On Concrete Universals:
A Modern Treatment using Category Theory

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Abstract

Today it would be considered "bad Platonic metaphysics" to think that among all the concrete instances of a property there could be a universal instance so that all instances had the property by virtue of participating in that concrete universal. Yet there is a mathematical theory, category theory, dating from the mid-20th century that shows how to precisely model concrete universals within the "Platonic Heaven" of mathematics. This paper, written for the philosophical logician, develops this category-theoretic treatment of concrete universals along with a new concept to abstractly model the functions of a brain.

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Introduction: "Bad Platonic Metaphysics"

Consider the following example of "bad metaphysics."

> Given all the entities that have a certain property, there is one entity among them that exemplifies the property in an absolutely perfect and universal way. It is called the "concrete universal." There is a relationship of "participation" or "resemblance" so that all the other entities that have the property "participate in" or "resemble" that perfect example, the concrete universal. And conversely, every entity that participates in or resembles the universal also has the property. The concrete universal represents the "essence" of the property. All the other instances of the property have "imperfections." There is a process of removing imperfections so that by removing all the imperfections, one arrives at the essence of the property, the concrete universal.

To the modern ear, all this sounds like the worst sort of "bad Platonic metaphysics." Yet there is a mathematical theory developed within the last seventy years, category theory [MacLane 1971], that provides precisely that treatment of concrete universals within mathematics.

A simple example using sets will illustrate the points. Given two sets A and B, consider the property of sets: $F(X) \equiv$ "X is contained in A and is contained in B." In other words, the property is the property of being both a subset of A and a subset of B. In this example, the *participation* relation is the subset inclusion relation. There is a set, namely the intersection or meet of A and B, denoted $A\cap B$, that has the property (so it is a "concrete" instance of the property), and it is universal in the sense that any other set has the property if and only if it participates in (i.e., is included in) the universal example:

concreteness: $F(A\cap B)$, i.e., $A\cap B\subseteq A$ and $A\cap B\subseteq B$, and
universality: X participates in $A\cap B$ if and only if $F(X)$, i.e., $X \subseteq A\cap B$ if and only if $X\subseteq A$ and $X\subseteq B$.

This example of a concrete universal is quite simple, but all this "bad metaphysical talk" has highly developed and precise models in category theory. This interpretation of the universals of category theory as concrete universals is the main point of this paper. We will briefly mention two other controversies in philosophy related to concrete universals: the Third Man Argument and the set theoretical paradoxes. Also we will consider one of the most important uses of concrete universals in pure mathematics, namely adjoint functors or adjunctions, and we show how the building blocks of adjunctions can be recombined in a new way to define the notion of a brain functor that abstractly models a brain.

Theories of Universals

In Plato's Theory of Ideas or Forms (ειδη), a property F has an entity associated with it, the *universal* u_F, which uniquely represents the property. An object x has the property F, i.e., F(x), if and only if (iff) the object x *participates* in the universal u_F. Let μ (from μεθεξις or methexis) represent the participation relation so

$$\text{"}x \,\mu\, u_F\text{" reads as "}x \text{ participates in } u_F\text{".}$$

Given a relation μ, an entity u_F is said to be *a universal for the property F* (with respect to μ) if it satisfies the following *universality condition*:

$$\text{for any } x,\ x \,\mu\, u_F \text{ if and only if } F(x).$$

A universal representing a property should be in some sense unique. Hence there should be an equivalence relation (≈) so that universals satisfy a *uniqueness condition:*

$$\text{if } u_F \text{ and } u_F' \text{ are universals for the same } F, \text{ then } u_F \approx u_F'.$$

A mathematical theory is said to be a *theory of universals* if it contains a binary relation μ and an equivalence relation ≈ so that with certain properties F there are associated entities u_F satisfying the following conditions:

(I) *Universality*: for any x, $x \,\mu\, u_F$ iff F(x), and
(II) *Uniqueness*: if u_F and u_F' are universals for the same F [i.e., satisfy (I) above], then $u_F \approx u_F'$.

A universal u_F is said to be *abstract* if it does not participate in itself, i.e., $\neg(u_F \,\mu\, u_F)$. Alternatively, a universal u_F is *concrete* if it is self-participating, i.e., $u_F \,\mu\, u_F$.

Set Theory as The Theory of Abstract Universals

There is a modern mathematical theory that readily qualifies as a theory of universals, namely set theory. The universal representing a property F is the set of all elements with the property:

$$u_F = \{ x \mid F(x) \}.$$

The participation relation is the set membership relation usually represented by $\in$. The universality condition in set theory is the equivalence called a (naive) *comprehension axiom*: there is a set y such that for any x, $x \in y$ iff F(x). Set theory also has an *extensionality axiom*, which states that two sets with the same members are identical:

$$\text{for all } x, (x \in y \text{ iff } x \in y') \text{ implies } y = y'.$$

Thus if y and y' both satisfy the comprehension axiom scheme for the same F then y and y' have the same members so y = y'. Hence in set theory the uniqueness condition on universals is satisfied with the equivalence relation ($\approx$) as equality (=) between sets. Thus naive set theory qualifies as a theory of universals.

The hope that naive set theory would provide a *general* theory of universals proved to be unfounded. The naive comprehension axiom lead to inconsistency for such properties as

$$F(x) \equiv \text{"x is not a member of x"} \equiv x \notin x$$

If R is the universal for that property, i.e., R is the set of all sets which are not members of themselves, the naive comprehension axiom yields a contradiction.

$$R \in R \text{ iff } R \notin R.$$

Russell's Paradox

The characteristic feature of Russell's Paradox and the other set theoretical paradoxes is the self-reference wherein the universal is allowed to qualify for the property represented by the universal, e.g., the Russell set R is allowed to be one of the x's in the universality relation: $x \in R$ iff $x \notin x$.

There are several ways to restrict the naive comprehension axiom to defeat the set theoretical paradoxes, e.g., as in Russell's type theory, Zermelo-Fraenkel set theory, or von Neumann-Bernays set theory. The various restrictions are based on an iterative concept of set [Boolos 1971] which forces a set y to be more "abstract", e.g., of higher type or rank, than the elements $x \in y$. Thus the universals provided by the various set theories are "abstract" universals in the intuitive sense that they are more abstract than the objects having the property represented by the universal. Sets may not be members of themselves.[1]

With the modifications to avoid the paradoxes, a set theory still qualifies as a theory of universals. The membership relation is the participation relation so that for suitably restricted predicates, there exists a set satisfying the universality condition. Set equality serves as the equivalence relation in the uniqueness conditions. But set theory cannot qualify as a *general* theory of universals. The paradox-induced modifications turn the various set theories into theories of *abstract* (i.e., non-self-participating) universals since they prohibit the self-membership of sets.

