Forthcoming in:

LOGIC JOURNAL of the IGPL

INTEREST GROUP IN PURE AND APPLIED LOGICS

The Quantum Logic of Direct-Sum Decompositions: The Dual to the Quantum Logic of Subspaces

David Ellerman*
University of California Riverside

June 19, 2017

Abstract

Since the pioneering work of Birkhoff and von Neumann, quantum logic has been interpreted as the logic of (closed) subspaces of a Hilbert space. There is a progression from the usual Boolean logic of subsets to the "quantum logic" of subspaces of a general vector space–which is then specialized to the closed subspaces of a Hilbert space. But there is a "dual" progression. The set notion of a partition (or quotient set or equivalence relation) is dual (in a category-theoretic sense) to the notion of a subset. Hence the Boolean logic of subsets has a dual logic of partitions. Then the dual progression is from that logic of set partitions to the quantum logic of direct-sum decompositions (i.e., the vector space version of a set partition) of a general vector space–which can then be specialized to the direct-sum decompositions of a Hilbert space. This allows the quantum logic of direct-sum decompositions to express measurement by any self-adjoint operators. The quantum logic of direct-sum decompositions is dual to the usual quantum logic of subspaces in the same sense that the logic of partitions is dual to the usual Boolean logic of subsets.

Contents

*The author would like to thank the referee whose careful review significantly improved the paper.

1 Introduction

This paper is an introduction to quantum logic based on direct-sum decompositions rather than on subspaces. Intuitively, a *direct-sum decomposition* (DSD) of a vector space V over a base field $\mathbb{K}$ is a set of (nonzero) subspaces $\{V_i\}_{i\in I}$ that are *disjoint* (i.e., their pair-wise intersections are the zero space $\{0\}$) and that span the space such that each vector $v \in V$ has a unique expression $v = \sum_{i\in I} v_i$ with each $v_i \in V_i$ (with only a finite number of v_i's nonzero). For introductory purposes, we assume V is finite dimensional. Each self-adjoint operator on a Hilbert space, and, in general, each diagonalizable operator, has eigenspaces that form a direct-sum decomposition of the vector space. But the notion of a direct-sum decomposition makes sense over arbitrary vector spaces independently of an operator.

For instance, in the pedagogical model of "quantum mechanics over sets" or QM/Sets [6], the vector space is $\mathbb{Z}_2^n$ so the only diagonalizable operators are projection operators $\hat{P} : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$. But given a set $U = \{v_1, ..., v_n\}$ of basis vectors for $\mathbb{Z}_2^n$, any real-valued function $f : U \to \mathbb{R}$ determines a DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$ of $\mathbb{Z}_2^n$ (where $\wp()$ is the power-set and $f(U)$ is the image or "spectrum" of "eigenvalues" of the numerical attribute f). Thus the concept of a direct-sum decomposition of a vector space allows one to capture many of the relevant properties of such a real-valued "observable" even though it does *not* take values in the base field (which is only $\mathbb{Z}_2$ in QM/Sets). It is only as the base field is increased up to the complex numbers that all real-valued observables can be "internalized" as self-adjoint operators taking values in the base field.

By way of background, "logic" is usually seen as being about propositions, and Birkhoff and von Neumann [2] kept that focus in their development of quantum logic. But going "back to Boole," Boolean logic was the logic of subsets of a universe set–with the propositional case being a very important special case. A *Boolean tautology* was a formula such that no matter what subsets of the universe set were substituted for the variables, the whole formula evaluated to the universe set. It was then a theorem, not a definition as in most modern logic texts, that it was sufficient for validity to consider only the "propositional" special case (truth-table validity) where the universe was, in effect, a one element set (with subsets symbolized as 0 and 1 or F and T). One advantage in going "back to Boole" and considering the logic of subsets (instead of only the propositional special case) is that the concept of a subset has a (category-theoretic) dual in the concept of a quotient set, equivalence relation, or partition–and hence the development of the logic of partitions ([4]; [5]).[1]

"Subsets of a universe set" linearize to "subspaces of a vector space" and thus the usual quantum logic can also be seen as being about subspaces (e.g., the closed subspaces of a Hilbert space)–or the associated propositions about a vector being in the subspaces (just as the variables in the Boolean logic of subsets can also be interpreted in the usual way about propositions, e.g., that a generic element is in a subset). Dually, "partitions on a universe set" linearize to "direct-sum decompositions on a vector space" (which can then be specialized to a Hilbert space). The focus of this paper is that vector space version of a partition, namely a direct-sum decomposition (N.B.: not a quotient space). The dual of a partial Boolean algebra (pBA) [9] is then a "partial partition algebra" of DSDs on an arbitrary vector space (our topic here)–which can then be specialized to a Hilbert space for the strictly quantum mechanical interpretation (or specialized to a vector space over $\mathbb{Z}_2$ for pedagogical purposes).

[1] As Gian-Carlo Rota put it: "categorically speaking, the Boolean σ-algebra of events and the lattice Σ of all Boolean σ-subalgebras are dual notions" [14, p. 65] using the characterization of partitions by Boolean subalgbras [11, p. 43] that goes back to Ore [13]. The category theorist, F. William Lawvere, called subobjects "parts" and then noted that: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [12, p. 85]

Linearize → / ↓ Dualize	Linearizes to vector-space concept	Quantum logic case: (Hilbert space)	Pedagogical model of QM/Sets (space over $\mathbb{Z}_2$)
Boolean algebra of *subsets* of a set	Lattice of *subspaces* of a vector space or pBA of projection operators	Orthomodular lattice of closed *subspaces* or pBA of projection operators	Partial BA of projection operators on vector space over $\mathbb{Z}_2$
Partition algebra of *partitions* on a set	Partial partition algebra of *direct-sum decompositions* of a vector space	Special quantum partition algebra of *DSDs* on a Hilbert space	Partial partition algebra of DSDs on vector space over $\mathbb{Z}_2$

Figure 1: Progressions from sets to vector spaces using the dual concepts of subset and partition.

There is a natural partial order ("refinement" as with partitions on sets) on the DSDs of a vector space V and there is a minimum element $\mathbf{0} = \{V\}$, the *indiscrete* DSD (nicknamed the "blob") which consists of the whole space V. A DSD is *atomic* in the partial order if there is no DSD between it and the minimum DSD $\mathbf{0}$, and the atomic DSDs are the binary ones consisting of just two nontrivial subspaces. Each atomic DSD determines a pair of projection operators, and the indiscrete DSD also determines a pair of projection operators, namely the identity operator $\hat{I}$ and the zero operator $\hat{I} - \hat{I} = \hat{0}$. Conversely, each projection operator $\hat{P} : V \to V$ (other than the identity or zero operator) determines an atomic DSD consisting of the image of $\hat{P}$ and the image of $\hat{I} - \hat{P}$, while the identity and zero operators determine the indiscrete DSD.

