Hindawi
Complexity
Volume 2018, Article ID 8714694, 6 pages
https://doi.org/10.1155/2018/8714694

*Research Article*

# Weak Solutions of a Coupled System of Urysohn-Stieltjes Functional (Delayed) Integral Equations

**A. M. A. El-Sayed [1] and M. M. A. Al-Fadel [2]**

[1] *Faculty of Science, Alexandria University, Alexandria, Egypt*
[2] *Faculty of Science, Omar Al-Mukhtar University, Al Bayda, Libya*

Correspondence should be addressed to A. M. A. El-Sayed; amasayed@gmail.com

Received 16 April 2018; Accepted 31 July 2018; Published 4 September 2018

Academic Editor: Lucia Valentina Gambuzza



We study the existence of weak solutions for the coupled system of functional integral equations of Urysohn-Stieltjes type in the reflexive Banach space $E$. As an application, the coupled system of Hammerstien-Stieltjes functional integral equations is also studied.

## 1. Introduction and Preliminaries

Consider the Urysohn-Stieltjes integral equation:

$$x(t) = p(t) + \int_0^1 f(t, s, x(s))\, d_s g(t, s), \quad t \in I = [0, 1], \tag{1}$$

where $g : I \times I \to R$ is nondecreasing in the second argument (see [1]) and the symbol $d_s$ indicates the integration with respect to $s$. Equations of type (1) and some of their generalizations were considered in the paper (see [2]). We remark here that when $E = R$, these types of equations have been studied by Banaś (see [1–5]) and also by some other authors, for example (see [6–10] and for coupled systems [11]). For the solutions in a reflexive Banach space (see [12, 13]).

In this paper, let $\psi_i(t) \le t$ be continuous functions on $[0, 1]$. We generalize these results to study the existence of weak solutions $(x, y) \in C[I, E] \times C[I, E]$ for the coupled system of Urysohn-Stieltjes functional (delayed) integral equations:

$$\begin{aligned} x(t) &= a_1(t) + \int_0^1 f_1(t, s, y(\psi_1(s)))\, d_s g_1(t, s), \quad t \in I, \\ y(t) &= a_2(t) + \int_0^1 f_2(t, s, x(\psi_2(s)))\, d_s g_2(t, s), \quad t \in I, \end{aligned} \tag{2}$$

in the reflexive Banach space $E$ under the weak-weak continuity assumption imposed on $f_i : I \times I \times E \to E, i = 1, 2$.

As an application, we study the existence of weak solutions $x, y \in C[I, E]$ for the coupled system of Hammerstien-Stieltjes functional integral equations:

$$\begin{aligned} x(t) &= a_1(t) + \int_0^1 k_1(t, s) h_1(s, y(\psi_1(s)))\, d_s g_1(t, s), \quad t \in I, \\ y(t) &= a_2(t) + \int_0^1 k_2(t, s) h_2(s, x(\psi_2(s)))\, d_s g_2(t, s), \quad t \in I. \end{aligned} \tag{3}$$

For the definition, background, and properties of the Stieltjes integral, we refer to Banaś [1]. However, the coupled system of integral equations has been studied, recently, by some authors (see [14, 15]).

Throughout this paper, otherwise stated, $E$ denotes a reflexive Banach space with norm $\|\cdot\|$ and dual $E^*$. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

Let $X = C[I, E] \times C[I, E] = \{u(t) = (x(t), y(t)) : x \in C[I, E], y \in C[I, E], t \in I\}$ be a Banach space with the norm defined as

$$\|(x, y)\|_X = \|x\|_{C[I,E]} + \|y\|_{C[I,E]}, \quad \forall (x, y) \in X. \tag{4}$$

Now, we shall present some auxiliary results that will be needed in this work. Let $E$ be a Banach space (need not be reflexive) and let $x : [a, b] \to E$; then

(1) $x(.)$ is said to be weakly continuous (measurable) on $[a, b]$ if for every $\phi \in E^*, \phi(x(.))$ is continuous (measurable) on $[a, b]$.

