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ORIGINAL RESEARCH

Reichenbach's empirical axiomatization of relativity

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Abstract
A well known conception of axiomatization has it that an axiomatized theory must be interpreted, or otherwise coordinated with reality, in order to acquire empirical content. An early version of this account is often ascribed to key figures in the logical empiricist movement, and to central figures in the early "formalist" tradition in mathematics as well. In this context, Reichenbach's "coordinative definitions" are regarded as investing abstract propositions with empirical significance. We argue that over-emphasis on the abstract elements of this approach fails to appreciate a rich tradition of empirical axiomatization in the late nineteenth and early twentieth centuries, evident in particular in the work of Moritz Pasch, Heinrich Hertz, David Hilbert, and Reichenbach himself. We claim that such over-emphasis leads to a misunderstanding of the role of empirical facts in Reichenbach's approach to the axiomatization of a physical theory, and of the role of Reichenbach's coordinative definitions in particular.

Keywords Reichenbach · Pasch · Hilbert · Hertz · Axiomatization · Relativity

1 Introduction

Accounts of the logical empiricist movement often emphasize the idea of a strict separation between the theoretical and empirical content of theories.[1] This idea carries with it a particular role for the axiomatization of a physical theory: to establish a collection of formal, structural relationships which can then be hooked up to reality via a series of definitions. With such a role for axiomatization in mind, Feigl famously summarized

[1] For a recent account of the fundamental methods of logical empiricism, see Lutz (2012).

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the "picturesque but illuminating elucidations used, e.g. by Schlick, Carnap, Hempel, and Margenau" as follows:

> the "pure calculus," i.e., the uninterpreted postulate system, "floats" or "hovers" freely above the plane of empirical facts. It is only through the "connecting links," i.e., the "coordinative definitions" (Reichenbach's terms, roughly synonymous with the "correspondence rules" of Margenau and Carmap, or the "epistemic correlations" of Northrop, and only related to but not strictly identical with Bridgman's "operational definitions"), that the postulate system acquires empirical meaning. (Feigl, 1970, p. 5)

Here we see Feigl assimilating Reichenbach, in particular, within a broader interpretation of logical empiricism. On this view, Reichenbach's coordinative definitions play the role of anchoring an "uninterpreted postulate system" to the empirical facts; hence an axiomatized theory is construed as consisting, in the main, of an entirely formal structure—a "pure calculus".

Such a reading of Reichenbach coincides with a robustly "formalist" reading of David Hilbert, who is cited in the first footnote of Reichenbach's *Axiomatization of the Theory of Relativity*. The strictly formalist or abstract line on Hilbert, which we will discuss below, has it that an axiomatized theory has no specific content. Instead, the axioms implicitly specify their own range of application.[2] According to this interpretation, Hilbert and Reichenbach view axioms, and structural relations of a theory generally, as mathematical or abstract.[3] Theoretical abstracta—e.g., relations, concepts, axioms—have no relationship with empirical content until we provide them with it, by specifying conventional definitions that link abstract structures to the world.[4]

Our particular goal in this paper is to oppose the idea that either Reichenbach or Hilbert had a strictly formal or abstract understanding of an axiomatized physical theory along these lines. As we hope to make evident in what follows, Reichenbach did not regard an axiomatized theory of physics as an "uninterpreted postulate system" floating freely "above the plane of empirical facts". More positively, we want to draw attention to the commonalities between Reichenbach's approach and a tradition of

[2] The abstract or strictly formalist reading emphasizes the fact that Hilbert's axiomatization of geometry provides "implicit definitions" of the terms of geometrical theories ("point", "line", "plane"), which do not specify a unique referent for these terms. The term "implicit definition" was not often used by Hilbert himself, despite its centrality to formalist readings of Hilbert (our thanks to an anonymous reviewer for this point).

[3] Hilary Putnam (1974) argued that Reichenbach's view does not show how relations between abstract and physical elements of theories are controlled. If the axioms and definitions of a theory are arrived at by free choice, and if they specify the class of models under which the claims of the theory are mapped to "true", then it seems that the fundamental relations of the theory float free from the theory's empirical content. Lionel Shapiro argues in response that "Reichenbach's much-maligned 'conventionalism' reduces to a forceful illustration of the fact that mathematical structures remain physically contentless pending specification of the real-world correlates of their abstract elements" (Shapiro, 1994, p. 296).

[4] Unsurprisingly, this is particularly applicable to modern theoretical physics. In the case of general relativity in particular, Michael Friedman has argued that "the four dimensional, variably-curved geometry of general relativity is an entirely non-intuitive representation having no intrinsic connection whatever to ordinary human sense experience", and that this prompts "both the logical empiricists and Einstein himself" to "discern an intimate and essential relationship between the general theory of relativity, on the one hand, and the modern 'formal' or 'axiomatic' conception of geometry associated with David Hilbert, on the other" (Friedman, 2001, p. 78).

empirical axiomatization that developed in the 19th and early twentieth centuries.[5] This does not mean that Reichenbach expunged all formal elements from his axiomatized theory, or that he required a reduction of all theoretical elements to material terms. Instead, in the tradition we will explore, axiomatization is a means of clarifying the results of a theory: demonstrating which elements are formal, and which are empirical.

As we will argue in section two, David Hilbert, Heinrich Hertz, and Moritz Pasch exemplify the 19th and early twentieth century tradition of empirical axiomatization that is our focus. Section three is devoted to a study of the "constructive" axiomatization project carried out in Reichenbach's *Axiomatization of the Theory of Relativity* (henceforth ATR), which will show that Reichenbach belongs to this tradition as well. Reading Reichenbach as standing in this tradition allows us to appreciate the significance of Reichenbach's use of "elementary facts" as axioms, which in turn allows for a more nuanced understanding of the role of Reichenbach's coordinative definitions and of his constructive methods in general. Section four will turn to a brief consideration of the evolution of Reichenbach's notion of coordinative definitions in light of the preceding discussion, and section five concludes.

2 A tradition of empirical axiomatization

Reichenbach's education was embedded in a tradition of axiomatic reasoning that had been established in the mid-nineteenth century and flourished in the work of Pasch, Hertz, Noether, Hilbert, and Einstein.[6] In this tradition, the need for axiomatization arises because of the organic and messy way in which theories naturally develop. In mathematical physics in particular it may be difficult to clearly distinguish between empirical claims, conventional choices, and mathematical artifacts. The axiomatized version of the theory must re-establish all of the results of the pre-axiomatized theory, and a successful axiomatization allows for a proper assessment of theoretical virtues such as simplicity, empirical adequacy, consistency, and completeness.

In order to sketch this tradition of axiomatization here, we will present a series of vignettes of the work of Pasch, Hertz, and Hilbert. Importantly, in this tradition formal methods are employed alongside a robust empiricism, as is particularly evident in Pasch's philosophy of geometry. Moving to Hertz's treatment of classical mechanics,

[5] Richardson (2021) suggests a reading of Reichenbach's (1920) *The Theory of Relativity and* A Priori*Knowledge* according to which "a scientific method in philosophy would show that attention to the formal features of knowledge in the exact sciences of nature requires that philosophy of science move from scientific neo-Kantianism to a theory of knowledge that is closer to empiricism. This was the main argument of the 1920 book" (p. 158). Our reading of Reichenbach's (1924) book is consistent with this account of his philosophical development at the time (without attributing our reading to Richardson). Richardson's account of Reichenbach's development after 1920 (and the influence of Schlick) is significant context for the 1924 book.