To the extent that the usual quantum logic of subspaces can be viewed as representing measurement, it is the measurement of projection operators $\hat{P}$ and $\hat{I} - \hat{P}$ whose images form an atomic DSD. The quantum logic of DSDs is the more natural setting to represent measurement of all self-adjoint operators–since measurement in any case involves DSDs, atomic or otherwise. But it would be a misperception to see the quantum logic of DSDs as a "generalization" of the quantum logic of subspaces because self-adjoint operators generalize projection operators on Hilbert spaces. Instead, the two quantum logics should be seen as dual formations, i.e., as the two dual vector-space or "linearized" versions of the dual logics of the Boolean logic of subsets and the logic of set partitions as in Figure 1. Symbolically,

Logic of subsets : Logic of partitions :: QL of subspaces : QL of DSDs.

2 The partial partition algebra of direct-sum decompositions

Definition 1 *Let V be a finite dimensional vector space over a field $\mathbb{K}$. A direct sum decomposition (DSD) of V is a set of subspaces $\{V_i\}_{i \in I}$ such that $V_i \cap \sum_{i' \neq i} V_{i'} = \{0\}$ (the zero space) for $i \in I$ and which span the space, i.e., $\oplus_{i \in I} V_i = V$.*

Let $DSD(V)$ be the set of DSDs of V. To fix notation, let $\pi = \{V_i\}_{i \in I}$, $\sigma = \{W_j\}_{j \in J}$, and $\tau = \{X_k\}_{k \in K}$ be three arbitrary DSDs of V.

2.1 Compatibility of DSDs

In the algebra of partitions on a fixed set, the operations of join, meet, and implication are always defined, but in the context of "vector space partitions," i.e., DSDs, we need to define a notion of compatibility. Intuitively, for vector spaces:

diagonalizable operator = DSD + scalars (eigenvalues) associated with subspaces.

Since the quantum logic of DSDs abstracts away from the specific eigenvalues, we need the DSD-version of the commutativity of operators.

Given two DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$, their *proto-join* is the set of non-zero subspaces $\{V_i \cap W_j | V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$ (which do not necessarily form a DSD). If the two DSDs π and σ were defined as the eigenspace DSDs of two diagonalizable operators, then the space spanned by the proto-join would be the space spanned by the simultaneous eigenvectors of the two operators, and that space is the kernel of the commutator of the two operators [6]. If the two operators commuted, then their commutator is the zero operator whose kernel is the whole space so the proto-join would span the whole space. Hence the natural definition of compatibility, without any mention of operators, is:

Definition 2 *π and σ are compatible, written $\pi \leftrightarrow \sigma$, if the proto-join spans the whole space V (and is thus a DSD).*

The *indiscrete DSD* $\mathbf{0} = \{V\}$ (the "blob") is compatible with all DSDs, i.e., $\mathbf{0} \leftrightarrow \pi$ for any π.

2.2 The join of compatible DSDs

When two DSDs π and σ are compatible, $\pi \leftrightarrow \sigma$, their proto-join is the *join:*

$$\pi \vee \sigma = \{V_i \cap W_j | V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$$

Join of DSDs when $\pi \leftrightarrow \sigma$.

The binary relation of compatibility on DSDs is reflexive and symmetric. The indiscrete DSD $\mathbf{0} = \{V\}$ acts as the identity for the join: $\mathbf{0} \vee \pi = \pi$ for any DSD π.

In a set of mutually compatible DSDs, we need to show that the join operation preserves compatibility. If $\pi \leftrightarrow \sigma$, it is trivial that $(\pi \vee \sigma) \leftrightarrow \pi$ and $(\pi \vee \sigma) \leftrightarrow \sigma$, but for a third DSD τ with $\pi \leftrightarrow \tau$ and $\sigma \leftrightarrow \tau$, does $(\pi \vee \sigma) \leftrightarrow \tau$?

Lemma 3 *Let the DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$ be compatible so that $\pi\vee\sigma$ is a DSD and thus any $v \in V$ has a unique expression $v = \sum_{(i,j)\in I\times J} v_{ij}$ where $v_{ij} \in V_i \cap W_j$. Let $v_i = \sum_{j\in J} v_{ij} \in V_i$ so that $v = \sum_{i\in I} v_i$. If $v \in V_i$, then $v = v_i$.*

Proof. Let $\widehat{v_i} = \sum_{i'\in I, i'\neq i} v_{i'}$ so that $v = v_i + \widehat{v_i}$. Hence if $v \in V_i$, then $v - v_i = \widehat{v_i} \in V_i$. Since $\widehat{v_i} \in \oplus_{i'\in I,i'\neq i} V_{i'}$, $\widehat{v_i} \in V_i \cap \oplus_{i'\in I,i'\neq i} V_{i'}$ so $\widehat{v_i} = 0$ since $\pi = \{V_i\}_{i\in I}$ is a DSD which implies $V_i \cap \oplus_{i'\in I,i'\neq i} V_{i'} = \{0\}$. ∎

Theorem 4 *Given three DSDs, $\pi = \{V_i\}_{i\in I}$, $\sigma = \{W_j\}_{j\in J}$, and $\tau = \{X_k\}_{k\in K}$ that are mutually compatible, i.e., $\pi \leftrightarrow \sigma$, $\pi \leftrightarrow \tau$, and $\sigma \leftrightarrow \tau$, then $(\pi \vee \sigma) \leftrightarrow \tau$.*

Proof. We need to prove $\pi \vee \sigma \leftrightarrow \tau = \{X_k\}_{k\in K}$, i.e., that $\oplus_{(i,j,k)\in I\times J\times K} (V_i \cap W_j \cap X_k) = V$. Consider any nonzero $v \in V$ where since $\pi \leftrightarrow \sigma$, there are $v_{ij} \in V_i \cap W_j$ for each $i \in I$ and $j \in J$ such that $v = \sum_{(i,j)\in I\times J} v_{ij}$. Consider any such nonzero v_{ij}. Now since $\pi \leftrightarrow \tau$, there are $v_{ij,i'k} \in V_{i'} \cap X_k$ for each $i' \in I$ and $k \in K$ such that $v_{ij} = \sum_{(i',k)\in I\times K} v_{ij,i'k}$. But since $v_{ij} \in V_i$, by the Lemma, only $v_{ij,ik}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,ik}$. Symmetrically, since $\sigma \leftrightarrow \tau$, there are $v_{ij,j'k} \in W_{j'} \cap X_k$ for each $j' \in J$ and $k \in K$ such that $v_{ij} = \sum_{(j',k)\in J\times K} v_{ij,j'k}$. But since $v_{ij} \in W_j$, by the Lemma, only $v_{ij,jk}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,jk}$. Now since $\{X_k\}_{k\in K}$ is a DSD, there is a unique expression for $v_{ij} = \sum_{k\in K} v_{ijk}$ where $v_{ijk} \in X_k$. Hence by uniqueness: $v_{ijk} = v_{ij,ik} = v_{ij,jk}$. But since $v_{ij,ik} \in V_i$ and $v_{ij,jk} \in W_j$ and $v_{ij,ik} = v_{ijk} = v_{ij,jk}$, we have $v_{ijk} \in V_i \cap W_j \cap X_k$. Thus $v = \sum_{(i,j)\in I\times J} v_{ij} = \sum_{(i,j)\in I\times J} \sum_{k\in K} v_{ijk} = \sum_{(i,j,k)\in I\times J\times K} v_{ijk}$. Since v was arbitrary, $\oplus_{(i,j,k)\in I\times J\times K} (V_i \cap W_j \cap X_k) = V$. ∎

2.3 The meet of two DSDs

Definition 5 *For any two DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$, the meet $\pi \wedge \sigma$ is the DSD whose subspaces are direct sums of subspaces from π and the direct sum of subspaces from σ and are minimal subspaces in that regard. That is, $\{Y_l\}_{l\in L}$ is the meet if there is a set partition $\{I_l\}_{l\in L}$ on I and a set partition $\{J_l\}_{l\in L}$ on J such that for all $l \in L$: $Y_l = \oplus_{i\in I_l} V_i = \oplus_{j\in J_l} W_j$ and that holds for no more refined partitions on the index sets.*

Note that for the blob $\mathbf{0} = \{V\}$, $V = \oplus_{i\in I} V_i = \oplus_{j\in J} W_j$ using the blob set partitions $\{I\}$ and $\{J\}$, but in general the meet will use more refined partitions on I and J. If $\pi \leftrightarrow \tau$ and $\sigma \leftrightarrow \tau$, then it is trivial that $(\pi \wedge \sigma) \leftrightarrow \tau$.

