

# Condorcet's principle and the strong no-show paradoxes

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**Abstract** We consider two no-show paradoxes, in which a voter obtains a preferable outcome by abstaining from a vote. One arises when the casting of a ballot that ranks a candidate in first place causes that candidate to lose the election, superseded by a lower-ranked candidate. The other arises when a ballot that ranks a candidate in last place causes that candidate to win, superseding a higher-ranked candidate. We show that when there are at least four candidates and when voters may express indifference, every voting rule satisfying Condorcet's principle must generate both of these paradoxes.

**Keywords** Voting · No-show · Paradox · Condorcet · Participation

## 1 Introduction

Condorcet's principle, proposed in the eighteenth century by the Marquis of Condorcet, is one of the most important normative principles in the theory of voting. A Condorcet winner is a candidate for election who is preferred by a majority in all pairwise comparisons with the other candidates. Condorcet's principle says that a Condorcet winner must be elected whenever there is one (Condorcet, Marquis de 1785).<sup>1</sup>

However, Moulin (1988) shows that Condorcet's principle entails a surprising and troubling paradox for voting rules, called the no-show paradox. This paradox arises when the addition of a ballot that ranks candidate  $x$  above candidate  $y$  may take victory away from  $x$  and give it to  $y$ .

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<sup>1</sup> A voting rule consistent with this principle was proposed as early as the thirteenth century (see Colomer 2013).

A voting rule that is free from this paradox, so that no voter is made worse off for having voted sincerely rather than abstaining, is said to satisfy the participation principle.<sup>2</sup> Moulin proves that a voting rule cannot satisfy both Condorcet's principle and the participation principle when there are four or more candidates.

If we are to satisfy Condorcet's principle then we must tolerate the no-show paradox. However, we may consider some instances of the paradox to be more severe than others. In that case, we may have reason to prefer some "Condorcet-consistent" voting rules over others, since some of them might at least be free from these severe cases of the paradox.

In this paper we consider two special cases of the no-show paradox. These would seem to be especially bizarre instances. One arises when the casting of a ballot that ranks a candidate in first place causes that candidate to lose the election, superseded by a lower-ranked candidate. The other arises when a ballot that ranks a candidate in last place causes that candidate to win, superseding a higher-ranked candidate. We call these the strong no-show paradoxes after Pérez (2001) and Nurmi (2002). One or both of these special cases of the paradox are also considered by Smith (1973), Richelson (1978), Brams and Fishburn (1983), Saari (1995), and Lepelley and Merlin (2001).<sup>3</sup>

Pérez (2001) demonstrates that Condorcet-consistent voting rules do exist that are free from one or even both of the strong no-show paradoxes, no matter the number of candidates. One of these is the Simpson–Kramer Min–Max rule. That rule is free from both of the strong no-show paradoxes. Young's rule is free from one of the two paradoxes; a ballot that ranks a candidate in last place can never cause that candidate to win under Young's rule. See Pérez (2001) for definitions of both of these rules and several other Condorcet-consistent rules.

### 1.1 Weak orderings

Moulin (1988) considers the aggregation of linear orderings or, in other words, the case where voters do not express indifference. An inspection of Moulin's proof is sufficient to confirm that his impossibility result continues to hold true for the aggregation of weak orderings. It is unsurprising that expanding the domain of voting rules by permitting voter indifference does not lead to a possibility result. Typically, in social choice theory, the emergence of possibility is associated with the contraction of a domain rather than the expansion of one (see Gaertner 2001).<sup>4</sup>

Pérez (2001) considers the aggregation of linear orderings and the aggregation of weak orderings. It is only in the case of linear orderings that he finds compatibility between Condorcet's principle and freedom from the strong no-show paradoxes. He

<sup>2</sup> Logical relations between the participation principle and other monotonicity conditions are examined by Campbell and Kelly (2002), Nurmi (2004), Sanver and Zwicker (2009, 2012), and Felsenthal and Tideman (2013).

