Bell's Theorem, Uncertainty, and Conditional Events

Ian T. Durham

ABSTRACT

This article explores a relationship between the generalized form of Heisenberg's uncertainty relations and Bell-type inequalities in the context of their associated algebras. I begin by exploring the algebraic and logical background for each, drawing parallels and a noticeable symmetry. In addition I describe a thought experiment linking the conceptual foundation of one to a mathematical representation of the other. Finally, I explore the requirements for a more inscrutable relationship between the two pointing out the tantalizing questions this suggestion raises as well as potential answers. The purpose of this article is to show that there is more to this relationship than meets the eye and suggests that a very general Bell-like theorem can be interpreted as a limiting case of the broader generalized uncertainty principle.

1 Introduction

Conditional events involve two propositions where one specifies a condition that, if met, can lead to the second being met. As Milne points out in his overview, if the truth or falsity of the conditional assertion is dependent upon the truth or falsity of the second assertion. Only if this second assertion is fulfilled can the truth or falsity of the conditional assertion be judged (Milne [1997]). The phenomenon of quantum entanglement gives rise to conditional events in the form of correlated observables, i.e. given a pair of entangled particles, regardless of their separation distance, the measurement of the spin of one immediately determines the spin of the other. Though the original development of Bell's theorem involved hidden variables, generalization of this theorem does not *a priori* require the invocation of such variables. What the generalization does require is the presence of some inequality in instances of local realism. To some extent, Heisenberg's Uncertainty Principle is wrapped up in much of this discussion since it forms one of the fundamental building blocks of the 'new' quantum theory that Einstein had such trouble with, leading, ultimately, to the EPR paper in 1935 (Einstein, Podolsky, and Rosen [1935]). But just how wrapped up in these proceedings is it? Perhaps it is *too* wrapped up in these proceedings for certain analyses to be undertaken. As I shall demonstrate, there is more here than initially meets the eye.

Taking a step back, then, let us first assume we are initially ignorant to Bell's theorem and all subsequent work. Such is the freedom felt by Heisenberg himself whilst deriving the uncertainty relations. The familiar derivations, of course, can be approached in various ways but, ultimately, relate to the commutation relation, first invoked in 1925 by Born and Jordan (for details see Jammer [1966] and Beller [1999]). Of course this was the dawn of matrix mechanics and the algebra contained therein serves as the backbone of the uncertainty relations.

What if, however, we are, rather, ignorant to Heisenberg's methods and only know the overarching philosophical implications of his principle, yet we are familiar with Bell's theorem and many of the various inequalities built upon this work. Again, the roots of this work appear in Bell's well-known article of 1964 (reprinted in Bell [1987]). The governing algebra for the inequalities that followed was distinctly Boolean. And herein lies the first point of mathematical separation between these two types of inequalities – or is it? It certainly warrants a closer look.

2 Foundations

2.1 Matrix Algebra, Operators, and Uncertainty

The general rules for matrix algebra are initially very similar to the rules for ordinary algebra. Values and identities are fairly straightforward. Matrices, however, are multi-valued where the positions of the values within the matrices determine the ultimate nature of the matrices themselves. This results in the fact that matrix algebra is non-commutative, meaning the order of the matrices in certain operations matters. There are non-commutative forms of ordinary algebra, but the existence of commutative forms immediately forces matrix algebra into a direct relation with only a *sub*set of ordinary algebra. In a three-valued system, ordinary non-commutative algebra obeys the following,

$$a(b+c) = ab + ac \qquad \textbf{left}$$
$$(b+c)a = ba + ca \qquad \textbf{right}$$

while ordinary commutative algebra does not distinguish between these. Generally, matrix algebra obeys similar rules,

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

where **A**, **B**, and **C** are matrices. Matrix algebra, then, is distributive and associative, but not necessarily commutative. The semantics here can be a bit confusing when we consider that, though this is really a non-commutative result, matrices are said to have a *commutation relation* when $\mathbf{AB} + \mathbf{BA} \neq 0$ and an *anticommutation relation* when $\mathbf{AB} - \mathbf{BA} \neq 0$.