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

If $x$ is weakly continuous on $I$, then $x$ is strongly measurable and hence weakly measurable (see [16, 17]). It is evident that in reflexive Banach spaces, if $x$ is weakly continuous function on $[a, b]$, then $x$ is weakly Riemann integrable (see [17]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of the coupled system (2) in the space $C[I, E] \times C[I, E]$.

*Definition 1.* Let $f : I \times E \to E$. Then $f(t, u)$ is said to be weakly-weakly continuous at $(t_0, u_0)$; if given $\epsilon > 0, \phi \in E^*$; there exists $\delta > 0$ and a weakly open set $U$ containing $u_0$ such that

$$|\phi(f(t, u) - f(t_0, u_0))| < \epsilon, \tag{5}$$

whenever

$$\begin{aligned} &|t - t_0| < \delta, \\ &u \in U. \end{aligned} \tag{6}$$

Now, we have the following fixed point theorem, due to O'Regan, in the Banach space (see [18]).

**Theorem 1.** *Let $E$ be a Banach space, let $Q$ be a nonempty, bounded, closed, and convex subset of $C[I, E]$, and let $F : Q \to Q$ be weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in $E$ for each $t \in I$. Then $F$ has a fixed point in set $Q$.*

Recall [19] that a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology. Thus, putting in mind that $TQ(t)$ is a bounded subset of $E$, then the condition $TQ$ $(t)$ is weakly relatively compact and is automatically satisfied. Accordingly, we immediately have the following theorem.

**Theorem 2.** *Let $E$ be a reflexive Banach space with $Q$ a nonempty, closed, convex, and equicontinuous subset of $C[I, E]$. Assume that $T : Q \to Q$ is weakly sequentially continuous. Then $T$ has a fixed point in $Q$.*

**Proposition 1.** *In the reflexive Banach space, the subset is weakly relatively compact if and only if it is bounded in the norm topology.*

**Proposition 2.** *Let $E$ be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|y\| = \varphi(y)$.*

## 2. Main Results

In this section, we present our main result by proving the existence of weak solutions for the coupled system of Urysohn-Stieltjes integral equation (2) in the reflexive Banach space. Let us first state the following assumptions:

*Assumption 1.* $a_i \in C[I, E], i = 1, 2$.

*Assumption 2.* $\psi_i : I \to I$ is a continuous function such that $\psi_i(t) \leq t$.

*Assumption 3.* $f_i : I \times I \times E \to E, i = 1, 2$ satisfy the following conditions:

(1) $f_i(\ldots, s, x(\psi_i(s)))$ are continuous functions on $I$ for every $x \in E$.

(2) $f_i(t, \ldots)$ are weakly-weakly continuous functions, $\forall t \in I$.

(3) There exist two continuous functions $m_i(t), m_i : I \times I \to I$ and two positive constants $b_i$, such that $\|f_i(t, s, x)\| \leq m_i(t, s) + b_i\|x\|, i = 1, 2$.

*Assumption 4.* The functions $t \to g_i(t, 1)$ and $t \to g_i(t, 0)$ are continuous on $I$.

*Assumption 5.* For all $t_1, t_2 \in I$ such that $t_1 < t_2$, the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ are nondecreasing on $I$.

*Assumption 6.* $g_i(0, s) = 0$, for any $s \in I$.

*Remark 1.* Observe that Assumptions 5 and 6 imply that the function $s \to g(t, s)$ is nondecreasing on the interval $I$, for any fixed $t \in I$ (Remark 1 in [5]). Indeed, putting $t_2 = t$, $t_1 = 0$ in Assumption 5 and keeping in mind Assumption 6, we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on $I$.

