Badiou and Philosophy

Edited by Sean Bowden and Simon Duffy

EDINBURGH
University Press

Edinburgh University Press Ltd
22 George Square, Edinburgh EH8 9LF

www.euppublishing.com

Typeset in 11/13 Adobe Sabon
by Servis Filmsetting Ltd, Stockport, Cheshire, and
printed and bound in Great Britain by
CPI Group (UK) Ltd, Croydon, CR0 4YY

A CIP record for this book is available from the British Library

ISBN 978 0 7486 4352 3 (hardback)
ISBN 978 0 7486 4351 6 (paperback)
ISBN 978 0 7486 4353 0 (webready PDF)
ISBN 978 0 7486 6833 5 (epub)
ISBN 978 0 7486 6834 2 (Amazon ebook)

Contents

4

Badiou's Platonism: The Mathematical Ideas of Post-Cantorian Set Theory

Simon Duffy

Plato's philosophy is important to Badiou for a number of reasons, chief among which is that Badiou considered Plato to have recognised that mathematics provides the only sound or adequate basis for ontology. The mathematical basis of ontology is central to Badiou's philosophy, and his engagement with Plato is instrumental in determining how he positions his philosophy in relation to those approaches to the philosophy of mathematics that endorse an orthodox Platonic realism, i.e. the independent existence of a realm of mathematical objects. The Platonism that Badiou makes claim to bears little resemblance to this orthodoxy. Like Plato, Badiou insists on the primacy of the eternal and immutable abstraction of the mathematico-ontological Idea; however, Badiou's reconstructed Platonism champions the mathematics of post-Cantorian set theory, which itself affirms the irreducible multiplicity of being. Badiou in this way reconfigures the Platonic notion of the relation between the one and the multiple in terms of the multiple-without-one as represented in the axiom of the void or empty set. Rather than engage with the Plato that is figured in the ontological realism of the orthodox Platonic approach to the philosophy of mathematics, Badiou is intent on characterising the Plato that responds to the demands of a post-Cantorian set theory, and he considers Plato's philosophy to provide a response to such a challenge. In effect, Badiou reorients mathematical Platonism from an epistemological to an ontological problematic, a move that relies on the plausibility of rejecting the empiricist ontology underlying orthodox mathematical Platonism. To draw a connection between these two approaches to Platonism and to determine what sets them radically apart, this paper focuses on the use that they each make of model theory to further their respective arguments.

Orthodox Platonism in Mathematics and Its Problems

Orthodox Platonism in mathematics advances an ontological realism according to which mathematical objects, like numbers, functions and sets, exist. These mathematical objects are considered to be abstract, causally inert and eternal. The problem that accompanies orthodox Platonism is an epistemological one. If mathematical objects are causally inert, how do we know anything about them?[1] Any such knowledge would require epistemic access to an acausal, eternal and detached mathematical realm.

The epistemic problem for realism in mathematics presumes something like a causal theory of knowledge, according to which claims to knowledge of particular objects is grounded in some account of the causal link between knower and object known. While this empiricist framework may account for knowledge of ordinary objects in the physical world, this sets up a problem for the orthodox Platonist as it doesn't account for knowledge of mathematical objects.

A further issue that can be raised is the question of the applicability of the abstract mathematical realm to the ordinary physical world. Generally, mathematics is applied when a given area of the physical world is postulated as exemplifying a certain mathematical structure. In nearly all scientific theories, the structures of physical systems are described or modelled in terms of mathematical structures.[2] But this doesn't explain how the eternal, acausal, detached mathematical universe relates to the material world, which is the subject matter of science and everyday language. The challenge to the orthodox mathematical Platonist is to provide an account of how it is that mathematical knowledge is utilised or deployed in scientific discourse, and of how it seems to function as an essential part of it.

One realist approach, which begins with the latter problem of the relation between mathematics and science in order to attempt to provide a response to the epistemic problem, is that presented in the Quine-Putnam indispensability argument. Quine and Putnam considered mathematics to be indispensable for science, and, on the basis of the understanding that the best scientific theories determine what one ought to believe to exist, it follows that one ought to believe that the mathematical entities implicated in these theories exist.[3] While this approach does seem to provide a response to the epistemic problem, it fails to address the issue of exactly how mathematics can be applied to science, that is while noting the indispensability of mathematics for science, it fails to provide an account of the nature of this relation. The response to the epistemic problem provided by the indispensability argument can therefore not be sustained, or at least, from a realist perspective, not until an adequate response is provided to the question of the nature of this relation.[4]

One way of addressing the nature of this relation is to actually attempt to provide a uniform semantics for both mathematical and scientific languages, rather than merely presuming this to be the case which is all that is required for the indispensability argument. This could be achieved by developing a model-theoretic framework according to which the relationship between mathematical language and mathematical reality is modelled on the relationship between a formal language and model-theoretic interpretations of it. The point is that if realism is correct, then model theory provides the picture, or 'model', of how mathematical languages describe mathematical reality.