The multi-valued, ordered aspect of matrices also results in non-trivial inverses and transposes, the latter created by swapping the rows and columns of the original matrices. In addition, none of this is even possible unless the matrices involved are conformable to the operations used. For example, for matrix multiplication, when multiplying **A** and **B** in that order, **A** must have the same number of columns as there are rows in **B**.

The generalized uncertainty relations cannot, however, be applied just yet. Two more conditions must be fulfilled: the matrices involved must be *operators* and they must represent *observable quantities*. Once those conditions are fulfilled, the following if-then statement is applicable:

If there exist two observables **A** *and* **B** *such that*

$$\left[\mathbf{A},\mathbf{B}\right]=\mathbf{C}\neq 0$$

then, if the measurement of **A** *is uncertain in some amount, the measurement of* **B** *will also be uncertain in some amount such that*

$$\Delta\mathbf{A}\cdot\Delta\mathbf{B}\geq\tfrac{1}{2}\left|\langle\mathbf{C}\rangle\right|$$

where $\langle\mathbf{C}\rangle$ *is the expectation value and is a function of the probability of finding* **C** *in some interval.*

The above if-then statement connects the generalized commutation relation from matrix algebra to the inequality that represents the generalized uncertainty relations (see also Liboff [1998] and Durham [2004]). Liboff further clarifies the relationship between the commutation relation, the uncertainty relations, and associated probabilities:

> If **A** and **B** do not commute, then the eigenstate φ_a of **A** which the system goes into on measurement of **A** is not necessarily an eigenstate of **B**. Subsequent measurements of **B** will give any spectrum of eigenvalues of **B** with a corresponding probability distribution *P*(*b*).[1]

The probability amplitude, meaning the probability of some measurement of **B** occurring, is $P(b)=\left|\langle\varphi_b|\varphi_a\rangle\right|^2$. The generalization presented here in modern notation is actually a result of work by Schrödinger building on the results of Robertson and Condon.

2.2 Boolean Algebra and Bell's Theorem

In contrast to matrix algebra, Boolean algebra is usually presented as being commutative (see for instance Gullberg [1997]), though non-commutative operations certainly exist. I will raise questions about this a bit later. For the moment, let's assume there's nothing unusual happening. As such, Boolean algebra has the simple commutativity rules

[1] Liboff [1998], p. 146. I have modified the notation from Liboff a bit in an effort to remain consistent.

$$a \cup b = b \cup a$$
$$a \cap b = b \cap a.$$

Boolean algebra, unlike matrix algebra, is not only discrete but also binary. Boolean algebra also has a few unique rules, however, since it is ultimately a part of set theory. For instance, it obeys idempotent laws such as

$$x \cap x (= x^2) = x$$

which simply says that the intersection of *x* with itself is simply the set having the properties of set *x*.

Bell's original analysis of the EPR paper and von Neumann's proof of the non-existence of hidden variables was built around a simple thought experiment that included two possible measurements, each with only two possible outcomes, thus making the associated algebra Boolean. Bell's original inequality was an examination of correlated measurements by two independent observers (reprinted in Bell [1987]). Two electrons in a singlet state (i.e. entangled) serve as the standard example, though most experimental tests have used photons. The original inequality took the form

$$1 + C(b,c) \geq \left| C(a,b) - C(a,c) \right|$$

where *C* is known as the quantum correlation of the particle pairs and is simply the expectation value of the product of the outcomes. The settings on the apparatus are given by *a*, *b*, and *c*. Experimental tests of Bell's theorem employ modified versions of Bell's inequalities and are all likewise Boolean. Of particular interest to us later is the Clauser-Horne or CH74 inequalities, here in the form given by van Fraassen,

$$-1 \leq P(AB) + P(AB') + P(A'B') - P(A'B) - P(A) - P(B') \leq 0$$

as well as seven other inequalities formed simply by permuting *A* with *A*', *B* with *B*', and finally both the previous permutations together, where *P* is a surface state.