[6] In his student days, "Reichenbach studied mathematics, philosophy, and physics under such teachers as Born, Cassirer, Hilbert, Planck, and Sommerfeld at the Universities of Berlin, Göttingen, and Munich" (Salmon, 1979, p. 5), and in 1918-1919, Reichenbach "was one of five intrepid attendees of Einstein's first seminar on general relativity given at [the Humboldt-Universität in Berlin] in the tumultuous winter" (Ryckman, 2018, Sect. 4.1).

we see how an axiomatized framework can be devised with the explicit aim of transforming a confused formulation of a theory into a logically perspicuous one. Finally, in Hilbert's work on both geometry and physics we find echoes of Pasch and Hertz respectively. For all three figures, empirical content constrains the development of the axiomatic framework from the beginning.

2.1 Moritz Pasch

Moritz Pasch's (1882) *Vorlesungen über neuere Geometrie* is prominent in the development of early twentieth century axiomatization.[7] In tandem with his empiricism, Pasch strongly defends the methods of "formalization" and "deductivism" in geometry; however, as is the case with Hilbert (Sect. 2.3 below), it is crucial to understand Pasch's formal methods in the context of his overall approach.[8]

Pasch presents an overarching conception of the discipline of mathematics and its intimate relationship with underlying "philosophical" investigations.[9] Within mathematics proper, there is a distinction between the rough ("*derb*") work of obtaining new results, which takes up most of the energy of practicing mathematicians, and the delicate ("*heikel*") work of clarifying the structural relationships among known mathematical propositions and establishing clear foundations—the work that we would now recognize as the axiomatization of a mathematical theory. Undergirding these two aspects of mathematical practice are certain philosophical investigations, again subdivided into two broad groups. The first consists in establishing the *meaning* of mathematical terms, and here is where Pasch's empiricism is evident: according to Pasch, all mathematical notions have their meanings fixed by empirical concepts. He insists "that the basic terms of a mathematical theory can neither be defined nor can they be reduced to other concepts, but that we can only understand them through reference to appropriate physical objects ('*den Hinweis auf geeignete Naturobjecte*')".[10] The second type of philosophical investigation, significantly broader in scope than the first, concerns the necessary conditions for any mathematical practice; indeed, the origin ("*Ursprung*") of all thinking in general.

To convey how these philosophical and mathematical investigations are related to one another, Pasch employs the following botanical metaphor. The delicate task of clarifying a mathematical theory corresponds to the search for "stem" (*Stam*) concepts and propositions, whereas the investigation into its empirical underpinnings corresponds to the search for "core" ("*Kern*") concepts and propositions. The deeper philosophical investigations into the underlying necessary conditions for mathematics is referred to as an "area of roots" (*Wurzelgebiet*). Presumably, then, the "rough" work that is the

[7] See Eder and Schiemer (2018), Sect. 2.3.

[8] "The two main aspects of Pasch's philosophy are a formal stance with regard to the validity of mathematical deductions and a strong commitment to an empiricist understanding of the basic concepts of mathematics" (Schlimm, 2010, p. 94). While, as Schlimm notes, these two elements may seem to be in tension with each other, the basic aim of Pasch's methods of axiomatization is to distinguish them carefully.

[9] Here we follow Schlimm (2010).

[10] Schlimm (2010, p. 100); quotation is from Pasch (1882), p. 16.

focus of most practicing mathematicians corresponds to the outgrowth of branches, twigs, and leaves.

For present purposes, the point that bears emphasizing is that, for Pasch, the mathematical task of identifying the stem propositions is distinct from the philosophical task of "determining the meanings of the mathematical terms and of giving an account of their applicability to the world" (Schlimm, 2010, p. 96). This distinctive combination of empiricist and formalist traits is made vivid in his conception of relative consistency proofs—switching out one empirical core for another within a given formal stem:

> As a method for establishing consistency Pasch explains how a given set of meaningful propositions can be 'formalized', resulting in an 'empty stem', which in turn can be 'realized' by replacing the meaningless symbols by meaningful concepts, yielding a 'filled stem' (Pasch, 1926a, p. 11 [1915]). If a realization of a formalized stem is consistent, Pasch argues, then the original stem is also consistent. In modern terminology, Pasch here describes the notion of relative consistency proofs. (Schlimm, 2010, p. 105)

Pasch emphasizes that geometrical concepts and relations originate in reasoning about physical objects. As geometrical theories develop, a "network of artificial concepts" is used to support further results, but this network can become entangled and in need of clarification. To reveal the impact of these artificial concepts on the theory, Pasch recommends that the theory be made even more artificial via the methods of formalization. His empiricism is thus paired with one of the most forceful early formalist axiomatizations of geometry: Pasch argues that it is desirable for the terms representing physical objects in geometrical theories to be replaced with "meaningless signs" and for proofs to go through independently of the physical sense of the terms.[11] This means, in turn, that the deductions of results from stem concepts and propositions within the theory must be shown to be entirely independent of the theory's empirical content.

The ultimate aim of Pasch's formalization is to show the extent to which the formal structures and relations that have been built up over time either reflect the empirical content of the theory, or are independent of that content. Pasch's early work on projective geometry and the principle of duality in particular revealed to him the extent to which mathematical stem propositions can float free of the core propositions and concepts of a theory: that is, the extent to which the formal structure of a theory can be independent of its empirical content.[12] But Pasch never wavered in his commitment to the view that the geometrical concepts and propositions of a theory originally encode empirical content and that geometrical axioms should be developed from empirical core concepts and basic empirical facts.

[11] "[M]athematical proofs must remain valid if the basic concepts are replaced throughout 'by any concepts or by meaningless signs'" (Schlimm, 2010, p. 103; his trans. of Pasch, 1914, p. 120).

[12] As Pasch notes in his early work on the principle of duality in projective geometry, formal concepts may refer to more than one thing, and axioms may be duals (equivalent, but distinct formal statements of an axiom with different content). Thus, without further elaboration, neither can be used to provide a definition of the *core* concepts of a theory, only the *stem* concepts. See (Pasch, 1914, p. 143) and discussion in (Schlimm, 2010, p. 100). See (Eder & Schiemer, 2018, Sects. 2.2 and 2.3) for the principle of duality and its importance for Pasch and Hilbert.

2.2 Heinrich Hertz

Hertz's reformulation of classical mechanics in his 1894 treatise *Principles of Mechanics* bears all the major hallmarks of an empirical axiomatization. In his preface, Hertz describes his aim as to "give a complete and definite presentation of the laws of mechanics", from which one can "more clearly perceive the physical meaning of mechanical principles, how they are related to each other, and how far they hold good" (Hertz, 1894/1899, p. xxii). To achieve this, Hertz throws mechanics into "the older synthetic form," a form which "has the merit of compelling us to specify beforehand, definitely even if monotonously, the logical value which every important statement is intended to have" (Hertz, 1894/1899, p. 35).

In *Principles,* Hertz claims to distil the empirical content of classical mechanics into a single variational principle—the "fundamental law":

> Every free system persists in its state of rest or of uniform motion in a straightest path. (Hertz, 1894/1899, p. 309)

Principles provides the analytical resources to construct a (high dimensional) configuration space representation of a given mechanical system, such that the path traced out by a single point in configuration space represents that system's motion.[13] The claim of the fundamental law is that this path through configuration space is always a straightest path, i.e. a path of minimum curvature. Although Hertz argues that the entire empirical content of classical mechanics is captured in this single claim, he isn't primarily concerned with representing the content of mechanics in maximally economic fashion. Rather, he regards the major advantage of his work to be its clarification of the theory.[14]

Much of Hertz's explanation of his motivations to reformulate mechanics is found in his sustained criticism of the traditional presentation of mechanics; a presentation which Hertz regards as severely lacking in clarity. In this connection, Hertz discusses the existence of disagreements over the rigor of supposedly elementary theorems—disagreements which "in a logically complete science, such as pure mathematics… [are] utterly inconceivable" (Hertz, 1894/1899, p. 7). Another consideration that Hertz raises concerns "the statements which one hears with wearisome frequency, that the nature of force is still a mystery" (Hertz, 1894/1899, p. 7). From Hertz's perspective, this situation has arisen precisely because the traditional presentation of mechanics is insufficiently clear.[15]

It is in order to avoid this kind of confusion that Hertz engages in his axiomatization project. Given the remarkable successes of the theory, however, Hertz wants

13 Note that Hertz himself did not use the expression "configuration space". For some relevant discussion of this point, see Lützen (2005, pp. 129–131 and 154–156).