Model theory is the branch of logic developed to study (or model) mathematical structures by considering first-order sentences which are true of those structures and the sets which are definable in those structures by first-order formulas.[5] In model theory, there are three different languages that are in operation: (1) the mathematical language itself, which is informal; (2) the object language, which is the set of first-order sentences of a formal language that 'models' the first; and (3) the metalanguage, which is the informal or semi-formalised language in which the semantics is carried out, i.e. it is the language used to describe what is happening in the object language. The assumption is that standard first-order sentences of a formal language capture something about real mathematical languages. A first-order sentence is a formula that has well-defined truth values under an interpretation. For example, given the formula $P(x)$, which states that the predicate P is true of x, whether $P(x)$ is true depends on what x represents, and the first-order sentence $\exists x P(x)$ will be either true or false in a given interpretation. An interpretation of the set of sentences of a first-order language assigns a denotation to all non-logical constants in that language, for example what is denoted by P. It also determines a domain of discourse that specifies the range of the universal ($\forall$) and existential ($\exists$) quantifiers, where the domain of discourse generally refers to the set of entities that a model is based on. The result is that each term, x, is assigned an object that it represents, and each sentence, for example $\exists x P(x)$, is assigned a truth value. In this way, a model-theoretic interpretation determines the satisfaction conditions for the formal sentences and thereby provides semantic meaning to the terms and formulas of the language.[6] The metalanguage, which is a 'fully developed language',[7] must contain a faithful representation of the object language and should have the resources to make substantial assertions about the ontology that is attributed to the object language. In this way, the central notion of model theory is 'truth in a model'. The conditions for truth in the proposed model represent truth conditions, and it follows that truth in a model is a model of truth. What this means is that the truth of the existence of mathematical objects in the model, or in the object language, is a model of the truth of the existence of mathematical objects for the mathematical language itself. One criticism of this approach is that the best that can be achieved is that all models of a theory are isomorphic, in which case the ontology is only determined up to isomorphism, i.e. metaphysical realists do not really have any access to the correspondence they postulate.[8]

The structuralist approach to the programme of realism in the philosophy of mathematics, represented in the work of Stewart Shapiro, draws upon Plato to set up a response to this criticism, a response which is an extension of the model-theoretic approach. Shapiro argues that Plato distinguishes between two different approaches to natural numbers: arithmetic and logistic. Arithmetic 'deals with the even and the odd, with reference to how much each happens to be'.[9] According to Plato, if 'one becomes perfect in the arithmetical art', then 'he knows also all of the numbers.'[10] Logistic differs from arithmetic 'in so far as it studies the even and the odd with respect to the multitude they make both with themselves and with each other.'[11] So while arithmetic deals straightforwardly with the natural numbers, Shapiro argues that theoretical logistic concerns 'the relations among the numbers'.[12] Drawing upon the work of Klein, who argues that theoretical logistic 'raises to an explicit science that knowledge of relations among numbers which . . . precedes, and indeed must precede, all calculation,'[13] Shapiro argues that 'the structuralist rejects this distinction between Plato's arithmetic and theoretical logistic.' He maintains that 'there is no more to the individual numbers "in themselves" than the relations they bear to each other.'[14] Shapiro turns to the *Republic* to find the ultimate Platonic endorsement of this move. He argues that 'in the *Republic* (525C–D), Plato said that guardians should pursue *logistic* for the sake of knowing. It is through this study of the *relations* among numbers that their soul is able to grasp the nature of numbers as they are in themselves. We structuralists agree.'[15]

In order to overcome the criticism of the problem of isomorphism in the model-theoretic framework, the structuralist program of realism in the philosophy of mathematics deploys the model-theoretic framework in relation to the problem of mathematical structures, which it can more directly address. In this respect, as Shapiro argues, 'Structure is all that matters.'[16] Mathematical objects are defined as structureless points or positions in structures that have no identity or features outside of a structure. And a structure is defined as the abstract form of a system, which highlights the interrelationships among its objects.[17] The aim of Shapiro's structuralist approach is to develop a language in which to interpret the mathematics done by real mathematicians, which can then be used to try to make progress on philosophical questions.

The 'Modern Platonist' Response and Its Reformulation of the Question

Another avowedly Platonic approach that redeploys the model-theoretic framework is that provided by Alain Badiou in *Being and Event*, and subsequently elaborated upon in *Logics of Worlds*. The main point of distinction between the approaches of Badiou and Shapiro that sets their projects apart and at odds with one another is that Badiou rejects the empiricist framework that characterises the epistemic problem for the orthodox Platonist.

Badiou considers himself to be a 'modern Platonist' (TW 54), and draws upon three crucial aspects of Plato's work to set up this transformation.

First, Badiou maintains that 'the independent existence of mathematical structures is entirely relative for Plato' (TW 49), the claim being that Plato's account of anamnesis[18] does not set up the 'criterion of the exteriority (or transcendence) of mathematical structures (or objects)' (TW 49). On the contrary, it designates that 'thought is never confronted with "objectivities" from which it is supposedly separated' (TW 49). Badiou considers a mathematical structure to be an 'Idea' that is 'always already there and would remain unthinkable were one not able to "activate" it in thought' (TW 49). He maintains that 'Plato's fundamental concern is to declare the immanent identity, the co-belonging, of the knowing mind and the known, their essential ontological commensurability' (TW 49). So the problem for Badiou in this respect is to provide an account of how these Ideas are activated in thought, which is facilitated by providing an account of this 'essential ontological commensurability'.

