of an atom is certainly not strengthened by receiving a denial on the truth-state of an atom, that forces a move from **T** or **F** to **BOTH**. While **NONE** is an epistemic vacuum, **BOTH** is an epistemic hell. So **BOTH** might as well be viewed as epistemically less comfortable (hence weaker) than **T** or **F** since hearing that p is false when p is already known to be **T** tends to harm the agent's certainty about the truth of p and to make its falsity dubious as well. One might even consider that **BOTH** is a less comfortable epistemic value than **NONE**, as the simultaneous presence of opposite arguments may lead to more perplexity than the absence thereof. While **NONE** means "in expectation of more information", **BOTH** may mean "in expectation of clarification". So, **BOTH** is an epistemic value where the strength of belief in a proposition is weak.

When a proposition is in the **BOTH** status, an agent should be careful before drawing consequences from it. The idea that inconsistency should prevent inferences, just like the lack of knowledge, is an alternative to the more usual assumption that anything follows in the presence of inconsistency. It stresses the indistinguishability between **NONE** and **BOTH** according to the logical lattice order $\succ$. For instance, the treatment of local inconsistencies in propositional logic using variable forgetting [36] goes along this line of equating contradiction with ignorance, while taking dependence between literals into account.

6 Beyond 4

It seems that a meaningful handling of incompleteness and inconsistency in classical logic cannot be properly achieved by Belnap 4-valued logic, because it does not take into account semantic constraints bearing on Boolean variables in complex formulas. There is no doubt that Belnap logic is correct in its formal development, and that the ensuing bilattice structure is of interest. But the truth-functionality assumption (insofar as the truth-tables are applied without any restriction) is not appropriate when dealing with inconsistency and incomplete knowledge in classical logic. Actually, there is a stream of papers generated by Belnap's logic, and the bilattice structure [9], with applications to logic programming [23] and nonmonotonic reasoning [27]. However, these approaches seem to avoid the difficulties pointed out here because they do not always take truth-functionality for granted. In this section, we briefly outline possible extensions or refinements of Belnap approach and point out to some existing literature.

6.1 Refining epistemic truth-values

The four "epistemic truth-values" are actually not enough to capture the set of possible situations induced by the presence of only two sources of information S_1 and S_2 forming a society semantics [13] for the logic. Indeed, since each source has three possible attitudes (to say 1, 0 or nothing (?)) there are nine situations. Assuming symmetry between (equally reliable) sources yields six possible situations

$$(0,0),(0,1),(0,?),(1,1),(1,?),(?,?)$$

and Belnap assumes four (see also the discussion by Slaney [43]). In fact, **T** stands for any of the two situations $(1,1)$ and $(1,?)$, the former being a stronger epistemic state than the latter. Indeed when sources provide $(1,?)$, one cannot prevent the mute one from supplying 0, eventually. So that **T** represents a very brittle notion of truth. This would call for a many-valued logic with more than 4 truth-values, even if treating the epistemic value $(?,?)$ as a genuine truth-value would still lead to paradoxes.

6.2 Inconsistency-tolerant inference from multiple sources

One may reformulate the problem posed by Belnap in the setting of classical logic as follows: Given a set of (independent) propositional variables $a_1,\ldots,a_p$ informed by n sources, what can be said about the truth-status of a given formula p? The type of answer expected in Belnap setting concerns the potential assertions of the set of sources regarding formulas p, based on information possessed by each source about the truth-value of propositional variables. As previously, a source declares each atom a_i true, false or does not provide information. Each source j is then characterized by a partial model (a coherent situation s_j) based on information on atoms. Then, exploiting the synergy between sources, one of the four following responses (corresponding to the 4 epistemic truth-values) can be expected:

- at least one consistent group of sources asserts p and no other one asserts $\neg p$;
- at least one consistent group of sources asserts $\neg p$ and no other one asserts p;
- there is a consistent group of sources that asserts p and another that asserts $\neg p$;
- no consistent group of source asserts p nor $\neg p$.

It corresponds to a mode of argumentative inference under inconsistency proposed by Benferhat et al [6], applied to the (generally inconsistent) knowledge base $B=\{s_i, i=1,\ldots,p\}$. Denote by C, C' consistent subbases of B (hence of sources). The assignment of epistemic values to complex propositions is then carried out as follows:

- p is considered **T** if $\exists C\models p$, but $C'\models\neg p$ for no C';
- p is considered **F** if $\exists C\models\neg p$, but $C'\models p$ for no C';
- p is considered **BOTH** if $\exists C\models p$, and $\exists C'\models\neg p$;
- p is considered **NONE** if $\nexists C\models p$, and $\nexists C'\models\neg p$.

