On Adjoint and Brain Functors

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Abstract

There is some consensus among orthodox category theorists that the concept of adjoint functors is the most important concept contributed to mathematics by category theory. We give a heterodox treatment of adjoints using heteromorphisms (object-to-object morphisms between objects of different categories) that parses an adjunction into two separate parts (left and right representations of heteromorphisms). Then these separate parts can be recombined in a new way to define a cognate concept, the brain functor, to abstractly model the functions of perception and action of a brain. The treatment uses the simplest possible mathematics and is focused on the interpretation and application of the mathematical concepts.

Contents

1 Category theory in the life and cognitive sciences

There is already a considerable but widely varying literature on the application of category theory to the life and cognitive sciences–such as the work of Robert Rosen ([25], [26]) and his followers[1] as well as Andrée Ehresmann and Jean-Paul Vanbremeersch [3] and their commentators.[2]

[1] See [31] and [17] and their references.

[2] See [12] for Kainen's comments on the Ehresmann-Vanbremeersch approach, Kainen's own approach, and a broad bibliography of relevant papers.

The approach taken here is based on a specific use of the characteristic concepts of category theory, namely universal mapping properties. One such approach in the literature is that of François Magnan and Gonzalo Reyes which emphasizes that "Category theory provides means to circumscribe and study what is universal in mathematics and other scientific disciplines." [21, p. 57]. Their intended field of application is cognitive science.

> We may even suggest that universals of the mind may be expressed by means of universal properties in the theory of categories and much of the work done up to now in this area seems to bear out this suggestion....
>
> By discussing the process of counting in some detail, we give evidence that this universal ability of the human mind may be conveniently conceptualized in terms of this theory of universals which is category theory. [21, p. 59]

Another current approach that emphasizes universal mapping properties ("universal constructions") is that of S. Phillips, W. H. Wilson, and G. S. Halford ([9], [24], [23]).

In addition to the focus on universals, the approach here is distinctive in the use of heteromorphisms–which are object-to-object morphisms between objects if different categories–in contrast to the usual homomorphisms or homs between objects in the same category. By explicitly adding heteromorphisms to the usual homs-only presentation of category theory, this approach can directly represent interactions between the objects of different categories (intuitively, between an organism and the environment). But it is still early days, and many approaches need to be tried to find out "where theory lives."

2 The ubiquity and importance of adjoints

Before developing the concept of a brain functor, we need to consider the related concept of a pair of adjoint functors. The developers of category theory, Saunders Mac Lane and Samuel Eilenberg, famously said that categories were defined in order to define functors, and functors were defined in order to define natural transformations [4]. A few years later, the concept of universal constructions or universal mapping properties was isolated ([18] and [28]). Adjoints were defined a decade later by Daniel Kan [13] and the realization of their ubiquity ("Adjoint functors arise everywhere" [19, p. v]) and their foundational importance has steadily increased over time (Lawvere [15] and Lambek [14]). Now it would perhaps not be too much of an exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. As Steven Awodey put it:

> The notion of adjoint functor applies everything that we have learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [1, p. 179]

Other category theorists have given similar testimonials.

> To some, including this writer, adjunction is the most important concept in category theory. [30, p. 6]
>
> The isolation and explication of the notion of adjointness is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas." [8, p. 438]
>
> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [29, p. 367]

3 Adjoints and universals

How do the ubiquitous and important adjoint functors relate to the universal constructions? Mac Lane and Birkhoff succinctly state the idea of the universals of category theory and note that adjunctions can be analyzed in terms of those universals.

> The construction of a new algebraic object will often solve a specific problem in a universal way, in the sense that every other solution of the given problem is obtained from this one by a unique homomorphism. The basic idea of an adjoint functor arises from the analysis of such universals. [20, p. v]

We can use some old language from Plato's theory of universals to describe those universals of category theory (Ellerman [5]) that solve a problem in a universal or paradigmatic way so that "every other solution of the given problem is obtained from this one" in a unique way.

In Plato's Theory of Ideas or Forms ($\varepsilon\iota\delta\eta$), a property F has an entity associated with it, the universal u_F, which uniquely represents the property. An object x has the property F, i.e., $F(x)$, if and only if (iff) the object x *participates* in the universal u_F. Let μ (from $\mu\varepsilon\theta\varepsilon\xi\iota\varsigma$ or *methexis*) represent the participation relation so

$$\text{"}x\ \mu\ u_F\text{" reads as "}x \text{ participates in } u_F\text{".}$$

Given a relation μ, an entity u_F is said to be a *universal* for the property F (with respect to μ) if it satisfies the following universality condition:

$$\text{for any } x,\ x\ \mu\ u_F \text{ if and only if } F(x).$$

A universal representing a property should be in some sense unique. Hence there should be an equivalence relation ($\approx$) so that universals satisfy a uniqueness condition:

$$\text{if } u_F \text{ and } u'_F \text{ are universals for the same } F, \text{ then } u_F \approx u'_F.$$

The two criteria for a *theory of universals* is that it contains a binary relation μ and an equivalence relation $\approx$ so that with certain properties F there are associated entities u_F satisfying the following conditions:

(1) *Universality condition*: for any x, $x\ \mu\ u_F$ iff $F(x)$, and

(2) *Uniqueness condition*: if u_F and u'_F are universals for the same F [i.e., satisfy (1)], then $u_F \approx u'_F$.

