Translators' Introduction

The importance of *Grundgesetze der Arithmetik*

Gottlob Frege's *Grundgesetze der Arithmetik* was originally published in two volumes: the first in 1893, the second in 1903. It was to be the pinnacle of Frege's life's work. The aim was to demonstrate that arithmetic and analysis are reducible to logic—a position later called "Logicism". Frege's project began with the publication of *Begriffsschrift* in 1879, which contains the first version of his logical system, also named '*Begriffsschrift*' (concept-script), the mature formulation of which Frege would present in *Grundgesetze*. His work was groundbreaking in many ways: it contained the first occurrence of the quantifier in formal logic, including a treatment of second-order quantification; the first formal treatment of multiple and embedded generality; and it also offered the first formulation of a logical system containing relations. *Begriffsschrift* is widely acknowledged to constitute one of the greatest advances in logic since Aristotle. As W.V. Quine put it: "Logic is an old subject, and since 1879 it has been a great one" (Quine 1950, p. vii).

Begriffsschrift was followed by *Die Grundlagen der Arithmetik* in 1884. Having previously completed a manuscript of a more formal treatment of Logicism around 1882—a lost ancestor of *Grundgesetze*—Frege developed a philosophical foundation for his position in *Grundlagen*. It was likely on the advice of Carl Stumpf[1] that Frege decided to write a less technical and more accessible introduction to Logicism. *Grundlagen* is regarded by many as a philosophical masterpiece and by some, most notably Sir Michael Dummett, as "the most brilliant piece of philosophical writing of its length ever penned" (Auxier and Hahn, 2007, p. 9). The first part contains devastating criticisms of well-known approaches to the philosophy of arithmetic, including those of John Stuart Mill and Immanuel Kant, as well as those of Ernst Schröder, Hermann Hankel, and others. In the second part, Frege develops his Logicism. In *Grundlagen*, Frege eschews the use of a formal system, merely offering non-formal sketches of how a version of the Peano–Dedekind axioms for arithmetic can be derived from pure logic. He closes *Grundlagen* by suggesting that the Logicist approach is not restricted to arithmetic: it also has the potential to account for higher mathematics, in particular,

[1] Stumpf was a German philosopher and psychologist, a student of Franz Brentano and Hermann Lotze, who at the time was professor of philosophy in Prague. In a letter by Stumpf (September 9, 1882)—in response to a letter that Frege wrote to him (but, it seems, wrongly filed as a letter to Anton Marty who was Stumpf's colleague in Prague)—Frege is encouraged to spell out his Logicist programme in a more accessible manner, without recourse to his formal language. See Gabriel et al. (1976), p. 257, and the editor's note on p. 162.

real and complex analysis. At the end of *Grundlagen*, Frege was thus left with the monumental task properly to establish Logicism: he needed to identify a small number of basic laws of logic; offer a small number of indisputably sound rules of inference; and, finally, provide gapless proofs in his formal system of the basic laws of arithmetic, using only the identified basic laws and rules of logic together with suitable explicit definitions. This was to be the task of his *magnum opus*: *Grundgesetze der Arithmetik*.

The first volume of *Grundgesetze* is structured to reflect these three main tasks and includes a substantial philosophical foreword. In this foreword, Frege provides extensive criticism of psychological approaches to logic—in particular, against Benno Erdmann's *Logik* (1892)—and also offers an explanation of how his logical system has changed since 1879 and why the publication of *Grundgesetze* was delayed. Frege notes that a nearly completed manuscript was discarded in light of "a deep-reaching development" (vol. I, p. X) in his logical views, in particular his adoption of the now infamous Basic Law V. The remainder of the volume is divided into two parts. In part I, Frege introduces the language of concept-script, his modes of inference, his basic laws, and a number of important explicit definitions. Part II of *Grundgesetze* contains the proofs of the basic laws of cardinal number—i.e., the most important theorems of arithmetic proven within Frege's logical system—as well as Frege's treatment of recursion and countable infinity (see Heck (2012) for details). The proofs are contained in sections labelled "Construction". Each such section is preceded by a section called "Analysis" in which Frege outlines in non-technical prose the general proof strategy in order to facilitate understanding of the subsequent formal proofs. Volume I finishes rather abruptly. One may speculate that Frege was given a page limit by the publisher.

It was to be another ten years until the publication of volume II. It seamlessly continues where volume I left off: with no introduction, but merely a brief reminder of two theorems of volume I that had not previously been indexed for further use but which are employed in the subsequent proofs. Frege thus finishes part II of *Grundgesetze*, providing sixty-eight further pages of proofs of arithmetical theorems. Part III occupies the rest of the second volume. Following a strategy analogous to the one he uses in *Grundlagen*, Frege first provides strong, and in parts polemical, criticisms of alternative approaches to real analysis of many of his contemporaries. One particular focus is his criticism of (game) formalism, as advocated by Hermann Hankel and Frege's colleague in Jena, Johannes Thomae. The critical sections of part III are followed by a brief non-formal description of the approximate shape of Frege's approach to a logicist foundation of real analysis as based on the notion of "magnitude". (Frege points out that he will not "follow this path in every detail" (vol. II, p. 162).) Part III.2 contains the beginnings of a formal treatment of real analysis, and finishes with an outline of what remains to be accomplished at the close of volume II. Evidently Frege had plans for, or may even had already written parts of, a third volume in order to finish part III. Perhaps it was also to contain a part IV dedicated—as already noted in *Grundlagen*—to a logicist treatment of the complex numbers.[2]

[2] Whether and how much of the continuation of part III was written is unfortunately unclear. We may again speculate that Frege was, in turn, pushed to the publication of volume II by having exceeded a certain page limit. It seems natural to assume that he had some of the proofs that volume III was meant to contain. Indeed there is evidence that Frege worked on further aspects of his theory of irrational numbers—see Dummett (1991), p. 242, fn. 3. According to the inventory of Frege's *Nachlaß*, published in Veraart (1976), there were manuscripts that might have contained such material. Frege's *Nachlaß*, except for a small part that was transcribed by Heinrich Scholz

However, it was not to be. As is well known, in 1902 Frege received a letter from Bertrand Russell, when the second volume was already in press. In this letter, Russell proposes his famous antinomy, which made Frege realise that his Basic Law V, governing the identity of value-ranges, leads into inconsistency.[3] Frege discusses a revision to Basic Law V—which he labels: V′—in the afterword to volume II of *Grundgesetze*. However, it can be surmised that Frege himself realised (probably sometime after 1906) that V′ was unsuitable for his project—it is inconsistent with the assumption that there are at least two distinct objects.[4] Frege did not publish a third volume. Given the inconsistency of his Basic Law V and Frege's inability to find a suitable substitute that could be regarded as a basic law of logic, Frege gave up on Logicism late in his life.[5]

A recent resurgence of Logicism, and so-called Neo-Fregeanism, has again sparked interest in Frege's original writings and, in particular, in *Grundgesetze*. Despite the inconsistency of the formal system of *Grundgesetze*, Frege's proof strategy and technical accomplishment have come under renewed investigation. There is, for example, the recent "discovery" of so-called *Frege's Theorem*:[6] the proof that the axioms of arithmetic can be derived in second-order logic using *Hume's Principle*—a principle governing the identity of cardinal numbers: the number of Fs equals the number of Gs if, and only if, the Fs and the Gs are in one-to-one correspondence. Frege first introduced Hume's Principle in *Grundlagen* (see §§63 and 72) but rejected it as a foundation for arithmetic because of the infamous Julius Caesar Problem (*Grundlagen*, §§56 and 66): it cannot be decided on the basis of Hume's Principle alone, whether Julius Caesar (or any other object that is not given as a number) is identical to the number Two, say. In *Grundgesetze*, Frege proves both directions of Hume's Principle (vol. I, propositions (32), §65, and (49), §69; see also §38 where Frege mentions the principle). Once the two directions of Hume's Principle are proven Frege makes no further essential use of Basic Law V in the development of arithmetic.[7] There have also been attempts to offer revisions to Basic Law V, e.g., in Boolos (1998); lastly, there has been recent work on identifying consistent fragments of the logic of *Grundgesetze*.[8] Yet it is not just these formal aspects of *Grundgesetze* that called for a new engagement with Frege's *magnum opus*. The philosophical arguments in part III, in particular, Frege's account of definitions, his conception of Basic Law V, his critical assessment of contemporary accounts of real analysis, and Frege's own approach to real analysis have again become relevant to current debates in the philosophy of logic and mathematics. Given that only parts of *Grundgesetze* had previously been translated, the need for a complete English translation became more and more pressing.