As in the old movie of the same name, "The Blob" absorbs everything it meets:

$$\mathbf{0} \wedge \pi = \mathbf{0}.$$

2.4 The refinement partial order on DSDs

The *partial order* on the DSDs of V is defined as for set partitions but with subspaces replacing subsets:

Definition 6 *π refines σ, written $\sigma \preceq \pi$, if for every $V_i \in \pi$, $\exists W_j \in \sigma$ such that $V_i \subseteq W_j$.*

It is clear that refinement is reflexive and transitive. For anti-symmetry, suppose $\sigma \preceq \pi$ and $\pi \preceq \sigma$. Fixing $V_i \in \pi$ there is W_j with $V_i \subseteq W_j$, and then for that W_j, there is a $V_{i'}$ such that $W_j \subseteq V_{i'}$. Hence $V_i \cap V_{i'} = V_i$ so $V_i = V_{i'} = W_j$ (since if $V_i \neq V_{i'}$, then $V_i \cap V_{i'} = \{0\}$) . And by symmetry, any $W_{j'}$ must equal the $V_{i'}$ where $W_{j'} \subseteq V_{i'}$ so $\pi = \sigma$.

Lemma 7 *If $\sigma \preceq \pi$, then each $W_j = \oplus\{V_i : V_i \subseteq W_j\}$.*

Proof. Consider any nonzero vector $v \in W_j$. Since π is a DSD, $v = \sum_{i\in I} v_i$ where $v_i \in V_i$ so we can divide v into two parts: $v = \sum_{V_i \subseteq W_j} v_i + \sum_{V_{i'} \nsubseteq W_j} v_{i'}$. Now $\sigma \preceq \pi$, so for each $v_{i'} \in V_{i'} \nsubseteq W_j$, there is a $W_{j'}$ such that $v_{i'} \in V_{i'} \subseteq W_{j'}$ so $\sum_{V_{i'} \nsubseteq W_j} v_{i'} \in \sum_{j'\neq j} W_{j'}$. But $\sum_{V_{i'} \nsubseteq W_j} v_{i'} = v - \sum_{V_i \subseteq W_j} v_i \in W_j$ and $W_j \cap \sum_{j'\neq j} W_{j'} = \{0\}$ since σ is a DSD. Thus $v - \sum_{V_i \subseteq W_j} v_i = 0$ so $v \in \oplus\{V_i : V_i \subseteq W_j\}$. ∎

Hence $\sigma \preceq \pi$ implies $\pi \leftrightarrow \sigma$ and $\pi \vee \sigma = \pi$ as well as $\pi \wedge \sigma = \sigma$ as expected.

Proposition 1 *For any two DSDs π and σ, if they a common upper bound τ, i.e., $\pi, \sigma \preceq \tau$, then (i) $\pi \leftrightarrow \sigma$, and (ii) the join $\pi \vee \sigma$ is defined and is the least upper bound of π and σ.*

Proof. If $\pi, \sigma \preceq \tau = \{X_k\}_{k\in K}$, then for each X_k, there is a V_i such that $X_k \subseteq V_i$ and there is a W_j such that $X_k \subseteq W_j$ so $X_k \subseteq V_i \cap W_j$. Since the $\{X_k\}_{k\in K}$ span the space so must the nonzero $V_i \cap W_j$ so $\pi \leftrightarrow \sigma$ which proves (i) and makes $\pi \vee \sigma = \{V_i \cap W_j \neq \{0\}\}_{(i,j)\in I\times J}$ into a DSD. To prove (ii), as just shown, for any given X_k, there is a V_i and W_j such that $X_k \subseteq V_i \cap W_j$ so $\pi \vee \sigma \preceq \tau$. Hence $\pi \vee \sigma$ is the least upper bound of π and σ in the refinement partial order. ∎

Two DSDs π and σ need not have a common upper bound so $DSD(V)$ is not a join-semilattice.

Lemma 8 *Given a DSD $\pi = \{V_i\}_{i\in I}$, let $X = \oplus_{i\in I_X} V_i$ and $Y = \oplus_{i\in I_Y} V_i$ both be direct sums of some V_i's. If $X \cap Y$ is nonzero, then $X \cap Y = \oplus_{i\in I_X \cap I_Y} V_i$.*

Proof. Consider a nonzero $v \in X \cap Y$ so there is a unique expression $v = \sum_{i\in I_X} v_{i,X}$ where $v_{i,X} \in V_i \subseteq X$ and a unique expression $v = \sum_{i\in I_Y} v_{i,Y}$ where $v_{i,Y} \in V_i \subseteq Y$. Since π is a DSD, there is also a unique expression $v = \sum_{i\in I} v_i$ so, for each nonzero v_i, $v_i = v_{i,X} = v_{i,Y} \in V_i \cap X \cap Y$. Thus for any such i, V_i is a common direct summand to X and Y, so $V_i \subseteq X \cap Y$. Thus every nonzero element $v \in X \cap Y$ is in a direct sum of V_i's for $V_i \subseteq X \cap Y$ and thus $X \cap Y$ is the direct sum of V_i that are common direct summands of X and Y. ∎

Proposition 2 *The meet $\pi \wedge \sigma$ is the greatest lower bound of π and σ.*

Proof. If $\tau \preceq \pi, \sigma$ then each $X_k = \oplus \{V_i : V_i \subseteq X_k\} = \oplus \{W_j : W_j \subseteq X_k\}$. By the construction of $\pi \wedge \sigma$, there is a set partition $\{I_l\}_{l \in L}$ on I and a set partition $\{J_l\}_{l \in L}$ on J such that each subspace in the meet $\pi \wedge \sigma = \{Y_l\}$ satisfies: $Y_l = \oplus_{i \in I_l} V_i = \oplus_{j \in J_l} W_j$, and where no subsets of I smaller than I_l and subsets of J smaller than J_l have that property. Since each V_i is contained in some X_k, if $i \in I_l$, then $V_i \subseteq Y_l \cap X_k$. Since both Y_l and X_k are direct sums of some V_i, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of the common direct summand V_i's. Symmetrically, since the same Y_l and X_k are direct sums of some W_j's, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of the common direct summand W_j's. But since Y_l is the smallest direct sum of both V_i's and W_j's, $Y_l \cap X_k = Y_l$, i.e., $Y_l \subseteq X_k$, and thus $\pi \wedge \sigma$ is the greatest (in the refinement partial ordering) lower bound on π and σ. ■

As the blob is compatible with all DSDs, it is the minimum element in the ordering: $\mathbf{0} \preceq \pi$ for any $\pi \in DSD(V)$. Hence any two DSDs π and σ always have a common lower bound, so they always have a meet $\pi \wedge \sigma$, i.e., $DSD(V)$ is a meet-semilattice. Thus the *partial partition algebra* $DSD(V)$ could also be called the *meet-semi-lattice of DSDs* on a vector space V.