A universal u_F is said to be *non-self-predicative* if it does not participate in itself, i.e., $\neg(u_F\ \mu\ u_F)$. A universal u_F is *self-predicative* if it participates in itself, i.e., $u_F\ \mu\ u_F$. For the sets in an iterative set theory (Boolos [2]), set membership is the participation relation, set equality is the equivalence relation, and those sets are never-self-predicative (since the set of instances of a property is always of higher type or rank than the instances). The universals of category theory form the "other bookend" as always-self-predicative universals. The set-theoretical paradoxes arose from trying to have *one* theory of universals ("Frege's Paradise") where the universals could be *either* self-predicative or non-self-predicative,[3] instead of having two opposite "bookend" theories, one for never-self-predicative universals (set theory) and one for always always-self-predicative universals (category theory).

For the self-predicative universals of category theory,[4] the participation relation is the *uniquely-factors-through* relation. It can always be formulated in a suitable category as:

[3] Then the universal for all the non-self-predicative universals would give rise to Russell's Paradox since it could not be self-predicative or non-self-predicative (Russell [27, p. 80]).

[4] In the general case, a category is usually defined as follows (e.g., [20] or [19]):

A category C consists of

(a) a set of objects $a, b, c, ...$,

$$"x\ \mu\ u_F" \text{ means "there exists a unique arrow } x \Rightarrow u_F\text{".}$$

Then x is said to *uniquely factor through* u_F, and the arrow $x \Rightarrow u_F$ is the unique factor or participation morphism. In the universality condition,

$$\text{for any } x,\ x\ \mu\ u_F \text{ if and only if } F(x),$$

the existence of the identity arrow $1_{u_F} : u_F \Rightarrow u_F$ is the self-participation of the self-predicative universal that corresponds with $F(u_F)$, the self-predication of the property to u_F. In category theory, the equivalence relation used in the uniqueness condition is the isomorphism ($\cong$).[5]

4 Adjunctions for partial orders

We will use a specific heterodox treatment of adjunctions, first developed by Pareigis [22] and later rediscovered and developed by Ellerman ([6], [7]), which shows that adjoints arise by gluing together in a certain way two universals (left and right representations). But for illustration, we start with adjunctions for partial orders, the simplest non-trivial example of a category (where adjunctions are called "Galois connections").[6]

The set of subsets or powerset $\wp(U)$ of a universe set U is a category whose objects are the subsets $c \subseteq U$ and whose morphisms are the inclusions ($\subseteq$). Consider the set of all ordered pairs of subsets $\langle a, b\rangle$ from the Cartesian product $\wp(U) \times \wp(U)$ where the partial order (using the same symbol $\subseteq$) is defined by pairwise inclusion. That is, given the two ordered pairs $\langle a', b'\rangle$ and $\langle a, b\rangle$, we define

$$\langle a', b'\rangle \subseteq \langle a, b\rangle \text{ if } a' \subseteq a \text{ and } b' \subseteq b.$$

Functors between these partial-order categories are just order-preserving maps. From $\wp(U)$ to $\wp(U)\times\wp(U)$, there is the diagonal functor $\Delta(x) = \langle x, x\rangle$, and from $\wp(U)\times\wp(U)$ to $\wp(U)$, there is the meet functor $\cap(\langle a, b\rangle) = a \cap b$. Consider now the following *adjointness relation* or *Galois connection* between the two partial orders:[7]

$$\Delta(c) \subseteq \langle a, b\rangle \text{ iff } c \subseteq \cap(\langle a, b\rangle)$$
$$\text{Adjointness Equivalence}$$

for sets a, b, and c in $\wp(U)$.

It has a certain symmetry that can be exploited. If we fix $\langle a, b\rangle$, then we have the universality condition for the meet of a and b: for any c in $\wp(U)$,

(b) for each pair of objects a, b, a set $hom_C(a, b) = C(a, b)$ whose elements are represented as homomorphisms $f : a \Rightarrow b$,

(c) for any $f \in hom_C(a, b)$ and $g \in hom_C(b, c)$, there is the composition $gf : a \Rightarrow b \Rightarrow c$ in $hom_C(a, c)$,

(d) composition of arrows is an associative operation, and

(e) for each object a, there is an arrow $1_a \in hom_C(a, a)$, called the *identity* of a, such that for any $f : a \Rightarrow b$ and $g : c \Rightarrow a$, $f1_a = f$ and $1_a g = g$.

An arrow $f : a \Rightarrow b$ is an *isomorphism*, $a \cong b$, if there is an arrow $g : b \Rightarrow a$ such that $fg = 1_b$ and $gf = 1_a$. A *functor* is a map from one category to another that preserves composition and identities.

[5] Thus it must be verified that two concrete universals for the same property are isomorphic. By the universality condition, two concrete universals u and u' for the same property must participate in each other. Let $f : u' \Rightarrow u$ and $g : u \Rightarrow u'$ be the unique arrows given by the mutual participation. Then by composition $gf : u' \Rightarrow u'$ is the unique arrow $u' \Rightarrow u'$ but $1_{u'}$ is another such arrow so by uniqueness, $gf = 1'_u$. Similarly, $fg : u \Rightarrow u$ is the unique self-participation arrow for u so $fg = 1_u$. Thus mutual participation of u and u' implies the isomorphism $u \cong u'$.

[6] A binary relation $\leq$ on U is a *partial order* if for all $u, u', u'' \in U$, it is reflexive ($u \leq u$), transitive ($u \leq u'$ and $u' \leq u''$ imply $u \leq u''$), and anti-symmetric ($u \leq u'$ and $u' \leq u$ imply $u = u'$).