Clearly, as this approach relies on classical inference steps, classical tautologies will be assigned **T** and contradictions **F**, contrary to what results from the Belnap truth-tables. Moreover, if the procedure assigns **BOTH** to p it does so for its negation, and similarly for **NONE**. Moreover, because the above setting follows a kind of "society semantics" view [13], we do find that if p is **BOTH** and q is **NONE**, where p and q are logically independent, then

$p \wedge q$ will be **F**. Indeed since $\nexists C \models q$, it follows $\nexists C \models p \wedge q$; and since $\exists C \models \neg p$, $C \models \neg p \vee \neg q$ as well. Contrary to what happens with Belnap logic, if p is **F** and q is **F**, it does not imply $p \vee q$ is **F**, because when $\exists C \models \neg p$ and $\exists C' \models \neg q$, maybe $\nexists C'' \models \neg p \wedge \neg q$. Similarly, when p is **T** along with q is **T**, it does not imply $p \wedge q$ is **T**, because when $\exists C \models p$ and $\exists C' \models q$, maybe $\nexists C'' \models p \wedge q$. So, this approach, close to argumentation, seems to avoid the pitfalls of Belnap truth-tables.

6.3 Probabilistic interpretations of Belnap setting

One may also adopt a probabilistic view of the refinement when more than 2 sources are involved. There is indeed an intuitively appealing connection between the epistemic values and (frequentist) probabilities. If we decide to count the number of times an atomic proposition a is said to be true or false by independent external sources, it is clear that **BOTH** could be refined by attaching frequentist probabilities to a and $\neg a$, like in a coin flipping game. Then **NONE** corresponds to the idea of total uncertainty (no toss of the coin made yet, so no probability is available), **T** to the case where the coin has so far only shown heads, **F** to the case where the coin has so far only shown tails. In practical situations **BOTH** can be expected to be by far the most likely situation; it summarizes all different situations where the probabilities are positive on both sides. Then, the other epistemic values might not be very often met in practice. In the same vein as our present criticism, it is well-known that probabilistic logic cannot be construed as a truth-functional many-valued logic [19].

Due to the presence of ignorance as one possible epistemic value, a frequentist extension of Belnap's approach would actually lead to an imprecise probability setting. The weakest probabilistic interpretation of Belnap epistemic values could be as in table 5. The reader can check that the truth-table for the conjunction (of logically independent propositions only!) obtained with such qualitative probabilities (Table 6) differs from Belnap conjunction in several respects: while it confirms that **BOTH**$\cap$**NONE**$=$**F** (since from $0 < Prob(p) < 1$ and $Prob(q)$ unknown, if follows that $Prob(p \wedge q) < 1$), the following discrepancies compared to Belnap truth-tables can be observed:

- **BOTH**$\cap$**BOTH**$=$**F**; indeed, from $0 < Prob(p) < 1$ and $0 < Prob(q) < 1$, it follows that $Prob(p \wedge q) < 1$, since $Prob(p \wedge q) \leq \min(Prob(p), Prob(q))$ only[15].
- **BOTH**$\cap$**T**$=$**F**, since there is less information about q, and from $0 < Prob(p) < 1$ it always follows that $Prob(p \wedge q) < 1$.
- **T**$\cap$**T**$=$**NONE**. Indeed the only knowledge of the positivity of $Prob(p)$ and $Prob(q)$ is not sufficient to ensure any information concerning $Prob(p \wedge q)$.

Note that the epistemic value **BOTH** is not preserved by conjunction. We leave a similar treatment of disjunction to the reader. In fact, the argumentative approach presented in subsection 6.2 is in agreement with Table 6. Just replace $Prob(p) > 0$ by "there is a consistent argument for p". For instance, finding an argument for p and an argument for q does not provide for the existence of an argument for their conjunction. The above considerations cast doubts on the intuitive assumption that the combination of epistemic values **T** and **F** should coincide with the Boolean calculus, letting them behave as 1 and 0.

[15] The lower Frechet bound $\max(0, Prob(p) + Prob(q) - 1)$ is zero in this case.

