On classical finite probability theory as a quantum probability calculus

David Ellerman
Department of Philosophy
U. of California/Riverside

September 23, 2014

Abstract

This paper shows how the classical finite probability theory (with equiprobable outcomes) can be reinterpreted and recast as the quantum probability calculus of a pedagogical or "toy" model of *quantum mechanics over sets* (QM/sets). There are two parts. The notion of an "event" is reinterpreted from being an epistemological state of indefiniteness to being an objective state of indefiniteness. And the mathematical framework of finite probability theory is recast as the quantum probability calculus for QM/sets. The point is not to clarify finite probability theory but to elucidate quantum mechanics itself by seeing some of its quantum features in a classical setting.

Contents

1 Introduction

This paper develops an interpretation of ordinary Laplace-Boole finite logical probability theory ([22], [2]) where the events are interpreted as objective states that have "objective indefiniteness" [29, p. 27] as in quantum mechanics (QM). The probabilities are then the probabilities, given one objectively indefinite state, to make the transition to another (more definite) objective state when "sampling a random variable" (analogous to a measurement of an observable in QM). In this manner,the Laplace-Boole probability calculus is presented as the probability calculus for a "quantum mechanics over sets" where the usual vector spaces over $\mathbb{C}$ for QM are replaced with vector spaces over $\mathbb{Z}_2$.[1] Quantum mechanics over sets (QM/sets) is a bare-bones "logical" (e.g., non-physical) version of QM, e.g., with spectral decomposition, the Dirac brackets, ket-bra resolution, the norm, observable-attributes, and the Born rule all in the simple classical setting of sets, that nevertheless provides models of characteristically quantum results such as the double-slit experiment, Bell's Theorem, and much more. In that manner, QM/sets can serve not only as a pedagogical (or "toy") model of QM but as an engine to better elucidate QM itself.

2 Laplace-Boole probability theory

Since our purpose is conceptual rather than mathematical, we will stick to the simplest case of finite probability theory with a finite sample space $U = \{u_1, ..., u_n\}$ of n equiprobable outcomes and to finite dimensional QM. The *events* in the usual interpretation are the subsets $S \subseteq U$, and the *probability* of an event S occurring in a trial is the ratio of the cardinalities: $\Pr(S) = \frac{|S|}{|U|}$. Given that a conditioning event $S \subseteq U$ occurs, the *conditional probability* that $T \subseteq U$ occurs is: $\Pr(T|S) = \frac{\Pr(T\cap S)}{\Pr(S)} = \frac{|T\cap S|}{|S|}$. The ordinary probability $\Pr(T)$ of an event T can be taken as the conditional probability with U as the conditioning event so all probabilities can be seen as conditional probabilities. Given a (real-valued) random variable, i.e., a *numerical attribute* $f : U \to \mathbb{R}$ on the elements of U, the *probability of observing a value r given an event S* is the conditional probability of the event $f^{-1}(r)$ given S:

$$\Pr(r|S) = \frac{|f^{-1}(r)\cap S|}{|S|}.$$

That is all the probability theory we will need here.

There are two parts to developing the quantum interpretation of the Laplace-Boole finite probability theory:

1. reinterpret the notion of an "event" as an ontological state of indefiniteness rather than an epistemological state of ignorance, and
2. show how the mathematics of finite probability theory can be recast using the mathematical notions of quantum mechanics except that the base field of $\mathbb{Z}_2$ is substituted, *mutatis mutandis*, for the complex numbers $\mathbb{C}$.

3 The objective interpretation of states

3.1 Objective indefiniteness in the QM literature

The usual interpretation of probability theory is about epistemological ignorance or indefiniteness rather than ontological indefiniteness. The "states" are states of knowledge. Given the state of

[1] Thus this treatment differs significantly from the previous attempts ([26], [17]) to develop a quantum theory with $\mathbb{C}$ replaced by $\mathbb{Z}_2$ since those attempts do "not make use of the idea of probability."[26, p. 919]

knowledge that event S occurs, the probability $\Pr(T|S)$ is the probability of making the epistemological "quantum jump" from the indefinite state of knowledge that S occurs to the more definite state of knowledge that $S \cap T$ occurs. The interpretation of probability theory developed here and used in QM/sets uses an objective or ontological notion of indefiniteness instead of epistemological indefiniteness. The events become objective states rather than states of knowledge.

The notion of objective indefiniteness is hardly supposed to be clear and distinct: indeed much of the difficulty in interpreting QM seems to be based on the difficulty (of the human mind) in grasping an objectively indefinite state. The mind always wants to assume that our macroscopic notion of definiteness will still apply at smaller and smaller scales so that, for instance, a particle still has a definite position and momentum regardless of what our indelicate measurements might reveal. The inherent impossibility of such definiteness in quantum mechanics, as specified in Heisenberg's indeterminacy principle,[2] has led many scientists and philosophers to try to flesh out some notion of ontological indefiniteness. Abner Shimony has been the most insistent on the notion of "objective indefiniteness" ([27], [28], [29]) [the phrase also used by Shimony's student Gregg Jaeger [19] and used here[3]], but other philosophers of physics have suggested related ideas such as:

- Peter Mittelstaedt's "incompletely determined" quantum states with "objective indeterminateness" [25],
- Paul Busch and Gregg Jaeger's "unsharp quantum reality" [3],
- Paul Feyerabend's "inherent indefiniteness" [13],
- Allen Stairs' "value indefiniteness" and "disjunctive facts" [31],
- E. J. Lowe's "vague identity" and "indeterminacy" that is "ontic" [23],
- Steven French and Decio Krause's "ontic vagueness" [16],
- Paul Teller's "relational holism" [32], and so forth.

3.2 Objective indefiniteness in probability theory

The elements u of the "sample space" U considered as the singletons $\{u\}$ are the definite states, the *eigenstates* of definiteness.[4] Collecting together a number of eigenstates into a multiple-element subset $S \subseteq U$ is the *superposition* of those definite eigenstates $\{u\} \subseteq S$. Thus a multiple-element subset or "event" S is interpreted as an object or objective state that is *objectively indefinite* between the definite eigenstates $\{u\} \subseteq S$.

Instead of being given the epistemological state of the conditioning event S, we are always are given an objective state S which could be U.[5] Then the conditional probability $\Pr(T|S) = \Pr(T \cap S|S)$ is interpreted as the probability that S will reduce or "collapse" to the more definite objective state $T \cap S \subseteq S$ when an experiment is made that is a "measurement" of a numerical attribute on U.

In the usual presentation of probability theory, the numerical attribute associated with an event T is left implicit but it can be taken as the characteristic function $\chi_T : U \to \{0,1\} \subseteq \mathbb{R}$ so that the conditional probability $\Pr(T|S) = \Pr(T \cap S|S)$ is the probability that the measurement of the attribute χ_T returns the value of 1, i.e.,

[2] Heisenberg's principle is often called the "uncertainty principle" as if the indefiniteness was only epistomological rather than ontological.

[3] Full disclosure: Shimony was my undergraduate thesis advisor at MIT.

[4] However, when we later consider the singletons of the U-elements as just one basis set among many in the vector space $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$, then we will see that the $\{u\}$ are definite for some attributes but may be completely indefinite for other attributes.

[5] The empty subset $\emptyset$ is not considered as an objective state so $S \neq \emptyset$.

$$\Pr(1|S) = \frac{|\chi_T^{-1}(1) \cap S|}{|S|} = \frac{|T \cap S|}{|S|} = \Pr(T \cap S|S) = \Pr(T|S).$$

In this manner, the "trial" or "experiment" in the usual epistemological interpretation of finite probability theory can always be seen as a "measurement" of a numerical attribute that "reduces" or "collapses" the state of knowledge from S to $T \cap S$. In the objective indefiniteness interpretation, a state reduction is also made but it is an objective state rather than a state of knowledge that is reduced or "collapsed" when a measurement-experiment is performed.

3.3 Some mental imagery for objective indefiniteness

There is no pretension that we have a clear and distinct mental image of a "blurred" and indistinct objective state. But that does not prevent one from trying to build *some* imagery no matter how inadequate.

In Boole's logic of subsets [2], each element u of the universe set U either definitely has or does not have a given property P (represented as a subset S of the universe). Moreover an element $u \in U$ has properties "all the way down" so that it is uniquely determined by the subset S containing u as in Leibniz's principle of the identity of indiscernibles. Change takes place by the definite properties changing. For a hound to go from point A to point B, there must be some trajectory of definite ground locations from A to B. One might be subjectively or epistemologically indefinite about the exact positions along the hound's path even though the path is objectively definite.