<sup>3</sup> Brams and Fishburn note that paradoxes of this kind were remarked upon by the Royal Commission Appointed to Enquire into Electoral Systems (1910, p. 21) and Meredith (1913, p. 93).

<sup>4</sup> Nevertheless, Barberà (2007) shows that the introduction of indifferences can sometimes complicate the statement of both positive and negative results in social choice theory.

**Table 1** A cyclical profile with 15 voters

1	1	1	2	2	2	2	2	2
<i>a</i>	<i>b</i>	<i>c</i>	<i>bc</i>	<i>ac</i>	<i>ab</i>	<i>a</i>	<i>b</i>	<i>c</i>
<i>b</i>	<i>c</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>bc</i>	<i>ac</i>	<i>ab</i>
<i>c</i>	<i>a</i>	<i>b</i>						

shows that this compatibility vanishes when we move from linear to weak orderings. In fact, even when there are just three candidates there is an impossibility. Recall that Moulin’s impossibility result applies just when there are four or more candidates.

Voting rules that are free from the strong no-show paradoxes satisfy conditions called Positive Involvement and Negative Involvement. In the case of weak orderings, Positive Involvement says that if candidate *a* is a winning candidate and we add a voter who ranks *a* in first place then *a* must remain a winning candidate. Negative Involvement says that if candidate *a* is a losing candidate and we add a voter who ranks *a* in last place then *a* must remain a losing candidate.

To help motivate what comes next, we reproduce here Pérez’s proof of this impossibility in the case of weak orderings. Suppose that there are three candidates *a*, *b*, and *c*, and 15 voters. The voters’ preferences are described in Table 1. Each number above the horizontal line indicates the number of voters with the preference ordering given below that number. Letters are written next to each other to indicate indifference. This profile of preferences is “cyclical” in the sense that each candidate pairwise defeats one of the others by a margin of one vote: *a* beats *b*, *b* beats *c*, and *c* beats *a*. Suppose, without loss of generality, that *a* is the unique winning candidate or is tied for victory with one or both of the other candidates. In the case of a tie, all of the tied candidates are said to be winning candidates. A tie-breaking mechanism such as a lottery may be used to elect one from among the winning candidates.

Suppose that we add two new voters who are indifferent between *a* and *c*, and prefer both of those to *b*. Let us write that ordering as  $a \sim c \succ b$ . Then *c* becomes the Condorcet winner. So *a* is no longer a winning candidate, despite *a* being ranked in first place by the new voters.

Now suppose that instead of adding those two voters we exclude the two voters with preference ordering  $b \succ a \sim c$ . Then, again, *c* becomes the Condorcet winner. So when we readmit these two voters *a* becomes a winning candidate despite both voters ranking *a* in last place. This completes the proof of the impossibility.

In that first case, candidate *a* may well object to being removed from the set of winners as a result of the addition of two new voters who rank *a* in first place. However, from the perspective of the new voters, or of a welfarist social planner, it is not clear that there is really a problem. The new voters are indifferent between *a* and *c* and so they are no worse off. Indeed, if the original result was a tie between all three candidates and if a lottery was to be used to break that tie, then the new voters are better off for having voted. They have ensured victory for one of their most preferred candidates, ruling out the chance that *b* might be elected by lottery. Forbidding a scenario in which the new voters are better off for having voted is arguably not in keeping with the spirit of the participation principle.

Let us return to the second scenario where we removed two voters with preference ordering  $b > a \sim c$ . When those two voters abstain  $c$  is the unique winner. When they cast their ballots the set of winning candidates contains  $a$ . These two voters then are no worse off for having voted. Indeed, they may well be better off. Suppose, again, that the result at the 15-voter profile is a three-way tie with a lottery to be used for tie-breaking. Then these two voters benefit from voting. They have given  $b$  a chance of being elected by lottery. An impartial observer may see no reason to object to this change in the outcome, and these voters do not appear to have a strategic incentive to abstain in this case.