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[1] Quine's system ML [1955b] allows "$V \in V$" for the universal class V, but no standard model of ML has ever been found where "$\in$" is interpreted as set membership [viz. Hatcher 1982, Chapter 7]. We are concerned with theories that are "set theories" in the sense that "$\in$" can be interpreted as set membership.

Concrete Universals

Philosophy contemplates another type of universal, a *concrete universal.* The intuitive idea of a concrete universal for a property is that it is an object that has the property and has it in such a universal sense that all other objects with the property resemble or participate in that paradigmatic or archetypal instance. The concrete universal u_F for a property F is *concrete* in the sense that it has the property itself, i.e., $F(u_F)$. It is *universal* in the intuitive sense that it represents F-ness is such a perfect and exemplary manner that any object resembles or participates in the universal u_F if and only if it has the property F.

The intuitive notion of a concrete universal occurs in ordinary language (the "all-American boy"), in Greek-inspired Christian theology (the Word made flesh together with *imitatio Christi* as the participation or resemblence relation to the concrete universal), in the arts and literature (the old idea that great art uses a concrete instance to universally exemplify certain human conditions), and in philosophy (the perfect example of F-ness with no imperfections, only those attributes necessary for F-ness).

The notion of a concrete universal occurred in Plato's Theory of Forms [Malcolm 1991]. Plato's forms are often considered to be abstract or non-self-participating universals quite distinct and "above" the concrete instances. In the words of one Plato scholar, "not even God can scratch Doghood behind the Ears" [Allen 1960]. But Plato did give examples of self-participation or self-predication, e.g., that Justice is just [Protagoras 330]. Moreover, Plato often used expressions that indicated self-predication of universals.

> But Plato also used language which suggests not only that the Forms exist separately (χωριστα) from all the particulars, but also that each Form is a peculiarly accurate or good particular of its own kind, i.e., the standard particular of the kind in question or the model (παραδειγμα) to which other particulars approximate. [Kneale and Kneale 1962, 19]

But many scholars regard the notion of a Form as *paradeigma* or concrete universal as an error.

> For general characters are not characterized by themselves: humanity is not human. The mistake is encouraged by the fact that in Greek the same phrase may signify both the concrete and the abstract, e.g. το λευκον (literally "the white") both "the white thing" and "whiteness", so that it is doubtful whether αυτο το λευκον (literally "the white itself") means "the superlatively white thing" or "whiteness in abstraction". [Kneale and Kneale 1962, 19-20]

Thus some Platonic language is ambivalent between interpreting a form as a concrete universal ("the superlatively white thing") and an abstract universal ("whiteness in abstraction").

The literature on Plato has reached no resolution on the question of self-predication. Scholarship has left Plato on both sides of the fence; many universals are not self-participating but some are. It is fitting that Plato should exhibit this ambivalence since the self-predication issue has only come to a head in the 20^{th} century with the set theoretical antinomies. Set theory had to be reconstructed as a theory of universals that were rigidly non-self-participating.

The reconstruction of set theory as the theory of abstract universals cleared the ground for a separate theory of universals that are always self-participating. Such a theory of concrete universals would realize the self-predicative strand of Plato's Theory of Forms.

A theory of concrete universals would have an appropriate participation relation μ so that for certain properties F, there are entities u_F satisfying the *universality condition*:

$$\text{for any } x,\ x\ \mu\ u_F \text{ if and only if } F(x).$$

The universality condition and $F(u_F)$ imply that u_F is a *concrete* universal in the previously defined sense of being self-participating, $u_F\ \mu\ u_F$. A theory of concrete universals would also have to have an equivalence relation so the concrete universals for the same property would be *the* universal up to that equivalence relation.

Is there a precise mathematical theory of concrete universals? Our claim is that category theory is precisely that theory.

To keep matters simple and intuitive, all our examples will use one of the simplest examples of categories, namely partially ordered sets.[2] Consider the universe of subsets or power set P(U) of a set U with the inclusion relation $\subseteq$ as the partial ordering relation. Given sets a and b, consider the property

$$F(x) \equiv x \subseteq a \ \&\ x \subseteq b.$$

The participation relation is set inclusion $\subseteq$ and the intersection $a \cap b$ is the universal u_F for this property F(x). The universality relation states that the intersection is the greatest lower bound of a and b in the inclusion ordering:

$$\text{for any } x,\ x \subseteq a\cap b \text{ iff } x \subseteq a \ \&\ x \subseteq b.$$

The universal has the property it represents, i.e., $a\cap b \subseteq a \ \&\ a\cap b \subseteq b$, so it is a self-participating or concrete universal. Two concrete universals for the same property must participate in each other. In partially ordered sets, the antisymmetry condition, $y \subseteq y' \ \&\ y' \subseteq y$ implies $y = y'$, means that equality can serve as the equivalence relation in the uniqueness condition for universals in a partial order.

Concrete Universals in more general categories

For the concrete universals of category theory,[3] the *participation relation* is the *uniquely-factors-through* relation. It can always be formulated in a suitable category as:

$$\text{"}x\ \mu\ u\text{" means "there exists a unique arrow } x\to u\text{".}$$

Then x is said to *uniquely factor through* u, and the arrow $x\to u$ is the unique factor or participation morphism. In the universality condition,

$$\text{for any } x,\ x\ \mu\ u \text{ if and only if } F(x),$$

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[2] A binary relation $\leq$ on U is a *partial order* if for all $u,u',u''\in U$, it is reflexive ($u\leq u$), transitive ($u\leq u'$ and $u'\leq u''$ imply $u\leq u''$), and anti-symmetric ($u\leq u'$ and $u'\leq u$ imply $u=u'$). For less trivial examples with more of a category-theoretic flavor, see Ellerman 1988.

[3] In the general case, a category may be defined as follows [e.g., MacLane and Birkhoff 1967 or MacLane 1971]:

A *category* C consists of

(a) a set of *objects* a, b, c, ...,

(b) for each pair of objects a,b, a set $\hom_C(a,b) = C(a,b)$ whose elements are represented as *arrows* or *morphsims* $f: a \to b$,

(c) for any $f \in \hom_C(a,b)$ and $g \in \hom_C(b,c)$, there is the *composition* $gf: a\to b\to c$ in $\hom_C(a,c)$,

(d) composition of arrows is an associative operation, and

(e) for each object a, there is an arrow $1_a \in \hom_C(a,a)$, called the *identity* of a, such that for any $f: a\to b$ and $g:c\to a$, $f1_a = f$ and $1_a g = g$.