In the dual logic of partitions ([7], [11]), a partition $\pi = \{B\}$ is made up of disjoint blocks B whose union is the universe set U (the blocks are also thought of as the equivalence classes in the associated equivalence relation). The blocks in a partition have been distinguished from each other by the partition, but the elements within each block have not been distinguished from each other; instead they are identified by the associated equivalence relation. Each block B represents the objectively indefinite (pure) state obtained by superposing the definite singletons $\{u\} \subseteq B$. When more distinctions are made (the QM/sets-version of a measurement), the blocks get smaller and the partitions (QM/sets-version of mixed states) become more refined until the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is reached where each block is a singleton (the QM/sets-version of a non-degenerate measurement yielding a completely decoherent mixed state). Change takes place by some attributes becoming more definite and other (incompatible) attributes becoming less definite. For a hawk, as opposed to a hound, to go from point A to point B, it would go from a definite perch at A into a flight of indefinite ground locations, and then would have a definite perch again at B.[6]

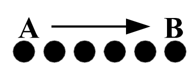
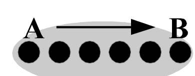
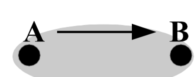
Classical trajectory from A to B. How a hound goes from A to B.	
Subjective indefiniteness about classical trajectory ("cloud of ignorance").	
Objective indefiniteness of quantum trajectory: definite position at A, indefinite position in transition, and definite position at B. How a hawk goes from A to B.	

Figure 1: How a hound and a hawk go from A to B

[6] The "flights and perchings" metaphor is from William James [20, p. 158] and according to Max Jammer, that description "was one of the major factors which influenced, wittingly or unwittingly, Bohr's formation of new conceptions in physics." [21, p. 178] The hawks and hounds pairing comes from Shakespeare's Sonnet 91.

The imagery of having a sharp focus versus being out-of-focus could also be used if one is clear that it is the reality itself that is in-focus or out-of-focus, not just the image through, say, a microscope. A classical trajectory is like a moving picture of sharp or definite in-focus realities, whereas the quantum trajectory starts with a sharply focused reality, goes out of focus, and then returns to an in-focus reality (by a measurement).

In the objective indefiniteness interpretation, a subset $S \subseteq U$ of a universe set U should be thought of as a single indefinite object S that is *represented* as the superposition of the definite objects $\{u\} \subseteq S$–just as a single superposition vector is represented as a weighted vector sum of certain basis of eigenvectors ("eigen" should be translated as "definite" here). Abner Shimony ([27] and [28]), in his description of a superposition state as being objectively indefinite, sometimes used Heisenberg's [18] language of "potentiality" and "actuality" to describe the relationship of the eigenvectors that are superposed to give an objectively indefinite state. This terminology could be adapted to the case of the sets. The singletons $\{u\} \subseteq S$ are "potential" in the objectively indefinite superposition S, and, with further distinctions, the indefinite entity S might "actualize" to $\{u\}$ for one of the "potential" $\{u\} \subseteq S$. Starting with S, the other $\{u\} \subsetneqq S$ (i.e., $u \notin S$) are not "potentialities" that could be "actualized" with further distinctions.

This terminology is, however, somewhat misleading since the indefinite entity S is perfectly actual (in the objectively indefinite interpretation); it is only the multiple eigenstates $\{u\} \subseteq S$ that are "potential" until "actualized" by some further distinctions. A non-degenerate measurement is not a process of a potential entity becoming an actual entity, it is a process of an actual indefinite entity becomes an actual definite entity. Since a distinction-creating measurement goes from actual indefinite to actual definite, the potential-to-actual language of Heisenberg should only be used with proper care–if at all.

Consider a three-element universe $U = \{a, b, c\}$ and a partition $\pi = \{\{a\}, \{b, c\}\}$. The block $S = \{b, c\}$ is objectively indefinite between $\{b\}$ and $\{c\}$ so those singletons are its "potentialities" in the sense that a distinction could result in either $\{b\}$ or $\{c\}$ being "actualized" in place of $\{b, c\}$. However $\{a\}$ is not a "potentiality" when one is starting with the indefinite entity $\{b, c\}$.

Note that this objective indefiniteness of $\{b, c\}$ is *not* well-described as saying that indefinite pre-distinction entity is "simultaneously both $\{b\}$and $\{c\}$" (like the common misdescription of the undetected particle "going through both slits" in the double-slit experiment); instead it is indefinite between $\{b\}$ and $\{c\}$. It is like saying that the 45° unit vector $(1, 1)/\sqrt{2}$ on the real x, y-plane is simultaneously on the x-axis and on the y-axis. A superposition of two sharp eigen-alternatives should *not* be thought of like a double-exposure photograph which has two fully definite images (e.g., simultaneously a picture of say $\{b\}$ and $\{c\}$). Instead of a double-exposure photograph, the superposition should be thought of as *representing* a blurred or indefinite reality that with further distinctions could sharpen to either of the sharp realities (mathematically, the distinctions project the 45° unit vector to either the x or y axis). But there must be some way to indicate which definite realities could be obtained by making further distinctions (measurements), and *that* is why the blurred or cloud-like indefinite reality is *represented* by mathematically superposing the definite possibilities.

Instead of a double-exposure photograph, a superposition representation might be thought of as "a photograph of clouds or patches of fog." (Schrödinger quoted in: [15, p. 66]) Schrödinger distinguishes a "photograph of clouds" from a blurry photograph presumably because the latter might imply that it was only the photograph that was blurry while the underlying objective reality was sharp. The "photograph of clouds" imagery for a superposition connotes a clear and complete photograph of an objectively "cloudy" or indefinite reality. Regardless of the (imperfect) imagery, one needs some way to indicate what are the definite eigenstates that could be "actualized" from a single indefinite entity S, and *that* is the role of conceptualizing a subset S as a collecting together or "superposing" certain "potential" eigenstates, i.e., the singletons $\{u\} \subseteq S$.

This point might be illustrated using some Guy Fawkes masks. Suppose there are two "orthogonal" eigenstates of having a goatee or a mustache, Mask 1 and Mask 2, represented formally by

$|goatee\rangle$ and $|mustache\rangle$.

Eigenstate 1: Guy Fawkes with goatee	 Mask 1
Eigenstate 2: Guy Fawkes with mustache	 Mask 2
This is the objectively indistinct state before (facial hair) distinctions to make an eigenstate.	 Mask 3
But that objectively indistinct state may be *represented* by superposition of possible distinct alternatives, the set {goatee, mustache} or vector $\lvert goatee\rangle + \lvert mustache\rangle$	 Mask 4

Figure 2: Objectively indefinite Mask 3 (not Mask 4) *represented* by superposition of distinct eigen-alternatives $|goatee\rangle + |mustache\rangle$

The objectively indefinite state is the distinction-less Mask 3 without facial hair, but it is formally *represented* as the superposition $|goatee\rangle + |mustache\rangle$ of the possible definite states. That superposition is unfortunately usually interpreted as representing the double exposure Mask 4 which, like the "particle going through both slits," is actually an impossible state since we have assumed that the definite states $|goatee\rangle$ (Mask 1) and $|mustache\rangle$ (Mask 2) are orthogonal.

The most important consequence is that in quantum dynamics without measurement, since the objectively indefinite states are *represented* by the linear superposition of the possible definite states, the evolution of the indefinite states is thus *represented* as the linear superposition of the evolution of the definite states. That is the source of the usual wave imagery in QM (e.g., as in Fourier analysis). But the point is that the evolving "wave function" or state vector as a superposition of evolving eigenstates, is only the way to *describe* the evolution of the *indefinite state* that is indefinite between those evolving eigenstates. Since the indefinite state is not actually the (impossible) "multiple exposure" of actual orthogonal definite states, the usual wave imagery of superposition and interference, as if there were actual waves of some sort, is rather misleading. The superposition and interference of evolving possible definite states is just how to *represent* the evolution of objectively indefinite states that are indefinite between those definite possibilities.