14 At the close of his preface, Hertz writes, "What I hope is new, and to this alone I attach value, is the arrangement and collocation of the whole [*die Anordnung und Zusammenstellung des Ganzen*]—the logical or philosophical aspect of the matter. According as it marks an advance in this direction or not, my work will attain or fail of its object." (Hertz, 1894/1899, p. xxiv).

15 With regard to the concept of force in particular, Hertz claims that "we have accumulated around the [term] 'force'… more relations than can be completely reconciled amongst themselves" (Hertz, 1894/1899, p. 7). For a recent discussion of Hertz's treatment of force, see Eisenthal (2021).

to emphasize that he does not regard this confusion as stemming from the *essential* content of the theory. Rather, this confusion must lie "in the unessential characteristics which we have ourselves arbitrarily worked into the essential content given by nature" (Hertz, 1894/1899, p. 8). Indeed, on Hertz's view, the traditional formulation of mechanics "fails to distinguish thoroughly and sharply between the elements in the image which arise from the necessities of thought, from experience, and from arbitrary choice" (Hertz, 1894/1899, p. 8). Thus the virtues of a perspicuous axiomatization of the theory are stated once more: "all indistinctness and uncertainty can be avoided by suitable arrangement of definitions and notations, and by due care in the mode of expression" (Hertz, 1894/1899, p. 9).

While emphasizing the virtues of logical perspicuity, Hertz also notes in passing that axiomatization is only necessary for mature sciences. In fact, Hertz notes that it may be better *not* to prioritize logical perspicuity during a theory's early stages. Hertz thus distinguishes the attitude one should take towards a mature scientific theory from the attitude one should take towards a nascent one:

> the logical indefiniteness of the representation, which we have just censured, has one advantage. It gives the foundations an appearance of immutability; and perhaps it was wise to introduce it in the beginnings of the science and to allow it to remain for a while. (Hertz, 1894/1899, p. 9)

Achieving a logically perspicuous axiomatization is of particular importance for a mature theory, one that has been substantially developed over a long period of time.[16] Hertz's project in *Principles* exemplifies the central features of an empirical axiomatization. The results of the pre-axiomatized theory are put forward from the start as the target—Hertz aims to give "a presentation of the laws of mechanics which shall be consistent with the state of our present knowledge, being neither too restricted nor too extensive in relation to the scope of this knowledge" (Hertz, 1894/1899, p. xxi). This need for this axiomatization arises because of the organic way in which the theory naturally developed, as a result of which it is difficult to clearly distinguish between empirical claims, conventional choices, and artifacts of the mathematical apparatus. A successful axiomatization clarifies the theory, making these distinctions easily recognizable.[17]

[16] "Mature knowledge regards logical clearness as of prime importance: only logically clear images does it test as to correctness; only correct images does it compare as to appropriateness. By pressure of circumstances the process is often reversed. Images are found to be suitable for a certain purpose; are next tested as to their correctness; and only in the last place purged of implied contradictions" (Hertz, 1894/1899, p. 10).

[17] In this sense, the tradition of empirical axiomatization provides a way to deepen our understanding of Reichenbach's well-known distinction between the contexts of discovery and of justification in *Experience and Prediction* (1938). Theories develop in an organic and messy way, and axiomatization clarifies the theory, analyzes it into its components, and elucidates the relations of dependence between the elements of the theory. It is possible that Reichenbach's contact with the tradition of empirical axiomatization informed the development of his view on discovery versus justification, over the 14 years between ATR and *Experience and Prediction*. For more on discovery and justification, including on the historical development of Reichenbach's views, see Schickore and Steinle (2006), perhaps especially the essay by Gregor Schiemann.

2.3 David Hilbert

Hilbert is a scion of the axiomatic method, perhaps best known for his axiomatization of geometry (1894, 1899). Besides this, he also offered his own reconstruction of the foundations of physics (1915, 1916) as well as a more general discussion of axiomatic methods (1918). As we will see, parallels with Pasch and Hertz are evident in Hilbert's work in geometry and physics respectively, the uncovering of which helps to counter any abstractly formal reading of Hilbert. We will also see that there were historical reasons for Reichenbach himself to have (mistakenly) ascribed an excessively formalist method to Hilbert. This obscured, or so we will argue, important commonalities of method between Hilbert and Reichenbach.

2.3.1 The foundations of geometry

Hilbert's work on the foundations of geometry first appeared in the 1890s. From the beginning, Hilbert distinguished geometry, which—following Pasch[18]—he considered a mathematical but also a physical science,[19] from analysis and number theory:

> The results of these domains (number theory, algebra, function theory) can be achieved by pure thinking.... Geometry, however, is completely different. I can never fathom the properties of space by mere thinking, just as little as I can recognize the basic laws of mechanics, the law of gravitation, or any other physical law in this way. Space is not a product of my thinking, but is rather given to me through my senses. Therefore I require my senses for the establishment of its properties. I require intuition and experiment, just as with the establishment of physical laws.[20]

The axiomatic method begins with intuition, but it does not end there. Once the basic intuitive-representational facts have been established, Hilbert moves on to the axiomatic clarification of the theory through the construction of a framework of concepts (*Fachwerk von Begriffen*). The aim of this clarification is not to depart entirely from the facts given in representation. It is to establish which mathematical relationships can be elucidated independently of the empirical content of the theory, and to provide a more secure foundation for the future development of a science.[21]

18 In a letter to Felix Klein (23 May 1893), Hilbert notes "I think that Pasch's ingenious book [*Vorlesungen*] is the best way to gain insight about the controversy among geometers over the axioms" (Toepell, 1986, pp. 44–45; trans. Eder and Schiemer 2018, Sect. 2.3).

19 As Schlimm (2010), 99 notes, Pasch's view from the *Vorlesungen* "that geometry is a natural science is frequently echoed by Hilbert". On this point, Schlimm cites Hallett & Majer (2004).

20 Hilbert (1891), trans. Majer (1995, p. 143).

21 Hilbert offered the following vivid metaphor for the role of axiomatization in the development of science: "The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development" (Hilbert, 1905, p. 102; quoted from translation in Corry (2018, p. 5).

Hilbert famously claims that the result of the axiomatization of a geometrical theory is a “scaffolding or schema” of concepts that can be applied to multiple systems of things.[22] Hilbert’s claim here must be understood in the context of his broader goals for axiomatization. One of these goals is to demonstrate more clearly which results of a theory are independent of the others, and to reveal significant dualities (e.g., Desargues’s theorem in projective geometry). Another significant aim of axiomatization is *to demonstrate geometrical facts* and to clarify formally how such a demonstration is possible in each case.