[7] The "iff" or "if and only if" is replaced by a natural isomorphism in the general case of an adjunction or a representation (defined later).

$$\Delta(c) \subseteq \langle a, b\rangle \text{ iff } c \subseteq a \cap b.$$
Universality Condition for $\cap(a, b)$.

The defining property F on elements c of $\wp(U)$ is that $F(c) \equiv \Delta(c) \subseteq \langle a, b\rangle$ so the universality condition just says: $F(c)$ iff c "participates in" $\cap(a, b) = a \cap b$ (where inclusion is the participation relation). Moreover $a \cap b$ is the self-predicative universal since, of course, $a \cap b \subseteq a \cap b$.

But using the symmetry, we could fix c and have another universality condition using the reverse inclusion in $\wp(U) \times \wp(U)$ as the relation: for any $\langle a, b\rangle$ in $\wp(U) \times \wp(U)$,

$$\langle a, b\rangle \supseteq \langle c, c\rangle \text{ iff } c \subseteq \cap(a, b)$$
Universality Condition for $\Delta(c)$.

Here the defining property G on elements $\langle a, b\rangle$ of $\wp(U) \times \wp(U)$ is that $G(\langle a, b\rangle) \equiv c \subseteq \cap(a, b)$. The self-predicative universal for that property is the image of c under the diagonal functor $\Delta(c) = \langle c, c\rangle$. In the Platonic language, $\langle a, b\rangle$ participates in $\langle c, c\rangle$ iff $G(\langle a, b\rangle)$ (where the reverse inclusion is the participation relation). Moreover, $\langle c, c\rangle$ is the self-predicative universal since $\langle c, c\rangle \supseteq \langle c, c\rangle$.

Thus in this adjoint situation between the two partial order categories $\wp(U)$ and $\wp(U) \times \wp(U)$, we have a pair of order-preserving maps ("adjoint functors") going each way between the categories such that each element in a category defines a certain property in the other category and the map carries the element to the self-predicative universal for that property.

$$\Delta : \wp(U) \to \wp(U) \times \wp(U) \text{ and } \cap : \wp(U) \times \wp(U) \to \wp(U)$$
Example of Adjoint Functors Between Partial Orders.

To further analyze adjoints, we need the notion of a "heteromorphism" so that we can define the self-predicative universal in each category as given by a left or right representation.

5 Heteromorphisms and adjunctions

We have seen that there are two self-predicative universals (often one is trivial like $\Delta(c)$ in the above example) involved in an adjunction and that the object-to-object maps or relations were always within one category (or partial order), e.g., the "hom-sets" in a category where "hom" is short for homomorphism (a morphism between objects in the same category). Using object-to-object maps between objects of *different* categories (properly called "heteromorphisms" or "chimera morphisms"), the notion of an adjunction can be factored into two representations (or "half-adjunctions" in Ellerman [6, p. 158]), each of which isolates a self-predicative universal, a solution to a universal mapping problem.

This heteromorphic treatment of adjoints will be illustrated using the above example. The objects $c \in \wp(U)$ in the partial order $\wp(U)$ are single subsets c of U and the objects $\langle a, b\rangle$ in the partial order $\wp(U) \times \wp(U)$ are pairs of subsets of U. A *heteromorphism* or *het* from a single subset c to the pair of subsets $\langle a, b\rangle$, which are objects in different categories, is given by the "cone" or "fork" $c \subseteq a$ and $c \subseteq b$ which could be symbolized $c \to \langle a, b\rangle$.[8] Fixing $\langle a, b\rangle$, there is a single subset $\cap(\langle a, b\rangle) = a \cap b$ with a canonical het $\cap(\langle a, b\rangle) = a \cap b \to \langle a, b\rangle$. Then the meet functor that takes $\langle a, b\rangle$ to $a \cap b$ gives a *right representation* if for every het from $c \to \langle a, b\rangle$, there is a (unique) hom $c \subseteq \cap(a, b) = a \cap b$ that gives us the following (commutative) diagram where the arrows are hets.

[8] The hets between objects of different categories are represented as single arrows ($\to$) while the homomorphisms or homs between objects in the same category are represented by double arrows ($\Rightarrow$) or inclusions ($\subseteq$) in the case of partial orders. The functors or maps between whole categories are also represented by single arrows ($\to$).

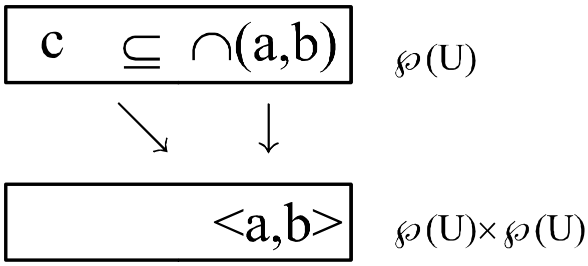

Fig. 1: Right representation diagram.

Hence the canonical het $\cap(a,b) = a \cap b \to \langle a,b\rangle$ is not only a universal for the property $c \to \langle a,b\rangle$ but is self-predicative by the identity morphism $a \cap b \subseteq a \cap b$. This gives the following if-and-only-if equivalence between the diagonal het and the horizontal hom:

$$c \to \langle a,b\rangle \text{ iff } c \subseteq a \cap b.$$

Universality condition for right representation

which we previously encountered as the universality condition for $\cap(a,b)$.