An arrow $f:a\to b$ is an *isomorphism*, $a \cong b$, if there is an arrow $g:b\to a$ such that $fg = 1_b$ and $gf = 1_a$. A *functor* is a map from one category to another that preserves composition and identities.

the existence of the identity arrow 1_u:u→u is the self-participation of the concrete universal that corresponds with F(u), the application of the property to u. In category theory, the equivalence relation used in the uniqueness condition is the isomorphism (≅).[4]

It is sometimes convenient to "turn the arrows around" and use the dual definition where "x µ u" means "there exists a unique arrow u→x" that can also be viewed as the original definition stated in the dual or opposite category.

Category theory qualifies as a theory of universals with participation defined as "uniquely factors through" and the equivalence relation taken as isomorphism. The universals of category theory are self-participating or concrete; a universal u uniquely factors through itself by the identity morphism.

Category theory as the theory of concrete universals has quite a different flavor from set theory, the theory of abstract universals. Given the collection of all the elements with a property, set theory can postulate a more abstract entity, the set of those elements, to be the universal. But category theory cannot postulate its universals because those universals are concrete. Category theory must find its universals, if at all, among the entities with the property.

Universals as Essences

The concrete universal for a property represents the essential characteristics of the property without any imperfections (to use some language of an Aristotelian stamp). All the objects in category theory with universal mapping properties such as limits and colimits [viz. Schubert 1972, Chaps. 7-8] are concrete universals for universal properties. Thus the universals of category theory can typically be presented as the limit (or colimit) of a process of filtering out or eliminating imperfections to arrive at the pure essence of the property.

Consider the previous example of the intersection a∩b of sets a and b as the concrete universal for the property F of being contained in a and in b. Given a set x with the property of "being a subset of both a and b," an *imperfection* of x is another set x' with the property but which is not contained in x.

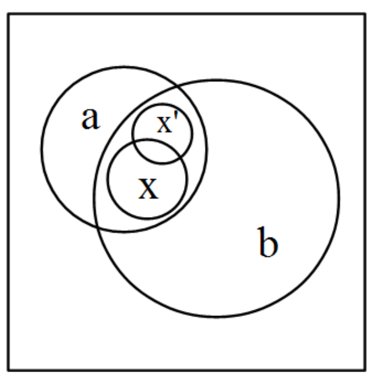

Fig. 1: The set x' is an *imperfection of the set* x
relative to the property of being a subset of both a and b.

[4] Thus it must be verified that two concrete universals for the same property are isomorphic. By the universality condition, two concrete universals u and u' for the same property must participate in each other. Let f:u'→u and g:u→u' be the unique arrows given by the mutual participation. Then by composition gf:u'→u' is the unique arrow u'→u' but $1_{u'}$ is another such arrow so by uniqueness, gf = $1_{u'}$. Similarly, fg:u→u is the unique self-participation arrow for u so fg = 1_u. Thus mutual participation of u and u' implies u ≅ u'.

If sets x and y both have the property, and x is contained in y then y is said to be *more essential* (in the sense of being "equally or more of the essence") than x.

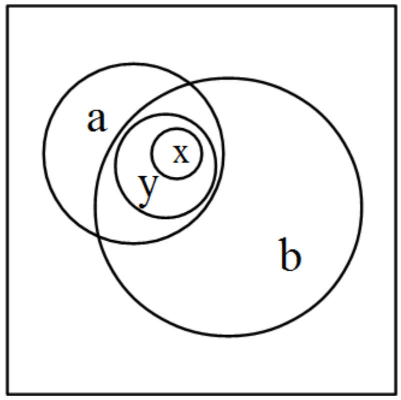

Fig. 2: The set y is *more essential* than x
relative to the property of being a subset of both a and b.

If y is more essential than x then any imperfection of y is an imperfection of x, and x may have a few other imperfections of its own. In this case, the process of eliminating or filtering out imperfections and becoming "more essential" is the process of taking the union of sets. If we remove all the imperfections, i.e., add to x all the other elements common to a and b, then we arrive at the "essence" of the property, the concrete universal $a \cap b$ for the property.

The property $F(x) \equiv x \subseteq a \ \& \ x \subseteq b$ is preserved under arbitrary unions:

$$\text{if } F(x_ß) \text{ for any } x_ß \text{ in } \{x_ß | ß \in B\}, \text{ then } F(\cup_ß x_ß).$$

Hence given any collection of instances $\{x_ß | ß \in B\}$ of the property F, their union is more essentially F than the instances. None of the sets in the collection are imperfections of the union. Thus the limit of this process, the "essence of F-ness," can be obtained as the union of *all* the instances of F:

$$\cup\{x \mid x \subseteq a \ \& \ x \subseteq b\} = a \cap b.$$

The Essence of being a Subset of Set a and a Subset of Set b
Obtained by Filtering Out All Imperfections.

It has *no* imperfections relative to the property F so it is the concrete universal. Moreover, since the universal is concrete, the set $a \cap b$ is among the sets x involved in the union and it contains all the other such sets x . Thus the union is "taken on," i.e., is equal to one of the sets in the union.

All the category theory examples can be dualized by "reversing the arrows." Reversing the inclusion relation in the definition of F yields the property:

$$G(x) \equiv a \subseteq x \ \& \ b \subseteq x.$$

The participation relation μ for G is the reverse of inclusion $\supseteq$ and the union of a and b is the concrete universal. The universality condition is:

$$\text{for all } x,\ x \supseteq a \cup b \text{ iff } a \subseteq x \ \& \ b \subseteq x.$$

If x has the property G but is not the universal, then x has certain imperfections. An imperfection of x (relative to the G property) would be given by an another set x' containing both a and b but not containing x. A set of instances of G could be purified of some imperfections by taking the intersection of the set. G-ness is preserved under arbitrary intersections. The intersection of a collection of sets with the property G is (equally or) more essential than the sets in the collection. None of the sets in the collection are imperfections of the intersection. Thus the universal or essence of G-ness can be obtained as the intersection of *all* the sets with the property G:

$$\cap\{x \mid a \subseteq x \ \& \ b \subseteq x\} = a \cup b.$$

The union of a and b has no imperfections relative to the property G.