Bell's theorem assumes these correlation inequalities hold under local realism but *not* under quantum mechanics. This can actually be formulated as an if-then statement in a very general form similar to the one formed above for the generalized uncertainty relations:

If local realism exists, then there also exists some inequality.

Unlike the previous if-then statement, however, this does not involve a connection between a commutation relation and an inequality. Rather, the connection is between *local realism* and an inequality (or inequalities). The relationship between local realism and the generalized commutation relation will be explored later.

3 Can Position and Momentum Satisfy a Bell-type Inequality?

Imagine an experiment designed to measure the properties of single electrons, perhaps in an atom or perhaps loosely bound or even free. The experiment is designed to include two analyzers that are independent of one another. The first analyzer is set to measure the position of a given electron at a specific point in time. The second analyzer is set to measure the momentum of a given electron at the same exact time that the first analyzer measures the position for that same electron. In this experiment I am not attempting to derive the uncertainty relations. Rather I assume that the principle is prior knowledge. As such we know immediately that the second analyzer cannot measure momentum simultaneous to the first analyzer measuring position. As an added constraint let's assume that the analyzers will return a 1 for a successful measurement of their associated variable with perfect precision (obviously this is not entirely realistic, but it is an extreme case and thus provides insight into more realistic cases) and a 0 for an unsuccessful measurement. I will now label a measurement of 1 by the first analyzer as A and a measurement of 0 as A'. Similarly I will label a measurement of 1 by the second analyzer as B and a measurement of 0 as B'. We can now begin to fit our model to the CH74 inequality in order to determine if it is consistent with uncertainty.

Let us begin by assuming a perfect measuring apparatus in both cases (i.e. there will be no false positives due to detector error, human response problems, etc.). Since P(AB) represents the probability that the two analyzers will simultaneously return a 1, we know immediately that this (the probability) is 0 due to the constraint of the uncertainty principle. In addition we know that a measurement of A always accompanies a measurement of B' and a measurement of A' always accompanies a measurement of B (again, assuming these are true simultaneous measurements). Therefore,

$$P(AB') + P(A'B) = 1.$$

In addition the assumption of a perfect measuring apparatus requires a measurement of 1 on one of the two channels making $P(A'B') = 0$. In a perfect system free from any external errors $P(A) = P(B')$. Thus the CH74 inequalities for a perfect system reduce to

$$-1 \leq (2P(AB') - 1) - 2P(A) \leq 0.$$

But a perfect system would also not recognize any difference between P(AB') and P(A) and would make the inequality trivially true for that extreme since $(2P(AB') - 1) - 2P(A)$ would be -1.

The other extreme is a system that simply doesn't function and can't return any values, 0 or 1. In this case all the probabilities are 0. As such, a model system such as the one I have suggested, has its apparent extremes at 0 and -1 which is consistent with the CH74 inequalities.

But ultimately the original CH74 inequalities were derived for imperfect systems and in order to be certain that our application of CH74 to the model system that I have suggested I will now consider an imperfect but operable system. Let us then return to the original inequalities and include inherent errors in the system in calculating our probabilities. Let us also assume that the system has been designed to err on the side of caution by only returning a 1 if the measurement of a given observable is to within an extraordinarily (though arbitrarily) high degree of accuracy such that we can expect P(AB) ~ 0. We then can start with

$$-1 \leq P(AB') + P(A'B') - P(A'B) - P(A) - P(B') \leq 0.$$

Is it necessarily still true that P(AB') + P(A'B) = 1? Yes, since they are opposites (i.e. a measurement of AB' is a measurement of 1–0 and a measurement of A'B is a measurement of 0–1 and it is impossible for the probabilities of these two occurances combined to be greater than 1). Therefore we have