The binary DSDs $\alpha = \{A_1, A_2\}$ are the atoms of the meet-semi-lattice $DSD(V)$. A meet-semi-lattice is said to be *atomistic* if every element is the join of the atoms below it.

Proposition 3 *The meet-semi-lattice $DSD(V)$ is atomistic.*

Proof. Consider a non-blob DSD $\pi = \{V_i\}_{i \in I}$. If $\alpha = \{A_1, A_2\} \preceq \pi = \{V_i\}_{i \in I}$, then $A_k = \oplus \{V_i : V_i \subseteq A_k\}$ for $k = 1, 2$. Thus for any other atom $\alpha' = \{A'_1, A'_2\} \preceq \pi$, the join $\alpha \vee \alpha'$ is defined and $\alpha \vee \alpha' \preceq \pi$, and each nonzero subspace $A_k \cap A'_{k'}$ is the direct sum of the common direct summand V_i's. If a join of atoms had a subspace $V_{i_1} \oplus V_{i_2}$, $i_1, i_2 \in I$, then the join with the atom $\{V_{i_1}, \oplus_{i' \neq i_1, i' \in I} V_{i'}\}$ would split apart $V_{i_1} \oplus V_{i_2}$, so the join of all the atoms below π gives the $V_i \in \pi$. ■

3 Partition logics in a partial partition algebra

3.1 The partition lattice determined by a maximal DSD

Just as a partial Boolean algebra is made up of overlapping Boolean algebras, so the partial partition algebra $DSD(V)$ is made up of overlapping partition logics or algebras. There is no maximum DSD, only maximal DSDs. Each maximal element in the partial ordering is a *discrete* (or "non-degenerate") DSD $\omega = \{U_z\}_{z \in Z}$ of one-dimensional subspaces (rays) of V (so $|Z|$ is the dimension of V).[2] A *partition lattice* is determined by the set of DSDs

$$\prod(\omega) = \{\pi | \pi \preceq \omega\} = [\mathbf{0}, \omega] \subseteq DSD(V)$$

compatible with a maximal element ω and with the induced ordering and operations (which is analogous to the way in which a complete set of one-dimensional subspaces determines a Boolean algebra in a partial Boolean algebra [8, p. 193]).

[2] Choosing a basis vector for each one-dimensional U_z would give a basis for V but the focus on DSDs means working only with the rays U_z.

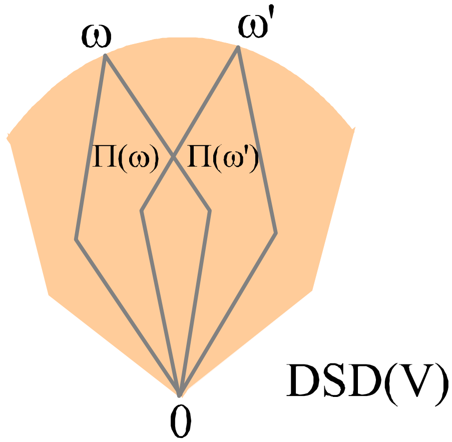

Figure 2: Partial Partition Algebra or Meet-Semi-Lattice of DSDs of V with partition logics $\prod(\omega)$ and $\prod(\omega')$.

It might be noted that much of the lattice-theoretic literature refers to the lattice of equivalence relations as the "lattice of partitions" where the partial order however "corresponds to set inclusion for the corresponding equivalence relations" [7, p. 251] so instead of being refinement, it is actually "reverse refinement" [11, p. 30]. The refinement partial order on $DSD(V)$ corresponds to set inclusion of the binary relations that are the complements of equivalence relations and are called *partition relations* [5] or *apartness relations*. In the lattice of equivalence relations, the top is the biggest (indiscrete) equivalence relation (where everything is identified) and the bottom is the smallest (discrete) equivalence relation where each element is identified only with itself–whereas the partition lattice $\prod(\omega)$ uses the opposite partial order.[3] With either partial order, the lattice is complete and relatively complemented but not distributive.

For any $\pi \in \prod(\omega)$, $\pi \preceq \omega$ so ω is (by definition) the maximum or *top* DSD in $\prod(\omega)$ and thus might be symbolized as the discrete DSD $\mathbf{1}_\omega = \omega$. Each subspace $V_i \in \pi \preceq \omega$ has $V_i = \oplus\{U_z : U_z \subseteq V_i, z \in Z\}$ so $\mathbf{1}_\omega$ absorbs what it joins and is the unit element for meets within $\prod(\omega)$:

$$\pi \vee \mathbf{1}_\omega = \mathbf{1}_\omega \text{ and } \pi \wedge \mathbf{1}_\omega = \pi.$$

All the DSDs π and σ compatible with ω, i.e., $\pi, \sigma \in \prod(\omega)$, are compatible with each other since they have a common upper bound.

Fixing a maximal DSD ω reduces much of the reasoning in $\prod(\omega)$ to reasoning about sets. If we just take $\omega = \{U_z\}_{z\in Z}$ as a set, then each DSD $\pi = \{V_i\}_{i\in I}$ in $\prod(\omega) = [\mathbf{0}, \omega]$ defines a set partition $\pi(\omega) = \{B_i\}_{i\in I}$ on ω where $B_i = \{U_z | U_z \subseteq V_i\}$ for $i \in I$ so that $V_i = \oplus B_i$. Also $|\prod(\omega)| = B(|Z|) = B(\dim(V))$, the Bell number for the dimension of V.

Indeed, given any DSD $\pi = \{V_i\}_{i\in I}$, each subspace W_j of $\sigma \in [\mathbf{0}, \pi]$ determines a subset $C_j = \{V_i : V_i \subseteq W_j\}$ so σ defines a set partition $\sigma(\pi) = \{C_j\}_{j\in J}$ on π as a set so $W_j = \oplus C_j$ for $j \in J$. Thus the lower segment $[\mathbf{0}, \pi]$ is isomorphic to the set-based partition lattice (join and meet operations) on that set π [5], and, in particular, $\prod(\omega) = [\mathbf{0}, \omega]$ is isomorphic to the lattice of set partitions on the set ω. As a partition lattice, $\prod(\omega)$, or in general $[\mathbf{0}, \pi]$, have the usual properties of partition lattices ([13]; [1]; [7, Chapter IV, section 4]). Many theorems about set partitions can then be transferred over in an appropriate form to $\prod(\omega)$.

For example, taking a *distinction* or *dit* of a DSD $\pi \in \prod(\omega)$ for $\omega = \{U_z\}_{z\in Z}$ as a pair $(U_z, U_{z'})$ in distinct subspaces, i.e., $U_z \subseteq V_i$ and $U_{z'} \subseteq V_{i'}$ for some distinct $V_i, V_{i'} \in \pi$, the common-dits property of non-blob set partitions [5, p. 106] carries over to $\prod(\omega)$.