Second, Badiou reinterprets the famous passage in the *Republic* where Plato opposes mathematics to the dialectic.

> The theorizing concerning being and the intelligible which is sustained by the science [*épistémè*] of the dialectic is clearer than that sustained by what are known as the sciences [*techné*] . . . It seems to me you characterize the [latter] procedure of geometers and their ilk as discursive [*dianoia*], while you do not characterize intellection thus, in so far as that discursiveness is established between [*metaxu*] opinion [*doxa*] and intellect [*nous*].[19]

In this passage, Plato singles out the procedures of the geometer, having in mind here the axioms of Euclidian geometry, as operating externally to the norms of thought, i.e. the dialectic. Badiou's modern move here is to embrace the axiomatic approach specifically because of this externality, which addresses that aspect of the problem mentioned above of how these Ideas are activated in thought. Badiou here also reveals his formalist leanings by endorsing the understanding that the theorem follows logically from its axioms, although it is a formalism without the implicit finitism that accompanies its usual presentation in the philosophy of mathematics as the manipulation and interpretation of finite sequences of symbols.

Third, in the *Parmenides* Badiou notes with approval what he considers to be the formulation, in the account of a speculative dream, of 'being' as pure or inconsistent multiplicity [*plethos*] (BE 34). However, he considers Plato to capitulate to the fact that 'there is no form of object for thought which is capable of gathering together the pure multiple, the multiple-without-one, and making it consist' (BE 34). The multiple, in this respect, can only be thought in terms of the One, and thus as consistent or structured multiplicity [*polla*]. Plato writes: 'It is necessary that the entirety of disseminated being [as inconsistent multiplicity] shatter apart, as soon as it is grasped by discursive thought' (BE 34). Badiou considers this to be where Plato is pre-modern, by which he specifically means pre-Cantorian, because it is Cantor who was the first to 'elucidate the thinking of being as pure multiplicity' (TW 55), an account of which will be given in the next section. In order to maintain the distinction between the two types of multiplicity, *plethos* and *polla*, Badiou suggests transcribing Plato's statement: 'If the one is not, nothing is', to 'If the one is not, (the) nothing is' (BE 35). This then aligns the Platonic text with the 'axiomatic decision' with which Badiou's 'entire discourse originates': 'that of the non-being of the one' (BE 31). According to Badiou, 'under the hypothesis of the non-being of the one, there is a fundamental asymmetry between the analytic of the multiple and the analytic of the one itself' (BE 32). It is only in relation to the 'non-being of the one' that multiplicity as pure or inconsistent, the multiple-without-one, is presentable. In axiomatic set theory, which is the first-order formal language that Badiou deploys in his model theoretic approach, the 'non-being of the one' is characteristic of the void or empty set, $\varnothing$ (BE 69).

In support of these moves, and of the claim that the status of mathematical objects is a secondary problem, Badiou draws upon comments made by Kurt Gödel about axiomatic set theory and Cantor's continuum hypothesis:

> The question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis.[20]

With this, Badiou positions Cantor's continuum hypothesis, and the development of transfinite numbers that underpins it, as of central importance to his approach. Badiou argues that

> With Cantor we move from a restricted ontology, in which the multiple is still tied to the metaphysical theme of the representation of objects, numbers and figures, to a general ontology, in which the cornerstone and goal of all mathematics becomes thought's free apprehension of

> multiplicity as such, and the thinkable is definitively untethered from the restricted dimension of the object. (TW 46)

Badiou characterises this 'general ontology', which is nothing other than pure multiplicity, as 'being qua being', and, on the basis of Cantor's account of transfinite numbers, maintains that 'it is legitimate to say that ontology, the science of being qua being, is nothing other than mathematics itself' (BE xiii). Badiou then presents this 'general ontology' as modelled by the Zermelo-Fraenkel axiomatisation of set theory (abbreviated ZF) and the open series of extensions of them, including in particular those by Gödel and Paul Cohen. In response to Quine's famous formula: 'to be is to be the value of a variable',[21] Badiou responds that 'the ZF system postulates that there is only one type of presentation of being: the multiple' (BE 44). He maintains that 'mathematical "objects" and "structures" . . . can *all* be designated as pure multiplicities built, in a regulated manner, on the basis of the void-set alone' (BE 6), and that '[t]he question of the exact nature of the relation of mathematics to being is therefore entirely concentrated – for the epoch in which we find ourselves – in the axiomatic decision which authorizes set theory' (BE 6). In order to characterise this axiomatic decision, an account of the development of transfinite numbers, which Badiou considers 'to prompt us to think being qua being' (NN 98), is required.