## 1.2 Contribution

For simplicity, let us now restrict our attention to deterministic voting rules that always choose a single winner. Perhaps ties are broken by lexicographic order rather than by lottery, for example. Under this restriction, the Positive Involvement condition can be decomposed into two parts. Suppose that  $a$  is the original winner and we add a new voter who ranks  $a$  in first place. Part (i) says that  $a$  must not be superseded by another candidate also ranked in first place by the new voter. Part (ii) says that  $a$  must not be superseded by a candidate ranked below  $a$  by the new voter.

In the proof given above, part (ii) of the Positive Involvement condition is entirely redundant. Perhaps by deleting part (i) and retaining only part (ii), which we might call the welfarist part, we can regain the compatibility with Condorcet's principle that was lost in the move from linear orderings to weak orderings.

Similarly, Negative Involvement can be decomposed into two parts. Suppose that  $a$  is *not* the original winner and we add a new voter who ranks  $a$  in last place. Part (i) says that  $a$  must not take victory from another candidate also ranked in last place by the new voter. Part (ii) says that  $a$  must not take victory from a candidate ranked above  $a$  by the new voter. Again, only part (i) is invoked in the proof argument above.

When part (i) of each of these conditions is deleted we call the resulting conditions Weak Positive Involvement and Weak Negative Involvement. We find that Condorcet's principle is compatible with both of these conditions when there are three candidates, and that it is compatible with neither of them when there are at least four candidates.

The case of linear orderings that Moulin considers is standard in the theory of voting. However, in economic theory more generally it is usual to model preferences by weak orderings. Notably, individual indifference is permitted both in the case of Arrow's theorem (Arrow 1963) and the Gibbard–Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975).

Both of those theorems have sometimes been presented in the literature in their linear-ordering forms. This is done for the sake of simplicity as the differences between the two forms of either theorem are considered to be minor. Yet, it turns out that moving to the weak-ordering setting has an important consequence for Moulin's theorem. When individuals can be indifferent, the requirement that a voter must never be made worse off by voting sincerely than by abstaining is significantly more demanding than is necessary. This requirement can be weakened to either Weak Positive Involvement or Weak Negative Involvement, thereby strengthening Moulin's critique of Condorcet's principle.

## 2 Notation

Let  $A$  be a finite set of candidates, and let  $N_\infty$  be a finite or countably infinite set of potential voters. Every finite subset of  $N_\infty$  is called an electorate. Let  $W(A)$  be the set of all weak orderings on  $A$ . By a weak ordering we mean a binary relation on  $A$  that is transitive and complete.

A *profile* assigns a weak ordering to each voter in an electorate. For every electorate  $N$  there is a set of possible profiles  $W(A)^N$ . We write  $u_{-i}$  to denote the profile obtained by removing individual  $i$  from profile  $u$ .

A voting rule is a function  $S$  that assigns a candidate to every possible pair of electorate and profile. So, given an electorate  $N$  and a profile  $u$  in  $W(A)^N$ ,  $S(N, u)$  is the winning candidate.<sup>5</sup>

When discussing a given electorate and profile, we write  $n_{ab}$  for the number of voters who prefer  $a$  to  $b$  less the number of voters who prefer  $b$  to  $a$ . Let  $m_a$  be the greatest value taken by  $n_{ba}$  over all  $b$  in  $A \setminus \{a\}$ . If  $m_a > 0$  then  $m_a$  is the margin of  $a$ 's greatest pairwise defeat. Candidate  $a$  is a Condorcet winner if and only if  $m_a < 0$ . If  $m_a$  is zero then  $a$  does not suffer any pairwise defeat but does tie with another candidate in a pairwise comparison.

## 3 Results

We first define *Condorcet consistency*.

*Condorcet consistency* For all candidates  $a$ , all electorates  $N$  and all profiles  $u$  in  $W(A)^N$ ,  $m_a < 0$  implies  $S(N, u) = a$ .