[3]Instead of the usual DeMorgan complementation-duality relation within a Boolean algebra, there is a complementation-duality relation between the logic of partitions and the logic of equivalence relations [5].

Proposition 4 (Common dits) *Any two non-blob DSDs $\pi, \sigma \in \prod(\omega)$ have a dit in common.*

Proof. Since π is not the blob, there are U_z, $U_{z'}$ with $U_z \subseteq V_i$ and $U_{z'} \subseteq V_{i'}$ for $V_i \neq V_{i'}$. If $U_z \subseteq W_j \in \sigma$ and $U_{z'} \subseteq W_{j'} \in \sigma$ for $W_j \neq W_{j'}$ we are finished so assume $U_z \oplus U_{z'} \subseteq W_j$ for some $j \in J$. Since σ is also not the blob, there is a $U_{z''}$ contained in some $W_{j''}$ where $W_{j''} \neq W_j$. Then $U_{z''}$ cannot be in the same subspace of π as U_z and $U_{z'}$ since those two are in different subspaces of π, so either $(U_z, U_{z''})$ or $(U_{z'}, U_{z''})$ is a dit common to π and σ. ■

3.2 The implication operation on DSDs

In order to be properly called a "logic", each partition lattice $\prod(\omega)$ of DSDs has a natural implication operation inherited from the logic of set partitions so *partition logic* refers to a partition lattice plus the implication operation.

Definition 9 *For $\sigma, \pi \in \prod(\omega)$, implication is:*

$$\sigma \Rightarrow \pi = \{U_z | U_z \subseteq V_i \text{ if } \exists V_i \in \pi \text{ and } W_j \in \sigma, V_i \subseteq W_j\}$$
$$\cup \{V_i | V_i \in \pi \text{ and } \neg\exists W_j \in \sigma, V_i \subseteq W_j\}.$$

Since each $V_i = \oplus\{U_z : U_z \subseteq V_i\}$, the implication $\sigma \Rightarrow \pi$ is still a DSD in $\prod(\omega)$ in spite of some of the $V_i \in \pi$ being "discretized" into the U_z contained in it. In the implication DSD $\sigma \Rightarrow \pi$, each $V_i \in \pi$ either remains whole like a mini-blob $\mathbf{0}_{V_i} = \{V_i\}$ on the space V_i if V_i is not contained in any $W_j \in \sigma$, or it is discretized into the $U_z \subseteq V_i$ which in effect assigns a "$\mathbf{1}$" to V_i if $\exists W_j$ such that $V_i \subseteq W_j$. In other words, the implication $\sigma \Rightarrow \pi$ acts like an indicator or characteristic function assigning a $\mathbf{1}$ or $\mathbf{0}$ to each V_i depending respectively on whether or not $\exists W_j$ such that $V_i \subseteq W_j$. Thus trivially:

$$\sigma \Rightarrow \pi = \mathbf{1}_\omega \text{ iff } \sigma \preceq \pi.$$

The interpretation of the implication DSD $\sigma \Rightarrow \pi$ follows from the 'classical' case of the analogously-defined implication set partition $\sigma \Rightarrow \pi$. If σ and π are the inverse-image set partitions for random variables Y_σ and Y_π on a sample space U, then $\sigma \preceq \pi$ (i.e., $\sigma \Rightarrow \pi = \mathbf{1}_U$) means that Y_π is a "sufficient statistic" [11, p. 31] for Y_σ in the sense that the value of Y_π determines the value of Y_σ. In general, the singletons in the set partition $\sigma \Rightarrow \pi$ indicate the extent to which Y_π is sufficient for Y_σ, i.e., the singletons of $\sigma \Rightarrow \pi$ are the outcomes in the sample space where the Y_π-value determines Y_σ-value.

Translating to the quantum case, if σ and π in $\prod(\omega)$ are the eigenspace DSDs of observables $\hat{O}_\sigma$ and $\hat{O}_\pi$, and $\sigma \Rightarrow \pi = \mathbf{1}_\omega$, then not only is each π-eigenvector a σ-eigenvector, but the π-eigenvalue of a π-eigenvector determines the σ-eigenvalue as well. Restated without operators, $\sigma \Rightarrow \pi = \mathbf{1}_\omega$ means that π is *sufficient* for σ in the sense that if a given nonzero vector v_i is in $V_i \in \pi$, then $v_i \in V_i \subseteq W_j \in \sigma$ for some W_j.

More generally, the one-dimensional subspaces U_z in the DSD $\sigma \Rightarrow \pi$ give the $\hat{O}_\pi$ eigenvalues, i.e., $U_z \subseteq V_i$, that determine the $\hat{O}_\sigma$ eigenvalues. For instance if $\hat{O}_\sigma$ had degenerate eigenvalues and $\hat{O}_{\pi_1}$,...,$\hat{O}_{\pi_m}$ were observables with DSDs also in $\prod(\omega)$ (and thus compatible), then $\sigma \Rightarrow \vee_{i=1}^m \pi_i = \mathbf{1}_\omega$ implies that the eigenvalues of $\hat{O}_{\pi_1}$,...,$\hat{O}_{\pi_m}$ are sufficient to uniquely determine the eigenvalues of $\hat{O}_\sigma$. When $\vee_{i=1}^m \pi_i = \mathbf{1}_\omega$ as well, then the eigenvalues of $\hat{O}_{\pi_1}$,...,$\hat{O}_{\pi_m}$ are sufficient to uniquely label the rays $U_z \in \omega$.

3.3 Exploiting duality in quantum partition logic

In partition logic on sets ([4], [5]), the set partition operations (e.g., join, meet, and implication) on the partitions on a given universe set U can be represented as subset operations on certain subsets of $U \times U$, i.e., on certain binary relations on U. For a set partition $\pi = \{B_1, ..., B_m\}$ on U, a *distinction* or *dit of* π is an ordered pair $(u,u') \in U \times U$ of elements in distinct blocks of π. The ditset $\operatorname{dit}(\pi)$ of π is a binary relation on U (i.e., a subset of $U \times U$), and it is the complement in $U \times U$ of the equivalence relation associated with π. A partition relation on $U \times U$ is defined as the complement of an equivalence relation. The partition relations on $U \times U$ are in one-to-one correspondence with the partitions on U. Given a partition π on U, the ditset $\operatorname{dit}(\pi)$ is the corresponding partition relation, and given a partition relation, the equivalence classes in the complementary equivalence relation give the corresponding partition.