$$-1 \leq (2P(AB') - 1) + P(A'B') - P(A) - P(B') \leq 0.$$

Since we have biased the system in favor of 0 measurements a failure of the system will be more likely to produce a 0 measurement. In fact, we have biased the system to such a degree that P(AB) ~ 0 implying nearly all failures are 0 measurements (though not all – or even nearly all – 0 measurements are failures). |P(B')|, in addition to registering a 'real' 0, could register a 0 due to the failure of the second analyzer. The failure of both analyzers simultaneously, as measured by P(A'B'), that can also measure the failure of one while the other returns a 'real' 0, must be necessarily less than the failure of one independely of the other (i.e. it's more likely that only one will fail as opposed to both). Therefore, P(A'B') < |P(B')|. Likewise, P(A'B') < |P(A')|. Since P(A) + P(A') = 1, this implies P(A'B') < |P(A)|. These conditions considered together imply

$$-1 \leq P(A'B') - P(A) - P(B') \leq 0.$$

The problematic portion of (2) is the 2P(AB') – 1 term. Or is it? Can P(AB') measure a system failure? The answer depends on whether the analyzers themselves are correlated. However, if they were correlated there really would be no need for two since a measurement on one would systematically determine the measurement on the other regardless of the reality of the uncertainty principle. Thus we need to add another condition to our experiment: the two analyzers are not in any way systematically (mechanically, electronically, etc.) correlated. As such P(AB') measures outcomes based solely on the uncertainty principle and is independent of the analyzers meaning P(AB') = P(A'B) = 0.5 much like a Stern-Gerlach device. This then eliminates the 2P(AB') – 1 term leaving just (3). Therefore the principle of uncertainty preventing simultaneous measurements of position and momentum is consistent with the CH74 inequalities and, as a result, consistent with local realism.

4 Symmetry…or not?

I have shown in the above example that position and momentum can satisfy the CH74 inequalities. In fact, if position and momentum were replaced by any pair of observables, **A** and **B** that satisfy a generalized uncertainty relation, this example would still hold. Clearly the observables satisfying the generalized uncertainty relation can also be used in

Bell-type inequalities. That hints at a deeper relation between the two types of inequalities, or rather between Bell's theorem and a generalized uncertainty principle. Let us now try to flush out any such relations.

4.1 To Commute or To Not Commute

In a purely algebraic sense, there are actual fairly noticeable differences between the two, though let us not allow such differences to keep us from exploring further. Clearly, from my above discussion of the algebraic foundations of each the most noticeable difference is that Boolean algebra is commutative under multiplication while matrix algebra is not. But are the observables of a Bell-type inequality really commutative under multiplication? The point of commutative algebra is that the order of the values in an equation do not matter: $ab = ba$. As exemplified by the commutation and anti-commutation relations, this is not true of a non-commutative algebra. The problem here is that the *measurement* of observables in Bell-type inequalities involves conditional events meaning that, in short, order matters. For instance, if you first make a measurement on a right-hand analyzer, the result will determine your measurement on the left-hand analyzer. This, of course, is completely independent of the binary nature of the analyzers. Clearly there should be a non-commutative operation here due to the conditional nature of these events. Order does actually seem to matter.

What does this mean, exactly, in the context of set theory? Since the commutative laws of Boolean algebra refer to the union or intersection of sets rather than the addition or multiplication of numbers, the logic is a bit different. Nonetheless, there is a way to represent conditional events in set theory. The general notation is known as a set builder and can be demonstrated as follows,

$$\{x \in A | p(x)\}.$$

This statement reads as 'the set of those elements x of A for which the proposition $p(x)$ is true.' In a Bell-type experiment there are only two distinct entities for the Boolean measurement sets R and L representing the right and left analyzers respectively: 0 and 1. The zero element here corresponds to the null set of general set theory while the unit element corresponds to the universal set. In this case the universal set is $\mathbf{U} = \{0,1\}$. But can a set builder *really* provide a true set of ordered or conditional events?

Actually, a close study of the commutative laws at work in Boolean algebra and set builders leads to a positive response to that question. The reason? *Because Boolean algebra allows only two possible outcomes on a measurement*! For example, take the standard Boolean commutative relation,

$$R \cup L = L \cup R$$

where I have let R represent the set of measurements from a right analyzer and L represent the set of measurements from the left analyzer in a Bell-type experiment. The restriction that R and L cannot have the same value simultaneously says *nothing* about the conditional nature of R or L.