(published as Frege (1983)), is presumed lost due to an airstrike towards the end of the second world war—however, see Wehmeier and Schmidt am Busch (2005).

[3] The antinomy Russell suggests is not well formed in Frege's system—in his response to Russell, Frege provides the correct formulation. See the Frege–Russell correspondence from 1902, in Gabriel et al. (1976), and, in particular, Russell (1902) and Frege (1902).

[4] See Quine (1955) and Cook (2013).

[5] See his posthumously published note written around 1924/25 in Frege (1924/25).

[6] It was published in Wright (1983). More recent presentations of the proof can be found in Boolos (1987) (discursive), Boolos (1990) (rigourous), Boolos (1995), Boolos (1996), and Zalta (2012). A very insightful discussion of Frege's Theorem, including its origins and recent (re-)discovery, can be found in Heck (2011).

[7] For an excellent discussion see Heck (2012).

[8] See e.g. Wehmeier (1999), Heck (2000), Fine (2002), and Burgess (2005).

Translating Frege's *Grundgesetze*

The first translation of parts of *Grundgesetze* was published in 1915 by Johann Stachelroth and Philip E. B. Jordain. Between 1915 and 1917, they published three articles in the journal *The Monist*, translating Frege's philosophical foreword of *Grundgesetze* (Frege, 1915, 1916), the introduction, and §§1–7 of the main text (Frege, 1917). In 1952, Peter Geach and Max Black published their edition *Translations from the Philosophical Writings of Gottlob Frege*, reprinting Stachelroth and Jourdain's translation of the foreword, as well as producing English translations of parts of volume II. More specifically, they offer the first translation of "Frege on Definitions I" (vol. II, §§56–67), "Frege on Definitions II" (vol. II, §§139–144, 146–147), "Frege against the Formalists" (vol. II, §§86–137),[9] and "Frege on Russell's Paradox" (vol. II, Afterword). Their book was substantially revised for its third edition, published in 1980, with changes made to the translation of '*Bedeutung*' and its cognates (see below). In 1964, Montgomery Furth provided the first full translation of part I of volume I of *Grundgesetze* with a detailed translator's introduction (Frege, 1964). After these translations went out of print, Michael Beaney published the *Frege Reader* (Beaney, 1997) which contains, amongst other essential works by Frege, selected passages from *Grundgesetze*. Perhaps not surprisingly, there is no uniformity in the translation of Frege's technical terms across the various translations. Controversy about how to translate, for example, the term '*Bedeutung*' led Beaney to leave the term untranslated.

Similar difficulties naturally also affected the present translation. Deciding how to translate technical terms was not always easy. We had the good fortune, however, to be able to draw on the advice of a team of experts who assisted us extensively throughout the process and provided invaluable feedback on important translation decisions. What follows are elucidations regarding our choices for the translation of some important technical terms and a more extensive glossary.

Anzahl

We translate the German '*Anzahl*' by 'cardinal number', and '*Zahl*' by 'number'. This is in contrast to Furth's choice to translate '*Anzahl*' using the capitalised 'Number' while translating '*Zahl*' as 'number'. 'Cardinal number' exactly covers Frege's intended technical use of '*Anzahl*'; by '*Zahl*', on the other hand, Frege intends a wider class of numbers, at least including the reals (compare his characterisation of the contrast between '*Anzahl*' and '*Zahl*' in vol. II, §157, p. 155–156). There are, however, two special cases worth mentioning. Firstly, Frege once uses '*Anzahl*' in a non-technical context where the English 'cardinal number' seems inappropriate. In that case only we use 'number' instead (vol. I, p. VI): "One must strive to reduce the number ['*Anzahl*'] of these fundamental laws as far as possible by proving everything that is provable." Secondly, throughout *Grundgesetze* Frege only once uses the German expression '*Nummern*' (vol. I, p. 70, §53) which we translate as 'number' as well. These occurrences are marked by translators' notes—all other occurrences of 'number' correspond to the German '*Zahl*'; all occurrences of 'cardinal number' correspond to the German '*Anzahl*'.

[9] This was previously published in Black (1950).

Aussage

The German '*Aussage*' is to be translated as 'predication'. For example, Frege's insight of §46 of *Grundlagen*, which he repeats in vol. I, p. IX, thus reads 'a statement of number contains a predication about a concept' in our translation. Austin uses 'assertion' in his translation of *Grundlagen*, but his choice is not quite correct. An '*Aussage*' in the sense relevant here can be made even when no assertion is involved, for example in questions, hypotheses, or in the antecedent of conditionals. There are a number of occurrences of '*Aussage*' that clearly indicate that Frege's use of the term is best captured by 'predication' and not 'assertion' (see for example vol. I, p. XXI).[10] We made an exception on p. XX, where we use 'Erdmann's statements' to capture the phrase '*Erdmanns Aussagen*'.

When it comes to the verb '*aussagen*', however, we did not always use 'to predicate'. For example, we translate '*das Prädicat* algebraisch *von einer Curve ausgesagt*' (vol. I, p. XIX) as 'the predicate *algebraic* as applied to a curve'—in addition to the obvious problem, 'the predicate *algebraic* as predicated' gives the false impression that cognates are used in the original. Moreover, in vol. II, §65, p. 76, and on some other occasions, we use the verb 'to say' for '*aussagen*' since 'to predicate' does not quite capture the intended meaning in these passages; to wit: "Consequently, one could truthfully say [*aussagen*] neither that it coincides with the reference of 'One' nor that it does not coincide with it."

Bedeutung, bedeuten, andeuten

Translators of Frege's writings face the difficulty of translating '*Bedeutung*' and its cognates '*bedeuten*', '*gleichbedeutend*', '*bedeutungsvoll*', etc. Furth uses 'denotation' (and cognates) in his translation of part I of *Grundgesetze*. Geach and Black's volume uses 'meaning' (and cognates) from the third edition onwards; in the first two editions, 'reference' is used for '*Bedeutung*' and 'stand for' or 'designate' for '*bedeuten*'. The changes to Geach and Black's third edition, however, were not implemented consistently which led to some confusion.[11] Beaney discusses at length the problems of the various options and ultimately decides to leave the German '*Bedeutung*' untranslated in his *Frege Reader*.[12] The main difficulty that Beaney, as well as Geach and Black, face is that they translate works that span Frege's entire career. That means, in particular, that they translate writings both before and after his 1892 article "*Über Sinn und Bedeutung*", in which he draws his famous, eponymous distinction. Given that after 1892 Frege takes the *Bedeutung* of a term to be the object referred to/denoted by that term, we decided to make this aspect clear and use 'reference' instead of Geach and Black's 'meaning'. The reason we decided to adopt 'reference' and not to follow Furth in using 'denotation' is three-fold: firstly, 'reference' as a translation of '*Bedeutung*' in Frege's writings after 1892 is better entrenched in the literature than 'denotation' (it is Frege's famous "sense/reference distinction"); secondly, 'denotation' has the

[10] See Dummett (1991), p. 88, and Künne (2009), p. 422, for further criticism of Austin's choice of translation.

[11] See in particular Beaney (1997), p. 46, fn. 106, for a discussion of the "unsystematic nature" of these changes.

[12] See his extensive discussion on the difficulty of translating '*Bedeutung*' on pp. 36–46 in Beaney (1997).

ring of an artificial technical term that both 'reference' and '*Bedeutung*' lack; thirdly, some of the cognates of '*Bedeutung*' are more easily translated using cognates of 'reference'. '*Gleichbedeutend*', for example, can easily be translated as 'co-referential', while 'co-denotational' seems somewhat unnatural;[13] the same holds of the triple '*bedeutungsvoll*', 'referential', 'denotational'.