Similarly for given c, there is a canonical het $c \to \Delta(c) = \langle c,c\rangle$ given by the two identities. Then the functor that takes c to $\Delta(c)$ is a *left representation* if for any given het $c \to \langle a,b\rangle$, there is a (unique) $\Delta(c) \subseteq \langle a,b\rangle$ to make the following diagram commute.

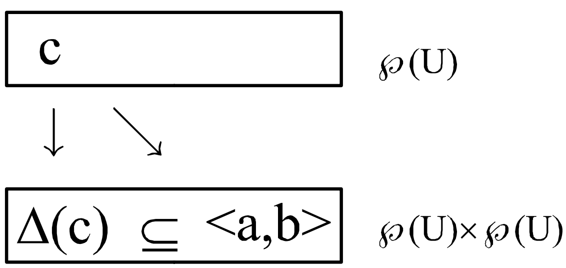

Fig. 2: Left representation diagram.

Hence the canonical het $c \to \Delta(c) = \langle c,c\rangle$ is not only a universal for the property $c \to \langle a,b\rangle$ but is self-predicative by the identities $c \subseteq c$. The corresponding universality equivalence is:

$$\Delta(c) \subseteq \langle a,b\rangle \text{ iff } c \to \langle a,b\rangle$$

Universality condition for left semi-adjunction

which we previously encountered as the universality condition for $\Delta(c)$.

The concept of a left or right representation of hets is the simplest and most general concept of a universal mapping property (self-predicative universal) in category theory. If there are both right and left representations (a bi-representation) with the *same* diagonal hets, $c \to \langle a,b\rangle$ in this case, then they combine to give an adjunction ([22, pp. 60-1]; [6, p. 130]):

$$\Delta(c) \subseteq \langle a,b\rangle \text{ iff } c \to \langle a,b\rangle \text{ iff } c \subseteq \cap(\langle a,b\rangle)$$

Adjunction equivalence with het middle term.

Gluing together the two left and right representation diagrams along the common het $c \to \langle a,b\rangle$ gives the *adjunctive square diagram* representing an adjunction.

$$\begin{tikzcd} c \arrow[d] \arrow[drr] & \subseteq & \cap(a,b) \arrow[d] & \wp(U) \\ \Delta(c) & \subseteq & \langle a,b \rangle & \wp(U)\times\wp(U) \end{tikzcd}$$

Fig. 3: Left and right bi-representation of hets = adjunction

In the orthodox (i.e., "heterophobic" or homs-only) treatment of adjunctions, the middle het term is left out since hets are not formally treated as a type of morphism in the usual presentation of category theory. Then we get the previously mentioned form (without any mention of hets) of the adjunction equivalence:

$$\Delta(c) \subseteq \langle a, b \rangle \text{ iff } c \subseteq \cap(\langle a, b \rangle)$$
Orthodox homs-only form of adjunction equivalence.

The left or right representations of hets are the most general form of self-predicative universals (universal mapping properties), and adjunctions are the special cases where left and right representations exist for the same hets.

6 Brain functors

If the self-predicative universals of category theory, which combine in one way to form an adjunction, serve to delineate the important, paradigmatic, canonical, or essential concepts and structures within pure mathematics, then one might well expect the self-predicative universals to also be important in applications.

In many adjunctions, the important fact is expressed by either the left or right representation (e.g., the free-group universal mapping property as a left representation), with no need for the "device" of the other representation used to express the adjunction in a het-free manner.

Another payoff from analyzing the important but "molecular" concept of an adjunction into two "atomic" representations is that we can then reassemble those "atomic" parts in a new way to define the cognate concept speculatively named a "brain functor."

The basic intuition is to think of one category in a representation as the "environment" and the other category as an "organism." Instead of representations within *each* category the hets going *one way* between the categories, suppose the hets going *both ways* were represented within *one* of the categories (the "organism").

A het from the environment to the organism is say, a visual or auditory stimulus. Then a left representation would play the role of the brain in providing the re-cognition or perception (expressed by the intentionality-of-perception slogan: "seeing is seeing-as") of the stimulus as a perception of, say, a tree where the internal re-cognition is represented by the morphism $\Rightarrow$ inside the "organism" category.

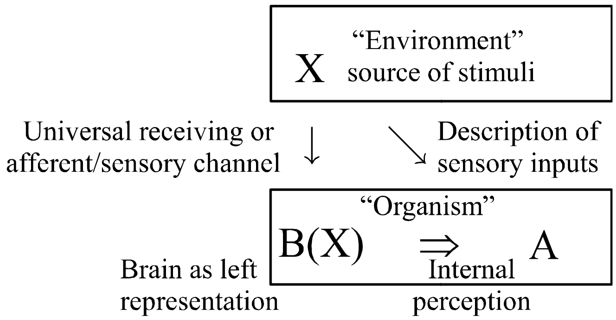

Fig. 4: Perceiving brain presented as a left representation.

Perhaps not surprisingly, this mathematically models the old philosophical theme in the Platonic tradition that external stimuli do not give knowledge; the stimuli only trigger the internal perception, recognition, or recollection (as in Plato's *Meno*) that is knowledge. In *De Magistro* (The Teacher), the neo-Platonic Christian philosopher Augustine of Hippo developed an argument (in the form of a dialogue with his son Adeodatus) that as teachers teach, it is only the student's internal appropriation of what is taught that gives understanding.