Let us then be more restrictive. Let us say that the elements of sets R and L are initially *undetermined* but that they cannot have the same value if they represent a single, actual measurement on an analyzer. A measurement of 1 on R then means a measurement of 0 on L. Conversely a measurement of 1 on L requires a measurement of 0 on R. In *both these cases*, the set representing the union of the two is the same,

$$R \cup L = L \cup R = \{0,1\}.$$

Likewise, the intersection of the sets in both cases is impossible (e.g. if $R = \{0\}$ and $L = \{1\}$ or vice-versa, there is no intersection).

We can conclude from this analysis that the *commutative* laws of Boolean algebra are sufficient for the normally *non*-commutative measurements in Bell-type experiments. Since I have shown that observables that follow a generalized uncertainty relation can be used in Bell-type experiments Boolean algebra is sufficient for treating such observables in such a situation. In fact, owing to the fact that normal Bell-type experiments generally use spin or polarization as their guiding physical directive, the Boolean algebra can be seen as a limiting case of the matrix algebra. Essentially, if observables are whittled down via various restrictions from a multi-valued matrix to a binary set the algebraic rules governing those observables transitions from non-commutative to commutative. Eddington actually developed a method for making this transition in the early 1940s (Eddington [1946]). On a macroscopic level, of course, most observables are treated as commuting, i.e. we take it for granted that we can measure position and momentum simultaneously.[2] As we work our way down to the microscopic question we transition to the non-commutative world of quantum mechanics – until we choose to simplify things so much that we are left with a binary system, in which case we return to a commutative relationship (though, admittedly, in the context of a different algebra entirely).

4.2 Origins of Observables in the Commutation Relation

As I have just shown, algebraically, the Boolean rules for Bell-type experiments are a limiting case of the matrix rules for Heisenberg-type problems. Here I suggest a further way to link the two situations beyond the context of algebra. This suggestion is only applicable to the position and momentum relation and the existence of a generalization still needs to be found if the relationship is to be even more firmly grounded.

The position and momentum observables appearing in the usual commutation relation as equations of motion are derivable from the Hamilton-Jacobi relations of classical mechanics. These equations, involving the Hamiltonian, represent integral curves of the Hamiltonian's vector fields on a symplectic manifold and the Hamiltonian itself is the Legendre transform of the Lagrangian. The Hamiltonians are smooth over this manifold for a Heisenberg group. This begs the question, then, does a binary form of the Hamilton-Jacobi relations exist?

---

[2] Never mind the obvious problem of momentum being *defined* in terms of a *change* in position. That brings up the age-old and ugly question of the existence of instantaneous velocities and accelerations.

Perhaps surprisingly, the answer is 'yes.' Richard Bellman was able to extend the Hamilton-Jacobi equations to a more generalized form sometimes known as the Hamilton-Jacobi-Bellman equation and subsequently found a *binary* version of this generalization used in dynamical programming now known as the Bellman equation. Further solidifying the relation between generalized uncertainty and Bell's theorem would benefit from a link between the Bellman equation (or something like it) and Bell-type inequalities and observables.

The reality of these observables also requires detailed study. Bell's theorem implies local realism. Does that immediately suggest uncertainty must obey local realism? Breaking local realism up into its constituent parts it is not a stretch to require a Lorentz invariant form for the Hamilton-Jacobi equations. But what about realism? The question, it turns out, may not be necessary. In a Heisenberg-type case there's room for error in measurement while in a Bell-type case the binary result seems to *imply* realism – i.e. you've *measured* your observables so they're obviously real! It's a bit analogous to wavefunction collapse. In the Bell case you've constrained your results to collapse to a binary result which guarantees realism (assuming the level of accuracy I described in my thought experiment in section 3). Realism, then, can be interpreted as a limiting case of a broader distribution.