There are two further important decisions we made in our translation that relate to '*Bedeutung*'. Firstly, when the term '*Bedeutung*' occurs in works of other authors such as Peano (vol. II, §58, fn. 1) or Thomae ('*formale Bedeutung*', vol. II, §97), we decided to opt for uniformity and use 'reference' ('formal reference' for Thomae) in our translation. The main reason is that Frege discusses these quotations in his own text and here too uses the term '*Bedeutung*'. It would seem awkward to have Frege use 'meaning', or whatever alternative translation may be used for '*Bedeutung*', in these passages. Moreover, it would obscure the fact that Frege takes the quoted author's use of '*Bedeutung*' to be in line with his specific use of '*Bedeutung*'. As a result, whenever the term 'reference' (and cognates) occurs in our translation it will correspond to the German '*Bedeutung*' (and cognates), whether it is Frege's writings or the writings of other authors' Frege is quoting. Indeed, our translation of '*Bedeutung*' is single-valued in both directions: no other English term is used as translation of '*Bedeutung*' and *vice versa*. Thus, although 'mean' occurs in our translation, it never occurs as a translation of '*bedeuten*'; instead, it translates '*heissen*' or '*meinen*'. Lastly, Frege uses '*andeuten*' to describe the function of Roman letters in order to draw a distinction between a name that *refers to* (*bedeutet*) an object, and a Roman letter (i.e., one of his devices for generality) that merely *indicates* (*andeutet*) an object. Unfortunately, the fact that 'be*deuten*' and 'an*deuten*' have the same stem is lost in translation.

Begriffsschrift

We translate the German '*Begriffsschrift*' as 'concept-script' when Frege is referring to his system of logic or the formal language it is formulated in but we leave the term untranslated when Frege refers to his 1879 book *Begriffsschrift*. Throughout our translation, we use original titles of Frege's works in italics, e.g., '*Die Grundlagen der Arithmetik*'; the same applies to titles of works by other writers that Frege is citing.

Begründung, Grundlage

The German word '*Begründung*' presents a difficulty. It can be translated either as 'foundation', or even 'basis' (see vol. II, p. 154), to indicate that Frege is offering a foundation for a theory, for example, for arithmetic; or it can be used to state that a justification is provided for a certain principle without necessarily also providing a foundation. As a result, we translate '*Begründung*' as 'foundation'/'basis' or 'justification', depending on context. A further complication arises, given that we also use 'foundation' as a translation of '*Grundlage*': to provide a *Grundlage* is to provide a *Begründung* in the foundational sense noted above.

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[13] Furth translates '*gleichbedeutend*' using either 'has the same denotation' or 'becomes the same in meaning' (see e.g. vol. I, §27, p. 45) suggesting that Frege is not using the term in the technical sense in the latter passage. We disagree with Furth's assessment and use 'co-referential' here as elsewhere.

Bestimmung, bestimmen

We translate '*Bestimmung*' (and cognates) either as 'determination' (and cognates) or as 'specification' (and cognates). In particular in passages where confusion is possible—e.g., when 'determination' could be misunderstood as 'resolution' or 'willpower'—we use 'specification' (and cognates), but we also do so where 'specify' is significantly more idiomatic than 'determine'.

Definition, Erklärung, Erläuterung

Most translators have not distinguished between Frege's uses of the terms '*Erklärung*' and '*Definition*', but use the English word 'definition' as a translation for both. Admittedly, Frege sometimes uses the verb '*erklären*' or the noun '*Erklärung*' to describe his definitions. However, there are other occurences of these terms where he does not intend to give a definition (cf. vol. I, §8, p. 12). There is no satisfactory account of what precisely distinguishes a Fregean '*Erklärung*' from a '*Definition*' and how they relate—nevertheless, Frege uses different words in the original text, and we decided to respect the distinction in our translation. Thus, in our translation, 'definition', 'define', always stand for '*Definition*', '*definieren*', respectively (and *vice versa*), while nearly all occurrences of '*Erklärung*', '*erklären*' are translated as 'explanation', 'explain'. The only exceptions appear in two passages (on p. XIII and in §6, vol. I) where we use 'to declare' as a translation of '*erklären*', as Frege here uses the verb in this sense (compare '*Erklärung der Menschenrechte*': 'declaration of human rights').

It is worth noting that both *Definition* and *Erklärung* are distinct from a Fregean *Erläuterung*, for which we choose the term 'elucidation'. An elucidation may be provided to assist the understanding of a definition or explanation: it is a mere heuristics that is, strictly speaking, inessential and can be imprecise without thereby undermining the correctness and precision of a proof that uses the technical notion (compare vol. I, §§34–35.) The purpose of an elucidations is simply to aid the reader and point in the right direction until the definition proper is given or when the technical notion in question is primitive (and thus governed by a basic law).

Lastly, we should note that both '*Definition*' as well as '*Definieren*'—the latter being a substantivised verb referring to the act of giving definitions—are translated as 'definition'. Only in cases in which the context does not clearly disambiguate between those two uses of 'definition', we use the gerund 'defining' as a translation of '*Definieren*' (see, for instance, vol. I, p. XXV: 'mathematician's defining').

Eindeutigkeit

We translate '*eindeutige Beziehung*' as 'single-valued relation', and likewise '*eindeutige Zuordnung*' as 'single-valued correlation'. In contrast, Furth uses 'many-one' (as in 'many-one relation' and 'many-one correspondence'). The phrase '*eindeutig bestimmt*', however, is translated as 'uniquely determined', as we wish to avoid the impossible phrase 'single-valuedly determined'. This should not lead to any misunderstandings, as 'unique relation' instead of 'single-valued relation' would. '*Beiderseits eindeutig*' is translated as 'single-valued in both directions' instead of the phrase 'one to one' that Austin uses in his translation of *Grundlagen*. We avoid introducing a new technical term where the same technical term is reused in the original.

Erkenntnis

'*Erkenntnis*' is standardly translated as 'knowledge' or 'cognition'. There are well-known difficulties with the translation of '*Erkenntnis*' in Kant (think, in particular, of the difference between '*Wissen*' and '*Erkenntnis*') as well as in other writings, such as Carnap's. One problem is that '*Erkenntnis*' can be used to indicate a certain process of coming to know—as in '*Erkenntnisthat*' ('act of cognition', vol. I, p. VII)—or the result of this process—'*Als Ziel muss die Erkenntnis dastehen*' ('Knowledge must stand as the goal', vol. II, p. 101). We thus did not translate '*Erkenntnis*' uniformly: we use 'knowledge' or 'cognition', and once 'insight' (vol. II, p. 85), depending on context. To allow the reader to track Frege's use of '*Erkenntnis*' (and to distinguish it from occurrences of 'knowledge' as a translation of '*Wissen*', as, e.g., in vol. II, §56), we mark each occurrence of a translation of '*Erkenntnis*' with a translators' note.

Festsetzung, festsetzen

There are two exceptions to our translation of '*Festsetzung*' and '*festsetzen*' as 'stipulation' and 'stipulate'. In the foreword, p. XV, we use 'legislate' to reflect a legal connotation of '*festsetzen*'. In this passage, Frege is discussing the normative aspect of laws of thought and the extent to which they 'legislate' ('*festsetzen*') how one ought to think. Given that laws do not, strictly speaking, 'stipulate' anything in the sense in which 'stipulate' is used elsewhere in *Grundgesetze*, Frege's use of '*festsetzen*' here is better captured by 'legislate'. In vol. II, p. 94, §83, we use 'fix' instead of 'stipulate' as a translation of '*festsetzen*': 'These definitions can fix the references of the new words and signs with at least the same right as [...]'. Using 'stipulate' here could give the impression that Frege intends to stipulate objects into existence, while the German original clearly does not invite this reading.

Ganze Zahl

'Integer' is usually chosen as a translation of '*ganze Zahl*', as, e.g., in Geach and Black's translation. In vol. II, §101, however, Frege quotes Thomae, who uses '*ganze Zahl*' intending the positive integers only. In other passages, Frege uses '*ganze Zahlen*' to encompass all integers (compare vol. II, §57, where Frege speaks of positive as well as negative *ganze Zahlen*). We chose 'whole number' as a translation since it has a similar ambiguity in English, despite some preference amongst mathematicians to use the term to designate only the positive integers. A further advantage is that 'number' is retained as a constituent of the expression 'whole number', as it is the case with '*Zahl*' and '*ganze Zahl*'.