As Hilbert sees it, the question for axiomatization in geometry is not: How can we conventionally specify the broadest possible class of structures for this theory? It is rather, in Hilbert’s own words: “What are the necessary and sufficient and mutually independent conditions a system of things has to satisfy, so that to each property of these things a geometric fact corresponds and conversely, thereby making it possible to completely describe and order all geometric facts by means of the above system of things?”[23] The aim of this axiomatization was to capture the content of geometry while revealing the structural relationships embedded within this content.[24] As noted by Schlimm (2010), Pasch’s distinctive approach, combining formal clarification with the goal of establishing empirical facts, is taken up by Hilbert in his work on the foundations of geometry. Indeed, when Hilbert was developing his axiomatic approach in the 1890s, he appealed frequently to Pasch’s lectures.[25]

Hilbert understood the geometric axioms “not only as characterizing a system of things that presents a ‘complete and simple image of geometric reality’, but viewed them also in a very traditional way: the axioms must allow us to purely logically establish all geometric facts” (Sieg, 2020, p. 141).[26] The aim of logical and axiomatic analysis in geometry is not to abstract away entirely from the intuitive facts that are the

22 As Hilbert says in a letter to Frege: “it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney sweep, … and then assume all my axioms relations between these things, then my propositions, e.g. Pythagoras’ theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements.” Letter from Hilbert to Frege, 1899; citation from excerpt in Frege (1980, pp. 40–41). Notably, Moritz Pasch also uses the metaphor of “scaffolding” in discussing geometrical concepts; see the section on Pasch above and Schlimm (2010, p. 103).

23 Hilbert (1894/2004, pp. 72–73); see Sieg (2020, p. 145).

24 As Hilbert put the matter: “The present investigation is a new attempt at formulating for geometry a simple and complete system of mutually independent axioms; it is also an attempt at deriving from them the most important geometric propositions in such a way that the significance of the different groups of axioms and the import of the consequences of the individual axioms is brought to light as clearly as possible” (Hilbert, 1899/2004, p. 436).

25 “Pasch’s lectures exerted a considerable direct influence on Hilbert’s thinking about geometry and axiomatics in general, as can be seen from the development of Hilbert’s lecture notes on geometry in the 1890s, which contain lengthy paraphrases of Pasch’s discussions, and also from remarks Hilbert made in his correspondence. This deep influence is not acknowledged properly in Hilbert’s seminal *Grundlagen der Geometrie* (1899), where Pasch is only credited in a footnote for the first ‘detailed investigations’ of the axioms of betweenness, in particular the axiom that became later known as Pasch’s axiom’” (Schlimm, 2010, p. 93).

26 The terminology “complete and simple” reflects the influence of Hertz’s *Principles* on Hilbert; see Patton (2014) for details. In a similar vein, just as Hertz had constructed a logically perspicuous “image” (*Bild*) of mechanics, Hilbert sought a complete image of “geometric reality”; see Sieg (2020), p. 14.

basis of geometry as a physical science, but rather to show clearly, once the theory has reached maturity, how geometrical results are established, and on what they depend.

2.3.2 The foundations of physics

In 1900, when Hilbert gave his famous lecture in Paris listing 23 extant mathematical problems (1900/1996), his sixth problem was to carry the axiomatic method beyond geometry.[27] Following this, Hilbert submitted his "First Communication" and "Second Communication" on the "Foundations of Physics" to the Proceedings of the Göttingen Academy of Science in 1915 and 1916.[28] These were later edited, revised, and published in the *Mathematische Annalen* in 1924. In his Communications, Hilbert presents what may initially seem to be an abstract or formalist axiomatization of relativity, one that derives generally covariant field equations from the variation of the action integral.[29] However, as we saw in the case of Hertz's reformulation of mechanics, showing that a branch of physics can be derived from a single variational principle is not necessarily an abstract method. Recall that Hertz 's aim in *Principles* was to "give a complete and definite presentation of the laws of mechanics", from which one can "more clearly perceive the physical meaning of mechanical principles, how they are related to each other, and how far they hold good" (Hertz, 1894/1899, xxii). Hilbert deployed the axiomatic method in the Communications of 1915 and 1916 in a similar way, this time elucidating the physical meaning of two theories: Gustav Mie's electrodynamics and Einstein's general theory of relativity.

Hilbert saw the formal derivation of fundamental equations from a variational principle as a task done in the service of clarifying and extending the physical content of the theories at stake.[30] Indeed, Hilbert thought he had demonstrated key physical results, writing in the First Communication:[31]

> that by means of the fundamental equations set up here, the most intimate and up to now hidden processes within the atom can be explained and that, in particular, a reduction of all physical constants to mathematical constants will be

[27] We are grateful to an anonymous reviewer for insightful suggestions that materially improved this section.

[28] Hilbert's First Communication was submitted to the Göttingen Academy of Science for publication in its Proceedings in November 1915 (1915/2009), and his Second Communication was submitted to the Academy a year later in December 1916 and published in 1917 (1917/2009). In 1924, Hilbert combined both papers into a single article to be printed (with some revisions) in the *Mathematische Annalen*—see Majer and Sauer (2009, p. 26) (and also Brading and Ryckman, 2008, p. 103).

[29] "The central idea of Hilbert's communication is to combine both Einstein's [general relativity] and Mie's [electrodynamic theory] in a variational framework with an invariant action integral. In the paper, Hilbert showed how to generalize Lorentz-covariant electrodynamics to a generally covariant theory in such a way that… generally covariant gravitational field equations follow from the variation of the action integral" (Sauer, 2002, p. 225). And: "In seeking a derivation of the field equations of gravitation from a variational principle, Hilbert upped the ante in postulating a single generally invariant 'world function', a Lagrangian for both the gravitational and the matter fields, from which the fundamental equations of a pure field physics might be derived" (Brading & Ryckman, 2008, p. 110).

[30] See Sauer (2006), pp. 227–228 and *passim*.

[31] Of course, we may be skeptical of whether Hilbert succeeded. For historical purposes, that question is less important than what Hilbert *thought* he was doing.

> possible—as, on the whole, the possibility comes near that physics turns into a science of the kind of geometry: certainly the most magnificent glory of the axiomatic method which here, as we see, puts the most powerful instruments of calculus, i.e. the calculus of variations and invariant theory, to its service.[32]

To understand Hilbert's axiomatic method in his 1915 and 1917 Communications, then, we should look to the method he elaborates both in geometry and in physics.[33] Brading and Ryckman (2008) emphasize that Hilbert's goals for the foundations of physics involve resolving "the tension between causality and general covariance" by pursuing the logical and axiomatic methods laid out in Hilbert's "Axiomatisches Denken" and in his *Foundations of Geometry*:[34]

> the axiomatic method is conceived as a logical analysis that begins with certain 'facts' presented to our finite intuition or experience. Both pure mathematics and natural science alike begin with 'facts', i.e., singular judgments about 'something... already... given to us in representation' (*in der Vorstellung*): certain extra-logical discrete objects that are intuitively present as an immediate experience prior to all thinking. (Brading & Ryckman, 2008, p. 107)

In physics, just as in the case of geometry, Hilbert's choice of axioms is constrained by factual content that a theory already has. As Corry notes, "the centrality attributed by Hilbert to the axiomatic method in mathematics and in science is strongly connected with thoroughgoing empiricist conceptions" (Corry, 2018, p. 5).[35]

2.3.3 Hilbert and logical empiricism

As we will argue in the next section, Reichenbach belongs squarely in this tradition of empirical axiomatics. Before turning to this argument, however, it is worth noting that Reichenbach himself did not recognise the empirical basis of Hilbert's axiomatization

[32] Hilbert (1915/2009, pp. 45–46; trans. Sauer 2006, p. 228).

[33] See Brading and Ryckman (2008, p. 105) and Hallett and Majer (2004, p. 66).

[34] As Brading and Ryckman note, "set within the logical and epistemological context of the 'axiomatic method', Hilbert's two notes may be seen to have the common goal of pinpointing, and then charting a path toward resolution of, the tension between causality and general covariance that, in the infamous 'hole argument', had stymied Einstein from 1913 to the autumn of 1915. Unlike Einstein's largely informal and heuristic extraction from the clutches of the 'hole argument', Hilbert stated the difficulty in a mathematically precise manner as an ill-posed Cauchy problem in the theory of partial differential equations, and then indicated how it can be resolved" (2008, p. 104).