TABLE 5. Belnap epistemic values as qualitative probabilities

T	**F**	**BOTH**	**NONE**
$Prob(p) > 0$	$Prob(\neg p) > 0$	$1 > Prob(p) > 0$	$1 \geq Prob(p) \geq 0$

TABLE 6. Conjunction of qualitative probabilities

$\cap$	**F**	**NONE**	**BOTH**	**T**
F	**F**	**F**	**F**	**F**
NONE	**F**	**NONE**	**F**	**NONE**
BOTH	**F**	**F**	**F**	**F**
T	**F**	**NONE**	**F**	**NONE**

6.4 Belnap set-up in the belief function framework

In a more informative setting, suppose the number of sources is known to be an integer n. Then, if i sources assert an atomic formula a is provably true, and j sources claim a is false, one may obtain the relative frequencies:

$$f_a(1) = \frac{i}{n}; f_a(0) = \frac{j}{n}; f_a(?) = 1 - \frac{i+j}{n}.$$

By construction $f_a(1)+f_a(0)+f_a(?)=1$. For instance, $f_a(1)$ is the probability that a is provably true, $f_a(?)$ the probability that it is unknown. One may identify this frequency assignment to a mass function m_a in the sense of Shafer [44] on the set of classical truth-values $\{0,1\}$. Namely, using the conventions of possibility theory, m_a assigns weights to $\{1\}$, $\{0\}$, $\{0,1\}$: $m_a(\{1\}) = f_a(1)$, $m_a(\{0\}) = f_a(0)$, $m_a(\{0,1\}) = f_a(?)$. Note that the epistemic values **T** and **F** become consonant belief functions, for which the mass function bears on nested pairs of subsets of truth-values. They correspond to possibility distributions $\pi_a(1) = 1, \pi_a(0) = 1 - \frac{i}{n}$ (when $f_a(0) = 0$, for **T**) and $\pi_a(1) = 1 - \frac{j}{n}, \pi_a(0) = 1$ (when $f_a(1) = 0$, for **F**). **BOTH** corresponds to when $f_a(0) > 0, f_a(1) > 0$, **NONE** to $f_a(?) = 1$.

We can also interpret the frequency assignment as a mass assignment on the set of interpretations Ω, bearing on a, $\neg a$ and $\top$, respectively. Then, we can compute the degrees of belief and plausibility of a as

$$Bel(a) = \frac{i}{n}; Pl(a) = 1 - Bel(\neg a) = 1 - \frac{j}{n}.$$

This model is not new, actually (see Dubois and Prade, [18]).

The problem of evaluating logical formulas can then be posed in a rigourous way. We can assume each propositional variable a_k is informed by all sources in the form of mass functions m_k, for $k = 1 \ldots \mathbf{k}$. A joint mass function can be obtained by merging the $\mathbf{k}$ mass functions on the set of $2^{\mathbf{k}}$ interpretations in Ω. Note that this approach would never assign positive masses to subsets of Ω other than partial (or complete) interpretations, i.e. coherent situations $s \in S$ in the sense of Section 4. For instance using Dempster rule of combination, the mass of a partial interpretation of the form $s = \wedge_{i \in I} a_i \bigwedge \wedge_{i \in J} \neg a_i$, where $I \cap J = \emptyset$ is

$$m(s) = \prod_{i \in I} m_i(a_i) \cdot \prod_{i \in J} m_i(\neg a_i) \cdot \prod_{i \notin I \cup J} m_i(\top).$$

Then, degrees of belief and plausibility of a given formula p can be computed as

$$Bel(p) = \sum_{s \models p} m(s); Pl(p) = \sum_{s \not\models \neg \phi} m(s).$$

This is clearly an extension of the supervaluationist approach to partial logic, where room for inconsistency is naturally provided for by allowing for conflicting sources. Note that the contradiction $\perp$ is, in some sense, tamed as it receives no positive mass while conflicting source opinions $(Bel(p) > 0, Bel(\neg p) > 0)$ are handled, and non-trivial inferences are made.

6.5 Belnap truth-values as modalities

If we stick to a qualitative non-truth-functional approach, it is natural to view **T**, **F**, **BOTH**, **NONE** as modalities rather than truth-values. The following translations are one possible choice:

- a is **T** modelled by $\Box a$,
- a is **F** modelled by $\Box \neg a$.