Entailment as Participation Between Concrete Universals

In Plato's Theory of Forms, a logical inference is valid because it follows the necessary connections between universals. Threeness entails oddness because the universal for threeness "brings on" [επιφερει or epipherei, viz. Vlastos 1981, 102; or Sayre 1969, Part IV] or "shares in" the universal for oddness. In a mathematical theory of universals, the "entailment" relation between universals is defined as follows: given universals u_F and u_G,

$$u_F \textit{ entails } u_G \text{ if for any x, if } x \ \mu \ u_F \text{ then } x \ \mu \ u_G.$$

In set theory, the participation relation μ is the membership relation $\in$ so the entailment relation between sets as abstract universals is the *inclusion* relation. Thus in set theory as the theory of abstract universals, the entailment relation (inclusion) between universals is not the same as the participation relation (membership). The difference between inclusion and membership is illustrated by the copulas in "All roses are beautiful" and "The rose is beautiful."

In category theory, the participation relation μ is the uniquely-factors-through relation and the universals are self-participating. If u_G entails u_F, then $x \ \mu \ u_G$ implies $x \ \mu \ u_F$. Since $u_G \ \mu \ u_G$ (a relationship that does not hold for abstract universals), it follows that $u_G \ \mu \ u_F$. In short, for the concrete universals of category theory,

Entailment relation = Participation relation restricted to concrete universals.

To speak in a Platonic philosophical mode for illustrative purposes, let "The Rose" and "The Beautiful" be the concrete universals for the respective properties. In the theory of concrete universals, the general statement "All roses are beautiful" and the singular statement "The Rose is beautiful" are *equivalent*. Both express the proposition that "The Rose participates in The Beautiful," and that proposition is distinct from the statement "The rose is beautiful" (about a rather imperfect flower in one's backyard).

For an example of entailment, let us first consider another universal in the partial order of subsets of some given universe set. Given sets a and b, the *complement of a relative to b* is the concrete universal for the property

$$H(x) \equiv a \cap x \subseteq b.$$

Let the concrete universal be symbolized as $a \supset b$ so by concreteness and universality we have:

$$a \cap (a \supset b) \subseteq b, \text{ and}$$
$$\text{for all x, } x \subseteq a \supset b \text{ iff } a \cap x \subseteq b.$$

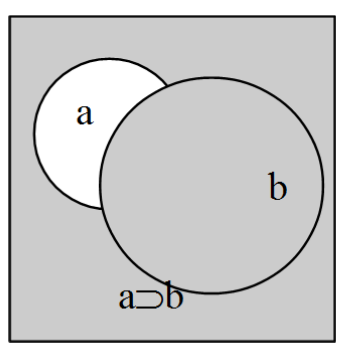

Fig. 3: Relative complement $a \supset b$ is the union of b with the complement of a.

The property $F(x) \equiv x \subseteq a \ \& \ x \subseteq b$ entails the property $H(x) \equiv a \cap x \subseteq b$. The entailment between the properties is realized concretely by the participation relationship between the two concrete universals for the respective properties:

$$a \cap b \subseteq a \supset b.$$

Universal for F Participates in Universal for H

The reasoning is that F(x) implies $x \subseteq a \cap b$, and $a \cap b \subseteq a \supset b$ so $x \subseteq (a \supset b)$ (by transitivity of inclusion) and thus H(x). Thus the entailment, "for all x, F(x) implies H(x)," is realized concretely by the participation of the concrete universal for F(x) in the concrete universal for H(x).

We can now pair together the statements in our intuitive example and the corresponding rigorous statements in the set theoretical example (using the correlation "The Rose" $\leftrightarrow$ $(a \cap b)$ and "The Beautiful" $\leftrightarrow$ $a \supset b$). The three statements in each column of the table are equivalent.

Intuitive Example	Corresponding Rigorous Statement in Set Example
All roses are beautiful.	For all subsets x, $x \subseteq a \ \& \ x \subseteq b$ implies $a \cap x \subseteq b$.
The Rose is beautiful.	$a \cap (a \cap b) \subseteq b$.
The Rose participates in The Beautiful.	$a \cap b \subseteq a \supset b$.

The Third Man Argument in Plato

Much of the modern Platonic literature on self-participation and self-predication [e.g., Malcolm 1991] stems from the work of Vlastos on the Third Man argument [1954, 1981]. The name derives from Aristotle, but the argument occurs in the dialogues.

> But now take largeness itself and the other things which are large. Suppose you look at all these in the same way in your mind's eye, will not yet another unity make its appearance—a largeness by virtue of which they all appear large?
>
> So it would seem.
>
> If so, a second form of largeness will present itself, over and above largeness itself and the things that share in it, and again, covering all these, yet another, which will make all of them large. So each of your forms will no longer be one, but an indefinite number. [Parmenides, 132]

If a form is self-predicative, the participation relation can be interpreted as "resemblance". An instance has the property F because it resembles the paradigmatic example of F-ness. But then, the Third Man argument contends, the common property shared by Largeness and other large things gives rise to a "One over the many", a form Largeness* such that Largeness and the large things share the common property by virtue of resembling Largeness*. And the argument repeats itself giving rise to an infinite regress of forms. A key part of the Third Man argument is what Vlastos calls the *Non-Identity thesis*:

> NI If anything has a given character by participating in a Form, it is not identical with that Form. [Vlastos 1981, 351]

It implies that Largeness* is not identical with Largeness.

P. T. Geach [1956] has developed a self-predicative interpretation of Forms as standards or norms, an idea he attributes to Wittgenstein. A stick is a yard long because it resembles, lengthwise, the standard yard measure. Geach avoids the Third Man regress with the exceptionalist device of holding the Form "separate" from the many so they could not be grouped together to give rise to a new "One over the many". Geach aptly notes the analogy with Frege's ad hoc and unsuccessful attempt to avoid the Russell-type paradoxes by allowing a set of all and only the sets which are not members of themselves—except for that set itself [viz. Quine 1955a, Geach 1980].

Category theory provides a mathematical model for the Third Man argument, and it shows how to avoid the regress. The category-theoretic model shows that the flaw in the Third Man argument lies not in self-predication but in the Non-Identity thesis [viz. Vlastos 1954, 326-329]. "The One" is not necessarily "over the many"; it can be (isomorphic to) one among the many. In mathematical terms, a colimit or limit can "take on" one of the elements in the diagram. In the special case of sets ordered by inclusion, the union or intersection of a collection of sets is not necessarily distinct from the sets in the collection; it could be one among the many.

For example, let $A = \cup\{A_ß\}$ be the One formed as the union of a collection of many sets $\{A_ß\}$. Then add A to the collection and form the new One* as

$$A^* = \cup\{A_ß\} \cup A.$$

This operation leads to no Third Man regress since $A^* = A$.

Whitehead described European philosophy as a series of footnotes to Plato, and the Theory of Forms was central to Plato's thought. We have seen a number of ways in which the interpretation of category theory as the theory of concrete universals provides a rigorous self-predicative mathematical model for Plato's Theory of Forms and for the intuitive notion of a concrete universal elsewhere in philosophy.