[35] Hilbert's axiomatic approach was "above all, a tool for retrospectively investigating the logical structure of *well-established and elaborated scientific theories*" (Corry, 2018, p. 5; emphasis in original). Corry (2018) has argued that it is important to distinguish between a narrower and broader conception of Hilbert's "formalist program". On the narrower construal, Hilbert's program is mainly concerned with demonstrating specific results like the consistency of mathematics using finitist arguments. On the broader construal, Hilbert's program carries with it an overarching conception of the essence of mathematics as *only* a system of conventional rules applied to arbitrary signs. Reichenbach's own reading of Hilbert may be due to the logical empiricists' focus on Hilbert's metamathematical program; see the discussion in Majer (2002) and Sieg (2020).

of geometry.[36] On the contrary, Reichenbach saw Hilbert's approach to axiomatization as appropriate for pure mathematics, and thus contrasted his own approach with Hilbert's by appealing to the essential difference between physics and mathematics. To Reichenbach, physical statements "are more than mere consequences of arbitrary definitions; they are supposed to describe the real world" (ATR, 4). Although axiomatizations in mathematics and physics have a number of things in common, Reichenbach emphasized the distinctive empirical demands on axioms in physics:

> The axiomatic exposition of a physical theory is at the outset subject to the same laws as that of a mathematical theory: it must satisfy the logical requirements of consistency, independence, uniqueness and completeness. Yet since the physical axioms also contain the whole theory implicitly, they must themselves be justified: they must not be arbitrary but true. (ATR, 4)

Reichenbach seems to have concluded that he and Hilbert, on these grounds, could not have agreed on how to establish truth in axiomatized physical theories.

Logical empiricists, including Schlick, Reichenbach and Carnap, did take on board some aspects of Hilbert's axiomatic method, and they were familiar with Hilbert's axiomatic method—or, at least, with elements of it. However, Majer observes that these figures missed "crucial aspects of Hilbert's work", and that "misapprehensions and wrong conclusions resulted from this failure" (2002, p. 213). In particular, Majer argues that Schlick, Reichenbach, and Carnap overlooked Hilbert's "tireless efforts to extend the axiomatic method beyond the domain of geometry [to] the totality of physical sciences" (p. 217).[37] He details how the early logical empiricists' reception of Hilbert focused on the debates about implicit definition and metamathematics to the exclusion of Hilbert's substantive application of the axiomatic method to physical theory.

Majer's account may explain why Reichenbach missed the common ground between his own account in ATR and Hilbert's methods in his Communications on "The Foundations of Physics". Indeed, perhaps Reichenbach was focusing, like Schlick and Carnap, on Hilbert's then-recent work on the New Foundations of mathematics and on metamathematics more generally.[38] Sieg (2020) diagnoses key differences between Hilbert and Bernays's existential or structural axiomatics, developed from 1900 on, and Hilbert's formal axiomatics, which started "already in 1913" with Hilbert's analysis of *Principia Mathematica* (p. 146).[39] The new research program led Hilbert away

[36] For a historical discussion of Reichenbach's relationship to Hilbert, including the "epistemological and institutional connection between Reichenbach's circle in Berlin and Hilbert's in Göttingen" (p. 33), see Benis Sinaceur (2018), Sect. 3.

[37] Majer contributes a nuanced analysis of Hilbert's research programs, and of the impact the logical empiricists' misunderstanding of Hilbert's aims for the axiomatic method had on their understanding of his epistemological achievements (Majer, 2002, pp. 218–222). It is beyond the scope of this paper to discuss these matters further here. Majer's paper does not focus on Reichenbach in particular, thus, our paper could provide a useful complement to Majer's analysis. See also Mayer 1995 for details on the logical empiricists' reception of Hilbert.

[38] Hilbert focused on the New Foundations and on metamathematics from 1918 to about 1922; see Majer (2002, p. 218).

[39] Sieg (2020, 146) notes that the 1917 Zurich address "Axiomatisches Denken" (HIlbert, 1918/1996) is a landmark in the turn from existential to formal axiomatics, for Hilbert.

from the focus on clarifying the empirical content of physical theories via structural definitions, characteristic of existential axiomatics, to a formal axiomatics aimed at contributing to the debates on the reduction of mathematics to logic following the publication of *Principia*.

Reichenbach's focus on Hilbert's formal axiomatics may have influenced the characterization of Hilbert's axiomatic method in ATR as being more focused on pure mathematics than on the truth sought in physical reasoning. While Hilbert thought geometry was a physical science, his approach to metamathematics and the New Foundations appears to have pushed him in the direction of formal axiomatics. However, Hilbert's "move" to formal axiomatics is not a wholesale shift: his empirical approach to axiomatics is present right up to the publication of Reichenbach's book. For instance, the position in "Natur und mathematisches Erkennen," one of three talks given in July 1923, is consistent with an empiricist reading of Hilbert.[40]

Thus, ironically, Reichenbach's description of axiomatized mathematics as purely formal in ATR may in fact point to a difference between Reichenbach and Hilbert.[41] Well into the 1920s, Hilbert saw geometry as a physical science, and argued that the axiomatic method is applied similarly in geometry and in physics. As seen above, Hilbert saw analysis and number theory as purely formal sciences, and not as physical sciences like geometry. Reichenbach maintained a clear distinction between (pure) mathematics and (empirical) physics, one that he thought distinguished his approach in ATR from Hilbert's axiomatic method.

On the contrary, however, the program of existential or structural axiomatics in geometry and physics which Hilbert and Bernays developed from around 1900 to the 1920s is reasonably viewed as a precursor to Reichenbach's approach in ATR, as we will see in the sections following. Existential axiomatics involves the assumption of elementary, representational facts, and proceeds, in the first instance, to a consistency proof built using those facts and the axioms of the system itself. This proof requires exhibiting a system.[42] Along with internal proofs of consistency, Hilbert allows for the "deepening of the foundations" of mathematics and physics via demonstrations that a fundamental axiom, or proposition, could apply to a novel domain, thus unifying distinct areas of research. Hilbert's Second Communication provides a key example of this method.

Reichenbach was aware of the general resemblance between his approach and Hilbert's, as he refers to Hilbert's axiomatic method several times throughout the 1924 text. However, he clearly did not see the parallels between Hilbert's existential or structural axiomatics and his own methods in ATR.[43] A central aim of this paper is to reveal those parallels, and to show how Reichenbach's empirical axiomatization of relativity in ATR and Hilbert's Communications on the Foundations of Physics share a deep kinship.

[40] This is supported by Majer and Sauer's introduction to the work (Majer & Sauer, 2009, p. 378).

[41] We are grateful to an anonymous reviewer for raising this point.

[42] "In the nineteenth century, logicians viewed the consistency of a notion from a semantic perspective as requiring a model. That is the way we put matters, whereas those earlier logicians, including Frege, saw themselves as facing the task of exhibiting a system that falls under the notion" (Sieg, 2020, p. 138).

[43] As Majer notes, this is a shame: "the logical empiricists would have done better, if they had paid somewhat more attention to Hilbert and his axiomatic approach to science" (2020, p. 213).