Cantor's Account of Transfinite Numbers or Ordinals

To begin with, an ordinal number describes the numerical position or order of an object, for example first, second, third, etc., as opposed to a cardinal number which is used in counting: one, two, three, etc. An ordinal number is defined as 'the order type of a well ordered set'.[22] There are finite ordinals, denoted using Arabic numerals, and transfinite ordinals, denoted using the lower case Greek letter ω (omega). While the ordinality and cardinality of finite sets is the same, this is not the case with transfinite ordinals and cardinals, as will be explained shortly. It was Cantor who developed transfinite ordinals as an extension of the whole numbers, i.e. transfinite ordinals are larger than any whole number. The smallest transfinite ordinal ω, is the set of all finite ordinals {0, 1, 2, . . .}, which is the countably infinite set **N** of natural numbers.[23] The cardinality of this set is denoted $\aleph_0$

(aleph-0).[24] Note that the cardinality of **Z**, the integers, and **Q**, the rational numbers, is also $\aleph_0$, whereas **R**, the set of real numbers, is uncountably infinite, and its cardinality is denoted by *c*, which is called the 'continuum' in set theory. Because **R** is the power set of **Z**, where the power set of any set is the set of all of its subsets, and because every set of size or cardinality *n* has a power set of cardinality 2^n, then $c = 2^{\aleph_0}$. While there is only one countably infinite cardinal, $\aleph_0$, there are uncountably many countable transfinite ordinals, because like other kinds of numbers, transfinite ordinals can be added, multiplied and exponentiated:[25]

$$\omega, \omega + 1, \omega + 2, \ldots, \omega \times 2, (\omega \times 2) + 1, \ldots, \omega^2, \omega^2 + 1, \ldots, \omega^3, \ldots, \omega^{\omega}, \ldots, \omega^{\omega^{\omega}}, \ldots, \varepsilon_0, \ldots$$

The cardinality of the ordinal that succeeds all countable transfinite ordinals, of which there are uncountably many, is denoted $\aleph_1$ (aleph-1).[26] Each ordinal is the well-ordered set of all smaller ordinals, i.e. every element of an ordinal is an ordinal. Any set of ordinals which contains all the predecessors of each of its elements has an ordinal number which is greater than any ordinal in the set, i.e. for any ordinal α, the union $\alpha \cup \{\alpha\}$ is a bigger ordinal $\alpha + 1$. For this reason, there is no largest ordinal. The ordinals therefore 'do not constitute a set: no multiple form can totalise them' (NN 98). What this means for Badiou is that the ordinals are the ontological schema of pure or inconsistent multiplicity.

Badiou argues that '[t]he anchoring of the ordinals in being as such is twofold' (NN 98). (1) The 'absolutely initial point . . . is the empty set,' which is an ordinal, and is 'decided axiomatically' as the empty set, $\varnothing$. In ZF, the axiom of the void or empty set states that the empty set exists. As the 'non-being of the one', the empty set provides set theory with its only existential link to being and thereby grounds all the forms constructible from it in existence. Badiou defers here to Zermelo's axiom of separation, which states that 'if the collection is a sub-collection of a given set, then it exists.'[27] Rather than using this axiom to prove the existence of the empty set by specifying a property that all sets do not have, which would be the orthodox Platonist approach since all sets already exist, Badiou argues that in order for the axiom of separation to separate some consistent multiplicity as a sub-collection, some pure multiple, as the multiple of multiples,[28] must already be presented, by which Badiou means the initial multiple, the empty set, which is guaranteed rather by the axiom of the empty set (BE 45). (2) 'The limit-point that "relaunches" the existence of the ordinals beyond . . . the whole natural finite numbers . . . is the first infinite set, ω,' which is also 'decided axiomatically'. The axiom that formalises the infinite set representing the natural numbers, **N**, is the axiom of infinity, which states that there exists an infinite set. These two axiomatic decisions, which Badiou considers to be crucial for modern thought, represent the ordinals as 'the modern scale of measurement' of pure or inconsistent multiplicity. He maintains that these two decisions determine that nothingness, the empty set, 'is a form of . . . numerable being, and that the infinite, far from being found in the One of a God, is omnipresent', as pure or inconsistent multiplicity, 'in every existing-situation' (NN 99). Before clarifying what Badiou means here by 'every existing-situation', which is dependent upon the model-theoretic implications of his approach, the Platonist implications of axiomatic set theory that Badiou is drawing upon require further explication.