*Definition 2.* By a weak solution for the coupled system (2), we mean the pair of functions $(x, y) \in C[I, E] \times C[I, E]$ such that

$$\begin{aligned} \varphi(x(t)) &= \varphi(a_1(t)) + \int_0^1 \varphi(f_1(t, s, y(\psi_1(s)))) d_s g_1(t, s), \quad t \in I, \\ \varphi(y(t)) &= \varphi(a_2(t)) + \int_0^1 \varphi(f_2(t, s, x(\psi_2(s)))) d_s g_2(t, s), \quad t \in I, \end{aligned} \tag{7}$$

for all $\varphi \in E^*$.

Now, let

$$\mu = \max\left\{\sup_t |g_i(t,1)| + \sup_t |g_i(t,0)|, \quad t \in I\right\}, i = 1, 2,$$
$$M = \max\{m_i(t,s) \colon t, \quad s \in I\}. \tag{8}$$

Then we have the following theorem.

**Theorem 3.** *Under the Assumptions 1–6, the coupled system of Urysohn-Stieltjes integral equation (2) has at least one weak solution $(x, y) \in C[I, E] \times C[I, E]$.*

*Proof 1.* Define an operator $A$ by

$$A(x, y) = (A_1 y, A_2 y), \tag{9}$$

where

$$A_1 y(t) = a_1(t) + \int_0^1 f_1(t, s, y(\psi_1(s))) d_s g_1(t, s), \quad t \in I,$$
$$A_2 x(t) = a_2(t) + \int_0^1 f_2(t, s, x(\psi_2(s))) d_s g_2(t, s), \quad t \in I. \tag{10}$$

For every $x_i \in C[I, E]$, $f_i(\dots, s, x(\psi_i(s)))$ is continuous on $I$, and $f_i(t, \dots, x(\psi_i(.)))$ are weakly continuous on $I$; then $\varphi(f_i(t, \dots, x(\psi_i(.))))$ are continuous for every $\varphi \in E^*$. Hence, in view of bounded variational of $g_i$ it follows, $f_i(t, s, x(\psi_i(s)))$ is weakly Riemann-Stieltjes integrable on $I$ with respect to $s \to g_i(t, s)$. Thus, $A_i$ make sense.

Now, we can prove that $A_1 : C[I, E] \to C[I, E]$ and for $\epsilon > 0$, $t_1, t_2 \in I$, $t_1 < t_2$, and $t_2 - t_1 < \epsilon$ (without loss of generality, assume that $A_1 y(t_2) - A_1 y(t_1) \neq 0$), and there exists $\varphi \in E^*$, such that

$$\begin{aligned}
&\|A_1 y(t_2) - A_1 y(t_1)\| \\
&\quad \le |\varphi(a_1(t_2) - a_1(t_1))| + \left|\int_0^1 \varphi(f_1(t_2, s, y(\psi_1(s)))) d_s g_1(t_2, s) \right.\\
&\qquad \left. - \int_0^1 \varphi(f_1(t_1, s, y(\psi_1(s)))) d_s g_1(t_1, s)\right| \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \left|\int_0^1 \varphi(f_1(t_2, s, y(\psi_1(s)))) d_s g_1(t_2, s)\right. \\
&\qquad \left. - \int_0^1 \varphi(f_1(t_1, s, y(\psi_1(s)))) d_s g_1(t_2, s)\right| \\
&\qquad + \left|\int_0^1 \varphi(f_1(t_1, s, y(\psi_1(s)))) d_s g_1(t_2, s)\right. \\
&\qquad \left. - \int_0^1 \varphi(f_1(t_1, s, y(\psi_1(s)))) d_s g_1(t_1, s)\right| \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \int_0^1 |\varphi(f_1(t_2, s, y(\psi_1(s))) \\
&\qquad - f_1(t_1, s, y(\psi_1(s))))| d_s\left(\bigvee_{z=0}^{s} g_1(t_2, z)\right) \\
&\qquad + \int_0^1 |\varphi(f_1(t_1, s, y(\psi_1(s))))| d_s\left(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)]\right) \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| \int_0^1 d_s g_1(t_2, s) \\
&\qquad + \int_0^1 \|f_1(t_1, s, y)\| d_s[g_1(t_2, s) - g_1(t_1, s)] \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| \\
&\qquad \cdot [g_1(t_2, 1) - g_1(t_2, 0)] + \int_0^1 m_1(t, s) d_s[g_1(t_2, s) - g_1(t_1, s)] \\
&\qquad + \int_0^1 b_1 \|y\| d_s[g_1(t_2, s) - g_1(t_1, s)] \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| [g_1(t_2, 1) \\
&\qquad - g_1(t_2, 0)] + M \int_0^1 d_s[g(t_2, s) - g(t_1, s)] \\
&\qquad + b_1 r_1 \int_0^1 d_s[g_1(t_2, s) - g_1(t_1, s)] \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| [g_1(t_2, 1) \\
&\qquad - g_1(t_2, 0)] + (M + b_1 r_1) \\
&\qquad \cdot [(g(t_2, 1) - g(t_1, 1)) - (g(t_2, 0) - g(t_1, 0))] \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| \\
&\qquad \cdot [g_1(t_2, 1) - g_1(t_2, 0)] + (M + b_1 r_1)[|g_1(t_2, 1) - g_1(t_1, 1)| \\
&\qquad + |g_1(t_2, 0) - g_1(t_1, 0)|].
\end{aligned} \tag{11}$$