> Then those who are called pupils consider within themselves whether what has been explained has been said truly; looking of course to that interior truth, according to the measure of which each is able. Thus they learn,.... But men are mistaken, so that they call those teachers who are not, merely because for the most part there is no delay between the time of speaking and the time of cognition. And since after the speaker has reminded them, the pupils quickly learn within, they think that they have been taught outwardly by him who prompts them. (Augustine, *De Magistro*, Chapter XIV)

The basic point is the active role of the mind in generating understanding (represented by the internal hom). This is clear even at the simple level of understanding spoken words. We hear the auditory sense data of words in a completely strange language as well as the words in our native language. But the strange words, like @#$%^, bounce off our minds with no resultant understanding while the words in a familiar language prompt an internal process of generating a meaning so that we understand the words. Thus it could be said that "understanding a language" means there is a left representation for the heard statements in that language, but there is no such internal re-cognition mechanism for the heard auditory inputs in a strange language.

Dually, there are also hets going the other way from the "organism" to the "environment" and there is a similar distinction between mere behavior and an action that expresses an intention. Mathematically that is described by dualizing or turning the arrows around which gives an acting brain presented as a right representation.

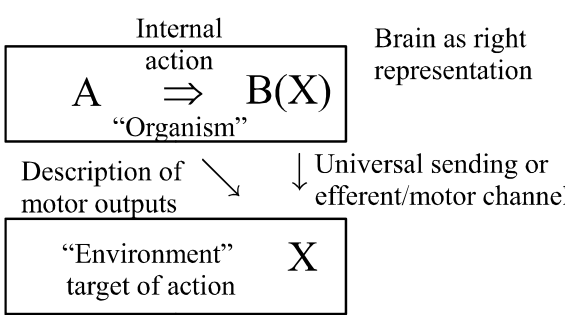

Fig. 5: Acting brain as a right representation.

In the heteromorphic treatment of adjunctions, an adjunction arises when the hets from one category to another, $Het(X, A)$, have a right representation, $Het(X, A) \cong Hom(X, G(A))$, and a left representation, $Hom(F(X), A) \cong Het(X, A)$. But instead of taking the same set of hets as being represented by two different functors on the right and left, suppose we consider a single functor $B(X)$ that represents the hets $Het(X, A)$ on the left:

$$Het(X, A) \cong Hom(B(X), A),$$

and represents the hets $Het(A, X)$ [going in the opposite direction] on the right:

$$Hom(A, B(X)) \cong Het(A, X).$$

If the hets each way between two categories are represented by the same functor $B(X)$ as left and right representations, then that functor is said to be a *brain functor*. Thus instead of a pair of functors being adjoint, we have a single functor $B(X)$ with values within one of the categories (the "organism") as representing the two-way interactions, "perception" and "action," between that category and another one (the "environment"). In particular, it should be noted how the "turn-around-the-arrows" category-theoretic duality matches perfectly the well-known "turn-around-the-arrows" duality between:

- sensory or afferent systems (brain furnishing the left representation of the environment to organism heteromorphisms), and
- motor or efferent systems (brain furnishing the right representation of the organism to environment heteromorphisms).

7 Diagramming adjoint and brain functors

An adjunction can be viewed as one way of putting together the building blocks of representations, and the concept of a brain functor is the cognate concept obtained by putting the building blocks of representations together in another way. The adjunctive diagram for an adjunction arises by gluing together the two diagrams for the left and right representations along the common diagonal het $X \to A$.

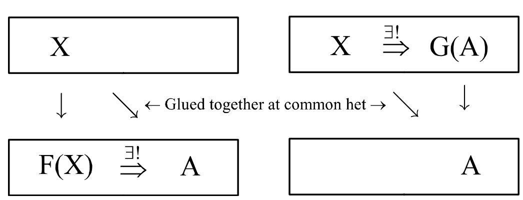

Fig. 6: Combining two representations make an adjunction.

The combined diagram is the *adjunctive square diagram* that represents an adjunction in the heteromorphic treatment of adjunctions.

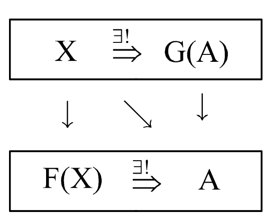

Fig. 7: Adjunctive Square Diagram.

The adjunctive square diagram might be compared to the usual "over and back" adjunction diagram in the het-free treatment which shows only homs ($\Rightarrow$) inside each category (represented by rectangles) with no hets ($\rightarrow$) between the objects of different categories. It is "over and back" since given a hom $f : X \Rightarrow G(A)$ in one category, there is a unique hom $\underline{f} : F(X) \Rightarrow A$ over in the other category whose functorial image $G\left(\underline{f}\right) : G(F(X)) \Rightarrow G(A)$ back in the original category makes the triangular diagram commute.

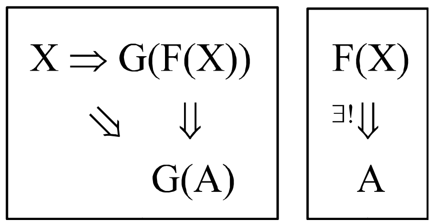

Fig. 8: Usual het-free or homs-only diagram for an adjunction.

The adjunctive square diagram is much simpler, and shows, one might argue, what is "really going on" in an adjunction, namely the *hets* $X \rightarrow A$ are represented by the *homs* $\mathrm{Hom}_{\mathbf{A}}(F(X), A)$ and $\mathrm{Hom}_{\mathbf{X}}(X, G(A)$ respectively on the left and right which determines the adjoint functors F and G [22, p. 47].