The operations on the set partitions (join, meet, and implication) have corresponding operations on partition relations. The simplest is that the join of partitions which corresponds to the union of ditsets: for set partitions π and σ on U, $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$. For the meet and implication operations, we need to use the reflexive-symmetric-transitive closure operation on subsets of $U \times U$ where for $S \subseteq U \times U$, the *RST-closure* $cl(S)$ is the equivalence relation that is the intersection of all the equivalence relations containing S.[4] Then the *interior*, $int(S)$, is the complement of the closure of the complement, i.e., $int(S) = cl(S^c)^c$(where $()^c$ is the set complement operation). Then the other operations on partition relations isomorphic to the partition operations are: $\operatorname{dit}(\pi \wedge \sigma) = int[\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)]$ and $\operatorname{dit}(\sigma \Rightarrow \pi) = int[\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)]$. The smallest partition relation is $\operatorname{dit}(\mathbf{0}_U) = \emptyset$ and the largest is $\operatorname{dit}(\mathbf{1}_U) = U \times U - \Delta$ (where $\Delta = \{(u,u) \,|u \in U\}$ is the diagonal, the smallest equivalence relation on U). Since $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$, the partial order on partition relations is just inclusion. In this manner the partition algebra $\prod(U)$ of partitions on U is represented as the algebra of the special subsets of $U \times U$ that are partition relations.

With ω fixed and playing the role of U, the above construction can be transferred to vector spaces. The operations on DSDs in $\prod(\omega)$ can be represented as subspace operations on certain subspaces of the tensor product $V \otimes V$ that are direct sums of the subspaces in the maximal DSD $\omega \otimes \omega = \{U_z \otimes U_{z'} \,|\, (U_z, U_{z'}) \in \omega \times \omega\}$ of one-dimensional subspaces on $V \otimes V$. The easiest translation uses the fact that a DSDs $\pi = \{V_i\}_{i \in I} \in \prod(\omega)$ defines a set partition $\pi(\omega) = \{B_i\}_{i \in I}$ on $\omega = \{U_z\}_{z \in Z}$ as a set where: $B_i = \{U_z | U_z \subseteq V_i\}$ and $V_i = \oplus B_i$ for $i \in I$. Then the *ditspace* defined by the DSD π is the subspace of $V \otimes V$:

$$\operatorname{Dit}(\pi) = \oplus \{U_z \otimes U_{z'} \,|\, (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega))\}.$$

Note that by the common-dits proposition, any two nonzero ditspaces, i.e., ditspaces for non-blob DSDs $\pi, \sigma \in \prod(\omega)$, have a nonzero intersection. The operations on the ditspaces are those induced by the operations on the ditsets. For $\pi, \sigma \in \prod(\omega)$,

$$\begin{aligned}
\operatorname{Dit}(\pi \vee \sigma) &= \oplus \{U_z \otimes U_{z'} \,|\, (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega) \vee \sigma(\omega))\} \\
\operatorname{Dit}(\pi \wedge \sigma) &= \oplus \{U_z \otimes U_{z'} \,|\, (U_z, U_{z'}) \in \operatorname{dit}(\pi(\omega) \wedge \sigma(\omega))\} \\
\operatorname{Dit}(\sigma \Rightarrow \pi) &= \oplus \{U_z \otimes U_{z'} \,|\, (U_z, U_{z'}) \in \operatorname{dit}(\sigma(\omega) \Rightarrow \pi(\omega))\}.
\end{aligned}$$

The smallest ditspace is $\operatorname{Dit}(\mathbf{0}) = \{0\}$ and the largest ditspace is $\operatorname{Dit}(\mathbf{1}_\omega) = \oplus \{U_z \otimes U_{z'} | U_z \neq U_{z'}\}$, and the partial ordering is inclusion. Then the partition algebra of DSDs in $\prod(\omega)$ is represented by the algebra of the ditspaces of $V \otimes V$ for DSDs in $\prod(\omega)$.

In view of the basic (category-theoretic) duality between subsets and partitions, this construction (using ditsets) to represent partition operations as subset operations (with the corresponding vector space version of the construction using ditspaces), has a dual construction to represent subset operations by partition operations. Instead of working with certain *subsets* of the *product* $U \times U$, the dual set construction works with certain *partitions* on the *coproduct* $U \uplus U$. And for the vector space version, instead of working with subspaces of the tensor product $V \otimes V$, the dual vector space construction works with DSDs on the coproduct or direct sum $V \oplus V^*$ (where V^* is a copy of V).

The set partition implication endows a rich structure on the partition algebra $\prod(U)$ of set partitions on U (always $|U| \geq 2$). For $\pi \in \prod(U)$, the π-*regular partitions* are the partitions of the

[4] NB: The closure operation is not topological since the union of two equivalence relations is not necessarily an equivalence relation.

form $\sigma \Rightarrow \pi$, which may be symbolized as $\overset{\pi}{\neg}\sigma$, for any $\sigma \in \prod(U)$. They are all in the segment $[\pi, \mathbf{1}_U]$ and they form a Boolean algebra, the *Boolean core* $\mathcal{B}_\pi$ of $[\pi, \mathbf{1}_U]$, under the partition operations of join, meet, and π-negation, where the π-negation of $\sigma \Rightarrow \pi = \overset{\pi}{\neg}\sigma$ is $(\sigma \Rightarrow \pi) \Rightarrow \pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. The dual construction uses this Boolean algebra based on partition operations.

Let's sketch the set version of the dual construction first and then go over the vector space version in more detail. Given a subset $S \subseteq U$, the *subset corelation* $\Delta(S)$ is the partition on $U \uplus U^*$ (U^* being a copy of U) whose blocks are the pairs $\{u, u^*\}$ for $u \in S$ and singletons $\{u\}$ and $\{u^*\}$ if $u \notin S$. The subset co-relations are partitions on the coproduct $U \uplus U$ defined by subsets of U, and they are dual to the partition relations $\operatorname{dit}(\pi)$ that are subsets of the product $U \times U$ defined by partitions on U. Then $\Delta(U)$ is the partition on $U \uplus U^*$ consisting of all pairs $\{u, u^*\}$ for $u \in U$, and $\Delta(\emptyset) = \mathbf{1}_{U \uplus U^*}$. The key lemma (see below) is that $\Delta(S) \Rightarrow \Delta(U) = \Delta(S^c)$ so the $\Delta(U)$-regular partitions on $U \uplus U^*$ are the same as the subset corelations. Then it can be seen (proof below) that the Boolean core $\mathcal{B}_{\Delta(U)}$ of $[\Delta(U), \mathbf{1}_{U \uplus U^*}]$ is a Boolean algebra using the partition operations of join, meet, and $\Delta(U)$-negation that is isomorphic to the powerset BA $\wp(U)$. In that manner, the Boolean subset operations on subsets of U are represented by partition operations on certain partitions on $U \uplus U^*$[4, p. 320].

For the vector space version of the dual construction, note that given a maximal DSD $\omega = \{U_z\}_{z \in Z}$, there is the associated powerset BA $\wp(\omega)$ or $\wp(Z)$ depending on whether we take ω or Z as playing the role of U. Choosing the latter option, for each $S \in \wp(Z)$, there is an associated subspace $A(S) = \oplus\{U_z | z \in S\}$ and an associated projection operator $P_S : V \to V$ to that subspace. Each atomic DSD $\{A, A'\}$ in $\prod(\omega)$ has the form $\{A(S), A(S^c)\}$ (where $S^c = Z - S$ is the complement in Z) with $V = A(Z)$ and $\{0\} = A(\emptyset)$. Thus there is an induced BA structure on the subspaces $\{A(S) | S \in \wp(Z)\}$ and on the projection operators $\{P_S | S \in \wp(Z)\}$ isomorphic to $\wp(Z)$. But how can this BA of certain subspaces of V be represented using the DSD operations of quantum partition logic?