Under this objectively-indefinite way of interpreting the "wave function" or state vector formalism, much of the literature on interpreting the "wave function," not to mention the imagery of an electron mysteriously going through both slits or a photon mysteriously going through both arms of an interferometer, is wrong-footed from the beginning. The difficult imagery (or "mystery") lies in imagining an objectively indefinite state, particularly when we try to force it into the space of definite states (like trying to locate Mask 3 in a space consisting of two definite states, Mask 1 and Mask 2).

4 Recasting finite probability theory as a quantum probability calculus

4.1 Vector spaces over $\mathbb{Z}_2$

To bring out the full quantum mechanical flavor in the classical Laplace-Boole finite probability theory, we recast it using the vector space mathematics of quantum theory.[7] But the vector spaces are over $\mathbb{Z}_2$ where the singletons $\{u_i\} \subseteq U$ of the finite "sample space" U are just one among many equicardinal basis sets for $\wp(U) \cong \mathbb{Z}_2^n$. This gives what might be called a "non-commutative" form of the classical Laplace-Boole finite probability theory.

The power set $\wp(U)$ of $U = (u_1, ..., u_n)$ is a vector space over $\mathbb{Z}_2 = \{0,1\}$, isomorphic to $\mathbb{Z}_2^n$, where the vector addition $S+T$ is the *symmetric difference* (or inequivalence) of subsets. For $S, T \subseteq U$,

$$S + T = (S - T) \cup (T - S) = S \cup T - S \cap T.$$

The U-*basis* in $\wp(U)$ is the set of singletons $\{u_1\}, \{u_2\}, ..., \{u_n\}$, i.e., the set $\{\{u\}\}_{u \in U}$. A vector $S \in \wp(U)$ is specified in the U-basis as $S = \sum_{u \in S} \{u\}$ and it is characterized by its $\mathbb{Z}_2$-valued characteristic function $\chi_S : U \to \mathbb{Z}_2 \subseteq \mathbb{R}$ of coefficients since $S = \sum_{u \in U} \chi_S(u) \{u\}$. Similarly, a vector v in $\mathbb{C}^n$ is specified in terms of an orthonormal basis $\{|v_i\rangle\}$ as $v = \sum_i c_i |v_i\rangle$ and is characterized by a $\mathbb{C}$-valued function $\langle \_ | v\rangle : \{v_i\} \to \mathbb{C}$ assigning a complex amplitude $\langle v_i | v\rangle = c_i$ to each basis vector $|v_i\rangle$. One of the key pieces of mathematical machinery in QM, namely the inner product, does not exist in vector spaces over finite fields but brackets can still be defined using $\langle \{u\} |_U S\rangle = \chi_S(u)$ (see below) and a norm can be defined to play a similar role in the probability calculus of QM/sets.

Seeing $\wp(U)$ as the abstract vector space $\mathbb{Z}_2^n$ allows different bases in which the vectors can be expressed (as well as the basis-free notion of a vector as a "ket"). Consider the simple case of $U = \{a,b,c\}$ where the U-basis is $\{a\}$, $\{b\}$, and $\{c\}$. But the three subsets $\{a,b\}$, $\{b,c\}$, and $\{a,b,c\}$ also form a basis since:

$\{b,c\} + \{a,b,c\} = \{a\}$;
$\{b,c\} + \{a,b\} + \{a,b,c\} = \{b\}$; and
$\{a,b\} + \{a,b,c\} = \{c\}$.

These new basis vectors could be considered as the basis-singletons in another equicardinal universe $U' = \{a', b', c'\}$ where $\{a'\}$, $\{b'\}$, and $\{c'\}$ refer to the same abstract vector as $\{a,b\}$, $\{b,c\}$, and $\{a,b,c\}$ respectively.

In the following *ket table*, each row is an abstract vector of $\mathbb{Z}_2^3$ expressed in the U-basis, the U'-basis, and a U''-basis.

$U = \{a,b,c\}$	$U' = \{a',b',c'\}$	$U'' = \{a'',b'',c''\}$
$\{a,b,c\}$	$\{c'\}$	$\{a'',b'',c''\}$
$\{a,b\}$	$\{a'\}$	$\{b''\}$
$\{b,c\}$	$\{b'\}$	$\{b'',c''\}$
$\{a,c\}$	$\{a',b'\}$	$\{c''\}$
$\{a\}$	$\{b',c'\}$	$\{a''\}$
$\{b\}$	$\{a',b',c'\}$	$\{a'',b''\}$
$\{c\}$	$\{a',c'\}$	$\{a'',c''\}$
$\emptyset$	$\emptyset$	$\emptyset$

Vector space isomorphism: $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$ where row = ket.

In the Dirac notation [5], the *ket* $|\{a,c\}\rangle$ represents the abstract vector that is represented in the U-basis as $\{a,c\}$. A row of the ket table gives the different representations of the *same* ket in the different bases, e.g., $|\{a,c\}\rangle = |\{a',b'\}\rangle = |\{c''\}\rangle$.

[7] We are assuming some basic familarity with the mathematics of finite dimensional QM.

4.2 The brackets

In a Hilbert space, the inner product is used to define the brackets $\langle v_i|v\rangle$ and the norm $|v| = \sqrt{\langle v|v\rangle}$. In a vector space over $\mathbb{Z}_2$, the Dirac notation can still be used to define the brackets and norm even though there is no inner product. For a singleton basis vector $\{u\} \subseteq U$, the *bra* $\langle\{u\}|_U : \wp(U) \to \mathbb{R}$ is defined by the *bracket*:

$$\langle\{u\}|_U S\rangle = \begin{cases} 1 \text{ if } u \in S \\ 0 \text{ if } u \notin S \end{cases} = |\{u\} \cap S| = \chi_S(u).$$

Note that the bracket is defined in terms of the U-basis and that is indicated by the U-subscript on the bra portion of the bracket. Then for $u_i, u_j \in U$, $\langle\{u_i\}|_U \{u_j\}\rangle = \chi_{\{u_j\}}(u_i) = \chi_{\{u_i\}}(u_j) = \delta_{ij}$ (the Kronecker delta function) which is the QM/sets-version of $\langle v_i|v_j\rangle = \delta_{ij}$ for an orthonormal basis $\{|v_i\rangle\}$ of $\mathbb{C}^n$. The bracket linearly extends to any two vectors $T, S \in \wp(U)$:

$$\langle T|_U S\rangle = |T \cap S|.$$ [8]

This is the QM/sets-version of the Dirac brackets in the mathematics of QM.

For more motivation, consider an orthonormal basis set $\{|v_i\rangle\}$ in a finite dimensional Hilbert space V. Given two subsets $T, S \subseteq \{|v_i\rangle\}$ of the basis set, consider the unnormalized superpositions $\psi_T = \sum_{|v_i\rangle \in T} |v_i\rangle$ and $\psi_S = \sum_{|v_i\rangle \in S} |v_i\rangle$. Then their inner product in the Hilbert space is $\langle\psi_T|\psi_S\rangle = |T \cap S|$ just as $\langle T|_U S\rangle = |T \cap S|$ for subsets $T, S \subseteq U$ of the U-basis of $\wp(U) \cong \mathbb{Z}_2^n$. In both cases, the bracket gives a measure of the overlap or indistinctness of the two vectors.

4.3 Ket-bra resolution

The *ket-bra* $|\{u\}\rangle\langle\{u\}|_U$ is defined as the one-dimensional projection operator:

$$|\{u\}\rangle\langle\{u\}|_U = \{u\} \cap () : \wp(U) \to \wp(U)$$

and the *ket-bra identity* holds as usual:

$$\sum_{u \in U} |\{u\}\rangle\langle\{u\}|_U = \sum_{u \in U} (\{u\} \cap ()) = I : \wp(U) \to \wp(U)$$

where the summation is the symmetric difference of sets in $\wp(U)$ and I is the identity map [as a linear operator on $\wp(U)$]. The overlap $\langle T|_U S\rangle$ can be resolved using the ket-bra identity in the same basis: $\langle T|_U S\rangle = \sum_u \langle T|_U \{u\}\rangle\langle\{u\}|_U S\rangle$. Similarly a ket $|S\rangle$ for $S \subseteq U$ can be resolved in the U-basis;

$$|S\rangle = \sum_{u \in U} |\{u\}\rangle\langle\{u\}|_U S\rangle = \sum_{u \in U} \langle\{u\}|_U S\rangle |\{u\}\rangle = \sum_{u \in U} |\{u\} \cap S| \, |\{u\}\rangle$$

where a subset $S \subseteq U$ is just expressed as the sum of the singletons $\{u\} \subseteq S$. That is *ket-bra resolution* in QM/sets. The ket $|S\rangle$ is the same as the ket $|S'\rangle$ for some subset $S' \subseteq U'$ in another U'-basis, but when the bra $\langle\{u\}|_U$ is applied to the ket $|S\rangle = |S'\rangle$, then it is the subset $S \subseteq U$, not $S' \subseteq U'$, that comes outside the ket symbol $|\rangle$ in $\langle\{u\}|_U S\rangle = |\{u\} \cap S|$.[9]

[8] In the other attempts to develop the mathematics of QM over $\mathbb{Z}_2$ [26], the fateful choice was made to have the brackets take values in the base field as in full QM over $\mathbb{C}$. Thus the result is a "modal" calculus (0 = impossibility and 1 = possibility) rather than the Laplace-Boole probability calculus of QM/sets. Similarly, the model of categorical quantum mechanics [1] in **Rel**, the category of sets and relations, has brackets with only the values of 0 and 1.