The diagram for a brain functor is obtained by gluing together the diagrams for the left and right representations at the common values of the brain functor $B(X)$.

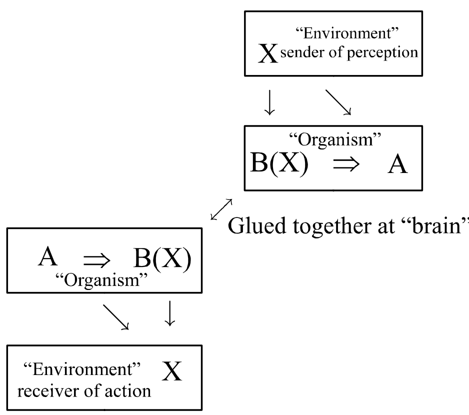

Fig. 9: Combining two representations to make a "brain".

If we think of the diagram for a representation as right triangle, then the adjunctive square diagram is obtained by gluing two triangles together on the hypotenuses (when they have the same hypotenuse), and the diagram for the brain functor is obtained by gluing two triangles together at the right angle vertices (when the vertices are the same) to form the *butterfly diagram.* If both the triangular "wings" could be filled-out as adjunctive squares, then the brain functor would have left and right adjoints.[9] In the butterfly diagram below, we have labelled the diagram for the brain as the language faculty for understanding and producing speech.

[9] The previous example of the diagonal functor $\Delta : \wp(U) \rightarrow \wp(U) \times \wp(U)$ is a brain functor since the meet functor $\cap(a, b) = a \cap b$ is the right adjoint, and the union functor $\cup(a, b) = a \cup b$ is the left adjoint.

The aforementioned underlying set functor that takes a group G to its underlying set $U[G]$ is a rather trivial example of a brain functor that does not arise from having both a left and right adjoint. It has a left adjoint (the free group functor) so U provides a right representation for the hets $X \rightarrow G$. Also it provides a left representation for the hets $G \rightarrow X$ but has no right adjoint.

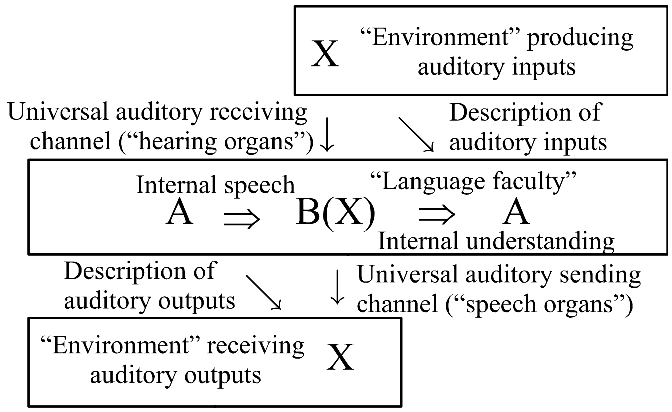

Fig. 10: Brain functor butterfly diagram interpreted as language faculty.

Wilhelm von Humboldt recognized the symmetry between the speaker and listener, which in the same person is abstractly represented as the dual functions of the "selfsame power" of the language faculty in the above butterfly diagram.

> Nothing can be present in the mind (Seele) that has not originated from one's own activity. Moreover understanding and speaking are but different effects of the selfsame power of speech. Speaking is never comparable to the transmission of mere matter (Stoff). In the person comprehending as well as in the speaker, the subject matter must be developed by the individual's own innate power. What the listener receives is merely the harmonious vocal stimulus.[10, p. 102]

8 A brain functor using partial orders

A simple example of a brain functor using partial orders will be developed first. In this setting, only the simplest "brain function" can be modeled, namely the building and functioning of an internal model of the external reality such as an internal coordinate system to map an external set of locations.[10] The external reality is given by a set of atomic points or locations Y, the atomic inside coordinates are the points in X, and the coordinate mapping function is the given function $f : X \to Y$. Just to keep the mathematics not completely trivial, we do not require f to be an isomorphism; multiple internal coordinates might refer to the same external point (i.e., f is not necessarily one-to-one) and some external points might not have internal coordinates (i.e., f is not necessarily onto). The two partial orders are the inclusion-ordered subsets of external points $\wp(Y)$ and the inclusion-ordered subsets of internal coordinates $\wp(X)$.

In the case where the "brain" is an "electronic brain" or computer, Y is the set of locations on an external input/output memory device such as a floppy disk, thumb drive, or any other external memory device. Each external location is marked with a 0 or 1, so the subsets $V \in \wp(Y)$ would be the external sets of 1's. The set X would be the set of internal memory locations which also contain either a 0 or 1, so the subsets $U \in \wp(X)$ are the internal sets of 1's. The coordinate function $f : X \to Y$ maps the internal memory locations to the external disk locations. The dual perception/action functions in the electronic brain would be the familiar read/write operations between the computer and the external memory device.

The brain functor in this example is $f^{-1} : \wp(Y) \to \wp(X)$ where for any subset $V \in \wp(Y)$, the value of the brain functor is:

[10] See also the coordinate-plot scheme in Lawvere and Schanuel [16, p. 86].

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

Given an external subset $V \subseteq Y$, what is the "best" internal subset $U \subseteq X$ that represents or recognizes V? The heteromorphism $V \to U$ is defined by the property, $F(U) =$ "U is complete for V" in the sense that all the $x \in X$ that map to V are contained in U, i.e.,

$$V \to U \text{ means } \forall x \in X, \text{ if } f(x) \in V, \text{ then } x \in U.$$

The left semi-adjunction for the property $F(U) =$ "U is complete for V" is given by the smallest complete subset $f^{-1}(V) \subseteq X$ and the universality condition: $f^{-1}(V) \subseteq U$ iff $V \to U$, is satisfied.