Next we define *Weak Positive Involvement* and *Weak Negative Involvement*.

*Weak Positive Involvement* For all electorates  $N$  containing at least two voters, all voters  $i$  in  $N$  and all profiles  $u$  in  $W(A)^N$ , if  $S(N \setminus \{i\}, u_{-i})$  is in the highest indifference class of  $i$ 's preference ordering then so is  $S(N, u)$ .

The Weak Positive Involvement criterion requires that if a candidate ranked in first place by individual  $i$  is elected when  $i$  does not participate, then a candidate ranked in first place by  $i$  should also be elected when  $i$  does participate.

*Weak Negative Involvement* For all electorates  $N$  containing at least two voters, all voters  $i$  in  $N$  and all profiles  $u$  in  $W(A)^N$ , if  $S(N \setminus \{i\}, u_{-i})$  is not in the lowest indifference class of  $i$ 's preference ordering then nor is  $S(N, u)$ .

The Weak Negative Involvement criterion requires that if a new voter  $i$  casts her ballot, bringing us from profile  $u_{-i}$  to profile  $u$ , and if the original winning candidate was not one of  $i$ 's least favorite candidates, then her ballot should not cause one of her

<sup>5</sup> A voting rule as defined here always takes a single candidate as its value. Ties are not permitted in the outcome. This is also the case in [Moulin \(1988\)](#). Moulin's theorem is extended to the case in which ties are permitted in the outcome by [Jimeno et al. \(2009\)](#). Their results are similar to some results found in the corresponding literature on extensions of the Gibbard–Satterthwaite theorem (see [Taylor 2005](#) for an overview of that literature).

least favorite candidates to win. If a candidate ranked in last place by  $i$  was already the winner at the original profile, then the Weak Negative Involvement criterion is silent.

Our first result establishes that, when there are three candidates, deleting part (i) from each of Positive Involvement and Negative Involvement, as discussed above, does achieve compatibility with Condorcet consistency.

**Theorem 1** *If there are three candidates or fewer then Condorcet consistency is compatible with both Weak Positive Involvement and Weak Negative Involvement.*

*Proof* Following [Moulin \(1988\)](#), we give the example of a voting rule that always elects a candidate from the Kramer set (see [Kramer 1977](#)).

Given  $(N, u)$ , let  $K$ , the Kramer set, be the set of all candidates  $a$  that minimize  $m_a$ . That is,  $K = \{a \in A \mid m_a \leq m_b \text{ for all } b \in A\}$ . If  $K$  is a singleton set then voting rule  $S$  elects that candidate in  $K$ . If  $K$  contains more than one candidate then  $S$  elects the candidate in  $K$  whose name comes first by lexicographic order.

If there is a Condorcet winner  $a$  then it follows that  $m_a < 0$  and  $m_b > 0$  for all other candidates  $b$ . Then  $K = \{a\}$  and  $a$  will be elected by  $S$ . So  $S$  is Condorcet-consistent. This is true no matter how many candidates there are.

We now establish that  $S$  satisfies Weak Positive Involvement and Weak Negative Involvement when there are three candidates. Let us label the candidates  $a$ ,  $b$ , and  $c$ . These labels mask the names of the candidates so that their lexicographic ordering by name is unknown to us. Take any electorate and profile. These will be the electorate  $N \setminus \{i\}$  and profile  $u_{-i}$  referred to in the definitions of Weak Positive Involvement and Weak Negative Involvement. Assume without loss of generality that  $S$  elects  $a$  at this profile. It follows that  $m_a \leq m_b$  and  $m_a \leq m_c$ . Let us now form the electorate  $N$  and profile  $u$  by adding a new voter.

We consider three cases. The first case is that the new voter strictly prefers  $a$  to both of the other candidates. The second case is that the new voter strictly prefers  $a$  to exactly one of the other candidates. Just for completeness, the final case is that the new voter does not strictly prefer  $a$  to either of the other candidates.