Adjoint Functors

One of the most important and beautiful notions in category theory is the notion of a pair of adjoint functors. The developers of category theory, Saunders MacLane and Samuel Eilenberg, famously said that categories were defined in order to define functors, and functors were defined in order to define natural transformations. Their original paper [Eilenberg and MacLane 1945] was entitled not "General Theory of Categories" but *General Theory of Natural Equivalences*. Adjoints were defined more than a decade later by Daniel Kan [1958] but the realization of their foundational importance has steadily increased over time [Lawvere 1969; Lambek 1981]. Now it would perhaps not be too much of an exaggeration to see categories, functors, and natural transformations as the prelude to defining adjoint functors. The notion of adjoint functors (and the constituent semi-adjunctions defined below) includes all the instances of concrete universal mapping properties discussed above. As Steven Awodey put it in his text:

> The notion of adjoint functor applies everything that we have learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. [Awodey 2006, 179]

Other category theorists have given similar testimonials.

> To some, including this writer, adjunction is the most important concept in category theory. [Wood 2004, 6]

> The isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas." [Goldblatt 2006, 438]

> Nowadays, every user of category theory agrees that [adjunction] is the concept which justifies the fundamental position of the subject in mathematics. [Taylor 1999, 367]

We will try to illustrate how adjoint functors relate to our theme of concrete universals while staying within the methodological restriction of using examples from partial orders (where adjunctions are called "Galois connections").

We have been working within the inclusion partial order on the set of subsets P(U) of a universe set U. Consider the set of all ordered pairs of subsets $\langle a,b\rangle$ from the Cartesian product $P(U)\times P(U)$ where the partial order (using the same symbol $\subseteq$) is defined by pairwise inclusion. That is, given the two ordered pairs $\langle a', b'\rangle$ and $\langle a,b\rangle$, we define

$$\langle a',b'\rangle \subseteq \langle a,b\rangle \text{ if } a' \subseteq a \text{ and } b' \subseteq b.$$

Order-preserving maps can be defined each way between these two partial orders. From P(U) to $P(U)\times P(U)$, there is the diagonal map $\Delta(x) = \langle x,x\rangle$, and from $P(U)\times P(U)$ to P(U), there is the meet map $\cap(\langle a,b\rangle) = a \cap b$. Consider now the following "*adjointness relation*" between the two partial orders:

$$\Delta(c) \subseteq \langle a,b\rangle \text{ iff } c \subseteq \cap(\langle a,b\rangle)$$

Adjointness Equivalence

for sets a, b, and c in P(U). It has a certain symmetry that can be exploited. If we fix $\langle a,b\rangle$, then we have the previous universality condition for the meet of a and b: for any c in P(U),

$$c \subseteq a \cap b \text{ iff } \Delta(c) \subseteq \langle a,b\rangle.$$

Universality Condition for Meet of Sets a and b

The defining property on elements c of P(U) is that $\Delta(c) \subseteq \langle a,b\rangle$ (just a fancy way of saying that c is a subset of both a and b). But using the symmetry, we could fix c and have another universality condition using the reverse inclusion in $P(U)\times P(U)$ as the participation relation: for any $\langle a,b\rangle$ in $P(U)\times P(U)$,

$$\langle a,b\rangle \supseteq \Delta(c) \text{ iff } c \subseteq a \cap b$$

Universality Condition for $\Delta(c)$.[5]

Here the defining property on elements $\langle a,b\rangle$ of $P(U)\times P(U)$ is that the meet of a and b is a superset of the given set c. The concrete universal for that property is the image of c under the diagonal map $\Delta(c) = \langle c,c\rangle$, just as the concrete universal for the other property defined given $\langle a,b\rangle$ was the image of $\langle a,b\rangle$ under the meet map $\cap(\langle a,b\rangle) = a \cap b$.

Thus in this adjoint situation between the two categories P(U) and $P(U)\times P(U)$, we have a pair of maps ("adjoint functors") going each way between the categories such that each element in a category defines a certain property in the other category and the map carries the element to the concrete universal for that property.

$$\Delta: P(U)\rightarrow P(U)\times P(U) \text{ and } \cap: P(U)\times P(U)\rightarrow P(U)$$

Example of Adjoint Functors Between Partial Orders

[5] We have written $\langle a,b\rangle \supseteq \Delta(c)$ "backwards" to be an instance of the participation relation "$x \mu u_F$" but it would more conventionally be written $\Delta(c) \subseteq \langle a,b\rangle$.

The notion of a pair of adjoint functors is ubiquitous; it is one of the main tools that highlights concrete universals throughout modern mathematics.

The Heteromorphic Analysis of Adjunctions

We have seen that there are two concrete universals (often one is trivial like Δ(c) in the above example) involved in an adjunctions and that the object-to-object maps or relations were always within one category (or partial order), e.g., the "hom-sets" in a category where "hom" is short for homomorphism (a morphism between objects in the same category). Using object-to-object maps between objects of different categories (properly called "heteromorphisms" or "chimera morphisms"), the notion of an adjunction can be factored into two semi- or half-adjunctions each of which isolates a concrete universal [Ellerman 2006, 2007].

This heteromorphic treatment of adjoints will be illustrated using the above example. The objects $c \in P(U)$ in the partial order P(U) are single subsets c of U and the objects $\langle a,b \rangle$ in the partial order $P(U)\times P(U)$ are pairs of subsets of U. A heteromorphism or *het* from a single subset c to the pair of subsets $\langle a,b \rangle$ is given by the relation $c \subseteq a$ and $c \subseteq b$ which could be symbolized $c \rightarrow \langle a,b \rangle$. Fixing $\langle a,b \rangle$, there is a single subset $\cap(\langle a,b \rangle) = a \cap b$ with a canonical het $a \cap b \rightarrow \langle a,b \rangle$. Then the functor that takes $\langle a,b \rangle$ to $a \cap b$ gives a *right semi-adjunction* if for every het from $c \rightarrow \langle a,b \rangle$, there is a (unique) hom $c \subseteq a \cap b$ that gives us the following (commutative) diagram where the arrows are hets.

$$\begin{tikzcd} c \arrow[r, phantom, "\subseteq"] \arrow[dr] & a \cap b \arrow[d] & P(U) \\ & \langle a,b \rangle & P(U)\times P(U) \end{tikzcd}$$

Fig. 4: Right semi-adjunction diagram.