Here $\langle T|_U S\rangle = |T \cap S|$ takes values outside the base field of $\mathbb{Z}_2$ just like, say, the Hamming distance function $d_H(T,S) = |T + S|$ on vector spaces over $\mathbb{Z}_2$ in coding theory. [24] The brackets taking values in the base field is a consequence of the base field being strengthened to $\mathbb{C}$. It is not a necessary feature of a quantum probability calculus as we see in QM/sets.

[9] The term "$\{u\} \cap S'$" is not even defined since it is the intersection of subsets $\{u\} \subseteq U$ and $S' \subseteq U'$ of two different universe sets U and U'.

4.4 The norm

The U-*norm* $\|S\|_U : \wp(U) \to \mathbb{R}$ is defined, as usual, as the square root of the bracket:[10]

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S \cap S|} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the QM/sets-version of the norm $|\psi| = \sqrt{\langle\psi|\psi\rangle}$ in ordinary QM. Note that a ket has to be expressed in the U-basis to apply the U-norm definition so, for example, $\|\{a'\}\|_U = \sqrt{2}$ since $|\{a'\}\rangle = |\{a,b\}\rangle$.

4.5 Numerical attributes and linear operators

In classical physics, the observables are numerical attributes, e.g., the assignment of a position and momentum to particles in phase space. One of the differences between classical and quantum physics is the replacement of these observable numerical attributes by linear operators associated with the observables where the values of the observables appear as eigenvalues of the operators. But this difference may be smaller than it would seem at first since a numerical attribute $f : U \to \mathbb{R}$ can be recast into an operator-like format in QM/sets, and there is even a QM/sets-analogue of spectral decomposition.

An observable, i.e., a Hermitian operator, on a Hilbert space V has a home basis set of orthonormal eigenvectors. In a similar manner, a real-valued attribute $f : U \to \mathbb{R}$ defined on U has the U-basis as its "home basis set." The connection between the numerical attributes $f : U \to \mathbb{R}$ of QM/sets and the Hermitian operators of full QM can be established by "seeing" the function f as being *like* an "operator" $f \upharpoonright ()$ on $\wp(U)$ in that it is used to define an eigenvalue equation [where $f \upharpoonright S$ is the *restriction* of f to $S \in \wp(U)$]. For any subset $S \in \wp(U)$, the definition of the equation is:

$$f \upharpoonright S = rS \text{ holds iff } f \text{ is constant on the subset } S \text{ with the value } r.$$

This is the QM/sets-version of an *eigenvalue equation* for numerical attributes $f : U \to \mathbb{R}$. Whenever S satisfies $f \upharpoonright S = rS$ for some r, then S is said to be an *eigenvector* in the vector space $\wp(U)$ of the numerical attribute $f : U \to \mathbb{R}$, and $r \in \mathbb{R}$ is the associated *eigenvalue*. Each eigenvalue r determines the *eigenspace* $\wp\left(f^{-1}(r)\right)$ of its eigenvectors which is a subspace of the vector space $\wp(U)$. The disjoint union $U = \biguplus \left\{f^{-1}(r)\right\}$ is expressed as the whole space being the direct sum of the eigenspaces: $\wp(U) = \sum_r \oplus\wp\left(f^{-1}(r)\right)$. Moreover, for distinct eigenvalues $r \neq r'$, any corresponding eigenvectors $S \in \wp\left(f^{-1}(r)\right)$ and $T \in \wp\left(f^{-1}(r')\right)$ are *orthogonal* in the sense that $\langle T|_U S\rangle = 0$. In general, for vectors $S, T \in \wp(U)$, orthogonality means zero overlap, i.e., disjointness.

The characteristic function $\chi_S : U \to \mathbb{R}$ for $S \subseteq U$ has the eigenvalues of 0 and 1 so it is a numerical attribute that *can* be represented as a linear operator $S \cap () : \wp(U) \to \wp(U)$. Hence in this case, the equation $f \upharpoonright T = rT$ for $f = \chi_S$ becomes an actual eigenvalue equation $S \cap T = rT$ for a linear operator $S \cap ()$ with the resulting eigenvalues and eigenvectors agreeing with those defined above for an arbitrary numerical attribute $f : U \to \mathbb{R}$. The numerical attributes $\chi_S : U \to \mathbb{R}$ are characterized by the property that their value-wise product, i.e., $(\chi_S \bullet \chi_S)(u) = \chi_S(u)\chi_S(u)$, is equal to the attribute value $\chi_S(u)$, and that is reflected in the idempotency of the corresponding operators:

$$\wp(U) \xrightarrow{S\cap()} \wp(U) \xrightarrow{S\cap()} \wp(U) = \wp(U) \xrightarrow{S\cap()} \wp(U).$$

[10] We use the double-line notation $\|S\|_U$ for the U-norm of a set to distinguish it from the single-line notation $|S|$ for the cardinality of a set, whereas the customary absolute value notation for the norm of a vector v in ordinary QM is $|v| = \sqrt{\langle v|v\rangle}$. The context should suffice to distinguish $|S|$ from $|v|$.

Thus the operators $S\cap()$ corresponding to the characteristic attributes χ_S are *projection operators*.[11]

The (maximal) eigenvectors $f^{-1}(r)$ for f, with r in the image or *spectrum* $f(U) \subseteq \mathbb{R}$, span the set U, i.e., $U = \sum_{r \in f(U)} f^{-1}(r)$. Hence the attribute $f : U \to \mathbb{R}$ has a spectral decomposition in terms of its (projection-defining) characteristic functions:

$$f = \sum_{r \in f(U)} r\chi_{f^{-1}(r)} : U \to \mathbb{R}$$

Spectral decomposition of set attribute $f : U \to \mathbb{R}$

which is the QM/sets-version of the spectral decomposition $L = \sum_\lambda \lambda P_\lambda$ of a Hermitian operator L in terms of the projection operators P_λ for its eigenvalues λ.

4.6 Completeness and orthogonality of projection operators

For any vector $S \in \wp(U)$, the operator $S \cap () : \wp(U) \to \wp(U)$ is the linear[12] projection operator to the subspace $\wp(S) \subseteq \wp(U)$. The usual completeness and orthogonality conditions on projection operators P_λ to the eigenspaces of an observable-operator have QM/sets-versions for numerical attributes $f : U \to \mathbb{R}$:

1. completeness: $\sum_\lambda P_\lambda = I : V \to V$ in QM has the QM/sets-version:

   $$\sum_r f^{-1}(r) \cap () = I : \wp(U) \to \wp(U), \text{ and}$$

2. orthogonality: for $\lambda \neq \mu$, $V \xrightarrow{P_\mu} V \xrightarrow{P_\lambda} V = V \xrightarrow{0} V$ (where 0 is the zero operator) has the QM/sets-version: for $r \neq r'$,

   $$\wp(U) \xrightarrow{f^{-1}(r')\cap()} \wp(U) \xrightarrow{f^{-1}(r)\cap()} \wp(U) = \wp(U) \xrightarrow{0} \wp(U).$$

Note that in spite of the lack of an inner product, the orthogonality of projection operators $S \cap ()$ is perfectly well-defined in QM/sets where it boils down to the disjointness of subsets, i.e., the cardinality of subsets' overlap (instead of their inner product) being 0.