And as a consequence,

- a is **NONE** modelled by $\neg \Box a \wedge \neg \Box \neg a$
- a is **BOTH** modelled by $\Box a \wedge \Box \neg a$

Since Belnap seems to intend a paraconsistent use of contradiction, a straightforward encoding in some usual modal logics would fail[16]. Indeed:

- in all systems where the axiom $\Box a \to a$ holds, $\Box a \wedge \Box \neg a$ is a contradiction since $\Box a$ true implies that a is true;
- in all systems (such as K) where $\Box a \wedge \Box b$ is equivalent to $\Box(a \wedge b)$, $\Box a \wedge \Box \neg a = \Box \perp$, which does not sound promising for a non-trivial translation of **BOTH**;
- the classical negation $\sim$ applied to $(a, \mathbf{T})$ in Belnap's logic is not coherent with the negation of $\Box a$, which is the modal encoding of $(a, \mathbf{T})$. Explained in terms of modalities: $\sim \Box a \equiv \Box \neg a$ differs from $\neg \Box a$.

Another possible translation[17] is that $\Box a$ means that all sources assert a and $\Diamond a$ means that at least one source asserts a. Hence **BOTH** reads $\Diamond a \wedge \Diamond \neg a$. But then the modeling of **NONE** becomes problematic for symmetric reasons since it is $\neg \Diamond a \wedge \neg \Diamond \neg a$, that is, $\Box a \wedge \Box \neg a = \Box \perp$. Nevertheless, $\Box \perp$ is not a contradiction in the modal system K. This has given rise to a literature on modal logics of inconsistent knowledge (see Meyer and Van der Hoek [38], for a survey of classical references). The multiple-expert modal logics studied by Fitting [24] are also worth pointing out as akin to Belnap's idea of accumulated knowledge from sources, where experts may disagree. Studying the precise positioning of Belnap logic with respect to modal logics of inconsistency would deserve a special study.

[16] A modal paraconsistent logic might be considered. However it is not clear that the distinction between the epistemic and the ontological status of propositions is much emphasised in such logics.

[17] suggested by a referee.

7 Conclusion

Many paradoxes in multivalued logics seem to be due to a confusion between truth-values and the epistemic values an agent may use to describe a state of knowledge : the former are compositional by assumption, the latter cannot be consistently so, at least under a Boolean view of the world[18]. It is clear that modal logic offers a more natural setting than partial logic for reasoning about incomplete information. In the Boolean setting, epistemic issues of ignorance and conflicts cannot be addressed by artificially augmenting the truth-set, that is, changing the domain of propositional variables. Actually, classical tautologies are the only thing a logically omniscient agent may know independently of the information that has been collected. So, in logical approaches to incompleteness and contradiction, the goal of preserving classical tautologies should supersede the goal of maintaining a truth-functional setting, when variables are two-valued. For instance, modal logics for reasoning about incomplete knowledge preserve classical tautologies. Of course, it does not mean that recognizing a complex syntactic entity as being a tautology is always feasible in practice. In any case, since a truth-functional use of the bilattice-based logic does not seem to be convincingly useful for reasoning under incompleteness and contradiction, the question of its domain of application remains pending.

Concerning other inconsistency-tolerant approaches, at least those relying on argumentation based on consistent subsets of a belief base escape the objections raised in this paper against Belnap four-valued logic. Whether these objections may be raised against other inconsistency-tolerant frameworks such as paraconsistent logics [7] or quasi-classical logic [8] is a matter of further research, as is the significance of many-valued truth-functional accounts of intuitionistic logic.

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[18] In fact, even in standard many-valued logics, compositionality does not resist to incomplete knowledge. For instance, in $[0,1]$-valued Lukasiewicz logic, where $t(\neg p) = 1 - t(p)$, it holds that $t(p \wedge \neg p) = \min(t(p), 1 - t(p)) \in [0, \frac{1}{2}]$. However if the truth-value $t(p)$ is only known to belong to some subinterval $[t_1, t_2]$ of the unit interval, the truth-functional calculus yields $t(p \wedge \neg p) \in [\min(t_1, 1 - t_2), \min(t_2, 1 - t_1)]$, not included in $[0, \frac{1}{2}]$, generally. But $\min(t(p), 1 - t(p)) \in [0, \frac{1}{2}]$ is known, due to the involutive negation connective, regardless of whether $t(p)$ is known or not.

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