5 Conclusion and Further Suggestions

I have thus painted a portrait of Bell's theorem and the associated Bell-type inequalities as being a limiting, binary (Boolean) case of the broader distributions represented by the generalized uncertainty principle and its associated inequalities. To briefly summarize the links between Bell's theorem and the Uncertainty Principle and their associated inequalities, let me begin by noting that the foundation for doing this lays in the fact that both situations deal with conditional events and an if-then statement that includes an inequality.

Algebraically, the matrix algebra that gives rise to the uncertainty relations is non-commutative while the Boolean algebra inherent in Bell-type inequalities in commutative. However, I have shown that in a binary situation the conditional nature of the observables involved is not compromised by this change. In fact the mere existence of a non-commutative binary algebra is questionable. Further solidifying this relation is the fact that one can design (as I have in section 3) a *binary* thought experiment involving observables obeying the generalized uncertainty relations. As such it appears that observables obeying an uncertainty relation will automatically obey a Bell-type inequality if they are measured in a binary system.

Physically, at least in the case of position and momentum, the mechanism that produces the non-commutative nature of the observables is derivable from a generalized form of the Hamilton-Jacobi equations known as the Hamilton-Jacobi-Bellman equation. The binary form of this equation, used in dynamical programming, is known as the Bellman equation. A relation between this equation and the observables in Bell-type inequalities similar to the relation between the Hamilton-Jacobi equations and position and momentum would solidify this relation.

Finally, ontologically, we see that while observables following uncertainty relations obey the Principle of Locality, the reduction to a binary measurement system *leads to* realism. The tantalizing suggestion inherent in all of this is that Bell's theorem is a Boolean limit of the Uncertainty Principle. It is my intention, through this paper, to induce further research in this area in an effort to shore up the bridge I have begun to build between the two. Recent work in quantum logic looks potentially promising as a foundation (see Marchetti & Rubele [2004], Baugh, Finkelstein, Galiautdinov, and Saller [2000], Rassias [2003], and Pitowsky [1994] as examples).

A note of thanks is due to Steven French who looked over the thought experiment in section 3.

Department of Physics
Saint Anselm College
Manchester, New Hampshire
USA
idurham@anselm.edu

References

Baugh, James, Finkelstein, David Ritz, Galiautdinov, Andrei, and Saller, Heinrich [2000]: 'Clifford Algebra as a Quantum Language', arXiv:hep-th/0009086 v1.

Bell, John [1987]: *Speakable and Unspeakable in Quantum Mechanics*, Cambridge: Cambridge University Press.

Beller, Mara [1999]: *A Quantum Dialogue: The Making of a Revolution*, Chicago: University of Chicago Press.

Durham, I.T. [2004]: *Sir Arthur Eddington and the Foundations of Modern Physics*, unpublished PhD thesis, University of St. Andrews.

Eddington, A.S. [1946]: *Fundamental Theory*, Cambridge: Cambridge University Press.

Einstein, A., Podolsky, B. and Rosen, N. [1935]: 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?', *Physical Review*, **47**, pp. 777-780.

Gullberg, Jan [1998]: *Mathematics: From the Birth of Numbers*, New York: Norton.

Jammer, Max [1966]: *The Conceptual Development of Quantum Mechanics*, New York: McGraw-Hill.

Liboff, R.L. [1998]: *Introductory Quantum Mechanics*, Reading: Addison Wesley.

Marchetti, P.A. and Rubele, R. [2004]: 'Quantum Logic and Non-Commutative Geometry', arXiv:quant-ph/0405159 v1.

Milne, Peter [1997]: 'Bruno de Finetti and the Logic of Conditional Events', *British Journal for the Philosophy of Science*, **48**, pp. 195-232.

Pitowsky, Itamar [1994]: 'George Boole's 'Conditions of Probable Experience' and the Quantum Puzzle', *British Journal for the Philosophy of Science*, **45**, pp. 95-125.

Rassias, John Michael [2003]: 'On the Heisenberg-Pauli-Weyl Inequality', *Journal of Inequalities in Pure and Applied Mathematics*, **5**, online.

Van Fraassen, Bas C. [1991]: *Quantum Mechanics: An Empiricist's View*, Oxford: Clarendon.