3 Reichenbach's constructive axiomatization of special relativity

Reichenbach's major goal in ATR is to cleanly separate properly empirical claims from conventional choices; the former captured in a set of *axioms* and the latter captured in a set of *definitions*.[44] Another more specific goal is to distinguish between the empirical assumptions concerning light signalling from those concerning rods and clocks.[45] Thus in the first part of ATR, which focuses on special relativity, Reichenbach presents thirteen "light axioms" and seventeen "light definitions" followed by six "matter axioms" and four "matter definitions". Although Reichenbach claims that "the space–time metric of special relativity is defined by light signals alone" (ATR p.13) this is an aspect of his approach that would be heavily criticized (most immediately in an unsympathetic review by Weyl).[46] Reichenbach also argues that as far as the kinematics that can be established through light signalling (the "*Lichtgeometrie*") is concerned, the choice between a classical or special relativistic spacetime is a matter of convention. Hence the novel empirical content of special relativity—captured in the matter axioms—is that "material structures do not adjust to the Galilean metrical determination but to the light-geometrical one" (ATR, 76). In the second part of ATR, Reichenbach goes on to present a further set of axioms and definitions for general relativity, elaborating on a remarkable (and false) claim that the special relativistic light axioms are sufficient to fix the metric in this context too.[47]

Reichenbach begins his introduction to ATR with the following statement of the value of axiomatization:

> The value of an axiomatic exposition consists in summarizing the content of a scientific theory in a small number of statements. Any evaluation of the theory may then be limited to an evaluation of the axioms, because every statement of the theory is implicitly contained in the axioms. (ATR, 3)

Noting that the axiomatic method has most frequently been applied in mathematics, Reichenbach turns immediately to a consideration of the particular demands on axiomatization in physics. In the case of mathematics it is evident that Reichenbach subscribes to a formalist view; mathematical axioms, for Reichenbach, are "arbitrary stipulations which are neither true nor false" (ATR, 3). Given that the physicist is concerned with truth, however, and given that the axioms of a physical theory should contain that whole theory implicitly, physical axioms "must themselves be justified: they must not be arbitrary but true" (ATR, 4).

This leads Reichenbach to distinguish a *deductive* approach to axiomatizing a physical theory from a *constructive* approach. On the deductive approach, the axioms which

44 Reichenbach's description of his definitions as conventional represents a shift from his earlier account in (Reichenbach, 1920/1965)—see below, Sect. 4, and Friedman (1999) §3.

45 Our thanks to an anonymous reviewer for prompting us to elaborate on this point.

46 Weyl (1924). For a detailed discussion of this matter see Rynasiewicz (2005). See also Ryckman (2005, pp. 96–97).

47 Although we do not discuss Reichenbach's treatment of general relativity in this paper, see Ryckman (2005) Section 4.4.5 for a criticism of Reichebach's "preposterous attempt to derive general relativity over finite regions from the infinitesimal validity of special relativity.".

summarize the content of the theory are highly general and abstract, such that particular observational statements can be derived from them and then confirmed via experiments. Reichenbach points to attempts to summarise physical theories with a single variational principle as examples of this deductive approach, noting that such a variational principle "can never be the direct object of experimentation, and yet, depending on the confirmation of its consequences, it may be called true or false with a certain degree of probability" (ATR, 4). Reichenbach surely has Hertz and Hilbert in mind as the most prominent examples of the deductive approach, whereas he himself opts to take the contrasting constructive approach instead:[48]

> It is possible to start with the observable facts and to end with the abstract conceptualization. A certain loss in formal elegance will be balanced by logical clarity. The empirical character of the axioms is immediately evident, and it is easy to see what consequences follow from their respective confirmations or disconfirmations. (ATR, 5)

Reichenbach claims that this approach is "more in line with physics… because it serves to carry out the primary aim of physics, the description of the physical world" (ATR, 5). Roughly speaking, the aim of a constructive approach is to choose as axioms statements that are as directly verifiable as possible, and then to derive the more abstract and abstruse elements of the theory from there. This approach is not without its attendant difficulties however, and Reichenbach immediately gives voice to concerns about the theory-ladenness of even the simplest of empirical claims:

Every factual statement, even the simplest one, contains more than an immediate perceptual experience; it is already an interpretation and therefore itself a theory. To infer the existence of a light source from the perception of light, or the existence of an external mass from the perception of pressure, is an extrapolation which only seems to be a matter of course, because its consistency has been tested innumerable times. It will be impossible, however, to make the content of a sense perception an axiom of a theory; it will always be necessary to extrapolate to a certain extent in order to obtain a factual statement from which consequences will be derivable. (ATR, 5)

Alongside general holistic worries concerning the difficulty of testing isolated empirical claims, Reichenbach addresses the challenge: "Will it still be advantageous, under such circumstances, to start an axiomatization with so-called empirical facts?" (ATR, 6) In the face of this, Reichenbach's strategy is to choose "elementary facts" as his axioms—sufficiently imprecise statements common to many different scientific theories, and which do not depend on the theory of relativity itself in particular.[49] For example, no matter what future investigations may reveal about the nature of

[48] Note that the issue of a deductive vs. constructive approach to axiomatization is orthogonal to the issue of an empirical vs. formal approach.

[49] "[T]his investigation starts with *elementary* facts as axioms; all are facts whose interpretation can be derived from certain experiments by means of simple theoretical considerations. Such a treatment (or separation) presents itself all the more readily, since the theory whose foundation is the subject of this book—Einstein's theory of relativity—constitutes an innovation in physics, and the elementary facts can be chosen in such a way as not to presuppose the new theory for their interpretation in connection with the experiments on which these facts are based" (ATR, 6).

light—and, in particular, no matter what "assumptions may be made about the motion of light in strong gravitational fields" (ATR, 6)—we can still employ classical optics when arranging the lenses within a telescope. Hence principles of classical optics applied within certain domains provide candidate elementary facts for Reichenbach's purposes:

In this sense, some facts are more elementary than others, namely, those whose interpretation within certain limits does not depend upon theoretical conceptions. This is why the facts of daily life appear so certain to us. Because of their imprecision they are invariant relative to all scientific theories. (ATR, 6)

As Reichenbach emphasised, "the axioms of our presentation are *empirical propositions*" (ATR, 7).

With this in view, let us now turn to consider the axioms themselves. Reichenbach begins with two notions: a *real point*, "a point at which material entities may be conceived to be at rest" (ATR, 26), and a *signal*, "a physical process that travels from a real point P to another point P' and has the following property: if this event is marked at P, the mark can also be observed at P'" (ATR, 27). From here, Reichenbach introduces every notion either in an axiom or in a definition. For present purposes, it will be sufficient to consider Reichenbach's Light Axioms for special relativity. These are collected together into five groups: (I) three axioms of temporal sequence; (II) six axioms concerning the connectivity, continuity, and positive definiteness of light signals; (III) "Fermat's axiom" that the maximum possible velocity is that of light; (IV) a stationarity and round trip axiom which imbue light signals with metrical properties; and finally, (V) an axiom of "Euclideanism" concerning spatial geometry. In brief, these constitute nine "topological" axioms (I,1–3; II, 1–6),[50] one light principle (III),[51] and three metrical axioms (IV,1–2; V). An immediate question is how we are to understand such axioms as elementary facts.

The first axiom states that a signal cannot return at the same instant that it departs, i.e. that no signals travel at an infinite velocity. The second axiom states that for any two events at a given point, a signal can always be found which connects them, departing from one and returning to the other or vice versa. The third axiom states that the events at a point form a linear continuum (in modern terminology: events at a point are homeomorphic to the real line). To be counted as elementary facts, these claims need to be intelligible without recourse to the theory of relativity and to be indisputably well-confirmed within a certain degree of precision. And indeed, it certainly seems to be the case—from the perspective of pre-relativistic physics as well as relativity—that instantaneous signaling is impossible (axiom I, 1), that a signal can always connect two events at a point (axiom I, 2), and that the events at a point form a linear continuum (axiom I, 3). Note, however, that if Reichenbach required elementary facts to be true in an absolute sense, i.e. true to an arbitrary degree of precision, then these axioms appear speculative and in need of further justification. Hence Reichenbach's strategy

50 See Ryckman (2005, pp. 99 and 105) for a brief discussion of the oddities of Reichenbach's use of the term "topological" in this context.

51 Note, however, that Reichenbach is careful to use the expression "Fermat's axiom" rather than "light principle" to label Axiom III; cf. ATR §21.

of using elementary facts which "because of their imprecision… are invariant relative to all scientific theories" (ATR, 6).