Gleichheit, Gleichung, gleich, Identität

All occurrences of '*Identität*', '*Gleichheit*', and '*Gleichung*' are translated as 'identity', 'equality', and 'equation', respectively. The adjective '*gleich*' is usually translated as 'equal', with some exceptions: for example, in vol. I, p. 62b, we use 'common', vol. I, p. 63b, 'the same as', and vol. II, p. 73, 'same'.

Relation

We translate '*Beziehung*' as 'relation' and the German term '*Relation*' using the capitalised English term 'Relation'. '*Relation*' is defined as '*Umfang einer Beziehung*' (extension of a relation). There simply is no other suitable English word apart from 'relation' that we could use as a translation of the German '*Relation*'—or indeed of '*Beziehung*'. We had no choice but to resort to the technique of capitalising an already used expression to capture an important difference in the original. The alternative, employed by Furth, as well as Geach and Black, in their translations of the afterword, is to use 'extension of a relation' as a translation of the German '*Relation*'. This might be acceptable given that the phrase only occurs twice in the afterword, but it is not a viable option for part III, where Frege introduces '*Relation*' as a technical term for 'extension of a relation' and uses it abundantly.

Satz

The translation of '*Satz*' was subject to much discussion with our advisors (see *Acknowledgments* below). Stachelroth and Jordain do not opt for a consistent translation and use 'theorem' as well as 'proposition'. Furth uses 'proposition' throughout; Geach and Black sometimes use 'sentence' and sometimes 'proposition'. We initially translated '*Satz*' as 'sentence', partly motivated by Frege's introduction of the term '*Satz*' in vol. I, §5, as short for '*Begriffsschriftsatz*'. He introduces the term to refer to the concept-script *representation* of a judgement—i.e., the judgement-stroke followed by a formula in concept-script notation—and in doing so gives '*Satz*' a decidedly syntactic flavour. Moreover, Frege's own index to the first volume of *Grundgesetze* contains '*Satz*' and refers to pages 9 and 44, and thus to passages where Frege describes *Sätze* as signs (*Zeichen*).

However, after much discussion, we decided to follow Furth and use 'proposition' rather than 'sentence' throughout volume I and II: it better captures the numerous ways in which Frege uses '*Satz*' while allowing for a uniform translation of the German term. Certain phrases—'sense of a proposition', or 'the proposition "Scylla had six dragon gullets"'—sound somewhat unusual to a modern ear, but readers should quickly get used to this type of phrasing. Obviously, Frege does not intend the modern sense of 'proposition' (i.e., what is expressed by a sentence), as he has his own terminology for objects of this sort: what a sentence expresses for Frege is its sense—that is, a *thought*.

In general, it seems that Frege's conception of '*Satz*' is something that essentially possesses both syntactic and semantic properties; nowadays, we might call this an 'interpreted sentence'. Yet, for Frege, a *Satz* is not something that could be characterised as a sentence that is interpreted, because this would suggest the possibility of an uninterpreted sentence or the reinterpretation of a sentence, which he dismisses.[14]

As Furth eloquently writes in his translator's introduction (Frege (1964), pp. lv–lvi),

> [t]he rendering of '*Satz*' presents great difficulties. In various contexts it can mean sentence, theorem, proposition, clause. In some of Frege's other writings (of 1891 and later) he uses it for that variety of expression (name) which, for him, denotes a truth-value; in such cases 'sentence'

[14] See, in particular, the so-called *Frege–Hilbert debate* in Gabriel et al. (1976), XV/3–9.

> would be appropriate. In this work however, '*Satz*' is almost without exception applied to expressions *with a judgment-stroke prefixed* and [...] such expressions, for Frege, are *not* names. 'Assertion' might then be considered, yet it seems that '*Satz*' ought to be rendered differently from '*Behauptung*'. 'Theorem' is ruled out as both too wide and too narrow: too narrow because Frege applies '*Satz*' quite generally, and not merely to theorems of his logical theory; too wide because, for example, in his discussion of the Russell paradox where he shows that a self-contradictory statement can in fact be derived from the axioms, Frege gives the derivation informally and does *not* prefix the suspect expressions with the judgment-stroke, apparently on the ground that although they are indeed (unhappily) theorems, he does not believe that they are *true*. Thus we are forced onto 'proposition'. Some later writers have used this word for the *sense* expressed by a sentence, Frege's *Gedanke*. Therefore the reader must take care here to understand "proposition" in something nearer to its vague English meaning of a 'propounding'. The situation is unsatisfactory, but Frege has left the translator little choice.

We should remark, however, that '*Satz*' also occurs in compound nouns such as '*Behauptungssatz*', '*Lehrsatz*', and '*Bedingungssatz*' (e.g., vol. II, §§65, 140). We translate these phrases as 'declarative sentence', 'theorem', and 'conditional clause', respectively.

Selbstverständlich, einleuchten

We translate '*selbstverständlich*' as 'self-evident' when it is used in a technical epistemic sense, and as 'evidently' when it is used colloquially. Further problems are raised by the translation of '*einleuchten*'. In contrast to other scholars,[15] we do not think the term is best rendered using 'self-evident' as well. Frege uses the term to indicate that something, such as a proof, is easily understood or accepted. For example, in vol. I, p. VIII, Frege writes: "*Man begnügt sich ja meistens damit, dass jeder Schritt im Beweise als richtig einleuchte*", which we render as "Mostly, no doubt, one contents oneself with the obvious correctness of each step in a proof" (similarly, on p. 1, vol. I., which is the only other occurrence of the term in volume I). In volume II, the phrase is used most prominently in §156, p. 154: "*Dass kein Widerspruch bestehe—meinen nun wohl Manche—leuchte unmittelbar ein, da* [...]", which we translate as "That there is no contradiction—some may now claim—is immediately obvious since [...]". Clearly, using 'self-evident' here would not express what Frege intends. Perhaps most importantly, Frege uses the term in the afterword when he writes about Basic Law V: "*Ich habe mir nie verhehlt, dass es nicht so einleuchtend ist, wie die andern, und wie es eigentlich von einem logischen Gesetze verlangt werden muss*", which we translate as "I have never concealed from myself that it is not as obvious as the others nor as obvious as must properly be required of a logical law". The only exception to our choice of translation occurs in §140, p. 142: "*die Gesetze in einleuchtender Weise zu entwickeln*" is translated as "a lucid development of the laws"—"obvious development" would be misleading.

[15] See for example Jeshion (2001).

vertreten

Frege uses '*vertreten*' in a formal context in relation to small Greek letters, i.e., such a letter *vertritt* an argument place. In such contexts, we use 'proxy for' as our translation. Frege also writes that functions can be "*vertreten durch ihre Werthverläufe*", a phrase we translate as 'represented by their value-range' (compare vol. I, §25). We here follow Furth in making this distinction. When Frege uses '*vertreten*' in a less formal context we use either 'stand in for' or, where appropriate, 'represent' (in particular, in the discussion of Thomae in vol. II, §§131–132). It is worth noting that we also use 'represent' as a translation of '*darstellen*'. In context, this conflation will not lead to any misunderstandings.

Vorhanden sein, Bestand haben, existieren

There is a temptation to use 'exist' to translate '*vorhanden sein*' or the phrase '*Bestand haben*'. However, since 'exists' is a second-level predicate according to Frege, expressions like 'a thing exists' are, strictly speaking, ill-formed. Moreover, the fact that Frege uses the German '*bestehen*' or '*vorhanden sein*' rather than '*existieren*' on many occasions is something we wanted our translation to reflect. We thus treat '*existieren*' (and cognates) as a technical term to be translated as 'exist' (and cognates). For the German '*vorhanden*' we sometimes use 'is present' or simply the verb 'to be', depending on context. In the case of '*Bestand haben*' (e.g., vol. I, p. XXIV) we use the phrase 'has being'. We also note that the German phrase '*Bestand ausmachen*' is translated as 'constitute'.

In vol. II, §155, p. 153, fn. 2, Frege refers to Kant's criticism of the ontological argument for the existence of God. Here, the German phrase we translate as 'existence of God' is not '*Existenz Gottes*', but rather '*Dasein Gottes*'.