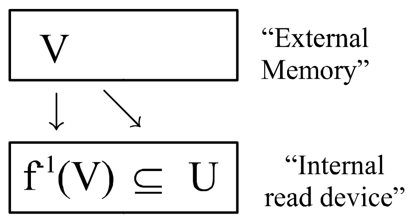

Fig. 11: Left representation to "read" V
with smallest complete subset $f^{-1}(V)$.

The self-predicative universal $f^{-1}(V) \subseteq X$ has the property, i.e., $V \to f^{-1}(V)$, and a subset $U \in \wp(X)$ has the property, i.e., $V \to U$, if and only U participates in the self-predicative universal $U \supseteq f^{-1}(V)$ (where "participation" is written as the reverse inclusion).

In the dual case of "action," the het $U' \to V$ going in the opposite direction from a subset U' of X to a subset V of Y is defined by the property, $G(U') =$ "U' is consistent with V " in the sense that no coordinate in U' maps outside of V, i.e.,

$$U' \to V \text{ means } f(U') \subseteq V.$$

The right semi-adjunction for the property $G(U') =$ "U' is consistent with V " is given by the largest consistent subset $f^{-1}(V) \in \wp(X)$, and the universality condition is: $U' \subseteq f^{-1}(V)$ iff $U' \to V$.

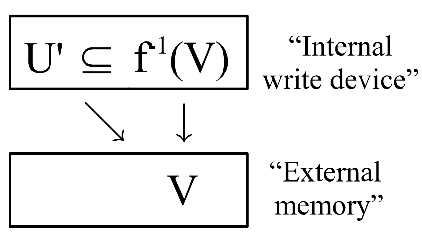

Fig. 12: Right representation to "write" V
with largest consistent subset $f^{-1}(V)$.

The self-predicative universal $f^{-1}(V) \in \wp(X)$ has the property, $f^{-1}(V) \to V$, and a subset $U' \in \wp(X)$ has the property, i.e., $U' \to V$, if and only if U' participates in the self-predicative universal, i.e., $U' \subseteq f^{-1}(V)$ (where "participation" is the inclusion). Combining the left and right representations at the common "brain" gives the butterfly diagram.

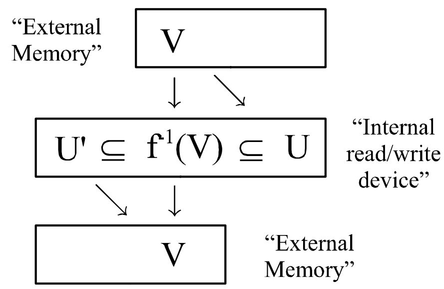

Fig. 13: Butterfly diagram for the brain functor $f^{-1}:\wp(Y)\to\wp(X)$.

Mathematically, this example is an instance of the general result that any functor that has both right and left adjoints is a brain functor. The right adjoint of $f^{-1}(V)$ is usually symbolized as:

$$\forall_f(U)=\{y\in Y:\forall x,\text{if } f(x)=y \text{ then } x\in U\}$$

with the adjunction equivalence:

$$f^{-1}(V)\subseteq U \text{ iff } V\subseteq \forall_f(U).$$

The left adjoint of $f^{-1}(V)$ is usually symbolized as:

$$\exists_f(U')=f(U')=\{y\in Y:\exists x\in U',f(x)=y\}$$

with the adjunction equivalence:

$$\exists_f(U')\subseteq V \text{ iff } U'\subseteq f^{-1}(V).$$

Since the f^{-1} brain functor has these right and left adjoints, the butterfly diagram for the brain functor can be completed to form two adjunctive square diagrams glued together at the value of the brain functor $f^{-1}(V)$.

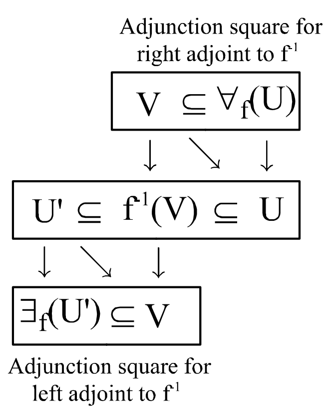

Fig. 14: Completing the butterfly diagram to form two adjunctive squares.

The quantifier notation is motivated by the special case where f is a projection $f=p_X:X\times Y\Rightarrow Y$ so for any binary relation $U\subseteq X\times Y$, then $\exists_f(U)=\{y\in Y:(\exists x)\,U(x,y)\}$ and $\forall_f(U)=\{y\in Y:(\forall x)\,U(x,y)\}$.

The fact that the read/write functions of an electronic brain can be modeled by the brain functor $f^{-1}:\wp(Y)\to\wp(X)$ is a simple example of modeling a brain at a conceptual level—which says little or nothing about the underlying (electronic) mechanisms. As complexity increases exponentially in animal and human brains, brain functors should be similarly seen as only modeling the brain functions at an abstract conceptual level (e.g., Figure 10 giving the scheme for the language faculty) while saying nothing about the underlying biological and chemical mechanisms.