This gives the following if-and-only-if (iff) equivalence between the diagonal het and the horizontal hom:

$$c \rightarrow \langle a,b \rangle \text{ iff } c \subseteq a \cap b.$$

Universality for right semi-adjunction.

Similarly for given c, there is a canonical het $c \rightarrow \Delta(c)$. Then the functor that takes c to $\Delta(c)$ is a *left semi-adjunction* if for any given het $c \rightarrow \langle a,b \rangle$, there is a (unique) $\Delta(c) \subseteq \langle a,b \rangle$ to make the following diagram commute.

$$\begin{tikzcd} c \arrow[d] \arrow[dr] & & P(U) \\ \Delta(c) \arrow[r, phantom, "\subseteq"] & \langle a,b \rangle & P(U)\times P(U) \end{tikzcd}$$

Fig. 5: Left semi-adjunction diagram.

The corresponding universality equivalence is:

$$\Delta(c) \subseteq \langle a,b \rangle \text{ iff } c \rightarrow \langle a,b \rangle$$

Universality for left semi-adjunction.

The concept of a semi-adjunction is the most general concept of a concrete universal in category theory. The name "semi-adjunction" (or "half-adjunction") is derivative from "adjunction" since two semi-adjunctions with the same diagonal hets, $c \rightarrow \langle a,b \rangle$ in this case, combine to give an adjunction:

$$\Delta(c) \subseteq \langle a,b \rangle \text{ iff } c \rightarrow \langle a,b \rangle \text{ iff } c \subseteq a \cap b$$

Adjunction equivalence with het middle term.

Gluing together the two left and right semi-adjunction diagrams along the common het $c \rightarrow \langle a,b \rangle$ gives the adjunctive square diagram representing an adjunction.

$$\begin{tikzcd} c \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr] & a \cap b \arrow[d] & P(U) \\ \Delta(c) \arrow[r, phantom, "\subseteq"] & \langle a,b \rangle & P(U) \times P(U) \end{tikzcd}$$

Fig. 6: Two semi-adjunctions = One adjunction

Ordinarily the middle het term is left out, so we get the usual form of the adjunction equivalence:

$$\Delta(c) \subseteq \langle a,b \rangle \text{ iff } c \subseteq a \cap b$$

Usual form of adjunction equivalence.

The left or right semi-adjunctions are the most general form of concrete universals, and the adjunctions or pairs of adjoint functors are the special cases where left and right semi-adjunctions exist for the same hets.

Brain functors

If the concrete universals of category theory, which combine in one way to form an adjunction, serve to describe the important and natural concepts within pure mathematics, then one might well expect the concrete universals to also be important in applications.

One payoff from analyzing the important concept of an adjunction into two semi-adjunctions is that we can then reassemble those parts in a different way to define the new concept that is speculatively named a "brain functor." The basic idea is to think of one category in a semi-adjunction as the "environment" and the other category as an "organism." Instead of semi-adjunctions representing within each category the hets going one way between the categories, suppose the hets going both ways were represented by semi-adjunctions within one of the categories (the "organism").

A het from the environment to the organism is, say, a visual stimulus. Then a left semi-adjunction would play the role of the brain in providing the re-cognition of the stimulus as, say, a perception of a tree where the internal recognition is represented by the morphism $\Rightarrow$ inside the "organism" category.

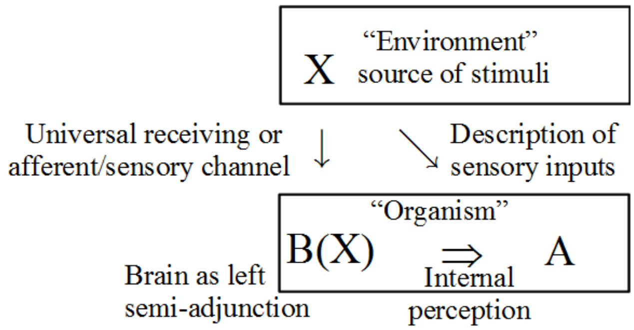

Fig. 7: Perceiving brain presented as a left semi-adjunction.

It is an old philosophical theme in the Platonic tradition that external stimuli do not give knowledge; the stimuli only trigger the internal perception or recognition that is knowledge. In *De Magistro* (The Teacher), the neo-Platonic Christian philosopher-theologian Augustine of Hippo (Annaba in modern Algeria) developed an argument (in the form of a dialogue with his son Adeodatus) that as teachers teach, it is only the student's internal appropriation of what is taught that gives understanding.

> Then those who are called pupils consider within themselves whether what has been explained has been said truly; looking of course to that interior truth, according to the measure of which each is able. Thus they learn,…. But men are mistaken, so that they call those teachers who are not, merely because for the most part there is no delay between the time of speaking and the time of cognition. And since after the speaker has reminded them, the pupils quickly learn within, they think that they have been taught outwardly by him who prompts them. (Augustine *De Magistro*, Chapter XIV)

The basic point is the active role of the mind in *generating* understanding. This is clear even at the simple level of understanding spoken words. We hear the auditory sense data of words in a completely strange language as well as the words in our native language. But the strange words 'bounce off' our minds with no resultant understanding while the words in a familiar language prompt an internal process of generating a meaning so that we understand the words.

There are also hets going the other way from the "organism" to the "environment" and there is a similar distinction between mere behavior and an action which expresses an intention. Mathematically that is described by dualizing or turning the arrows around which gives an acting brain presented as a right semi-adjunction.

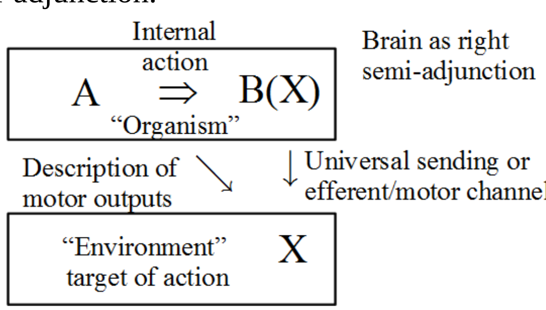

Fig. 8: Acting brain as a right semi-adjunction.

In the heteromorphic treatment of adjunctions, an adjunction arises when the hets from one category to another, Het(X,A), have a right semi-adjunction, Het(X,A) ≅ Hom(X,G(A)), *and* a left semi-adjunction, Hom(F(X),A) ≅ Het(X,A). But instead of taking the same set of hets as being represented by two different functors on the right and left, suppose we consider a single functor B(X) that represents the hets Het(X,A) on the left:

$$\mathrm{Het}(X,A) \cong \mathrm{Hom}(B(X),A),$$

and represents the hets Het(A,X) [going in the opposite direction] on the right:

$$\mathrm{Hom}(A,B(X)) \cong \mathrm{Het}(A,X).$$

If the hets each way between two categories are represented by the same functor B(X) as left and right semi-adjunctions, then that functor is said to be a *brain functor*. Thus instead of a pair of functors being adjoint, we have a single functor B(X) with values within one of the categories (the "organism") as representing the two-way interactions, "cognition" and "action," between that category and another one (the "environment").