Let $V \oplus V^*$ be the direct sum (coproduct) of V with a copy V^* of itself. Given a maximal element $\omega = \{U_z\}_{z \in Z}$ of V, then the union with the copy $\omega^* = \{U_z^*\}_{z \in Z}$ forms a maximal element $\omega \cup \omega^*$ in the refinement ordering of DSDs in $DSD(V \oplus V^*)$ so we can work in the partition logic $\prod(\omega \cup \omega^*)$.

Definition 10 *For $S \in \wp(Z)$ with the corresponding subspace $A(S)$, let $\Delta(A(S))$ or just $\Delta(S)$ be the DSD in $\prod(\omega \cup \omega^*)$, called a subspace corelation, consisting of all the one-dimensional subspaces U_z and U_z^* for $z \notin S$,i.e., $U_z \nsubseteq A(S)$, and $U_z \oplus U_z^*$ for $z \in S$, i.e., $U_z \subseteq A(S)$.*

Recall that due to the commutativity of vector addition, $U_z \oplus U_{z'}^* = U_{z'}^* \oplus U_z$. Then $\Delta(Z)$ is the DSD consisting of all the subspaces $U_z \oplus U_z^*$ for $z \in Z$ and $\Delta(\emptyset) = \mathbf{1}_{\omega \cup \omega^*} = \omega \cup \omega^*$.

Lemma 11 $\Delta(S) \Rightarrow \Delta(Z) = \Delta(S^c)$.

Proof. For any $z \in S$, we have $U_z \oplus U_z^*$ in both $\Delta(S)$ and $\Delta(Z)$, so $U_z \oplus U_z^*$ is discretized in $\Delta(S) \Rightarrow \Delta(Z)$ into U_z and U_z^* separately. For any $z \in S^c$, $U_z \oplus U_z^*$ is only in $\Delta(Z)$ so it remains whole in $\Delta(S) \Rightarrow \Delta(Z)$ so that implication DSD is $\Delta(S^c)$. ■

Thus the $\Delta(Z)$-regular DSDs $\Delta(S) \Rightarrow \Delta(Z)$ are the subspace corelations in $\prod(\omega \cup \omega^*)$. The Boolean core $\mathcal{B}_{\Delta(Z)}$ of the segment $[\Delta(Z), \omega \cup \omega^*]$ is a BA with the DSD operations of join, meet, implication, and $\Delta(Z)$-negation in $\prod(\omega \cup \omega^*)$.

Proposition 5 $\mathcal{B}_{\Delta(Z)} \cong \wp(Z)$.

Proof. The isomorphism associates $\Delta(S) \Rightarrow \Delta(Z) \in \mathcal{B}_{\Delta(Z)}$ with $S \in \wp(Z)$. For $S, T \in \wp(Z)$, the union $S \cup T$ is associated with the join $(\Delta(S) \Rightarrow \Delta(Z)) \vee (\Delta(T) \Rightarrow \Delta(Z)) = \Delta(S^c) \vee \Delta(T^c) = \Delta(S^c \cap T^c) = \Delta((S \cup T)^c) = \Delta(S \cup T) \Rightarrow \Delta(Z)$. The other Boolean operations of meet, implication, and $\Delta(Z)$-negation go in a similar manner. The null set $\emptyset \in \wp(Z)$ is associated with

$\Delta(\emptyset) \Rightarrow \Delta(Z) = \Delta(\emptyset^c) = \Delta(Z)$ which is the bottom of the BA $\mathcal{B}_{\Delta(Z)}$, and $Z \in \wp(Z)$ is associated with $\Delta(Z) \Rightarrow \Delta(Z) = \Delta(Z^c) = \Delta(\emptyset) = \mathbf{1}_{\omega\cup\omega^*}$ which is the top of $\mathcal{B}_{\Delta(Z)}$. If $S \subseteq T$ in $\wp(Z)$, then $T^c \subseteq S^c$ so $\Delta(S) \Rightarrow \Delta(Z) = \Delta(S^c) \preceq \Delta(T^c) = \Delta(T) \Rightarrow \Delta(Z)$ in the refinement ordering of $\prod(\omega \cup \omega^*)$. ■

The treatment of *DSD operations* on V as *subspace* operations on $V \otimes V$, and the dual treatment of *subspace operations* on V as *DSD operations* on $V \oplus V^*$ exhibit the dual relationship between the two quantum logics of DSDs and subspaces.

3.4 DSDs, CSCOs, and measurement

Given a self-adjoint operator $\hat{L}$ on a Hilbert space V (or diagonalizable operator on any V), the projections $\hat{P}_{\lambda_i}$ can be constructed from the DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of eigenspaces for the eigenvalues $\{\lambda_i\}_{i\in I}$, and then the operator can be reconstructed–given the eigenvalues–from the decomposition $\hat{L} = \sum_{i\in I} \lambda_i \hat{P}_{\lambda_i}$. What information about self-adjoint operators is lost by dealing only with their DSDs of eigenspaces? The information about which eigenvalues for eigenvectors are the same or different is retained by the distinct eigenspaces in the DSD. It is only the specific numerical values of the eigenvalues that is lost, and those numerical values are of little importance in QM. Any transformation into other real numbers that is one-to-one (thus avoiding "accidental" degeneracy) would do as well. Thus we can say that the essentials of the measurement process in QM can be translated into the language of the quantum logic of direct-sum decompositions.

Given a state ψ and a self-adjoint operator $\hat{L} : V \to V$ on a finite dimensional Hilbert space, the operator determines the DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of eigenspaces for the eigenvalues λ_i. The projective measurement operation uses the eigenspace DSD to decompose ψ into the unique parts given by the projections $\hat{P}_{\lambda_i}(\psi)$ into the eigenspaces V_{λ_i}, where $\hat{P}_{\lambda_i}(\psi)$ is the outcome of the projective measurement with probability $\Pr(\lambda_i|\psi) = \left\|\hat{P}_{\lambda_i}(\psi)\right\|^2 / \|\psi\|^2$.

The eigenspace DSD $\pi = \{V_{\lambda_i}\}_{i\in I}$ of $\hat{L}$ is refined by one or more maximal DSDs, i.e., $\pi = \{V_{\lambda_i}\}_{i\in I} \preceq \omega = \{U_z\}_{z\in Z}$. For each such ω, there is a set partition $\pi(\omega) = \{B_{\lambda_i}\}_{i\in I}$ on ω such that $V_{\lambda_i} = \oplus B_{\lambda_i}$. If some of the V_{λ_i} have dimension larger than one ("degeneracy"), then more measurements by commuting operators will be necessary to further decompose down to single eigenvectors. If two self-adjoint operators commute, then their eigenspace DSDs are compatible. Given another self-adjoint operator $\hat{M} : V \to V$ commuting with $\hat{L}$, its eigenspace DSD $\sigma = \{W_{\mu_j}\}_{j\in J}$ (for eigenvalues μ_j of $\hat{M}$) is compatible with $\pi = \{V_{\lambda_i}\}_{i\in I}$ and thus has a join DSD $\pi \vee \sigma$ in $DSD(V)$ which is also in $\prod(\omega)$ for one or more maximal ω each representing an orthonormal basis of simultaneous eigenvectors. The combined measurement by the two commuting operators is just the single measurement using the join DSD $\pi \vee \sigma$.