Only two of Reichenbach's Light Axioms are claims that are specifically relativistic: axiom II, 5 ("*If PP'P is a signal leaving P at the time t, there exists a* $t_1 > t$ *such that every signal PP'P leaving at t will return to P later than* t_1", ATR, 36) and axiom III ("*First signals are direct light signals*", ATR, 42). As Reichenbach notes, the novel character of these two axioms is due to the fact that they "formulate the limiting character of the velocity of light" (ATR, 92). All the same, it remains the case that these axioms are intelligible within, and well-confirmed by, pre-relativistic physics. Many of the remaining Light Axioms can be recognised as elementary facts in a similarly immediate way, and Reichenbach argues that in those cases where the axioms cannot be directly tested, "experiments from which the axioms are derivable can easily be indicated, again by means of pre-relativistic physics alone" (ATR, 7). Whether Reichenbach's arguments should be accepted on this matter is not our concern here; rather, we are interested in characterizing the kind of project that Reichenbach took himself to be attempting. And on this point we can be unequivocal: Reichenbach is engaged in a thoroughly empirical axiomatization project.

We should now turn to the role of Reichenbach's *definitions*. Reichenbach argues that, although physical axioms (unlike mathematical axioms) are empirical statements, physics and mathematics are alike in employing definitions which are "arbitrary" and "neither true nor false" (ATR, 8). However, an important distinction between mathematics and physics in this context is that mathematics uses *conceptual* definitions (which clarify "the meaning of a concept by means of other concepts", ATR, 8) whereas physics uses *coordinative* definitions:

> The physical definition takes the meaning of the concept for granted and coordinates to it a physical thing; it is a *coordinative definition*. Physical definitions, therefore, consist in the coordination of a mathematical definition to a "piece of reality"; one might call them *real definitions.* (ATR, 8)

Reichenbach's leading example of a coordinative definition is the one which "designates the Paris standard meter as the unit of length" (ATR, 8). Note that such a definition takes a corresponding conceptual (mathematical) definition—"that a certain particular interval is to serve as a [standard of] comparison for all other intervals"—for granted. Some of the difficulties facing elementary facts apply also to coordinative definitions: "the physical thing that is coordinated is not an immediate perceptual experience but must be constructed from such an experience by means of an interpretation" (ATR, 8). Unsurprisingly, Reichenbach simply deploys his earlier strategy again: "we use coordinative definitions whose degree of precision is not important and which, in particular, do not make use of relativistic definitions" (ATR, 8).

Interspersed with the Light Axioms, then, Reichenbach presents seventeen Light Definitions. Let us again consider the first three in particular. These are definitions which pertain to the relations "later than" and "earlier than", and to the notions of simultaneity and spatial neighborhood:

Definition 1 Of two events E_1 and E_2 happening at P, the event E_2 is called later than E_1 if a signal chain can be chosen in such a way that its departure coincides with E_1 and its return with E_2. In this case E_1 is called earlier than E_2. (ATR, 29)

Definition 2 Let a first signal, sent from O at time t_1 and reflected at P, return to O at time t_3. Then the instant of its arrival at P is to receive the time value

$t_2 = t_1 + \varepsilon\,(t_3 - t_1)\; 0 < \varepsilon < 1,$

where ε is an arbitrary factor which must have the same value for all points P. (ATR, 35)[52]

Definition 3. A point P' is said to be in the spatial neighborhood of P with the exactness η *if the time PP*
'P is smaller than an arbitrarily given small magnitude of η. (ATR, 40)

As expected, Reichenbach is using the definitions to make explicit the conventional choices that need to be made in the course of spelling out a theory. Hence Reichenbach specifies that the ε that appears in Definition 2 takes the standard value of 1/2 (Definition 8, "Einstein's definition of simultaneity", ATR, 44), that straight lines coincide with light rays (Definition 9, ATR, 51), and so on.[53]

By combining the elementary facts contained in his axioms with this series of definitions, Reichenbach claims to have thereby summarized the theory "in a small number of statements" so that any evaluation of the theory "may then be limited to an evaluation of the axioms, because every statement of the theory is implicitly contained in the axioms" (ATR, 3). The value of this exercise is to have achieved a clarification of the theory of relativity as it stands. Note how this diverges from a broadly formalist understanding of an axiomatized theory, according to which the theory already exists as an abstract (mathematical) structure before it is hooked up to the real world via some process of interpretation or coordination.

4 Coordinative definitions

A satisfactory interpretation of Reichenbach's account of axiomatization requires a careful understanding of the role of his coordinative definitions. Hence, at this juncture it will be useful to briefly consider how Reichenbach's treatment of coordinative definitions evolved between the publication of *The Theory of Relativity and* A Priori*Knowledge* in 1920 and that of *The Philosophy of Space and Time* in 1928, passing through ATR along the way.

Reichenbach's early discussion of "the problem of coordination in physics" can seem to invite just the kind of abstract or overly formal reading of Reichenbach that we oppose.[54] In particular, Reichenbach writes:

52 It is not until Definition 8 that Reichenbach introduces the standard value $\varepsilon = 1/2$, a stipulation which he labels "Einstein's definition of simultaneity" (ATR p. 44).

53 Recall that, although there may be good reasons to revise these definitions, they cannot themselves be evaluated with regard to their truth or falsity (they are "neither true nor false").

54 Note that there is a narrower view of formalism according to which Pasch, Hilbert, and Reichenbach would all be formalists. See Corry (2018).

> Physics has developed the method of defining one magnitude in terms of others by relating them to more and more general magnitudes and by ultimately arriving at "axioms," that is, the fundamental equations of physics. Yet what is obtained in this fashion is just a system of mathematical relations. What is lacking in such a system is a statement regarding the significance of physics, the assertion that the system of equations is *true for reality*. (Reichenbach, 1920/1965, p. 36)

Here it may appear that Reichenbach ascribes to an abstractly formal account of axiomatization—that the axioms of a physical theory are "just a system of mathematical relations" and that they need to be coordinated with reality in order to acquire empirical content. Indeed, Reichenbach immediately stresses: "Not only the totality of real things is coordinated to the total system of equations, but *individual* things are coordinated to *individual* equations" (Reichenbach 1921, p. 36, emphasis in original). This early discussion of coordination may well have contributed to the tendency to interpret Reichenbach in abstractly formalist terms. However, care must be taken here. In particular, Reichenbach prefaces the above by writing: "Offhand it looks as if the method of representing physical events by mathematical equations is the same as that of mathematics." Hence the ensuing remarks in which Reichenbach appears to endorse a formalist view of axiomatization should not be taken at face value—Reichenbach is evidently presenting the offhand observation according to which physics and mathematics *appear* more alike than they really are.[55]

As we hope to have made clear in section three of this paper, any overly abstract or formal interpretation of Reichenbach is undermined by an examination of Reichenbach's discussion in ATR. There, Reichenbach is perfectly explicit that the axioms he employs are "elementary facts"—empirical propositions that certainly do not stand in need of an interpretation in order to acquire empirical content. In the context of ATR, it becomes evident that coordinative definitions are needed insofar as it is necessary to make a number of conventional choices in the course of spelling out a physical theory. Hence Reichenbach offers the example of the designation of the standard meter in Paris as the paradigm case of a coordinative definition.