Hence,

$$\begin{aligned}
\|A_1 y(t_2) - A_1 y(t_1)\| \le{}& \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) \\
& - f_1(t_1, s, y)\| [g_1(t_2, 1) - g_1(t_2, 0)] \\
& + (M + b_1 r_1)[|g_1(t_2, 1) - g_1(t_1, 1)| \\
& + |g_1(t_2, 0) - g_1(t_1, 0)|].
\end{aligned} \tag{12}$$

Similarly, we can show that

$$\begin{aligned}
\|A_2 x(t_2) - A_2 x(t_1)\| \le{}& \|a_2(t_2) - a_2(t_1)\| + \|f_2(t_2, s, y) \\
& - f_2(t_1, s, y)\| [g_2(t_2, 1) - g_2(t_2, 0)] \\
& + (M + b_2 r_2)[|g_2(t_2, 1) - g_2(t_1, 1)| \\
& + |g_2(t_2, 0) - g_2(t_1, 0)|].
\end{aligned} \tag{13}$$

Now,

$$\begin{aligned}
A(x, y)(t_2) - A(x, y)(t_1) &= (A_1 y(t_2), A_2 x(t_2)) - (A_1 y(t_1), A_2 x(t_1)) \\
&= ((A_1 y(t_2) - A_1 y(t_1)), (A_2 x(t_2) - A_2 x(t_1))),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\|A(x,y)(t_2) - A(x,y)(t_1)\| \\
&\quad \le \|A_1 y(t_2) - A_1 y(t_1)\| + \|A_2 x(t_2) - A_2 x(t_1)\| \\
&\quad \le \|a_1(t_2) - a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| \\
&\qquad \cdot [g_1(t_2, 1) - g_1(t_2, 0)] + (M + b_1 r_1) \\
&\qquad \cdot [|g_1(t_2, 1) - g_1(t_1, 1)| + |g_1(t_2, 0) - g_1(t_1, 0)|] \\
&\qquad + \|a_2(t_2) - a_2(t_1)\| + \|f_2(t_2, s, y) - f_2(t_1, s, y)\| \\
&\qquad \cdot [g_2(t_2, 1) - g_2(t_2, 0)] + (M + b_2 r_2) \\
&\qquad \cdot [|g_2(t_2, 1) - g_2(t_1, 1)| + |g_2(t_2, 0) - g_2(t_1, 0)|].
\end{aligned} \tag{15}$$

Then from the continuity of the functions, $g_i$ is required in Assumption 4; we deduce that $A$ maps $X$ into $X$.