In the first case, Weak Positive Involvement requires that  $a$  remain the winner. On the other hand, Weak Negative Involvement does not necessarily require that  $a$  must remain the winner. It makes that demand only if  $b$  and  $c$  are in joint last place in the new voter's preference. Otherwise, it makes the milder demand that the candidate ranked last by the new voter does not become the winner. We have not specified who is ranked last by the new voter, but it is not necessary to do so since, as we will see, the winner will remain  $a$ .

In this first case  $m_a$  falls by one. Since we are adding just one new voter, all margins of pairwise defeat/victory can change by at most one. So  $m_b$  and  $m_c$  cannot fall by more than one. In other words,  $m_a$  remains equal to or falls below (or further below) each of  $m_b$  and  $m_c$  as a result of the additional voter. Therefore, the new Kramer set is a subset of the original Kramer set, and must contain  $a$ . Hence,  $S$  elects  $a$  at the new profile, as required.

For the second case, let us assume without loss of generality that  $b$  is the candidate who is weakly preferred to  $a$  by the new voter. It may be that  $a$  and  $b$  are both ranked first, or that  $b$  alone is. If  $a$  and  $b$  are both ranked first then Weak Positive Involvement requires that either  $a$  or  $b$  is the winner at the new profile. If  $b$  alone is ranked first

by the new voter then Weak Positive Involvement makes no requirement at all. On the other hand, regardless of who is ranked first by the new voter, Weak Negative Involvement requires that  $c$  must not become the winner since  $c$  is ranked in last place. So Weak Negative Involvement is either equivalent to or stronger than Weak Positive Involvement for this case. If the winner at the new profile is not  $c$  then both conditions are satisfied.

In this second case  $m_c$  increases by one, and  $m_a$  cannot increase by more than one. In other words,  $m_c$  remains equal to or rises above (or further above)  $m_a$ . So  $c$  can only be in the new Kramer set if that set also contains  $a$  and the original Kramer set contained  $c$  (implying that candidate  $a$ 's name comes before  $c$ 's by lexicographic order). Hence,  $S$  does not elect  $c$  at the new profile, as required.

In the final case both Weak Positive Involvement and Weak Negative Involvement permit any outcome at the new profile. A violation of either condition entails the new voter being made worse off as a result of having voted. In this final case, the new voter ranks  $a$  below or equal to each of the other candidates and, since  $a$  was already the winner at the original profile, this means that the new voter is not made worse off by any change in the outcome.

When there are just two candidates then it is clear that simple majority rule with lexicographic tie-breaking will satisfy Weak Positive Involvement and Weak Negative Involvement. □

Our second result states that when the number of candidates rises above three then no Condorcet-consistent rule can satisfy Weak Positive Involvement. We use 37 potential voters in the proof.

**Theorem 2** *If there are at least four candidates and at least 37 potential voters then Condorcet consistency is incompatible with Weak Positive Involvement.*

*Proof* This proof is based on the proof of statement (ii) in [Moulin \(1988\)](#). Let us assume that  $A$  contains at least four candidates and that  $S$  is Condorcet-consistent and satisfies Weak Positive Involvement. By way of contradiction, assume that  $N_\infty$  contains at least 37 potential voters. We make the following claim. For all distinct candidates  $a$  and  $b$ , every electorate  $N$  and every profile  $u$  in  $W(A)^N$ ,

$$m_a + 1 \leq 37 - |N| \text{ and } m_a + 2 \leq n_{ab} \text{ implies } S(N, u) \neq b. \tag{1}$$

To prove (1), take any electorate  $N$ , a profile  $u$  in  $W(A)^N$ , and candidates  $a$  and  $b$  such that  $m_a + 1 \leq 37 - |N|$  and  $m_a + 2 \leq n_{ab}$  and assume that  $S(N, u) = b$ . Since  $S$  is Condorcet-consistent,  $a$  cannot be a Condorcet winner. Therefore,  $m_a \geq 0$ . Let  $M$  and  $w$  be an electorate and profile obtained by adding  $m_a + 1$  voters to  $(N, u)$ , all of whom rank  $b$  alone in first place and  $a$  alone in second place.