The diagram for an adjunction arises by gluing together the two diagrams for the left and right semi-adjunctions along the common diagonal het X→A.

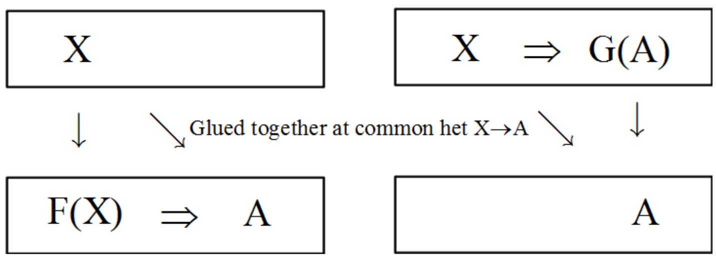

Fig. 9: Combining two semi-adjunctions make an adjunction

The combined diagram is the *adjunctive square* that represents an adjunction in the hetermorphic treatment of adjunctions.

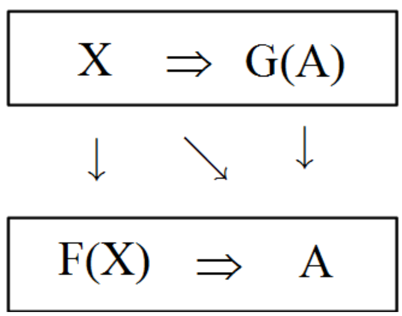

Fig. 10: Adjunctive Square Diagram.

The diagram for a brain functor is obtained by gluing together the diagrams for the left and right semi-adjunctions at the common values of the brain functor B(X).

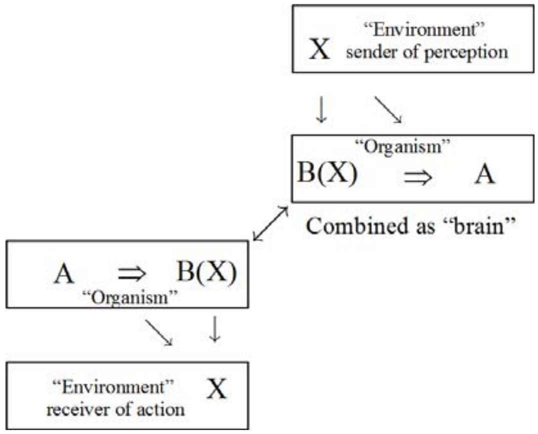

Fig. 11: Combining two semi-adjunctions to make a "brain"

This gives the brain functor "butterfly"[6] diagram—where we have taken the liberty to relabel the diagram for the brain as the language faculty for understanding and producing speech.

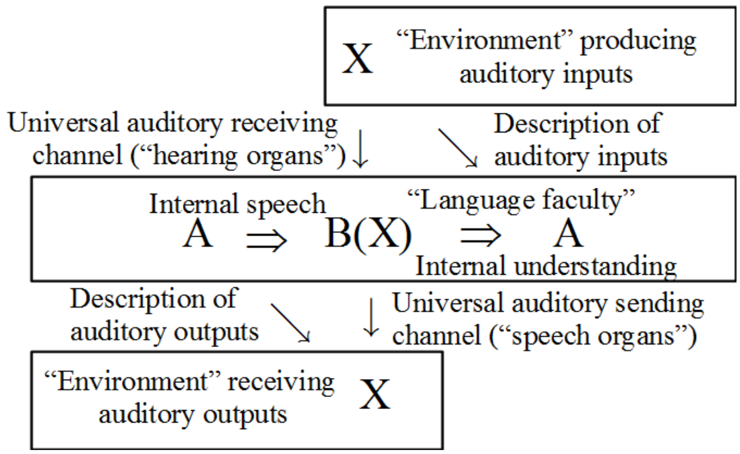

Fig. 12: Brain functor diagram interpreted as language faculty

A simple example of a brain functor, where the two categories are partial orders, is given in the Appendix.

Category Theory and Foundations

What is the relevance of category theory to the foundations of mathematics? Today, this question might be answered by pointing to Lawvere and Tierney's *theory of topoi* [e.g., Lawvere 1972, Lawvere et al. 1975, or Hatcher 1982]. Topos theory can be viewed as a categorically formulated generalization of set theory to abstract sheaf theory. A set can be viewed as a sheaf of sets on the one-point space, and much of the machinery of set theory can be generalized to sheaves [Ellerman 1974]. Since much of mathematics can be formulated in set theory, it can be reconstructed with many variations in topoi.

The concept of category theory as the logic of concrete universals presents quite a *different* picture of the foundational relevance of category theory. Topos theory is important in its own right as a generalization of set theory, but it does not exclusively capture category

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[6] Erase the labels and the arrows give the outline of a butterfly.

theory's foundational relevance. Concrete universals do not "generalize" abstract universals, so as the theory of concrete universals, category theory does not try to generalize set theory, the theory of abstract universals. Category theory presents the theory of the other type of universals, the self-participating or concrete universals.

Logic becomes concrete in category theory as the theory of concrete universals. *Facts become things*—at least in the Platonic Heaven of mathematics. Properties F can be realized concretely as things, the universals u_F. The fact that x is an F-instance is realized concretely by a thing, the unique participation morphism $x \to u_F$. A universal implication "for all x, F(x) implies H(x)" is realized concretely by another thing, the unique participation morphism $u_F \to u_H$ wherein one universal "brings on" or entails another universal.

Category theory is relevant to foundations in a different way than set theory. As the theory of concrete universals, category theory does not attempt to derive all of mathematics from a single theory. Instead, category theory's foundational relevance is that it provides universality concepts to characterize the important structures, schema, or Forms throughout mathematics (e.g., adjunctions).

Moreover, if the concrete universals characterize important concepts within pure mathematics, it should not be too surprising if they might also characterize, albeit at a very abstract level, important concepts and canonical schema (or Forms) in applications. The application scheme outlined here is the brain functor (obtained by rearranging the heteromorphic building blocks of adjunctions) which abstractly models the dual functions of perception and action.

The importance of category theory is that it provides a criterion of importance, concrete universality. Category theory provides the concepts to isolate the universal instance (where it exists) from among all the instances of a property. The Concrete Universal is the most important instance of a property because it represents the property in a paradigmatic way. All other instances have the property by virtue of participating in the Concrete Universal.