Vorstellung

'*Vorstellung*' presents another notorious difficulty when translating German texts into English. A natural choice is 'idea' but often a '*Vorstellung*' is a '*Vorstellung*' of something, which has led some translators to use 'representation' instead. The latter has an additional advantage: the verb '*vorstellen*' can be translated as 'represent', allowing cognates to be retained in the translation.

But this immediately gives rise to a difficulty: the German '*vertreten*' too is naturally translated as 'represent' (see above). Moreover, a '*Vorstellung*' is always subjective and requires a personal bearer; it is not clear that 'representation' carries this connotation as strongly as 'idea' does, if at all. An option may be 'mental representation', if it was not for the somewhat technical feel. Also, the noun '*das Vorstellen*' cannot easily be rendered using cognates of 'represent' (only the cumbersome 'the act of representing' could be entertained).

We opt for 'idea' and use the (admittedly uncommon) verb 'to ideate' as the translation of '*vorstellen*'.[16] Also, we translate '*Vorstellen*' as 'ideation' and '*das Vorgestelle'* as 'what is ideated', thereby respecting in our translation the fact that

[16] It is worth noting that 'ideate' was not "created" by Furth for his translation, as sometimes suggested. In fact, the verb 'to ideate' has been in use since the late 17th century and can be found in the Oxford English Dictionary.

the terms are cognates in German. These terms occur in the introduction to volume I, where Frege is criticising Benno Erdmann—the Erdmann quotes proved to be some of the most difficult passages to translate in all of *Grundgesetze*. Erdmann's arguments for strongly idealist conclusions provide further reason to use 'idea': it carries the right connotation.

der Wahrheitswerth davon, dass

The phrase '*der Wahrheitswerth davon, dass*' recurs on numerous occasions. For instance, in vol. I, §5, p. 10, Frege writes:

> *Wir können also sagen, dass*
>
> $$\Delta = (\text{—}\, \Delta)$$
>
> *der Wahrheitswerth davon ist, dass* Δ *ein Wahrheitswerth sei.*

Furth does not translate the phrase uniformly. Often, he adopts a gerundial construction. For instance, he translates the sentence above as:

> We can therefore say that
>
> $$\Delta = (\text{—}\, \Delta)$$
>
> is the truth-value of Δ's *being a truth-value.*

This, however, generates a number of difficulties. Firstly, there are stylistic reasons why this construction should be avoided. For example, vol. I, §8, p. 13, Furth translates:

> But if we want to designate the truth-value of the function
>
> $$(\xi + \xi = 2\,.\,\xi) = (\overset{\mathfrak{a}}{\smile}\, \xi = \mathfrak{a})\text{'s}$$
>
> *having the True as value for every argument*, then [...]

The possessive 's' appended to a formula is probably best avoided in any case; but here the gerund does not aid comprehension either. Secondly, this option requires the introduction of italics where there are none in the original. Finally, a gerund construction seems to refer to a concept, rather than a proposition, and so it would not normally be the bearer of a truth-value. As a result, we experimented with a number of alternatives: among them, 'is the truth-value of the proposition that ...'—which would have required the addition of 'proposition'—and Alonzo Church's 'is the truth-value thereof that ...' (as used in Church (1951), p. 108) which barely sounds English. In the end, we opted for the construction 'the truth-value of: that ...'. This adds a colon where there is none in the original, but it ensures making clear what the truth-value is a truth-value of, namely the ensuing proposition. Hence, the above two occurrences are now translated:

> We can accordingly say that
>
> $$\Delta = (\text{—}\, \Delta)$$
>
> is the truth-value of: that Δ is a truth value.

and

> If, however, one wants to designate the truth-value of: that the function
>
> $$(\xi + \xi = 2\,.\,\xi) = (\overset{\mathfrak{a}}{\smile}\, \xi = \mathfrak{a})$$
>
> has the True as value for every argument, then [. . .]

Werthverlauf

We decided to translate '*Werthverlauf*' and '*Werthverläufe*' using the English 'value-range' and 'value-ranges'. Furth uses the unwieldy 'course(s)-of-values'. Recent Frege scholarship also seems to prefer 'value-range'.[17] Moreover, we translate '*rechter Werthverlaufsname*' (vol. I, §31, p. 49) as 'regular value-range name' instead of 'fair course-of-values-name' (Furth).

Wortsprache

Literally, '*Wortsprache*' would be 'word-language', in contrast to an "artificial" symbolic language, and, in particular, to Frege's concept-script. Accordingly, 'natural language' might seem to be an option, but Esperanto, for instance, would count as a *Wortsprache*, while it is not a natural language. We thus translate '*Wortsprache*' as 'ordinary language'. While one may hesitate to call Esperanto an "ordinary" language, it is less jarring than calling it a "natural" language. Note that we also translate '*gewöhnliche Sprache*' as 'ordinary language'. Our translation is less than ideal; however, the text gains in readability. Nevertheless, note that some aspects of the sense of '*Wortsprache*' may be lost in its translation as 'ordinary language'.

Zeichen, bezeichnen

We generally use 'sign' rather than 'symbol' as a translation of '*Zeichen*'. This choice is in part motivated by our preference for using English cognates where cognates are used in the German original. We translate '*Bezeichnen*' as 'designate' and '*Bezeichnung*' as 'designation' (although sometimes the use of 'notation' was unavoidable), and hence 'sign' is the preferred translation for most occurrences. However, Frege also uses the plural '*Zeichen*' in phrases like '*in Zeichen*' or '*mit meinen/unseren Zeichen*' to indicate in what follows the thought in question is expressed in concept-script. We translate 'in signs' or 'in my/our symbolism', depending on the context (see for example vol. I, p. V).

Zuordnung, zuordnen

In part III, Frege discusses the theories of the real numbers by Cantor and by the formalists. According to these approaches, a certain real number is *assigned* to a series or sequence of numbers. The German word we translate as 'assign' here is '*zuordnen*'; indeed, we translate all occurrences of '*zuordnen*' as 'assign', and all occurrences of '*Zuordnung*' as 'assignment' in part III. Using 'correlate' and 'correlation' as a translation in these contexts would be inappropriate because '*Zuordnung*' here points

[17] See, for example, the recent Potter and Ricketts (2010).

to a priority—numbers are assigned to certain series or sequences and are thereby defined. This sense is better captured by 'assignment' than 'correlation'. However, in parts I and II, Frege uses '*zuordnen*' and '*Zuordnung*' in the context of his own theory, and we use 'correlate' and 'correlation' as translations. No priority is suggested here. Rather, objects falling under a certain concept are correlated (*zugeordnet*) with objects falling under a different concept. 'Correlation' rather than 'assignment' is here the appropriate translation of '*Zuordnung*'.

Note that 'assign' occurs on a number of occasions in vol. I, but only as a translation of German words other than '*zuordnen*'—for example, in §14, p. 25: "assign a label [to a proposition]" ("[*einem Satz*] *ein Abzeichen geben*"). This should not lead to any serious misunderstandings.