At  $(N, u)$  the greatest margin of defeat suffered by  $a$  is  $m_a$ , and  $a$  defeats  $b$  by a margin of at least  $m_a + 2$ . So the addition of  $m_a + 1$  voters who all rank  $b$  alone in first place and  $a$  alone in second place results in all of  $a$ 's pairwise defeats being reversed, while  $a$  continues to pairwise defeat  $b$ . So candidate  $a$  is a Condorcet winner at  $(M, w)$ . However, since  $b$  is elected at  $(N, u)$ , Weak Positive Involvement requires that  $b$  is elected at  $(M, w)$ . This contradiction establishes (1).

To complete the proof we construct two profiles. Take any four candidates  $a, b, c,$  and  $d$ . The first of the two profiles, let us call it profile  $u^1$ , is described in Table 2. Each number above the horizontal line indicates the number of voters who have submitted the ranking below that number. All other candidates (if there are any) are ranked below  $a, b, c,$  and  $d$  by the voters.

Figure 1 is a directed, weighted graph that indicates the margins of pairwise victory and defeat among the top four candidates. An edge is directed from  $b$  to  $a$  and carries a weight of six to indicate that  $b$  pairwise defeats  $a$  by a margin of six, and so on.

There are 24 participating voters, and we have  $m_a = 6$  and  $n_{ad} = 8$  so, by (1),  $S$  cannot elect  $d$ . We have  $m_d = 8$  and  $m_{db} = 10$  so  $S$  cannot elect  $b$ . For every candidate  $x$  in  $A \setminus \{a, b, c, d\}$  we have  $m_a = 6$  and  $n_{ax} = 24$  so  $S$  cannot elect  $x$ . Hence the winner must be  $a$  or  $c$ .

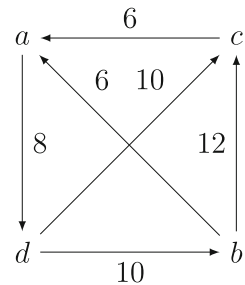
Now let us add eight voters to that first profile to create profile  $u^2$ . All eight of these new voters are indifferent between  $a$  and  $c$ , and rank those two candidates in joint first place. Their next most preferred candidate is  $b$ , followed by  $d$ , and they rank all other candidates (if there are any) below  $d$ .

Now the graph is as shown in Fig. 2.

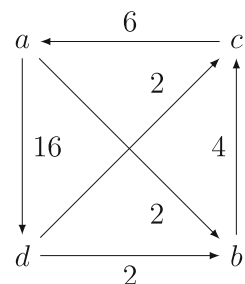
**Table 2** The preferences for profile  $u^1$

	6	3	8	7
$a$	$a$		$d$	$b$
$d$		$d$	$b$	$c$
$c$		$b$	$c$	$a$
$b$		$c$	$a$	$d$

**Fig. 1** The pairwise majority comparisons for profile  $u^1$



**Fig. 2** The pairwise majority comparisons for profile  $u^2$





There are 32 participating voters, and we find that  $m_c = 4$  and  $n_{ca} = 6$ , so, again by (1),  $S$  cannot elect  $a$ . We also have  $m_b = 2$  and  $n_{bc} = 4$ , so  $S$  cannot elect  $c$ . However, Weak Positive Involvement implies that  $S$  must elect  $a$  or  $c$ . This contradiction completes the proof.  $\square$

Next we consider the compatibility of Condorcet’s principle and Weak Negative Involvement. This is not symmetric to the case of Weak Positive Involvement. Recall, for instance, that under Young’s rule with linear orderings, as mentioned in the introduction, a voter may cause her favorite candidate to lose but will never cause her least favorite to win. For the case of weak orderings, however, we find that Weak Negative Involvement contradicts Condorcet’s principle when there are at least four candidates and at least 34 potential voters.