Appendix: A Brain Functor Between Partial Orders

A simple example of a brain functor using partial orders will be developed. In this setting, only the simplest "brain function" can be modeled, namely the building and functioning of an internal model of the external reality such as an internal coordinate system to map an external set of locations. The external reality is given by a set of atomic points or locations Y, the atomic coordinates are the points in X, and the coordinate mapping function is the given function $f:X \to Y$. Just to keep the mathematics not completely trivial, we do not require f to be an isomorphism; multiple coordinates might refer to the same point (i.e., f is not necessarily one-to-one) and some points might not have coordinates (i.e., f is not necessarily onto). The two partial orders are the inclusion-ordered subsets of points P(Y) and the inclusion-ordered subsets of coordinates P(X).

In the case where the "brain" is an "electronic brain" or computer, Y is the set of locations on an external input/output device such as a floppy disk or any other external memory device. Each location is marked with a 0 or 1, so the subsets $V \in P(Y)$ would be the external sets of 1s. The set X would be the set of internal memory locations which also contain either a 0 or 1, so the subsets $U \in P(X)$ are the internal sets of 1s. The coordinate function $f:X \to Y$ maps the internal memory locations to the external disk locations. The dual perception/action functions in the electronic brain would be the familiar read/write operations between the computer and the external input/output device.

The brain functor in this example is $f^{-1}:P(Y)\to P(X)$ where for any subset $V\in P(Y)$, the value of the brain functor is:

$$f^{-1}(V) = \{x\in X: f(x)\in V\}.$$

Given a subset $V\subseteq Y$, what is the "best" internal subset $U\subseteq X$ that represents or recognizes V? The heteromorphism $V\to U$ is defined by the property, F(U) = "U is *complete for V*" in the sense that all the $x\in X$ that map to V are contained in U, i.e.,

$$V\to U \text{ means } V \subseteq \{y\in Y:\forall x, \text{ if } f(x) = y \text{ then } x\in U\}.$$

The left semi-adjunction for the property F(U) = "U is complete for V" is given by the smallest complete subset $f^{-1}(V)\in P(X)$ and the universality condition: $f^{-1}(V) \subseteq U$ iff $V\to U$, is satisfied.

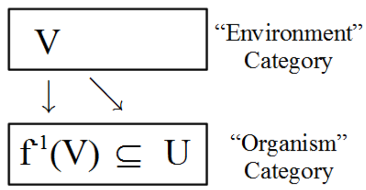

Fig. 13: Left semi-adjunction to "read" V
with smallest complete subset $f^{-1}(V)$

The concrete universal $f^{-1}(V) \in P(X)$ has the property, i.e., $V\to f^{-1}(V)$, and a subset $U\in P(X)$ has the property, i.e., $V\to U$, if and only U participates in the concrete universal $U \supseteq f^{-1}(V) = u_F$ (where "participation" is written as the reverse inclusion).

In the dual case of "action," the het $U\to V$ going in the opposite direction (relative to the perception het $V\to U$) from a subset of X to a subset of Y is defined by the property, G(U) = "U is *consistent with V* " in the sense that no coordinate in U maps outside of V, i.e.,

$$U\to V \text{ means } f(U) \subseteq V.$$

The right semi-adjunction for the property G(U) = "U is consistent with V" is given by the largest consistent subset $f^{-1}(V)\in P(X)$, and the universality condition is: $U \subseteq f^{-1}(V)$ iff $U\to V$.

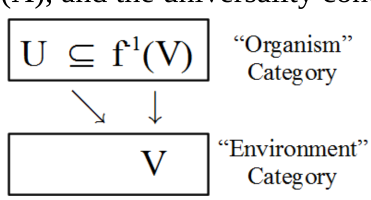

Fig. 14: Right semi-adjunction to "write" V
with largest consistent subset $f^{-1}(V)$.

The concrete universal $f^{-1}(V)\in P(X)$ has the property, $f^{-1}(V)\to V$, and a subset $U\in P(X)$ has the property, i.e., $U\to V$, if and only if U participates in the concrete universal, i.e., $U \subseteq f^{-1}(V) = u_G$ (where "participation" is the inclusion). Combining the left and right semi-adjunctions at the common "brain" gives the butterfly diagram.

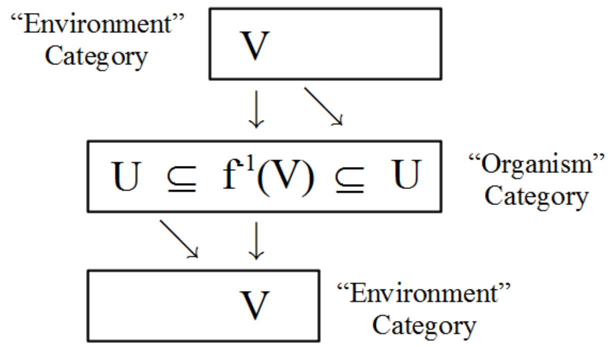

Fig. 15: Butterfly diagram for the brain functor $f^{-1}:P(Y)\to P(X)$

Mathematically, this example is an instance of the general result that any functor that has both right and left adjoints is a brain functor. The right adjoint of $f^{-1}(V)$ is usually symbolized as:

$$\forall_f(U) = \{y\in Y:\forall x, \text{ if } f(x) = y \text{ then } x\in U\}$$

with the adjunction equivalence:

$$f^{-1}(V) \subseteq U \text{ iff } V \subseteq \forall_f(U)$$

while the left adjoint of $f^{-1}(V)$ is usually symbolized as:

$$\exists_f(U) = f(U) = \{ y\in Y: \exists x\in U, f(x) = y\}$$

with the adjunction equivalence:

$$\exists_f(U) \subseteq V \text{ iff } U \subseteq f^{-1}(V).$$

The quantifier notation is motivated by the special case where f is the projection $f = p_X :X\times Y\to Y$ so for any binary relation $U \subseteq X\times Y$, then $\exists_f(U) = \{y\in Y: \exists xU(x,y)\}$ and $\forall_f(U) = \{y\in Y:\forall xU(x,y)\}$.

The fact that the read/write functions of an electronic brain can be modeled by the brain functor $f^{-1}:P(Y)\to P(X)$ might be seen as an artifact of the simplicity of the case. As complexity increases exponentially in animal and human brains, this sort of precise modeling is not to be expected. The point, in such an empirical application, is the canonical scheme or Form that describes the functions of a brain at a conceptual level (e.g., Figure 12 giving the scheme for the language faculty).

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