Glossary of technical terms

Allgemeingültigkeit	general validity
Allgemeingewissheit	general certainty
andeuten (*unbestimmt*)	indicate (indeterminately)
Anzahl	cardinal number
Anzahlreihe	cardinal number series
Aussage	predication
Bedeutung	reference
bedeutungslos	without reference
bedeutungsvoll	referential
Bedingungsstrich	conditional stroke
Begriff	concept
Begriffsschrift [book]	*Begriffsschrift*
Begriffsschrift [system, language]	concept-script
bestimmen	determine, specify
bezeichnen	designate
Bezeichnung	designation, notation
Beziehung	relation
Definition	definition
Definitionsdoppelstrich	double-stroke of definition
Definitionsstrich	definition-stroke
deutscher Buchstabe	German letter
Doppelwerthverlauf	double value-range
eindeutig(*e Beziehung*)	single-valued (relation)
eindeutig bestimmt	uniquely determined
Eigenname	proper name
endlich	finite
Endlos	Endlos
endlos fortlaufen	proceed endlessly
erfüllen	fill in, instantiate (for concepts)
ergänzen	complete (for argument-places)
Erkenntnis	knowledge, cognition, insight
Erklärung	explanation
Erläuterung	elucidation

festsetzen	stipulate, fix
Festsetzung	stipulation
Folge	sequence
folgen (*in einer Reihe*)	following (in a series)
Forderungssatz	postulate
formale Arithmetik	formal arithmetic
Function	function
Fürwahrhalten	taking to be true
ganze Zahl	whole number
Gegenstand	object
gekoppelt(*e Beziehung*)	coupled (relation)
gesättigt	saturated
Gestalt, *gleichgestaltet*	shape, equal-shaped
gleichbedeutend	co-referential
Gleichheit	equality
Gleichung	equation
gleichstufig(*e Beziehung*)	equal-levelled (relation)
gleichzahlig	equinumerous
griechischer Vokalbuchstabe	Greek vowel
Grösse	magnitude
Grössengebiet	domain of magnitudes
Grössenverhältnis	magnitude-ratio
Höhlung	concavity
Inhalt	content
inhaltliche Arithmetik	contentual arithmetic
Identität	identity
Kennzeichen (*zur Wiedererkennung*)	criterion (for recognition)
Klasse	class
lateinischer Buchstabe	Roman letter
Lehre	theory
Marke (*Functions-*, *Gegenstands-*)	marker (function-, object-)
Maasszahl	measuring number
Menge	collection, set
Merkmal	characteristic mark
Null	Zero
Nullrelation	null Relation
Oberglied	supercomponent
Positivalklasse	positival class
Positivklasse	positive class
Quantität	quantity
Relation	Relation [capitalised]
Reihe	series
reihende Beziehung	series-forming relation
Satz	proposition
Sinn	sense
Spiritus lenis	smooth breathing
Stufe	level
Theorie	theory

übergeordnet	superordinate
Umfang	extension
Umkehrung (*einer Beziehung*)	converse (of a relation)
unendlich	infinite
ungesättigt	unsaturated
ungleichstufig(*e Beziehung*)	unequal-levelled (relation)
unscharf begrenzt	without sharp boundaries
untergeordnet	subordinate
Unterglied	subcomponent
Urtheil	judgement
Urtheilsstrich	judgement-stroke
vertreten	represent, stand in for
Verneinungsstrich	negation-stroke
Verschmelzung	fusion
Vorstellung	idea
Wendung	contraposition
Werthverlauf	value-range
Wortsprache	ordinary language
Zahl	number
Zahlangabe	statement of number
Zahlgrösse, *Zahlengrösse*	numerical magnitude
Zahlzeichen	number-sign
Zeichen	sign
zugehörige Function	corresponding function
zuordnen	correlate, assign
zusammengesetzte Beziehung	composite relation
zusammengesetzter Name	complex name
zusammensetzen (*Beziehungen*)	compose (relations)
Zwischenzeichen	transition-sign

General remarks on the translation

We follow the original pagination of Frege's *Grundgesetze*, and we also respect Frege's use of columns. Given that our translation usually differs in length from the original, some pages contain more text than others, and sometimes there is a blank space between main text and footnotes. This minor cosmetic oddity is outweighed by the benefits of respecting the original pagination. We also follow Frege's original numbering of footnotes. As in the original, Frege's notes are listed at the bottom of the page and are numbered using arabic numerals, relative to page or indeed column. Translators' notes, in contrast, are indicated by small Roman letters and appear as endnotes. We corrected obvious and minor typos in the original without explicitly acknowledging it. Substantial typographical errors are corrected and listed in a section labelled "Corrections" which is appended to Frege's text. We made use of Frege's own *corrigenda* to both volumes, and also those suggested by Christian Thiel in a recent edition of *Grundgesetze* (Frege, 1998); moreover, we added a number of corrections. Two corrections suggested by Scholz and Bachmann which Thiel mentions have been omitted: they are incorrect.

We do not translate the titles of books or articles that Frege cites—with only one exception. In a footnote on p. 106, vol. II, Frege quotes the title of an article, and we translate the quotation into English. Frege is not merely referencing the article; the title contains the phrase "a function of given *letters*" ("*eine Function gegebener* Buchstaben"), which Frege uses as evidence to support his claim that the confusion of signs and their reference is rampant among the mathematicians of his time. The point would be lost if the title remained untranslated.

We added a bibliography containing all publications Frege cites, providing complete bibliographical information and, where available, references to English translations of these works. The translations of passages that Frege quotes from other writers are our own, regardless of whether English translations of these works exist.

We followed a principle of exegetical neutrality. As a result, our translation is usually close to the original. Technical terms are translated uniformly and not translated away. Passages that are ambiguous or otherwise unclear are often purposefully translated so as to retain the unclarities. We did not attempt to "improve" on the original; our goal was to translate the text so that it is suitable for scholarly work.

We made use of a number of dictionaries for our translation, in particular: Langenscheidt's *Fachwörterbuch Mathematik: Englisch–Deutsch–Französisch–Russisch*, fourth edition, 1996, for mathematical terminology. We drew on *The New and Complete Dictionary of the German and English Languages*, by Johann Ebers, published in 1796, H. E. Lloyd and G. H. Nöhden's *New Dictionary of the English and German Languages*, 1836, Georg W. Mentz's *New English–German and German–English Dictionary Containing All the Words in General Use*, 1841, and *Cassell's German and English Dictionary* by Karl Breul, 1909, to provide a better idea of 19th-century usage of German and English words. We consulted the *Oxford English Dictionary*, the *Oxford Thesaurus for English*, the *Oxford German Dictionary*, the *Langenscheidt English–German Dictionary*, the Macintosh OS X dictionary, and various online resources such as the *LEO German–English Dictionary* (`http://dict.leo.org`).

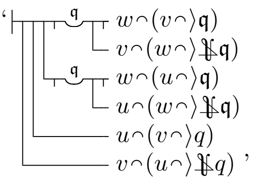
Typesetting *Grundgesetze*

The question might naturally arise: since we translated *Grundgesetze* prose into English, why did we not also "translate" Frege's formulae into modern notation? Early on in the project, we briefly considered such a change but decided against it. There are several reasons that decisively speak against such an endeavour.

Firstly, Frege elucidates his formalism in the first forty-six sections of volume I, and in doing so mentions (rather than uses) his notation and describes the formulae and their components. A rendering of his formalism in modern notation would have made these passages nonsensical. Keeping Frege's formalism up to §46 of the first volume and changing over to modern notation thereafter was obviously not an option. Rewriting the prose in which Frege mentions and describes his formalism—e.g., exchanging 'subcomponent' with 'antecedent'—would not have led to a translation suitable for scholarly purposes, and thus was not an option either.[18]

[18] This last strategy was adopted in a new German edition of *Grundgesetze*, see Frege (2009), whose purpose is different from ours.

Moreover, transforming Frege's notation into a more familiar formalism would generate the need for numerous parentheses which would hinder readability. For instance, the fairly easily readable proposition (25) of vol. I, p. 83:



turns into an unsurveyable forest of parentheses in modern notation:

$$\text{‘}\vdash (v\frown(u\frown)\not\backslash q) \supset (u\frown(v\frown)q) \supset (\neg\forall\mathfrak{q}(u\frown(w\frown)\not\backslash\mathfrak{q}) \supset \neg w\frown(u\frown)\mathfrak{q})) \supset \neg\forall\mathfrak{q}(v\frown(w\frown)\not\backslash\mathfrak{q}) \supset \neg w\frown(v\frown)\mathfrak{q})))))\text{’}.$$

Adopting the left-association convention for embedded conditionals in order to reduce the number of brackets provides little improvement:

$$\text{‘}\vdash v\frown(u\frown)\not\backslash q) \supset u\frown(v\frown)q) \supset \neg\forall\mathfrak{q}(u\frown(w\frown)\not\backslash\mathfrak{q}) \supset \neg w\frown(u\frown)\mathfrak{q})) \supset \neg\forall\mathfrak{q}(v\frown(w\frown)\not\backslash\mathfrak{q}) \supset \neg w\frown(v\frown)\mathfrak{q}))\text{’}.$$

Proposition (25) is far from being the longest or most complicated formula when it comes to the structure of embedded conditionals. There is little value in rendering Frege's formulae in this way as far as readability is concerned. Attempting to make the modern rendering more surveyable by choosing equivalent formulae that utilise signs for conjunction and disjunction, as well as those for the conditional and negation, would clash with Frege's rules of inference. In fact, Frege's rules may in any case seem bewildering, albeit valid, when applied to modern formulae, while they are natural and, dare we say, elegant in the concept-script system.