**Theorem 3** *If there are at least four candidates and at least 34 potential voters then Condorcet consistency is incompatible with Weak Negative Involvement.*

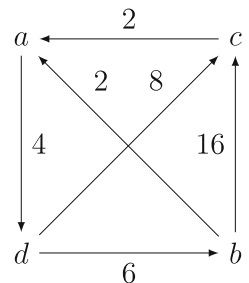
*Proof* Let us assume that there are at least four candidates and at least 34 potential voters and that  $S$  is Condorcet-consistent and satisfies Weak Negative Involvement. Take any four candidates  $a, b, c$ , and  $d$ . A profile is described in Table 3. Let us call this profile  $w^1$ . In this table we use Greek letters to label the six weak orderings that appear in the profile. We say that there are four  $\alpha$  voters, five  $\beta$  voters, and so on. The  $x$  in each column marks the position of all candidates  $x$  in  $A \setminus \{a, b, c, d\}$  (if there are any), and it is written next to another letter to indicate indifference. For example, the four  $\alpha$  voters are indifferent among the candidates in  $A \setminus \{a, b, c\}$ , and rank all of those candidates in last place.

Figure 3 indicates the margins of pairwise victory and defeat among the top four candidates.

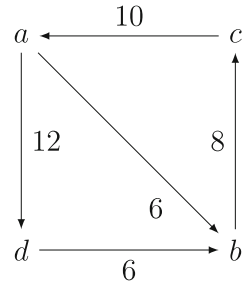
**Table 3** The preferences for profile  $w^1$

$\alpha$	$\beta$	$\gamma$	$\delta$	$\epsilon$	$\zeta$
4	5	1	6	4	6
$b$	$a$	$d$	$a$	$d$	$b$
$c$	$c$	$a$	$d$	$b$	$d$
$a$	$d$	$b$	$b$	$c$	$c$
$dx$	$bx$	$cx$	$cx$	$ax$	$ax$

**Fig. 3** The pairwise majority comparisons for profile  $w^1$



**Fig. 4** The pairwise majority comparisons for profile  $w^2$



There are 26 participating voters. If three of the four  $\alpha$  voters are deleted from profile  $w^1$  then  $a$ 's pairwise defeats to  $b$  and  $c$  are reversed and  $a$  becomes a Condorcet winner. So, by Weak Negative Involvement, a candidate in  $\{a, b, c\}$  must be elected at profile  $w^1$ . If instead we delete the five  $\beta$  voters then  $d$  becomes a Condorcet winner. So  $b$  (ranked in last by those voters) cannot be elected at profile  $u$ . The winner must be  $a$  or  $c$ .

Now let us add eight voters to create a final profile  $w^2$ . All eight of these new voters rank  $c$  alone in first place,  $a$  alone in second place, and rank all other candidates in joint last place.

Now the graph is as shown in Fig. 4 (there is a pairwise tie between  $c$  and  $d$ ).

There are 34 participating voters. If we delete the four  $\epsilon$  voters and five of the six  $\zeta$  voters then  $c$  becomes a Condorcet winner. So  $a$  cannot be elected at profile  $w^2$ . If instead we delete the single  $\gamma$  voter and the six  $\delta$  voters then  $b$  becomes a Condorcet winner. So  $c$  cannot be elected at profile  $w^2$ . However, Weak Negative Involvement implies that  $S$  must elect  $a$  or  $c$ . This contradiction completes the proof.  $\square$

As a final remark, we note that the numbers of potential voters in Theorems 2 and 3 may not be minimal. We do not know what upper bound is imposed on the number of potential voters by the conjunction of Condorcet consistency and Weak Positive Involvement (or Weak Negative Involvement) when there are four candidates. Indeed, the same goes for the conjunction of Condorcet consistency and the participation principle in the case of linear orderings. And these upper bounds may fall as the number of candidates rises. These are important open problems since voting is often conducted by small groups of individuals.

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