4.7 The Born Rule for measurement in QM and QM/sets

An orthogonal decomposition of a finite set U is just a partition $\pi = \{B\}$ of U since the blocks $B, B', \ldots$ are orthogonal (i.e., disjoint) and their sum is U. Given such an orthogonal decomposition of U, we have the:

$$\|U\|_U^2 = \sum_{B \in \pi} \|B\|_U^2$$

Pythagorean Theorem
for orthogonal decompositions of sets.

An old question is: "why the squaring of amplitudes in the Born rule of QM?" A state objectively indefinite between certain definite orthogonal alternatives A and B, where the latter are represented by vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, is represented by the vector sum $\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B}$. But what is the "strength,"

[11] For a general attribute $f : U \to \mathbb{R}$, the equation $f \upharpoonright T = rT$ cannot be interpreted as the customary eigenvalue equation in a vector space over $\mathbb{Z}_2$ since the values r are not in general in the base field. Hence a generalized interpretation of the eignevalue equation is used here for a general attribute f. Or, put the other way around, in order for general real-valued attributes to be interpreted as linear operators, in the way that characteristic functions χ_S were interpreted as projection operators $S \cap ()$, the base field would have to be strengthened to $\mathbb{C}$. That would take us, *mutatis mutandis*, from the probability calculus of QM/sets to that of full QM.

[12] It should be noted that the projection operator $S \cap () : \wp(U) \to \wp(U)$ is not only idempotent but linear, i.e., $(S \cap T_1) + (S \cap T_2) = S \cap (T_1 + T_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring with intersection as multiplication.

"intensity," or relative importance of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ in the vector sum $\overrightarrow{C}$? That question requires a *scalar* measure of strength or intensity. The magnitude or "length" given by the norm $\|\|$ does not answer the question since $\left\|\overrightarrow{A}\right\| + \left\|\overrightarrow{B}\right\| \neq \left\|\overrightarrow{C}\right\|$. But the Pythagorean Theorem shows that the norm-squared gives the scalar measure of "intensity" that answers the question: $\left\|\overrightarrow{A}\right\|^2 + \left\|\overrightarrow{B}\right\|^2 = \left\|\overrightarrow{C}\right\|^2$ in vector spaces over $\mathbb{Z}_2$ or over $\mathbb{C}$. And when the objectively indefinite superposition state is reduced by a measurement, then the *objective probability* that the indefinite state will reduce to one of the definite alternatives is given by that objective relative scalar measure of the eigen-alternative's "strength" or "intensity" in the indefinite state–and that is the Born Rule. In a slogan, Born is the off-spring of Pythagoras.

Given an orthogonal basis $\{|v_i\rangle\}$ in a finite dimensional Hilbert space and given the U-basis for the vector space $\wp(U)$, the Pythagorean results for the basis sets are:

$$|\psi|^2 = \sum_i \langle v_i|\psi\rangle^* \langle v_i|\psi\rangle = \sum_i |\langle v_i|\psi\rangle|^2 \text{ and } \|S\|_U^2 = \sum_{u\in U} \langle \{u\} |_U S\rangle^2.$$

Given an observable-operator in QM and a numerical attribute in QM/sets, the Pythagorean Theorems for the complete sets of orthogonal projection operators are:

$$|\psi|^2 = \sum_\lambda |P_\lambda(\psi)|^2 \text{ and } \|S\|_U^2 = \sum_r \left\|f^{-1}(r) \cap S\right\|_U^2 = \sum_r \left|f^{-1}(r) \cap S\right| = |S|.$$

Normalizing gives:

$$\sum_\lambda \frac{|P_\lambda(\psi)|^2}{|\psi|^2} = 1 \text{ and } \sum_r \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \sum_r \frac{\left|f^{-1}(r)\cap S\right|}{|S|} = 1$$

so the non-negative summands can be interpreted as probabilities–which is the Born rule in QM and in QM/sets.[13]

Here $\frac{|P_\lambda(\psi)|^2}{|\psi|^2}$ is the "mysterious" quantum probability of getting λ in an L-measurement of ψ, while $\frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ has the rather unmysterious interpretation in the pedagogical model, QM/sets, as the probability $\Pr(r|S)$ of the numerical attribute $f : U \to \mathbb{R}$ having the eigenvalue r when "measuring" $S \in \wp(U)$. Thus the QM/sets-version of the Born Rule is the perfectly ordinary Laplace-Boole rule for the conditional probability $\Pr(r|S) = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$, that given $S \subseteq U$, a random variable $f : U \to \mathbb{R}$ takes the value r.

In QM/sets, the indefinite object S is being "measured" using the observable f where the probability $\Pr(r|S)$ of getting the eigenvalue r is $\frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$ and where the "damned quantum jump" (Schrödinger) goes from S by the projection operator $f^{-1}(r) \cap ()$ to the projected resultant state $f^{-1}(r) \cap S$ which is in the eigenspace $\wp\left(f^{-1}(r)\right)$ for that eigenvalue r. The state resulting from the measurement represents a more-definite objective state $f^{-1}(r) \cap S$ that now has the definite f-value of r–so a second measurement would yield the same eigenvalue r with probability:

$$\Pr\left(r|f^{-1}(r) \cap S\right) = \frac{\left|f^{-1}(r)\cap\left[f^{-1}(r)\cap S\right]\right|}{\left|f^{-1}(r)\cap S\right|} = \frac{\left|f^{-1}(r)\cap S\right|}{\left|f^{-1}(r)\cap S\right|} = 1$$

and the same resulting vector $f^{-1}(r) \cap \left[f^{-1}(r) \cap S\right] = f^{-1}(r) \cap S$ using the idempotency of the projection operators.

Hence the treatment of measurement in QM/sets is *all* analogous to the treatment of measurement in standard Dirac-von-Neumann QM.

[13] Note that there is no notion of a normalized vector in a vector space over $\mathbb{Z}_2$ (another consequence of the lack of an inner product). The normalization is, as it were, postponed to the probability algorithm which is computed in the reals.

4.8 Summary of QM/sets and QM

The QM/set-versions of the corresponding QM notions are summarized in the following table for the finite U-basis of the $\mathbb{Z}_2$-vector space$\wp(U)$ and for an orthonormal basis $\{|v_i\rangle\}$ of a finite dimensional Hilbert space V.

QM/sets over $\mathbb{Z}_2$	Standard QM over $\mathbb{C}$
Projections: $S \cap () : \wp(U) \to \wp(U)$	$P : V \to V$ where $P^2 = P$
Spectral Decomposition.: $f = \sum_r r\chi_{f^{-1}(r)}$	$L = \sum_\lambda \lambda P_\lambda$
Completeness.: $\sum_r f^{-1}(r) \cap () = I$	$\sum_\lambda P_\lambda = I$
Orthog.: $r \neq r'$, $\left[f^{-1}(r) \cap ()\right]\left[f^{-1}(r') \cap ()\right] = \emptyset \cap ()$	$\lambda \neq \mu$, $P_\lambda P_\mu = 0$
Brackets: $\langle S\|_U T\rangle = \|S \cap T\|$ = overlap of $S, T \subseteq U$	$\langle\psi\|\varphi\rangle$ = overlap of ψ and φ
Ket-bra: $\sum_{u\in U} \|\{u\}\rangle \langle\{u\}\|_U = \sum_{u\in U} (\{u\} \cap ()) = I$	$\sum_i \|v_i\rangle \langle v_i\| = I$
Resolution: $\langle S\|_U T\rangle = \sum_u \langle S\|_U \{u\}\rangle \langle\{u\}\|_U T\rangle$	$\langle\psi\|\varphi\rangle = \sum_i \langle\psi\|v_i\rangle \langle v_i\|\varphi\rangle$
Norm: $\\|S\\|_U = \sqrt{\langle S\|_U S\rangle} = \sqrt{\|S\|}$ where $S \subseteq U$	$\|\psi\| = \sqrt{\langle\psi\|\psi\rangle}$
Basis Pythagoras: $\\|S\\|_U^2 = \sum_{u\in U} \langle\{u\}\|_U S\rangle^2 = \|S\|$	$\|\psi\|^2 = \sum_i \langle v_i\|\psi\rangle^* \langle v_i\|\psi\rangle$
Normalized: $\sum_{u\in U} \frac{\langle\{u\}\|_U S\rangle^2}{\\|S\\|_U^2} = \sum_{u\in S} \frac{1}{\|S\|} = 1$	$\sum_i \frac{\langle v_i\|\psi\rangle^* \langle v_i\|\psi\rangle}{\|\psi\|^2} = \sum_i \frac{\|\langle v_i\|\psi\rangle\|^2}{\|\psi\|^2} = 1$
Basis Born rule: $\Pr(\{u\}\|S) = \frac{\langle\{u\}\|_U S\rangle^2}{\\|S\\|_U^2}$	$\Pr(v_i\|\psi) = \frac{\|\langle v_i\|\psi\rangle\|^2}{\|\psi\|^2}$
Attribute Pythagoras: $\\|S\\|_U^2 = \sum_r \left\\|f^{-1}(r) \cap S\right\\|_U^2$	$\|\psi\|^2 = \sum_\lambda \|P_\lambda(\psi)\|^2$
Normalized: $\sum_r \frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \sum_r \frac{\|f^{-1}(r)\cap S\|}{\|S\|} = 1$	$\sum_\lambda \frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2} = 1$
Attribute Born rule: $\Pr(r\|S) = \frac{\\|f^{-1}(r)\cap S\\|_U^2}{\\|S\\|_U^2} = \frac{\|f^{-1}(r)\cap S\|}{\|S\|}$	$\Pr(\lambda\|\psi) = \frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2}$