9 A more complex brain functor

A more mathematically complex (beyond our restriction to partial orders) and adequate model of a brain (still at a conceptual level) is provided by the functor taking a finite set of vector spaces $\{V_i\}_{i=1,\ldots,n}$ over the same field (or R-modules over a ring R) to the product $\prod_i V_i$ of the vector spaces. Such a product is also the coproduct $\sum_i V_i$ [11, p. 173] and that space may be written as the biproduct:

$$V_1 \oplus \ldots \oplus V_n \cong \prod_i V_i \cong \sum_i V_i.$$

The het from a set of spaces $\{V_i\}$ to a single space V is a "*cocone*" of vector space maps $\{V_i \Rightarrow V\}$ and the canonical such het is the set of canonical injections $\{V_i \Rightarrow V_1 \oplus \ldots \oplus V_n\}$ with the "brain" at the point of the cocone. The perception left representation then might be taken as conceptually representing the function of the brain as integrating multiple sensory inputs into an interpreted perception.[11]

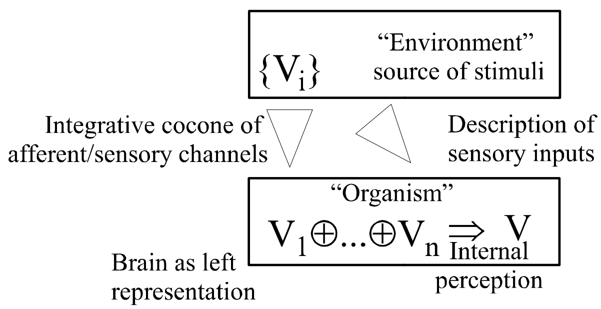

Fig. 15: Brain as integrating sensory inputs into a perception.

Dually, a het from single space V to a set of vector spaces $\{V_i\}$ is a *cone* $\{V \Rightarrow V_i\}$with the single space V at the point of the cone, and the canonical het is the set of canonical projections with the "brain" as the point of the cone: $\{V_1 \oplus \ldots \oplus V_n \Rightarrow Vi\}$. The action right representation then might be taken as conceptually representing the function of the brain as integrating or coordinating multiple motor outputs in the performance of an action.

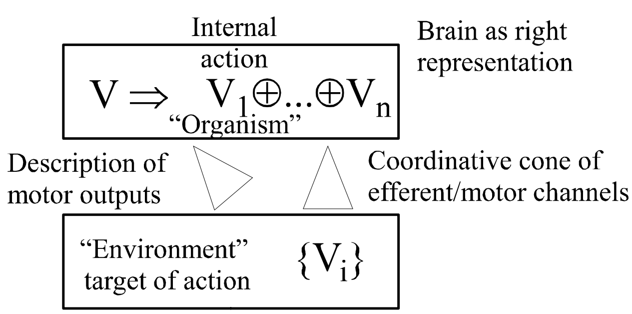

Fig. 16: Brain as coordinating motor outputs into an action.

Putting the two semi-adjunctions together gives the butterfly diagram for a brain.

[11] The cocones and cones are represented in the diagrams using cone shapes.

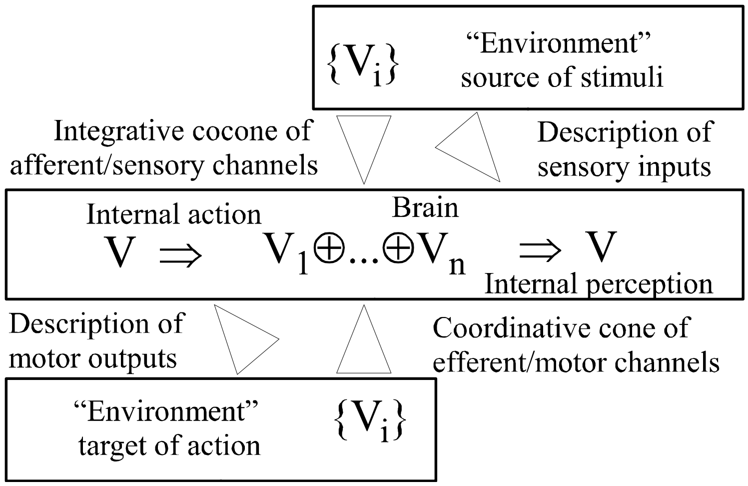

Fig. 17: Conceptual model of a perceiving and acting brain.

This gives a conceptual model of a single organ that integrates sensory inputs into a perception and coordinates motor outputs into an action, i.e., a brain.

10 Conclusion

In view of the success of category theory in modern mathematics, it is perfectly natural to try to apply it in the life and cognitive sciences. Many different approaches need to be tried to see which ones, if any, will find "where theory lives." The approach developed here differs from other approaches in several ways, but the most basic difference is the use of heteromorphisms to represent interactions between quite different entities (i.e., objects in different categories). Heteromorphisms also provide the natural setting to formulate universal mapping problems and their solutions as left or right representations of hets. In spite of abounding in the wilds of mathematical practice, hets are not recognized in the orthodox presentations of category theory. One consequence is that the notion of an adjunction appears as one atomic concept that cannot be factored into separate parts. But that is only a artifact of the homs-only treatment. The heteromorphic treatment shows that an adjunction factors naturally into a left and right representation of the hets going from one category to another–where, in general, one representation might exist without the other. One benefit of this heteromorphic factorization is that the two atomic concepts of a left and right representation can then be recombined in a new way to form the cognate concept of a brain functor. The main conclusion of the paper is that this concept of a brain functor seems to fit very well as an abstract but non-trivial description of the dual universal functions of a brain, perception (using the sensory or afferent systems) and action (using the motor or efferent systems).

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