The Platonist Implications of Axiomatic Set Theory

ZF and the extensions of it by Gödel and Cohen allow the Cantorian theory to be developed in full while avoiding all known paradoxical constructions, the simplest of which is Russell's paradoxical set of all sets, which Cantor called an inconsistent or absolutely infinite set.[29] The main problem left unanswered by Cantor's theory of transfinite numbers is the hypothesis, which tried to make sense of these inconsistent or absolutely infinite sets, referred to as the continuum hypothesis (abbreviated CH). CH proposes that there is no infinite set with a cardinal number between that of the 'small' countably infinite set of integers, denoted $\aleph_0$, and the 'large' uncountably infinite set of real numbers, denoted $2^{\aleph_0}$. CH therefore asserts that $\aleph_1 = 2^{\aleph_0}$, where $\aleph_1$ is the cardinality of the ordinal that succeeds all countable transfinite ordinals. Cantor believed CH to be true and spent many fruitless years trying to prove it. If CH is true, then $2^{\aleph_0}$ is the first cardinal larger than $\aleph_0$. However, independently of whether or not CH is true, the question remains as to whether such a cardinal $2^{\aleph_0}$ exists. Cantor argues for the existence of $2^{\aleph_0}$ by invoking the well-ordering principle (abbreviated WO), which simply states that a set is said to be well-ordered by a relation $<$ (less than) of ordering between its elements if every non-empty subset has a first element. This argument implies that every set can be well ordered and can therefore be associated with an ordinal number. The problem with Cantor's argument is that it assumes there to be a method for making an unlimited number of successive arbitrary choices for each subset to determine this first member. If the set is the set **N**, then there is no problem, since the standard ordering of **N** already provides well-ordering. But if the set is **R**, there is no known method to make the required choice. The assumption of the existence of such an infinite sequence of choices was considered by many to be unjustified.[30] In response to this problem, Zermelo provided a proof of WO on the basis of the axiom of choice (abbreviated AC, and indicated by the 'C' in ZFC), which proposes a function that provides for 'the *simultaneous choice* from each nonempty subset' of the first element.[31] This axiom 'reduces the construction of a transfinite sequence of successive choices', which in Cantor's argument appear to proceed through time, 'to the assumption of a single simultaneous collection of choices',[32] The main problem with AC for many mathematicians was that it presupposed the independent existence of the function that it proposes, i.e. it asserts existence without explicitly defining the function as a mathematical object and thus lays the axiomatic grounds for orthodox mathematical Platonism in set theory and the problems outlined above associated with it.

While a committed Platonic realist in the philosophy of mathematics who 'conceives sets to be arbitrary collections of entities existing independently of human consciousness and definitions' would consider AC to be 'immediately intuitively evident',[33] Badiou, on the contrary, considers the acceptance of AC to be solely the result of an axiomatic decision, the reasons for which will become evident once more of the history of dealing with CH is presented. So while both Badiou and the orthodox Platonist accept AC, and therefore that the cardinal $2^{\aleph_0}$ exists, the question that remains to be addressed is whether or not CH is true.

In 1937, Gödel proved that if ZF is consistent then it remains consistent if AC and the generalised continuum hypothesis (abbreviated GCH) are added to it as axioms. The GCH states that if an infinite set's cardinality lies between that of an infinite set and that of its power set, then it either has the same cardinality as the infinite set or the same cardinality as its power set. This is a generalisation of CH because the continuum, **R**, has the same cardinality as the power set of integers, **Z**. Gödel also introduced the notion of 'constructible set' to show that when the universe of sets, V, is restricted to the class of constructible sets, L, i.e. when V = L, then all the axioms of ZFC and GCH are proved.[34] What this consistency result showed was that any instance of GCH could not be disproved using ZFC.

The notion of constructible sets is problematic for the orthodox Platonist as the restriction to definable objects is contrary to the conception of an independently existing universe of arbitrary sets. Most Platonists would therefore reject V = L and the proof that relies on it. Badiou, on the contrary, affirms Gödel's notion of constructible sets, i.e. L, as another necessary axiomatic decision and the result that follows. Badiou argues that by 'considering constructible multiples *alone*, one stays within the framework of the Ideas of the multiple' (BE 300) elaborated above.[35]

This result, that GCH could not be disproved using ZFC, did not rule out that some instance of GCH could be proved in ZFC, even CH itself;[36] however, Gödel projected that CH would be independent or could not be derived from ZFC and that 'new axioms' might be required to decide it.[37]

Progress on this problem was not made until 1963 when Paul Cohen[38] proved that if ZF is consistent then: (1) AC is independent or cannot be derived from ZF; (2) CH is independent from ZFC; and (3) V = L is independent of ZFC + GCH.[39] The proof effectively showed that CH does not hold in all models of set theory. The technique he invented and called the method of forcing and generic sets involved building models of set theory. This method takes its point of departure in that used by Gödel. Rather than produce only one model by restricting a presumed model of set theory, V, to obtain that of the constructible sets, L, Cohen extended the model of constructible sets, L, by the adjunction of a variety of generic sets without altering the ordinals.[40] In fact, he adjoined sufficiently many generic subsets of $\omega = \{0, 1, 2, \ldots\}$ that the cardinality of this constructed model of ZFC, $\aleph_1$, was greater than $\aleph_0$ but less than c, thus violating CH.

The procedure of forcing starts with a countable transitive model M for any suitable finite list of axioms of ZFC + V = L.[41] The method of forcing is then used to construct a countable transitive model G, called a *generic extension* of M, for a finite list of axioms of ZFC + V = L, such that M contains G, abbreviated as M[G]. M[G] is 'the set of all sets which can be constructed from G by applying set-theoretic processes definable in M'.[42] As long as M doesn't equal G, G will satisfy $V \neq L$. G can also be made to satisfy ¬CH and 'a wide variety of other statements by varying certain details in [the] construction'.[43] While Gödel's method of constructibility established the consistency of statements true in L, specifically GCH, Cohen's method of forcing 'is a general technique for producing a wide variety of models satisfying diverse mathematical properties.'[44] It has since become the main method for showing statements to be independent of ZF or ZFC. Cohen's independence results are the basis of Badiou's claim that AC and $V = L$ are 'axiomatic decisions', as they are undecidable within the framework of ZF or of ZFC + GCH respectively. As for CH, it is 'demonstrable within the constructible universe, and refutable in certain generic extensions. It is therefore undecidable for set theory without restrictions' (BE 504).