Probability calculus for QM/sets over $\mathbb{Z}_2$ and for standard QM over $\mathbb{C}$

5 Measurement in QM/sets

5.1 Measurement as a partition join operation

In QM/sets, numerical attributes $f : U \to \mathbb{R}$ can be considered as random variables on a set of equiprobable objective states $\{u\} \subseteq U$. The inverse images of attributes (or random variables) define set partitions $\left\{f^{-1}(r)\right\}_{r\in f(U)}$ on the set U. Considered abstractly, the partitions on a set U are partially ordered by refinement where a partition $\pi = \{B\}$ *refines* a partition $\sigma = \{C\}$, written $\sigma \preceq \pi$, if for any block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. The principal logical operation needed here is the *partition join* where the join $\pi \vee \sigma$ is the partition whose blocks are the non-empty intersections $B \cap C$ for $B \in \pi$ and $C \in \sigma$.

Each partition π can be represented as a binary relation $\operatorname{dit}(\pi) \subseteq U \times U$ on U where the ordered pairs (u, u') in $\operatorname{dit}(\pi)$ are the *distinctions* or *dits* of π in the sense that u and u' are in distinct blocks of π. These *dit sets* $\operatorname{dit}(\pi)$ as binary relations might be called *partition relations* but they are also the "apartness relations" in computer science. An ordered pair (u, u') is an *indistinction* or *indit* of π if u and u' are in the same block of π. The set of indits, $\operatorname{indit}(\pi)$, as a binary relation is just the equivalence relation associated with the partition π, the complement of the dit set $\operatorname{dit}(\pi)$ in $U \times U$.

In the category-theoretic duality between *sub*-sets (which are the subject matter of Boole's subset logic, the latter being usually mis-specified as the special case of "propositional" logic) and *quotient*-sets or partitions ([7] or [11]), the *elements* of a subset and the *distinctions* of a partition are corresponding concepts.[14] The partial ordering of subsets in the Boolean lattice $\wp(U)$ is the

[14] Boole has been included along with Laplace in the name of classical finite probability theory since he developed it as the normalized counting measure on the elements of the subsets of his logic. Applying the same mathematical move to the dual logic of partitions results in developing the notion of *logical entropy* as the normalized counting measure on the distinctions of a partition. ([6], [8])

inclusion of elements, and the refinement partial ordering of partitions in the partition lattice $\prod(U)$ is just the inclusion of distinctions, i.e., $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. The top of the Boolean lattice is the subset U of all possible elements and the top of the partition lattice is the *discrete partition* $\mathbf{1} = \{\{u\}\}_{u \in U}$ of singletons which makes all possible distinctions: $\operatorname{dit}(\mathbf{1}) = U \times U - \Delta$ (where $\Delta = \{(u,u) : u \in U\}$ is the diagonal). The bottom of the Boolean lattice is the empty set $\emptyset$ of no elements and the bottom of the lattice of partitions is the *indiscrete partition* (or *blob*) $\mathbf{0} = \{U\}$ which makes no distinctions.

The two lattices can be illustrated in the case of $U = \{a, b, c\}$.

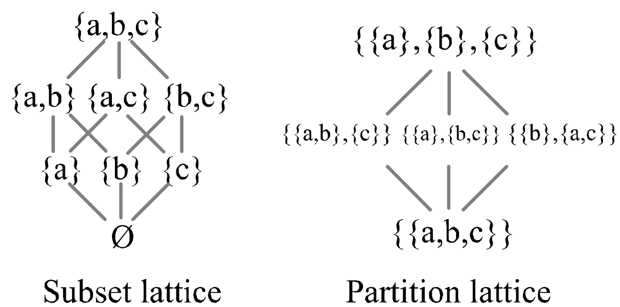


Figure 3: Subset and partition lattices

In the correspondences between QM/sets and QM, a block S in a partition on U [i.e., a vector $S \in \wp(U)$] corresponds to *pure* state in QM, and a partition $\pi = \{B\}$ on U is the *mixed state* of orthogonal pure states B with the probabilities $\Pr(B|U) = \frac{|B|}{|U|}$ given by the probability calculus on QM/sets. Given a pure state $S \subseteq U$, the possible results of a non-degenerate f-measurement, for (injective) $f : U \to \mathbb{R}$, are the blocks of the discrete partition $\{\{u\}\}_{u \in S}$ on S with each singleton being equiprobable. Each such measurement would have one of the potential eigenstates $\{u\} \subseteq S$ as the actual result.

Richard Feynman always emphasized the importance of distinctions.

> If you could, in principle, distinguish the alternative final states (even though you do not bother to do so), the total, final probability is obtained by calculating the probability for each state (not the amplitude) and then adding them together. If you cannot distinguish the final states even in principle, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[14, p. 3.9]

In QM, a measurement makes distinctions, i.e., makes alternatives distinguishable, and that turns a pure state into a mixture of probabilistic outcomes. A measurement in QM/sets is the distinction-creating process of turning a pure state $S \in \wp(U)$ into a mixed state partition $\{f^{-1}(r) \cap S\}_{r \in f(U)}$ on S. The distinction-creating process of measurement in QM/sets is the action on S of the partition join $\{S, S^c\} \vee \{f^{-1}(r)\}$ of the partition $\{S, S^c\}$ (where S^c is the complement of S) and the inverse-image partition $\{f^{-1}(r)\}_{r \in f(U)}$ of the numerical attribute $f : U \to \mathbb{R}$:

$$S \longrightarrow \{f^{-1}(r) \cap S\}_{r \in f(U)}$$

Action on the state S of an f-measurement-join with $\{f^{-1}(r)\}_{r \in f(U)}$.

The states $\{f^{-1}(r) \cap S\}_{r \in f(U)}$ are all possible or "potential" but the actual indefinite state S turns into one of the definite states with the probabilities given by the probability calculus: $\Pr(r|S) = \frac{\left\|f^{-1}(r) \cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r) \cap S\right|}{|S|}$.[15] When the objective state S turns into the objective state and eigenvector $f^{-1}(r) \cap S$, then the measurement returns the eigenvalue r (the *eigenvector-eigenvalue link*).

[15] Recall the Guy Fawkes mask without facial hair being distinguished with either a goatee or mustache.

That reduction of the state S to the state $f^{-1}(r) \cap S$ is mathematically described by applying the projection operator $f^{-1}(r) \cap ()$ and thus it is called a *projective* measurement.

Hermann Weyl touched on the relation between QM/sets and QM. He called a partition a "grating" or "sieve," and then considered both set partitions and vector space partitions (direct sum decompositions) as the respective types of gratings.[33, pp. 255-257] He started with a numerical attribute on a set, which defined the set partition or "grating" [33, p. 255] with blocks having the same attribute-value. Then he moved to the QM case where the universe set or "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [33, p. 256]. The appropriate notion of a vector space partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [33, p. 256], i.e., a direct sum decomposition of the space. After referring to a partition as a "grating" or "sieve," Weyl notes that "Measurement means application of a sieve or grating" [33, p. 259], e.g., in QM/sets, the application (i.e., join) of the set-grating $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ to the pure state $\{S\}$ to give the mixed state $\left\{f^{-1}(r) \cap S\right\}_{r \in f(U)}$.