There are important issues regarding Reichenbach's attitude to the problem of coordination in physics which have not been our primary focus here. In particular, in *The Theory of Relativity and* A Priori*Knowledge*, Reichenbach regards the axioms of coordination as playing a special *constitutive* role for empirical propositions. As Michael Friedman has recounted, this is an aspect of the notion of the relativized a priori that Reichenbach articulates at this time. Drawing on the developments in mathematics, geometry and physics that fed into Einstein's theories of relativity, Reichenbach distinguished between two aspects of Kant's notion of the synthetic a priori. On the one hand, Kant regarded synthetic a priori propositions as necessarily and unrevisably true; on the other hand, he regarded them as constitutive of the object of knowledge. The paradigm examples are the propositions of Euclidean geometry—a geometry which, according to Kant, necessarily applies to all phenomenal objects. For Reichenbach, as

[55] It is also important to note that Reichenbach's discussion of coordination in 1920 is preliminary. For one thing, Reichenbach has not yet articulated the expression "coordinative definition." Furthermore, the thrust of his discussion is concerned with the peculiar problem of comparing a mathematical structure with reality (in contrast with the well-defined problem of comparing two mathematical structures with one another); a problem which he addresses, early on, by fleshing out his distinctive neo-Kantian framework.

for us of course, it is simply untenable to regard the propositions of Euclidean geometry as necessarily true descriptions of physical space (indeed, we now regard them as false). But despite this, Reichenbach did not abandon Kant's notion of the *constitutive* role for geometric propositions, for we must first define our basic geometrical terms (*straight line*, *congruence*, and so on) if our physical theorizing is to be meaningful. The key point that Kant's account failed to accommodate was the idea that empirical pressures could lead us to give up one geometrical framework and replace it with another. Thus we arrive at Reichenbach's notion of the *relativized* a priori: relative to a given epoch in the development of physics, "at least some geometrical principles must be laid down antecedently as axioms of coordination before an empirical determination of space even makes sense" (Friedman, 1999, p. 61). Not long after the publication of *The Theory of Relativity and* A Priori*Knowledge*, however, Reichenbach moved away from his notion of the relativized a priori after Moritz Schlick persuaded him that axioms of coordination were best seen as purely conventional (see Friedman, 1999, pp. 63–64). Although these matters are not our central concern in this paper, this important and interesting story complements the central points that we wish to emphasize here: whether understood as conventions or as propositions of the relativized a priori, it is clear that Reichenbach's coordinative definitions do not play the role of connecting an entirely abstract mathematical theory (a "pure calculus", as Feigl put it) with empirical content.

With this in view, we can conclude this section with a brief reflection on Reichenbach's mature discussion in *The Philosophy of Space and Time*. Most importantly for present purposes, everything that Reichenbach says in this text is consistent with the empirical account of axiomatization put forward in ATR. Reichenbach's focus in *The Philosophy of Space and Time* is of course rather different, and although the first chapter begins with a discussion of the axiomatic method in geometry, there is no discussion of the role of axiomatization in physics. Notably, however, Reichenbach explicitly presupposes many of his earlier results in ATR:

> A considerable part of the necessary mathematical work was completed in the author's *Axiomatik der relativistische Raum-Zeit-Lehre* and detailed mathematical computations could therefore be omitted from this book. The philosophical interpretation of the theory of space and time presupposes the earlier work to which I have to refer the reader for rigorous proofs of many statements in the present book. (Reichenbach, 1928/1958, xv)

When it comes to his discussion of coordinative definitions, Reichenbach again begins by offering the example of the standard meter in Paris. The focus of his discussion is on the necessarily conventional nature of such a choice, and hence the importance of distinguishing the conventional features of a physical theory from properly empirical propositions. This leads into a lengthy discussion of the relation of congruence. According to Reichenbach, the conventional nature of the choice to employ rigid bodies in determining congruence relations is one of the most important lessons of the theory of relativity. And here we may recall one of the major insights that ATR makes available: that the distinction between a classical and special relativistic spacetime is merely a matter of convention when one is considering the *Lichtgeometry* built up from light signals. Hence it is the *matter* axioms, asserting that material rods and clocks "adjust" (*einstellen*) to the light geometry, that capture the novel empirical content of special relativity (see above, Sect. 3).[56]

The point that we wish to emphasize is that the project of distinguishing empirical from formal content is a key preoccupation of the tradition of empirical axiomatization we've examined. Pasch, Hertz, and Hilbert all focus on how to "distinguish thoroughly and sharply between the elements… which arise from the necessities of thought, from experience, and from arbitrary choice" (Hertz, 1894/1899, p. 8). The aim of this is precisely *not* to argue that *all* elements of an axiomatized theory are conventional. Rather, the point of distinguishing between conventional, a priori, and empirical elements of a theory via axiomatization is to identify what in a theory is a result of arbitrary choice, versus what is a claim about reality.

5 Conclusion

On an overly abstract or formalist interpretation, the role of Reichenbach's coordinative definitions is to invest an entirely abstract (mathematical) theory with empirical (physical) content.[57] The concern of this paper has been to show that such an interpretation fails to recognize the deep commonalities between Reichenbach's ATR and the rich tradition of empirical axiomatization, as is particularly evident in Reichenbach's axiomatization of relativity.

In section two, we explored how the work of Pasch, Hertz and Hilbert exemplifies this tradition of empirical axiomatization.[58] Beginning with Pasch, we saw how his distinction between mathematical stem concepts and philosophical core concepts allowed him to unite a formal approach to geometrical deduction with a robustly empiricist conception of geometry itself. Of particular relevance for our purposes, we noted that Pasch was engaged in an empirical axiomatization project: distinguishing the empirical content of geometry from any conventional elements introduced during its development. This kind of empirical axiomatization was even more vivid in Hertz's axiomatization of classical mechanics. Hertz saw that clarifying a mature scientific theory required disentangling the various strands that had evolved during the theory's development. His aim was to distil the core content of the theory and thereby avoid various confusions that might otherwise arise. Returning to the case of geometry, we saw that Hilbert also belongs in this tradition of empirical axiomatization. Not only did Hilbert regard geometry as resting on an empirical basis, he also saw the fundamental goal of axiomatization as clarificatory. This aspect of Hilbert's approach is again

56 Our thanks again to an anonymous reviewer for highlighting this point.

57 Similarly, one might argue that the axiomatic clarification of a theory, or the conventional specification of definitions or axioms, takes place independently of empirical reasoning. The above discussion focuses on how, for Reichenbach's axiomatization of relativity, these are deeply entwined.

58 The 19th and early twentieth century 'empirical axiomatization' we explore need not be read as 'empirical' in the sense employed in more contemporary theories. None of these figures require that the axioms of a theory be reduced entirely to material claims, for instance. Instead, the empirical tradition at stake in this paper begins with a mature theory and goes on to clarify which of its content is abstract or formal, and which is empirical—and which of the results of the theory depend on which. Throughout this process of analysis and clarification, it is necessary to appeal to empirical content—which can include concrete 'intuition', observation, and experiment.

apparent in his axiomatization of general relativity. Echoing Hertz, Hilbert distils the core empirical content of the theory into a single variational principle. In all these cases, the process of axiomatization does not lead to an "uninterpreted postulate system." Rather, the axiomatized theory succinctly captures the empirical content that the pre-axiomatized theory already had, identifies how abstract or conventional elements are employed in the theory, and analyzes how those elements might work to widen or restrict the theory's empirical scope.

Although Reichenbach did not himself recognize how closely he and Hilbert were aligned, situating Reichenbach in this historical tradition provides important context for understanding his own attempt to axiomatize the theory of relativity. As we have shown in section three, it is clear that Reichenbach is engaged in a thoroughly empirical axiomatization project in the sense discussed above. More generally, Reichenbach's approach to axiomatization has deep commonalities with the approaches that Pasch, Hertz, and Hilbert employ:

> The *axiomatic method* is the only method that will reveal the logical structure of the theory with perfect clarity. The distinction between axiom and definition leads to the separation of empirical content from arbitrary concept formation, and the derivations of particular statements clearly reveal the empirical and the logical components of every assertion. (ATR, xii–xiii)

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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