Finally, despite all similarities, concept-script differs in significant respects from modern logic—compare Roy Cook's appendix to this volume for details. Presenting Frege's logic in the formalism of modern second-order predicate logic would obscure this fact and fail to do justice to his system.

Thus, the only responsible way to render Frege's notation from a scholarly perspective—namely, rendering it in exactly the way Frege did—turned out to be the only sensible solution on the whole. Frege's concept-script is unfamiliar to most, and perhaps somewhat more difficult to learn than modern notation. But learning to understand Frege's formalism is certainly not an insurmountable task, and once some familiarity is achieved, Frege's system is surprisingly easy to read. Working through Frege's own introduction to the concept-script, the reader will soon appreciate this. Roy Cook's appendix will further aid the reader in this process and offer additional help in understanding the details of some of the more arcane features of Frege's system.

Typesetting concept-script could only take one form: using TEX. When we started our project, there was, however, no straightforward way to do so. TEX-expert Josh Parsons (post-doc at Arché, St Andrews, at the time) wrote the first LATEX-style for concept-script (the *begriff* package, which is now included in most TEX-distributions) that allowed us to typeset the formulae in an elegant way.[19] Josh's style-file renders

[19] LATEX is the widely used macro-package for the TEX typesetting system.

concept-script formulae as they appear in Frege's first book, *Begriffsschrift*. With the help of Richard Heck, J. J. Green, and Agustín Rayo, we wrote a style file based on Josh's that allowed us to render concept-script formulae in the way they appear in *Grundgesetze*. It required many changes and additions, but being able to build on Josh's work made the task significantly less demanding.

This type of free collaboration is of course characteristic of anything to do with TeX. TeX is open-source freeware, which is continually developed, improved, and supplemented by thousands of enthusiasts. For our translation, we used more TeX resources than we knew existed before we started. We are indebted to this great community without whom we could not have produced this edition. We would also like to thank the creators and community surrounding TeXShop and TeXLive.

All of Frege's formulae had to be rendered in LaTeX-code, and despite the convenience of LaTeX, doing so by typing up each formula using only a keyboard would have been even more daunting a task than it turned out to be. Robert MacInnis (then a computer-science student at St Andrews), under the supervision of Roy Dyckhoff, wrote a graphical user interface (GUI) that enables one to create concept-script formulae by mouse-click and outputs a choice of LaTeX- or XML-code (see MacInnis et al. (2004)). Some bug-fixes and adjustments of the software to suit our specific purposes were implemented by Guðmundur Andri Hjálmarsson (then a philosophy Ph.D. student at Arché, St Andrews).

In addition to the unfamiliar representation of the logic, Frege employs symbols for defined functions, such as ‘ꝑ’, ‘ℳ’, ‘ɟ’, ‘Λ̲’, and individual constants, ‘0’, ‘1’, ‘∾’. Frege chose these unfamiliar signs to follow his own advice regarding definitions. Referring to the abhorred practice of ‘piecemeal definition’, he writes: “Indeed, it would have been possible to replace the old signs and notations by new ones, and actually, this is what logic requires; but this is a decision that is hard to make. And this reluctance to introduce new signs or words is the cause of many unclarities in mathematics.” (vol. II, §58, pp. 70–71).

Frege chose his signs from whatever stock of metal types his publisher, Hermann Pohle, had in his printshop (the *Frommannsche Buchdruckerei*), but appears to have picked the signs, where possible, to be suggestive of the respective function: a sign for pound (‘℔’, an ‘lb’-ligature) is turned over to resemble a cursive ‘A’, or perhaps an ‘An’-ligature: ‘ꝑ’, and so serves as the sign for *Anzahl* (cardinal number); ‘ℳ’, an old currency sign for Mark is used for *Umkehrung* (converse); ‘∾’, apparently constructed from metrical signs (for the annotation of classical poetry) and a lying bracket, is importantly distinct from, but still suggestive of, John Wallis's ‘∞’ and Georg Cantor's ‘ω’—it is Frege's sign for *Endlos* (“Endless”), the transfinite cardinal number of the natural numbers.[20]

Frege found these symbols in his publisher's stock—we were not so lucky to find all of them in the stock of symbols LaTeX was able to provide at the time. Creating symbols, using Metafont, was beyond the abilities of everyone involved at that stage, but we could once more rely on the TeX community. Richard Heck sent out a plea for help online and found TeX-wizard J. J. Green who enthusiastically contributed his time and skills in TeX and Metafont. Jim created the *fge* package, a LaTeX-package that contains all of Frege's function symbols that were missing from the common stock.

[20] See Green et al. (2012) for a more detailed discussion of the typography of *Grundgesetze*.

All above mentioned LaTeX style-files (`begriff.sty`, `grundgesetze.sty`, `fge.sty`) are available on `www.ctan.org` and also on our website `www.frege.info`. The *Begriffsschrift* GUI mentioned above is also available on the latter website.

While we stuck closely to Frege's page-breaks as well as his formalism in all respects, we took more liberty with other typesetting features. The German original uses two different means of emphasis: italics (as in '*Begriffsschrift*') and letter-spacing (as in 'A n z a h l'). Letter-spacing is used for personal names, in the introduction of technical terms, and sometimes (vol. I, pp. 30–34) for the statement of rules; italics is used for Latin phrases and as a means to refer to concepts. We do not track the distinction between italics and letter-spacing, but set both as italics. No more confusion should arise from this than does from using italics for both Latin phrases and concepts. Moreover, the original is not consistent in applying the distinction: in vol. II, the introduction of technical terms sometimes uses italics instead of letter-spaced (see e.g., p. 171); the titles of works Frege cites (both articles and books) are often letter-spaced (e.g., vol. I, p. 5 fn. 1), but also sometimes italics (e.g., vol. I, p. IX–X; vol. 2, p. 152 fn. 1), and sometimes not emphasised at all (e.g., vol. I, p. XI fn. 1, p. 1 fn. 2 and 3, p. 3 fn. 4).

Another divergence from the original lies in the use of quotation marks. Frege uses single German-style quotation marks for logical and mathematical symbols, as in: ‚$— 2^2 = 4$'. German-style double quotation marks are used for quoting prose: „Unser Denken". It was an obvious decision to change these to English-style single and double quotation marks, respectively: '$— 2^2 = 4$', "our thinking". Larger formulae present a difficulty. Where Frege quotes concept-script propositions with one or more subcomponents, the opening quotation mark is vertically aligned with the lowest subcomponent, and the closing quotation mark with the supercomponent:

$$\begin{array}{ll} & \vdash\!\top\!\!\top\ \mathfrak{p}u = \mathfrak{p}v \quad ‘ \\ & \quad\ \llcorner\ u \frown (v \frown \rangle q) \\ , & \quad \llcorner\ v \frown (u \frown \rangle\!\not\ell\, q) \end{array}$$

Upon reflection, the most consistent rendering using English-style quotation was to reverse Frege's alignment of the quotation marks for displayed formulae:

$$\begin{array}{l} ‘\vdash\!\top\!\!\top\ \mathfrak{p}u = \mathfrak{p}v \\ \quad\ \llcorner\ u \frown (v \frown \rangle q) \\ \quad \llcorner\ v \frown (u \frown \rangle\!\not\ell\, q)’ \end{array}$$

and to use both opening and closing quotation marks aligned with the supercomponent where the proposition is in line with the prose: '$\begin{array}{l}\vdash\!\top\ \Gamma \\ \ \llcorner\ \Delta\end{array}$'.

Matters get more confusing in the second volume. The end of part II (the first fifty-four sections of vol. II) follows the quotation conventions described above. However, for almost all of part III.1 (§§55–164, pp. 69–162) French quotation marks replace the single quotation marks for logical and mathematical signs: »$(2 - 1) + 2$«, »$A \smile\!\!\not\ell B$«. The only exception in part III.1 is a long footnote on pp. 70–71, where single quotation marks are used, as in the first volume. Part III.2 uses single quotation marks as parts I and II do—up until the last four pages: §241, p. 240, uses French quotation marks again, and their use is continued in the afterword.