For some mental imagery of measurement, we might think of the grating as a series of regular-polygonal-shaped holes that might shape an indefinite blob of dough. In a measurement, the blob of dough falls through one of the polygonal holes with equal probability and then takes on that shape.

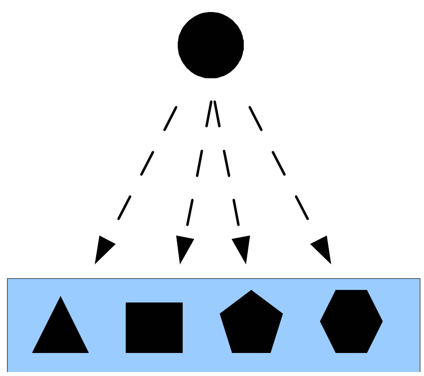


Figure 4: Measurement as randomly giving an indefinite blob of dough a definite polygonal shape.

5.2 Nondegenerate measurements

In the simple example illustrated below, we start at the one block or state of the indiscrete partition or blob which is the completely indistinct entity $\{a, b, c\}$. A measurement always uses some attribute that defines an inverse-image partition on $U = \{a, b, c\}$. In the case at hand, there are "essentially" four possible attributes that could be used to "measure" the indefinite entity $\{a, b, c\}$ (since there are four partitions that refine the indiscrete partition in Figure 3).

For an example of a nondegenerate measurement in QM/sets, consider any attribute $f : U \to \mathbb{R}$ which has the discrete partition as its inverse image (i.e., is injective), such as the ordinal number of the letter in the alphabet: $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. This attribute has three eigenvectors: $f \upharpoonright \{a\} = 1\{a\}$, $f \upharpoonright \{b\} = 2\{b\}$, and $f \upharpoonright \{c\} = 3\{c\}$ with the corresponding eigenvalues. The eigenvectors are $\{a\}$, $\{b\}$, and $\{c\}$, the blocks in the discrete partition of U. The nondegenerate measurement using the observable f acts on the pure state $U = \{a, b, c\}$ to give the mixed state $\mathbf{1}$:

$$U \to \left\{U \cap f^{-1}(r)\right\}_{r=1,2,3} = \mathbf{1}.$$

Each such measurement would return an eigenvalue r with the probability of $\Pr(r|S) = \frac{\left|f^{-1}(r) \cap S\right|}{|S|} = \frac{1}{3}$ for $r \in f(U) = \{1, 2, 3\}$.

A projective measurement makes distinctions in the measured state that are sufficient to induce the "quantum jump" or projection to the eigenvector associated with the observed eigenvalue. If the observed eigenvalue was 3, then the state $\{a,b,c\}$ projects to $f^{-1}(3) \cap \{a,b,c\} = \{c\} \cap \{a,b,c\} = \{c\}$ as pictured below.

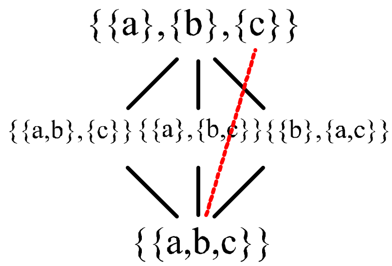


Figure 5: Nondegenerate measurement and resulting "quantum jump"

It might be emphasized that this is an objective state reduction (or "collapse of the wave packet") from the single indefinite objective state $\{a,b,c\}$ to the single definite state $\{c\}$, not a subjective removal of ignorance as if the state had all along been $\{c\}$.

5.3 Degenerate measurements

For an example of a degenerate measurement, we choose an attribute with a non-discrete inverse-image partition such as the partition $\pi = \{\{a\},\{b,c\}\}$. Hence the attribute could just be the characteristic function $\chi_{\{b,c\}}$ with the two eigenspaces $\wp(\{a\})$ and $\wp(\{b,c\})$ and the two eigenvalues 0 and 1 respectively. Since the eigenspace $\wp\left(\chi_{\{b,c\}}^{-1}(1)\right) = \wp(\{b,c\})$ is not one dimensional, the eigenvalue of 1 is a QM/sets-version of a *degenerate* eigenvalue. This attribute $\chi_{\{b,c\}}$ has four (non-zero) eigenvectors:

$$\chi_{\{b,c\}} \upharpoonright \{b,c\} = 1\{b,c\},\ \chi_{\{b,c\}} \upharpoonright \{b\} = 1\{b\},\ \chi_{\{b,c\}} \upharpoonright \{c\} = 1\{c\}, \text{ and } \chi_{\{b,c\}} \upharpoonright \{a\} = 0\{a\}.$$

The "measuring apparatus" makes distinctions by joining the attribute inverse-image partition

$$\chi_{\{b,c\}}^{-1} = \left\{\chi_{\{b,c\}}^{-1}(1), \chi_{\{b,c\}}^{-1}(0)\right\} = \{\{b,c\},\{a\}\}$$

with the pure state representing the indefinite entity $U = \{a,b,c\}$. The action on the pure state is:

$$U \to \{U\} \vee \chi_{\{b,c\}}^{-1} = \chi_{\{b,c\}}^{-1} = \{\{b,c\},\{a\}\}.$$

The measurement of that attribute returns one of the eigenvalues with the probabilities:

$$\Pr(0|U) = \frac{|\{a\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3} \text{ and } \Pr(1|U) = \frac{|\{b,c\}\cap\{a,b,c\}|}{|\{a,b,c\}|} = \frac{2}{3}.$$

Suppose it returns the eigenvalue 1. Then the indefinite entity $\{a,b,c\}$ reduces to the projected eigenstate $\chi_{\{b,c\}}^{-1}(1) \cap \{a,b,c\} = \{b,c\}$ for that eigenvalue [4, p. 221].

Since this is a degenerate result (i.e., the eigenspace $\wp\left(\chi_{\{b,c\}}^{-1}(1)\right) = \wp(\{b,c\})$ doesn't have dimension one), another measurement is needed to make more distinctions. Measurements by attributes, such as $\chi_{\{a,b\}}$ or $\chi_{\{a,c\}}$, that give either of the other two partitions, $\{\{a,b\},\{c\}\}$ or $\{\{b\},\{a,c\}\}$ as inverse images, would suffice to distinguish $\{b,c\}$ into $\{b\}$ or $\{c\}$. Hence either attribute together with the attribute $\chi_{\{b,c\}}$ would form a *Complete Set of Compatible Attributes* or

CSCA (i.e., the QM/sets-version of a Complete Set of Commuting Operators or CSCO [5]), where *complete* means that the join of the attributes' inverse-image partitions gives the discrete partition and where *compatible* means that all the attributes can be taken as defined on the same set of (simultaneous) basis eigenstates.

Taking, for example, the other attribute as $\chi_{\{a,b\}}$, the join of the two attributes' partitions is discrete:

$$\chi_{\{b,c\}}^{-1} \vee \chi_{\{a,b\}}^{-1} = \{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\} = \mathbf{1}.$$

Hence all the eigenstate singletons can be characterized by the ordered pairs of the eigenvalues of these two attributes: $\{a\} = |0,1\rangle$, $\{b\} = |1,1\rangle$, and $\{c\} = |1,0\rangle$ (using Dirac's ket-notation to give the ordered pairs and listing the eigenvalues of $\chi_{\{b,c\}}$ first on the left).

The second projective measurement of the indefinite entity $\{b,c\}$ using the attribute $\chi_{\{a,b\}}$ with the inverse-image partition $\chi_{\{a,b\}}^{-1} = \{\{a,b\},\{c\}\}$ would have the pure-to-mixed state action:

$$\{b,c\} \to \left\{\{b,c\} \cap \chi_{\{a,b\}}(1), \{b,c\} \cap \chi_{\{a,b\}}(0)\right\} = \{\{b\},\{c\}\}.$$

The distinction-making measurement would cause the indefinite entity $\{b,c\}$ to turn into one of the definite entities of $\{b\}$ or $\{c\}$ with the probabilities:

$$\Pr(1|\{b,c\}) = \frac{|\{a,b\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2} \text{ and } \Pr(0|\{b,c\}) = \frac{|\{c\}\cap\{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

If the measured eigenvalue is 0, then the state $\{b,c\}$ projects to $\chi_{\{a,b\}}^{-1}(0) \cap \{b,c\} = \{c\}$ as pictured below.