Dirac's notion of a Complete Set of Commuting Operators (CSCO) $\left\{\hat{O}_{\pi_l}\right\}_{l=1}^m$ [3] translates into the language of the quantum logic of DSDs as a *Complete Set of Compatible DSDs (CSCD)* $\{\pi_l\}_{l=1}^m$ whose join $\vee_{l=1}^m \pi_l$ *is* a maximal DSD $\omega = \mathbf{1}_\omega$ in $DSD(V)$ and thus is *the* maximum DSD $\mathbf{1}_\omega$ in $\prod(\omega)$. As noted above, the eigenvalues of the observables $\hat{O}_{\pi_l}$ can then be used to uniquely label the $U_z \in \mathbf{1}_\omega = \omega$. Without the operators to supply the eigenvalues, given a CSCD $\{\pi_l\}_{l=1}^m$ of $\prod(\omega)$, then for each $U_z \in \mathbf{1}_\omega$, there is a unique subspace $V_{z(l)} \in \pi_l$ for $l = 1, ..., m$ such that $\cap_{l=1}^m V_{z(l)} = U_z$ so each $U_z \in \mathbf{1}_\omega$ is uniquely determined or 'labelled' by a subspace $V_{z(l)} \in \pi_l$ (rather than its eigenvalue) for $l = 1, ..., m$.

In partition logic [5] on sets, a *valid formula*, i.e., a *partition tautology*, is a logical formula (using the partition operations of join, meet, and implication) so that when any partitions on the universe set U are substituted for the variables, the result is the discrete partition $\mathbf{1}_U$ on that set. Restated for DSDs, a *DSD tautology* in the partition logic $\prod(\omega)$ is any formula (in the language of join, meet, and implication) so that no matter which DSDs of $\prod(\omega)$ are substituted for the variables, the result is $\mathbf{1}_\omega$. For instance, *modus ponens* $\sigma \wedge (\sigma \Rightarrow \pi) \Rightarrow \pi$ is a DSD tautology in the partition logic $\prod(\omega)$, so for any DSDs $\pi, \sigma \in \prod(\omega)$, π is sufficient for $\sigma \wedge (\sigma \Rightarrow \pi)$. In the Boolean core $\mathcal{B}_\pi$ of $[\pi, \omega]$, the ordinary Boolean tautologies, like the law of excluded middle,

$$(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi) = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma,$$

hold for any $\pi, \sigma \in \prod(\omega)$, so they are DSD tautologies in the whole partition logic $\prod(\omega)$, where that formula is the *weak* law of excluded middle for π-negation. Thus for any DSDs $\pi, \sigma \in \prod(\omega)$, the DSDs $\sigma \Rightarrow \pi$ and $(\sigma \Rightarrow \pi) \Rightarrow \pi$ form a CSCD since their join is the discrete DSD $\mathbf{1}_\omega$. The law of excluded middle in $\mathcal{B}_\pi$ generalizes to the DSD tautology that is the disjunctive normal form decomposition of $\mathbf{1}_\omega$ for any number of variables. For instance, for any π, σ, and τ in $\prod(\omega)$, we have the DSD tautology:

$$\left(\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\tau\right) \vee \left(\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\tau\right) \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right)$$

so those four disjuncts form a CSCD. Assigning distinct real numbers to the subspaces of the disjunct DSDs defines commuting self-adjoint operators that form one of Dirac's CSCOs.

4 Final remarks

The usual subspace version of quantum logic can be viewed as the extension of the Boolean logic of subsets to the logic of subspaces of a vector space (specifically, closed subspaces of a Hilbert space). Since the notion of a set partition is the category-theoretic dual to the notion of a subset, the logic of set partitions is, in that sense, dual to the Boolean logic of subsets. Our topic has been the dual form of quantum logic that can be viewed as the extension of the logic of set partitions to the logic of direct-sum decompositions of a vector space (specifically, a Hilbert space).

The usual quantum logic of subspaces focuses on propositions, i.e., the proposition that a state vector is in a certain subspace, and the associated projection operators. Since a self-adjoint operator (observable) determines a direct-sum decomposition (losing only the specific numerical eigenvalues), the quantum logic of DSDs can be viewed as focusing on self-adjoint operators (abstracted from specific eigenvalues). The quantum logic of DSDs thus provides the natural setting to abstractly model projective measurement. As Weyl put it: "Measurement means application of a sieve or grating" [15, p. 259] (thinking of the eigenspace DSD as a "sieve"). Kolmogorov referred to the set partition given by the inverse-image of a random variable as the "experiment" ([10, p. 6], [11, p. 31]) so, in the same spirit, one might abstractly describe the vector-space partition or direct-sum decomposition of eigenspaces given by a self-adjoint operator as the "measurement."

References

[1] Birkhoff, Garrett 1948. *Lattice Theory.* New York: American Mathematical Society.

[2] Birkhoff, Garrett, and John Von Neumann. 1936. The Logic of Quantum Mechanics. *Annals of Mathematics* 37 (4): 823–43.

[3] Dirac, Paul A. M. 1958. *The Principles of Quantum Mechanics (4th ed.).* Oxford: Clarendon.

[4] Ellerman, David 2010. The Logic of Partitions: Introduction to the Dual of the Logic of Subsets. *Review of Symbolic Logic.* 3 (2 June): 287-350.

[5] Ellerman, David 2014. An Introduction of Partition Logic. *Logic Journal of the IGPL.* 22, no. 1: 94–125.

[6] Ellerman, David. 2016. Quantum Mechanics over Sets: A Pedagogical Model with Non-Commutative Finite Probability Theory as Its Quantum Probability Calculus. *Synthese Online First* (August): 34 pages. doi:DOI 10.1007/s11229-016-1175-0.

[7] Grätzer, George. 2003. *General Lattice Theory (2nd Ed.)*. Boston: Birkhäuser Verlag.

[8] Hughes, R. I. G. 1989. *The Structure and Interpretation of Quantum Mechanics*. Cambridge: Harvard University Press.

[9] Kochen, Simon, and E. P. Specker. 1967. The Problem of Hidden Variables in Quantum Mechanics. *Journal of Mathematics and Mechanics.* 17 (1): 59–87.

[10] Kolmogorov, A. N. 1956. *Foundations of the Theory of Probability.* Translated by Nathan Morrison. Second English ed. New York: Chelsea.

[11] Kung, Joseph P. S., Gian-Carlo Rota and Catherine H. Yan 2009. *Combinatorics: The Rota Way.* New York: Cambridge University Press.

[12] Lawvere, F. William, and Robert Rosebrugh. 2003. *Sets for Mathematics.* Cambridge UK: Cambridge University Press.

[13] Ore, Oystein. 1942. Theory of Equivalence Relations. *Duke Mathematical Journal,* 9: 573–627.

[14] Rota, Gian-Carlo. 2001. Twelve Problems in Probability No One Likes to Bring up. In *Algebraic Combinatorics and Computer Science: A Tribute to Gian-Carlo Rota*, edited by Henry Crapo and Domenico Senato, 57–93. Milano: Springer.

[15] Weyl, Hermann 1949. *Philosophy of Mathematics and Natural Science.* Princeton NJ: Princeton University Press.