Define the sets $Q_1$ and $Q_2$ by

$$Q_1 = \{y \in C[I, E] : \|y\| \le r_1\}, \quad r_1 = \frac{\|a_1\| + M\mu}{1 - b_1 \mu}, \tag{16}$$

and

$$Q_2 = \{x \in C[I, E] : \|x\| \le r_2\}, \quad r_2 = \frac{\|a_2\| + M\mu}{1 - b_2 \mu}. \tag{17}$$

Now, define the closed, convex, bounded, equicontinuous set $Q$ by

$$\begin{aligned}
Q = \{&u = (x, y) \in X : \|u\| \le r_1 + r_2, \|u(t_2) - u(t_1)\| \le \|a_1(t_2) \\
&- a_1(t_1)\| + \|f_1(t_2, s, y) - f_1(t_1, s, y)\| [g_1(t_2, 1) \\
&- g_1(t_2, 0)] + (M + b_1 r_1)[|g_1(t_2, 1) - g_1(t_1, 1)| \\
&+ |g_1(t_2, 0) - g_1(t_1, 0)|] + \|a_2(t_2) - a_2(t_1)\| \\
&+ \|f_2(t_2, s, y) - f_2(t_1, s, y)\| [g_2(t_2, 1) - g_2(t_2, 0)] \\
&+ (M + b_2 r_2)[|g_2(t_2, 1) - g_2(t_1, 1)| + |g_2(t_2, 0) - g_2(t_1, 0)|]\}.
\end{aligned} \tag{18}$$

Now, let $y \in Q_1$ and $x \in Q_2$; without loss of generality, we may assume $A_1 y \ne 0, A_2 x(t) \ne 0, t \in I$. By Proposition 2, we have

$$\begin{aligned}
\|A_1 y(t)\| &= \varphi(A_1 y(t)) \le |\varphi(a_1(t))| \\
&\quad + \left|\varphi\left(\int_0^1 f_1(t, s, y(\psi_1(s))) d_s g_1(t, s)\right)\right| \\
&\le \|a_1\| + \int_0^1 |\varphi(f_1(t, s, y(\psi_1(s))))| d_s \left(\bigvee_{z=0}^{s} g_1(t, z)\right) \\
&\le \|a_1\| + \int_0^1 \|f_1(t, s, y(\psi_1(s)))\| d_s \left(\bigvee_{z=0}^{s} g_1(t, z)\right) \\
&\le \|a_1\| + \int_0^1 m_1(t, s) d_s g_1(t, s) + \int_0^1 b_1 \|y\| d_s g_1(t, s) \\
&\le \|a_1\| + M \int_0^1 d_s g_1(t, s) + b_1 r_1 \int_0^1 d_s g_1(t, s) \\
&\le \|a_1\| + M[g_1(t, 1) - g_1(t, 0)] \\
&\quad + b_1 r_1 [g_1(t, 1) - g_1(t, 0)] \\
&\le \|a_1\| + (M + b_1 r_1)\left[\sup_{t \in I} |g_1(t, 1)| + \sup_{t \in I} |g_1(t, 0)|\right] \\
&\le \|a_1\| + (M + b_1 r_1)\mu.
\end{aligned} \tag{19}$$

Then

$$\|A_1 y(t)\| \le \|a_1\| + (M + b_1 r_1)\mu = r_1. \tag{20}$$

Similarly, we can prove that

$$\|A_2 x(t)\| \le \|a_2\| + (M + b_2 r_2)\mu = r_2. \tag{21}$$

Therefore, for any $u \in Q$,

$$\begin{aligned}
\|Au(t)\| &= \|A(x, y)(t)\| = \|(A_1 y(t), A_2 x(t))\| \\
&\le \|A_1 y(t)\| + \|A_2 x(t)\| \\
&\le \|a_1\| + (M + b_1 r_1)\mu + \|a_2\| + (M + b_2 r_2)\mu \\
&= r_1 + r_2,
\end{aligned} \tag{22}$$

that is, $\forall u \in Q \Rightarrow Au \in Q \Rightarrow AQ \subset Q$. Thus, $A : Q \to Q$.