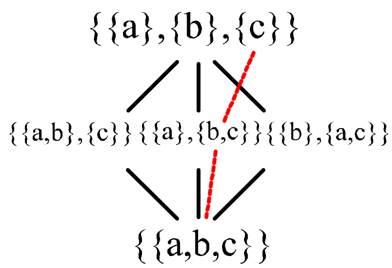


Figure 6: Degenerate measurement

The two projective measurements of $\{a,b,c\}$ using the complete set of compatible (e.g., both defined on U) attributes $\chi_{\{b,c\}}$ and $\chi_{\{a,b\}}$ produced the respective eigenvalues 1 and 0 so the resulting eigenstate was characterized by the eigenket $|1,0\rangle = \{c\}$.

Again, this is all analogous to standard Dirac-von-Neumann quantum mechanics.

6 Further steps

Showing that ordinary Laplace-Boole finite probability theory is the quantum probability calculus for the pedagogical or "toy" model, quantum mechanics over sets (QM/sets), is only an initial part of a research programme. The programme is to specify the objective indefiniteness interpretation or what Shimony calls "the Literal Interpretation" of quantum mechanics which results "from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete." [30, pp. 6-7]

QM/sets is one part of the programme and we have only scratched the surface of that model. For instance, we have not considered:

- the quantum dynamics of the QM/sets model which are the transformations $\mathbb{Z}_2^n \to \mathbb{Z}_2^n$ that preserve the distinctness of vectors (i.e., the non-singular transformations [26]) just as the dynamics in ordinary QM preserve the *degree* of distinctness or indistinctness, i.e., the overlap $\langle\psi|\varphi\rangle$, between vectors (i.e., the unitary transformations) and which allow simple models of the double-slit experiment;

- the whole "non-commutative" side of viewing the Laplace-Boole theory in the context of vector spaces over $\mathbb{Z}_2$ where the compatibility of numerical attributes $f : U \to \mathbb{R}$ and $g : U' \to \mathbb{R}$ defined on different equicardinal basis sets $\{u\} \subseteq U$ and $\{u'\} \subseteq U'$ can be analyzed in terms of the commutativity of all the associated projection operators $f^{-1}(r) \cap ()$ and $g^{-1}(s) \cap ()$ on $\mathbb{Z}_2^n$;

- the treatment of the mixed states in QM/sets using density matrices which allows a clear classical interpretation of the off-diagonal terms and how they change under measurement; or

- the treatment of entanglement in QM/sets which reduces to some old-fashioned correlation in the equiprobability distribution on the objective state that is a subset of a Cartesian product but which still allows a Bell-type result to be established ([9], [10], [12]).

Our purpose here is limited to showing how the perfectly classical Laplace-Boole finite probability theory is the quantum probability calculus of the pedagogical model of quantum mechanics over sets. The point is not to clarify finite probability theory but to elucidate quantum mechanics itself by seeing some of its quantum features formulated in a classical setting.

References

[1] Abramsky, Samson and Coecke, Bob 2004. A categorical semantics of quantum protocols. in: *Proceedings of the 19th IEEE Symposium on Logic in Computer Science (LiCS'04)*, IEEE Computer Science Press, 415–425 .

[2] Boole, George 1854. *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. Cambridge: Macmillan and Co.

[3] Busch, Paul and Gregg Jaeger 2010. Unsharp Quantum Reality. *Foundations of Physics*. 40: 1341-1367.

[4] Cohen-Tannoudji, Claude, Bernard Diu and Franck Laloë 2005. *Quantum Mechanics Vol. 1*. New York: John Wiley & Sons.

[5] Dirac, Paul A. M. 1958. *The Principles of Quantum Mechanics (4th ed.)*. Oxford: Clarendon.

[6] Ellerman, David 2009. Counting Distinctions: On the Conceptual Foundations of Shannon's Information Theory. *Synthese*. 168 (1 May): 119-149.

[7] Ellerman, David 2010. The Logic of Partitions: Introduction to the Dual of the Logic of Subsets. *Review of Symbolic Logic*. 3 (2 June): 287-350.

[8] Ellerman, David 2013. An Introduction to Logical Entropy and its Relation to Shannon Entropy. *International Journal of Semantic Computing*, 7(2): 121-145.

[9] Ellerman, David 2013. *The Objective Indefiniteness Interpretation of Quantum Mechanics*, arXiv:1210.7659 [quant-ph].

[10] Ellerman, David 2013. *Quantum mechanics over sets*. arXiv:1310.8221 [quant-ph].

[11] Ellerman, David 2014. An Introduction of Partition Logic. *Logic Journal of the IGPL.* 22, no. 1: 94–125.

[12] Ellerman, David 2014. *Partitions and Objective Indefiniteness in Quantum Mechanics*. arXiv:1401.2421v2 [quant-ph].

[13] Feyerabend, Paul 1983 (orig. 1962). Problems of Microphysics. In *Frontiers of Science and Philosophy*. Robert G. Colodny ed., Lanham MD: University Press of America: 189-283.

[14] Feynman, Richard P., Robert B. Leighton, and Matthew Sands 1965. *The Feynman Lectures on Physics: Quantum Mechanics (Vol. III)*. Reading MA: Addison-Wesley.

[15] Fine, Arthur 1986. *The Shaky Game: Einstein, Realism, and the Quantum Theory*. Chicago: University of Chicago Press.

[16] French, Steven and Decio Krause 2003. Quantum Vagueness. *Erkenntnis*. 59: 97-124.

[17] Hanson, Andrew J., Gerardo Ortiz, Amr Sabry, and Yu-Tsung Tai 2013. *Discrete Quantum Theories*. arXiv:1305.3292v1.

[18] Heisenberg, Werner 1958. *Physics & Philosophy: The Revolution in Modern Science*. New York: Harper Torchbooks.

[19] Jaeger, Gregg 2014. *Quantum Objects: Non-Local Correlation, Causality and Objective Indefiniteness in the Quantum World*. Heidelberg: Springer.

[20] James, William 1952 (1890). *The Principles of Psychology (Great Books Series #53)*. Chicago: Encyclopedia Britannica.

[21] Jammer, Max 1966. *The Conceptual Development of Quantum Mechanics*. New York: McGraw-Hill.

[22] Laplace, Pierre-Simon. 1995 (1825). *Philosophical Essay on Probabilities*. Translated by A. I. Dale. New York: Springer-Verlag.

[23] Lowe, E. J. 1994. Vague Identity and Quantum Indeterminacy. *Analysis*. 54 (2): 110-114.

[24] McEliece, Robert J. 1977. *The Theory of Information and Coding: A Mathematical Framework for Communication (Encyclopedia of Mathematics and its Applications, Vol. 3)*. Reading MA: Addison-Wesley.

[25] Mittelstaedt, Peter 1998. The Constitution of Objects in Kant's Philosophy and in Modern Physics. In *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*. Elena Castellani ed., Princeton NJ: Princeton University Press: 168-180.

[26] Schumacher, Benjamin and Michael Westmoreland 2012. Modal Quantum Theory. *Foundations of Physics*, 42, 918-925.

[27] Shimony, Abner 1988. The reality of the quantum world. *Scientific American*. 258 (1): 46-53.

[28] Shimony, Abner 1989. Conceptual foundations of quantum mechanics. In *The New Physics*. Paul Davies ed., Cambridge: Cambridge University Press: 373-395.

[29] Shimony, Abner 1989. Search for a Worldview Which Can Accommodate Our Knowledge of Microphysics. In *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem*. James T. Cushing and Ernan McMullin eds., Notre Dame, IN: University of Notre Dame Press: 25-37.

[30] Shimony, Abner 1999. Philosophical and Experimental Perspectives on Quantum Physics. In *Philosophical and Experimental Perspectives on Quantum Physics: Vienna Circle Institute Yearbook 7*, Dordrecht: Springer Science+Business Media, 1–18.

[31] Stairs, Allen 1983. Quantum Logic, Realism, and Value Definiteness. *Philosophy of Science*. 50 (4): 578-602.

[32] Teller, Paul 1989. Relativity, Relational Holism, and the Bell Inequalities. In *Philosophical Consequences of Quantum Theory: Reflections on Bell's Theorem*. James T. Cushing and Ernan McMullin eds., Notre Dame, IN: University of Notre Dame Press: 208-223.

[33] Weyl, Hermann 1949. *Philosophy of Mathematics and Natural Science*. Princeton NJ: Princeton University Press.