Building on Cohen's work, Easton[45] shows that for each regular transfinite cardinality of a set, the cardinality of its power set can be any cardinal provided that it is superior to the first and that 'it is a successor cardinal' (BE 279), where a successor cardinal is the smallest cardinal which is larger than the given cardinal.[46]

Consonant with Gödel's projection, a number of 'new axioms' called strong axioms of infinity, or large cardinal axioms, are candidates or have been newly proposed in the attempt to decide CH. These include the axioms that assert the existence of inaccessible cardinals, or Mahlo cardinals, and stronger axioms for the existence of measurable cardinals, compact cardinals, supercompact cardinals, huge cardinals.[47] What the large cardinal axioms attempt to do is 'to constitute within the infinite an abyss comparable to the one which distinguishes the first infinity, ω_0, from the finite multiples' (BE 311). It is in this way that the large cardinal axioms are considered to be 'strong axioms of infinity'. However, for each of these axioms, if it has been shown to be consistent with ZFC then it remains consistent regardless of whether CH or ¬CH is added. That is, 'CH is consistent with and independent from every large cardinal axiom that has been proposed as at all plausible.'[48] What this means is that 'none of them quite succeed' in deciding CH.

On a purely formal level, Kanamori and Magidor argue that interest in large cardinal axioms lies in the 'aesthetic intricacy of the net of consequences and interrelationships between them'. However, they go further to suggest that the adaptation of large cardinal axioms involves 'basic questions of belief concerning what is true about the universe,' and can therefore be characterised as a 'theological venture'.[49] Badiou endorses this suggestion and incorporates large cardinal axioms into his approach as approximations of the 'virtual being required by theologies' (BE 284).

The Model-Theoretic Implications of Badiou's 'Modern Platonism'

The definitive statement of Badiou's model-theoretic orientation in *Being and Event* is in the chapter on the 'Theory of the Pure Multiple', where he effectively states that 'the object-language (the formal language) . . . which will be that of the theory in which I operate' (BE 39) is axiomatic set theory, specifically ZFC, including, as indicated above, Gödel's axiom of constructibility, V = L. What this means is that the object-language that Badiou deploys is already itself a model of ZFC insofar as the acceptance of V = L, which in Cohen's terminology is the model M, indicates Badiou's decision to solely accept the existence of constructible sets, or as Badiou refers to them, 'constructible multiples' (BE 306). So Badiou's object-language already implicates the model M of ZFC that is determined in the first stage of the procedure of Cohen's method of forcing and generic sets.

The metalanguage with which Badiou discusses the object-language and that has the resources to make substantial assertions about the ontology attributed to the object language is the 'fully developed language' of philosophy itself, specifically Badiou's philosophy, which he refers to as a metaontology. For Badiou, mathematics doesn't recognise that it is ontology – this is left up to philosophy itself whose task is to explain how it is that mathematics is ontology.

The model-theoretic interpretations of the object language are the very generic extensions generated by Cohen's method of forcing, which constructs a *generic extension* G of M such that M contains G, i.e. M[G]. Cohen's generic extensions themselves are unknowable from the model M of which they are extensions, thus furnishing Badiou with the concept of the indiscernible multiple. This distinction between the indiscernible multiples of the generic extensions and the constructible multiples of M is also characteristic of their evental nature, in so far as 'the event does not exist' and is not decided (BE 305) in the latter but is decided and is a condition of the former. Badiou therefore characterises generic sets, indiscernible multiples, as the 'ontological schema of a truth' (BE 510). A procedure of fidelity to the truth of an indiscernible multiple is a *generic procedure* of which Badiou lists four types: artistic, scientific, political and amorous. He characterises these generic procedures as 'the four sources of truth' (BE 510). In addition to the role as metalanguage to the object language is the role of philosophy 'to propose a conceptual framework in which the contemporary compossibility' of these generic procedures 'can be grasped' (BE 4). These generic procedures are therefore characterised by Badiou as the conditions of philosophy. This marks an abrupt shift from talking about the sets of the model M as constructible multiples, to talking about specific constructible multiples, or as Badiou refers to them, 'situations' (BE 178), that are presentable by the model and its generic extensions. This is, however, consonant with Badiou's reorientation of the epistemic problem of the orthodox Platonist in mathematics. By claiming that mathematics is ontology, Badiou reorients the debate from an epistemological question about the nature of the relation between mathematical language and mathematical objects to an ontological question about how being is thought and how mathematics is implicated in this question. Badiou maintains that it 'is nothing new to philosophers – that there must be a link between the existence of mathematics and the question of being' (BE 7), and he singles out 'the Cantor-Gödel-Cohen-Easton symptom' (BE 280) of mathematics as providing the impetus for rethinking the nature of this link.

