Uncomputability and arithmetic

Adam Elga

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In class we very quickly sketched how one might represent claims about Turing machines in an augmented language of arithmetic.

The augmented language includes the quantifiers "For all" and "There exists", variables, parentheses, logical connectives, numerals, and addition, multiplication, exponentiation, and equality signs. (Note: including the exponentiation sign makes your task *much* easier.)

Explain how to represent claims about Turing machines (for example, claims of the form "machine m halts on input i") in the above language. The goal is a mechanical method for translating claims about TMs into arithmetical claims in a way that preserves truth value.

If that's the information you need, then go for it. Aim for a sweet spot between detail and conciseness/clarity—explain it in enough detail to convince yourself that you could work out all the details if given some time.

Otherwise, answer by reading on and answering the boldfaced questions.

Give this a shot, but don't agonize over it if you get stuck. This homework has no impact on your grade, but do write down your answers. Feel free to work on this with others in the class.

1

A specification for a *substitution game* is a start word, and a bunch of substitution rules. One plays the game by starting with the start word and making successive substitutions, each of which is licensed by one of the rules.

For example, one might have "EVIL" as one's start word, and the following rules:

1. Replace an occurrence of "EL" with "LE"
2. Replace an occurrence of "VIL" with "L"
3. Replace an occurrence of "E" with "EE"

One can play this game as follows:

$$EVIL \rightarrow EL \rightarrow LE \rightarrow LEE \rightarrow LEEE$$

Now suppose that we are given a particular turing machine. We can represent the state of the tape and head by a string of the form

**********HEAD(n)****

where the stars are replaced by symbols on the tape, the expression HEAD(n) is positioned to indicate the location of the machine head, and "n" is replaced by the current machine state. For example, one such string would be

drxvvzHEAD(25)rgmpppp

Given a particular machine, explain how to craft a substitution game so that the operation of the machine on a given tape is mirrored by successive legal moves in your substitution game.

2

We can mirror the operation of a given TM by an appropriate substition game. It remains to translate claims about that substitution game into arithmetic.

First we need an arithmetic expression that extracts a given substring of a string of digits. **How can one be defined?** (Hint 1: Imagine talking to someone over the phone. They are thinking of the number 537332008. Your job is to give them arithmetical operations to perform on their number to extract its 3rd through 5th digits–you want the result to be 733. What do you tell them to do? Now generalize that procedure. Hint: this is where it makes things easier that the augmented language includes an exponentiation sign.)

Next suppose that you are given a particular substitution game that only involves the digits 1 through 9.

How can you use the previous answer to express in arithmetic that the transition from x to y is legal in that game?

Now suppose that you are given a sequence of numbers, each of which only involves the digits 1-9. You can code that sequence as a big number by separating the members of the sequence by zeros:

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aaaaaaaaaaa0bbbb0ccccc0dddd
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How can you express in arithmetic that this big number codes for a sequence of legal moves in a given substitution game? (Hint: "Whenever u, v, and w are the positions of successive zeros in the big number, ...")

Finally, explain how to use all of the above to get arithmetic translations of claims such as: "Turing machine #k halts on input i."