We follow the quotation conventions of the first volume throughout the whole text and therefore replace all French quotation marks by single English-style quotation marks. Frege draws no distinction in the changing from single German to French quotation marks. It is merely a quirk.

We close this section with a few final oddities involving quotations within quotations in *Grundgesetze*. In vol. II, §85, p. 95, Frege quotes Cantor who, in turn, has a quoted expression in his sentence: Frege's original uses the French-style quotation marks for the inner quotation marks here; we replace them by single English quotation marks as per the convention above. In §127, p. 131, Frege quotes Thomae and again needs quotation within a quotation. The solution in the original is to use quadruple quotation marks:

> „Da alle Terme nicht angeschrieben werden können, so ist unter „„alle““ hier wie in ähnlichen Fällen zu verstehen [. . .].“

We follow this solution. Lastly, in vol. I, §54, p. 71b, Frege uses single quotation marks to quote his own, semi-formal expressions which, in turn, contain the quotation of logical symbols, *viz.* Roman letters. Idiosyncratic semi-circles are employed in lieu of quotation marks:

> Des bequemern Ausdrucks halber sage ich nun statt ‚Begriff, dessen Umfang durch ◟u◝ angedeutet wird‘ ‚u-Begriff‘ [. . .]

This use of semi-circles occurs thrice in this section (pp. 71b and 72b). We use single quotation marks for both inner and outer quotation marks.

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It is difficult properly to acknowledge, or even express in words, the amazing support and guidance we received in the many years since this project started. Philosophers around the world—many of whom we are now grateful to call our friends—have generously supported us by selflessly spending many hours giving us feedback on the translation. In describing how the translation came together, we hope to acknowledge all those who participated in the project.

From its very beginning, the translation project was a team effort. It was initiated in 2003 by Crispin Wright when he approached us with the proposal to work together on a full translation of Frege's *Grundgesetze*. We were enthusiastic and immediately agreed to it—after all, we thought, it couldn't take much longer than two years to translate this book. The project was inspired by the strongly collaborative working methods that prevailed at the Arché research centre in St Andrews. The work was initially supported and hosted under the auspices of the AHRC-funded project on The Logical and Metaphysical Foundations of Classical Mathematics, which was running at Arché from 2000 to 2005 under Crispin's leadership. Both of us participated in the project as graduate students.

The very first *Grundgesetze* translation meeting took place in April 2003. It was attended by Peter Clark (then Head of the Philosophy Departments at St Andrews), Roy Cook, Walter Pedriali, Stephen Read, Crispin Wright, Elia Zardini, and the two of us. The task was to discuss drafts of our translation, which were usually very close

to the original. During the very first meeting, the challenge lying ahead was brutally laid bare: we scarcely managed to discuss half a page of prose in a two hour session. The *Grundgesetze* translation project meetings continued to take place once or twice a week between 2003 and 2008, for two- to four-hour sessions. The work was gruelling, and the group shrank quickly to the core members Crispin, Walter, and the two of us. Walter was part of the group from 2003–5 and then rejoined us in the academic year 2006–7. Throughout these years, Walter provided feedback on the translation and helped by editing and collating corrections made during the meetings. We are very grateful for his help.

Given the difficulties in putting together a translation, and our lack of experience, Crispin suggested asking well-known Frege scholars to act as advisors to our project. The most we hoped for was that a few of the people we asked would help by providing feedback on specific questions regarding our translation and the choice of technical terms, or that they might assist with difficult passages. We were amazed by the enthusiasm with which our invitation was met, and by how much more of their time our advisors were willing to invest than we had expected. To organise the timing of feedback and advice more effectively, we held three workshops that brought many of our consultants together. We discussed a draft of volume I at the first *Grundgesetze* workshop in St Andrews in 2006, which was attended by Michael Beaney, Roy Cook, Gottfried Gabriel, Michael Hallett, Robert May, Eva Picardi, Stephen Read, Stewart Shapiro, William Stirton, Kai Wehmeier, and Elia Zardini. Volume II was discussed at the second workshop in December 2008 in St Andrews, where we received substantial feedback from Michael Beaney, Roy Cook, Gottfried Gabriel, Michael Hallett, Richard Heck, Robert May, Eva Picardi, William Stirton, Christian Thiel, and Kai Wehmeier. The last translation workshop, sponsored by the Northern Institute of Philosophy, Aberdeen, and New York University, took place in May 2010 in New York. We discussed a full draft of our translation (already completely revised several times) and received detailed, final comments from Michael Beaney, Roy Cook, Michael Hallett, Richard Heck, Robert May, William Stirton, and Kai Wehmeier. Many of our advisors not only invested a significant amount of their time in reading our translation and attending the workshops, but also gave us line-by-line comments on hundreds of pages of our translation. Richard Heck visited St Andrews several times in the early phases of the project and participated in the translation meetings. "Beyond the call of duty" scarcely captures the work our advisors invested.

Of course we did not always all agree on how various passages should best be translated, but this did not curtail the extremely fruitful, friendly, and collaborative working experience. The workshops were the most productive and collaborative events any of us has ever experienced—they were invaluable to improving the translation, but they also assured us that we would have strong support from our advisors during this long and difficult project. We would also like to acknowledge the feedback we received in written correspondence from Patricia Blanchette, Werner Holly, Michael Kremer, Wolfgang Künne, Matthias Schirn, Lionel Shapiro, Peter Simons, Peter Sullivan, Jamie Tappenden, and Joan Weiner.

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Special thanks go to Roy T. Cook. Roy was not only part of the translation project from its inception but also put together a detailed and superbly helpful technical appendix to this translation. The aim of the appendix is to help the reader grasp some of the formal aspects of Frege's work by providing a modern introduction to the symbolic language, logic, and defined notions.

The *Grundgesetze*-project would have never run as smoothly and as successfully had it not been for another core team-member: Sharon Coull. Her support and enthusiasm for the "*Grrrrrundgesetze*" was simply amazing. Many thanks, Sharon, for all your help.

Typesetting *Grundgesetze* proved to be another challenge. In the previous section we have already mentioned the many people who helped us representing Frege's formulae using LaTeX. First and foremost we owe a great debt to William Stirton, who rendered all of Frege's over four thousand formulae in LaTeX code and who also proof-read all of part II—formulae and prose. We would like to thank Charlie Siu who helped us extensively with the final typesetting of Frege's proofs (i.e., the sections entitled 'Construction'), and Nilanjan Bhowmick, Nora Hanson, and Daniel Massey, whose laborious task it was to proof-read the formulae of both volumes and who caught a number of mistakes we missed. Another arduous task was the compilation of the index. This was achieved by Thomas Hodgson, assisted by Michael Hughes and Thomas Cunningham—many thanks! And many thanks also to Kathleen Hanson for editing Frege's original title pages.

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This project would not have been possible had it not been for our colleagues' appreciation of the value of spending many years working on the translation. In a time when single-authored publications in top journals are a must for a successful tenure case; when the quantity of original articles published in top journals are used as the main guide to assessing the quality of one's research; when funding for British philosophy departments depends greatly on individual scores in the so-called *Research Excellence Framework*; when politicians force "buzz words" like *knowledge transfer* or *impact* high on the agenda of funding bodies and thereby on the agenda of vice-chancellors, deans, and heads of departments, we were extremely lucky to have had the continuing support from our heads of department who shared the long-term vision of this project. Thus, we would like to thank our heads of department: Peter Clark at the University of St Andrews, Antony Duff and Peter Sullivan at the University of Stirling, and Crawford Elder and Donald Baxter at the University of Connecticut. They not only supported our project financially but did so knowing that much of our individual research time would be spent on a translation of a 110 year-old book—work few will regard as *original research*, with no prospect of having *impact* outside of academia, with seemingly little material for *knowledge transfer*, and with an "output" that might not be *REF*-able. Thank you for thinking and acting "outside the box".

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Philip Ebert, Stirling, Scotland
Marcus Rossberg, Storrs, Conn., USA *November 2012*

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