In regards to the orthodox epistemic problem, Badiou refuses the reduction of the subject matter of mathematics to the status of objects on the model of empirical objects. In *Being and Event*, he maintains that:

> If the argument I present here holds up, the truth is that *there are no* mathematical objects. Strictly speaking, mathematics *presents nothing*, without constituting for all that an empty game, because not having anything to present, besides presentation itself – which is to say the Multiple –, and thereby never adopting the form of the object, this is certainly a condition of all discourse on being *qua being*. (BE 7)

He rather draws upon Plato's account of anamnesis to reinstate mathematical objects to the status of Ideas. He argues that: 'A mathematical idea is neither subjective ("the activity of the mathematician"), nor objective ("independently existing structures"). In one and the same gesture, it breaks with the sensible and posits the intelligible. In other words, it is an instance of thinking' (TW 50). Badiou draws upon Cohen's deployment of Gödel's idea of constructible sets to characterise what he refers to as 'the being of configurations of knowledge' (BE 284). Badiou argues that the axiom of constructibility is 'a veritable "Idea" of the multiple' and that the constructible universe that is a 'model' of the ZFC + V = L axioms is 'the framework of the Ideas of the multiple' (BE 426). It is the axioms of 'the Cantor-Gödel-Cohen-Easton symptom' (BE 280) that present this framework, and it is philosophy as metaontology that articulates how this framework should be thought in relation to the generic procedures. For this reason, Badiou maintains that:

> Mathematical ontology does not constitute, by itself, any orientation in thought, but it must be compatible with all of them: it must discern and propose the multiple-being which they have need of. (BE 284)

The ontology that Badiou proposes is dependent upon his axiomatic decision to present the empty set as the 'non-being of the one', which he characterises as the primitive name of being. This is a metaontological claim that cannot be derived mathematically. The ontology of the hierarchy of constructible sets, which is obtained by iterating the power-set operation on the empty set through the transfinite, 'is rooted in it' (TW 57). As Cassou-Noguès points out:

> Badiou can not found his axioms and establish that they are true propositions of the ontology of the multiple. But in the perspective that he puts in place, this foundation is not required. It is only necessary to remain faithful to . . . the event of Cantor's work and pursue a process that is thought to be producing truths, without ever being able to establish it.[50]

This is of course consonant with Badiou's own characterisation of philosophy as metaontology, and of ontology as 'a rich, complex, unfinishable science, submitted to the difficult constraint of a *fidelity* (deductive fidelity in this case)' (BE 8). The coherence of his approach rests solely upon the fidelity of his philosophy to this event. The consistency with which Badiou can continue to develop his philosophy in response to the ongoing engagement that mathematics has with the presentation of being qua being is the sole testament to this fidelity.

In this respect, Cohen's method of forcing is also behind the shift in focus that occurs in Badiou's second main text, *Logics of Worlds*, which exhibits an attempt to extend this fidelity by experimenting with the category theoretic extension of set theory, Heyting Algebra and Sheaf Theory. Kanamori points out that 'Forcing has been . . . adapted in a category theory context which is a casting of set theory in intuitionistic logic.'[51] Heyting algebra replaces Boolean algebra in intuitionistic logic, where Boolean algebra has become an important instrument in the interpretation of, and is deployed in an alternative approach to, Cohen's original procedures of the method of forcing. Kanamori also indicates that 'forcing can be interpreted as the construction of a certain topos of sheaves. The internal logic of the topos of presheaves over a partially ordered set is essentially Cohen's forcing . . .'[52] This move on Badiou's part can be seen as an attempt to address the fact of the ongoing engagement that mathematics has with the presentation of being qua being, and the potential limitations of the singular commitment to set theory in *Being and Event* as the definitive statement of this presentation.

It is not at all clear that this requirement of fidelity, which is characteristic of Badiou's metaontology, contributes anything to the debates about the realism of mathematical objects as conducted in the philosophy of mathematics. At best what Badiou is offering is an alternative way of formulating the question of fidelity, which for Badiou is to Cantorian set theory and the non-being of the one, rather than to the indispensability argument for Quine and Putnam, or to the existence of mathematical structures for Shapiro. The significant feature of this difference is that it entails accepting a radical alternative formulation of the relation between philosophy and mathematics that purports to render superfluous the empiricist framework within which these debates have to date been conducted. Whether or not Badiou's philosophy is robust enough to displace the indispensibility argument or the structuralist programme in realism has yet to be demonstrated in any convincing way.