Note that $Q$ is a nonempty, uniformly bounded, and strongly equicontinuous subset of $X$, by the uniform boundedness of $AQ$; according to Proposition 1, $AQ$ is relatively weakly compact.

It remains to prove that $A$ is weakly sequentially continuous.

Let $\{y_n(t)\}$ and $\{x_n(t)\}$ be sequences in $C[I, E]$ weakly convergent to $y(t)$ and $x(t)$, respectively $(\forall t \in I)$, since $f_1(t, s, \dots)$ and $f_2(t, s, \dots)$ are weakly continuous. Then $f_1(t, s, y_n(\psi_1(s)))$ and $f_2(t, s, x_n(\psi_2(s)))$ converge weakly to $f_1(t, s, y(\psi_1(s)))$ and $f_2(t, s, x(\psi_2(s)))$, respectively. Furthermore, $(\forall \varphi \in E^*)\varphi(f_1(t, s, y_n(\psi_1(s))))$ and $\varphi(f_2(t, s, x_n(\psi_2(s))))$ converge strongly to $\varphi(f_1(t, s, y(\psi_1(s))))$ and $\varphi(f_2(t, s, x(\psi_2(s))))$, respectively. Applying the Lebesgue-dominated convergence theorem, then we get

$$\begin{aligned}
&\varphi\left(\int_0^1 f_1(t, s, y_n(\psi_1(s))) d_s g_1(t, s)\right) \\
&\quad = \int_0^1 \varphi(f_1(t, s, y_n(\psi_1(s)))) d_s g_1(t, s) \longrightarrow \int_0^1 \varphi(f_1(t, s, y(\psi_1(s)))) d_s g_1(t, s), \\
&\qquad \forall \varphi \in E^*, t \in I,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\varphi\left(\int_0^1 f_2(t, s, x_n(\psi_2(s))) d_s g_2(t, s)\right) \\
&\quad = \int_0^1 \varphi(f_2(t, s, x_n(\psi_2(s)))) d_s g_2(t, s) \longrightarrow \int_0^1 \varphi(f_2(t, s, x(\psi_2(s)))) d_s g_2(t, s), \\
&\qquad \forall \varphi \in E^*, t \in I.
\end{aligned} \tag{24}$$

Thus, $A$ is weakly sequentially continuous on $Q$.

Since all conditions of Theorem 1 are satisfied, then the operator $A$ has at least one fixed point $(x, y) = u \in Q$ and the coupled system of Urysohn-Stieltjes integral equation (2) has at least one weak solution.

## 3. Hammerstein-Stieltjes Coupled System

This section, as an application, deals with the existence of weak continuous solution for the coupled system of Hammerstein-Stieltjes functional integral equation (3). Consider the following assumption:

*Assumption 7.* Let $h_i : I \times E \to E$ and $k_i : I \times I \to R_+$ assume that $h_i, k_i$ satisfy the following assumptions:

(1) $h_i(s, x(\psi_i(s)))$ are weakly-weakly continuous functions.

(2) There exist continuous functions $m_i^*(t)$ and constants $b_i > 0$ such that

$$\|h_i(t, x)\| \le m_i^*(t) + b_i\|x\|, \tag{25}$$

for $t, s \in I, x \in E$. Moreover, we put $M^* = \max\{m_i^*(t) : t \in I\}, M^* > 0$.

(3) $k_i(t, s)$ is a continuous function such that $K = \sup_t |k_i(t, s)|$, where $K$ is a positive constant.

*Definition 3.* By a weak solution for the coupled system (3), we mean the pair of functions $(x, y) \in C[I, E] \times C[I, E]$ such that

$$\begin{aligned} \varphi(x(t)) &= \varphi(a_1(t)) + \int_0^1 k_1(t, s)\varphi(h_1(s, y(\psi_1(s))))d_s g_1(t, s), \quad t \in I, \\ \varphi(y(t)) &= \varphi(a_2(t)) + \int_0^1 k_2(t, s)\varphi(h_2(s, x(\psi_2(s))))d_s g_2(t, s), \quad t \in I, \end{aligned} \tag{26}$$

for all $\varphi \in E^*$.