Notes

1. See Paul Benacerraf, 'Mathematical truth', *Journal of Philosophy*, 70: 19 (1973), pp. 661–79.
2. Stewart Shapiro, *Philosophy of Mathematics, Structure and Ontology* (Oxford: Oxford University Press, 2000), p. 17.
3. See Willard V. Quine, *From a Logical Point of View* (Cambridge, MA: Harvard University Press, 1964); *Theories and Things* (Cambridge, MA: Harvard University Press, 1981); Hilary Putnam, 'What is mathematical truth', in *Mathematics Matter and Method: Philosophical Papers, Volume 1*, 2nd edn (Cambridge: Cambridge University Press. 1979).
4. See Shapiro, *Philosophy of Mathematics*, p. 46.
5. David Marker, 'Model theory and exponentiation', *Notices of the American Mathematical Society*, 43 (1996), pp. 753–9, at p. 753.
6. For an account of a model-theoretic framework see Marker, 'Model theory', pp. 754–5; Shapiro, *Philosophy of Mathematics*, pp. 46–8.
7. Shapiro, *Philosophy of Mathematics*, p. 71.
8. Hilary Putnam, *Reason, Truth and History* (Cambridge: Cambridge University Press, 1981), pp. 72–4; Shapiro, *Philosophy of Mathematics*, p. 67
9. Plato, *Plato: Complete Works*, ed. John M. Cooper (Indianapolis, IN: Hackett, 1997), *Gorgias* 451A–C.
10. Ibid., *Theatetus* 198A–B; see also *Republic* VII 522C.
11. Ibid., *Gorgias* 451A–C; see also *Charmides* 165E–166B.
12. Shapiro, *Philosophy of Mathematics*, p. 73.
13. Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, MA: MIT Press 1968), p. 23.
14. Shapiro, *Philosophy of Mathematics*, p. 73.
15. Ibid.
16. Ibid., p. 56.
17. See ibid., pp. 76–7.
18. The Platonic doctrine of anamnesis holds that all learning is recollection, and that perception and enquiry remind us of what is innate in us (Plato, *Meno* 80A–86C; *Phaedo* 73C–78B).
19. Plato, *Republic* VI 511C–D. Badiou's translation. See TW 44.
20. Kurt Gödel, 'What is Cantor'scontinuum problem?' [1947, revised and expanded 1964], in Paul Benacerraf and Hilary Putnam (eds), *Philosophy of Mathematics: Selected Readings*, 2nd edn (Cambridge: Cambridge University Press, 1983), p. 485.
21. Quine, *Theories and Things*, p. 15.

22. Joseph W. Dauben, *Georg Cantor: His Mathematics and Philosophy of the Infinite* (Princeton, NJ: Princeton University Press, 1990), p. 199.
23. A countable set is any set that is either finite or the same size as **N**. An uncountable set is any set bigger than **N**.
24. Dauben, *Georg Cantor*, p. 179. Note that **N**, ω and $\aleph_0$ all name the same set, i.e. the set of natural numbers.
25. Ibid., pp. 103–11.
26. Ibid., p. 269.
27. Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs* (Amsterdam: North Holland, 1983), p. 12.
28. That is, the multiple from which all other multiples are constructed.
29. Russell's paradox raises the question of whether the set of all sets which are not members of themselves is a set. If the set exists, then it is included as one of its own sets, i.e. it is both a member and not a member of itself, which is a contradiction.
30. See Solomon Feferman, *In the Light of Logic* (Oxford: Oxford University Press, 1989), p. 37.
31. Ibid., p. 39.
32. Ibid.
33. Ibid., p. 40.
34. Gödel's 'constructible sets' are sets defined solely in terms of the subsets of the previous stage of construction that have already been constructed, rather than the set of all subsets as it is in V.
35. He maintains that 'the constructible universe is a *model* of these axioms [i.e. ZFC + V = L] in that if one applies the constructions and the guarantees of existence supported by the Ideas of the multiple, and if their domain of application is restricted to the constructible universe, then the constructible [universe] is generated in turn' (BE 300).
36. Feferman, *In the Light of Logic*, pp. 66–9.
37. Gödel, 'What is Cantor's continuum problem?', p. 476.
38. Paul. J. Cohen, *Set Theory and the Continuum Hypothesis* (New York: W. A. Benjamin, 1966).
39. The method of showing that a certain statement is not derivable from or is not a logical consequence of given axioms is to exhibit a model in which the axioms are true but the statement is false. This is indicted by the following notation: (1) ZF + ¬AC; (2) ZFC + ¬CH; (3) ZFC + GCH + V ≠ L. See Akihiro Kanamori, 'Cohen and set theory', *Bulletin of Symbolic Logic*, 14: 3 (2008), pp. 351–78, at p. 235.

40. See Ibid., p. 360.
41. According to Kunen, ‘Cohen’s original treatment made forcing seem very much related to the constructible hierarchy. His M was always a model for V = L’ (Kunen, *Set Theory*, p. 235).
42. Kunen, *Set Theory*, p. 188.
43. Ibid., p. 185.
44. Ibid., p. 184.
45. William B. Easton, ‘Powers of regular cardinals’, *Annals of Mathematical Logic*, 1 (1970), pp. 139–78.
46. See Judith Roitman, *Introduction to Modern Set Theory* (New York: Wiley, 1990), pp. 91–2.
47. See Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings* (New York: Springer, 1994), p. 472.
48. Feferman, *In the Light of Logic*, pp. 72–3.
49. Akihiro Kanamori and Menachem Magidor, ‘The evolution of large cardinal axioms in set theory’, in *Higher Set Theory: Lecture Notes in Mathematics*, 669 (New York: Springer, 1978), p. 104.
50. Pierre Cassou-Noguès, ‘L’excès de l’état par rapport à la situation dans *L’être et l’événement* de A. Badiou’, *Methodos*, 6 (2006), http://methodos.revues.org/471, para. 33.
51. Kanamori, ‘Cohen and set theory’, p. 371.
52. Ibid.