Being new for the existence of weak solutions of (3), we have the following theorem.

**Theorem 4:** *Let Assumptions 1, 2, 4, 5, and 7 be satisfied. Then the coupled system of Hammerstien-Stieltjes functional integral equation (3) has at least one weak solution* $(x, y) \in X$.

*Proof 2.* Let

$$f_i(t, s, x(s)) = k_i(t, s)h_i(s, x(\psi_i(s))). \tag{27}$$

Then from Assumption 7, we find that the assumptions of Theorem 3 are satisfied and the result follows.

For example, consider the functions $g_i : I \times I \to R$ defined by the formula

$$\begin{aligned} g_1(t, s) &= \begin{cases} t ln \dfrac{t+s}{t}, & \text{for } t \in (0, 1], s \in I, \\ 0, & \text{for } t = 0, s \in I, \end{cases} \\ g_2(t, s) &= t(t + s - 1), \quad t \in I. \end{aligned} \tag{28}$$

It can be easily seen that the functions $g_1(t, s)$ and $g_2(t, s)$ satisfy assumptions (iv)-(vi) given in Theorem given in Theorem 3. In this case, the coupled system of Urysohn-Stieltjes integral equation (2) has the following form:

$$x(t) = a_1(t) + \int_0^1 \frac{t}{t+s} f_1(t, s, y(\psi_1(s)))ds, \quad t \in I, \tag{29}$$

$$y(t) = a_2(t) + \int_0^1 t f_2(t, s, x(\psi_i(s)))ds, \quad t \in I. \tag{30}$$

Therefore, the coupled system (29) has at least one weak solution $u = (x, y) \in X$, if the functions $a_i$, $\psi_i$, and $f_i$ satisfy Assumptions 1–3.

## Data Availability

There is no data availability.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

## References


[1] J. Banaś, "Some properties of Urysohn-Stieltjes integral operators," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 1, pp. 79–88, 1998.

[2] J. Banaś and J. C. Mena, "Some properties of nonlinear Volterra-Stieltjes integral operators," *Computers & Mathematics with Applications*, vol. 49, no. 9-10, pp. 1565–1573, 2005.

[3] J. Banaś and J. Dronka, "Integral operators of Volterra-Stieltjes type, their properties and applications," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1321–1331, 2000.

[4] J. Banaś and K. Sadarangani, "Solvability of Volterra-Stieltjes operator-integral equations and their applications," *Computers & Mathematics with Applications*, vol. 41, no. 12, pp. 1535–1544, 2001.

[5] J. Banaś and D. O'Regan, "Volterra-Stieltjes integral operators," *Mathematical and Computer Modelling*, vol. 41, no. 2-3, pp. 335–344, 2005.

[6] C. W. Bitzer, "Stieltjes-Volterra integral equations," *Illinois Journal of Mathematics*, vol. 14, no. 3, pp. 434–451, 1970.

[7] S. Z. Chen, Q. G. Huang, and L. H. Erbe, "Bounded and zero-convergent solutions of a class of Stieltjes integro-differential equations," *Proceedings of the American Mathematical Society*, vol. 113, no. 4, pp. 999–1008, 1991.

[8] J. S. Macnerney, "Integral equations and semigroups," *Illinois Journal of Mathematics*, vol. 7, pp. 148–173, 1963.

[9] A. B. Mingarelli, *Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, vol. 989 of Lecture Notes in Mathematics, Springer, 1983.

[10] I. P. Natanson, *Theory of Functions of a Real Variable*, Ungar, New York, 1960.

[11] A. M. A. El-Sayed and M. M. A. Al-Fadel, "Existence of solution for a coupled system of Urysohn-Stieltjes functional integral equations," *Tbilisi Mathematical Journal*, vol. 11, no. 1, pp. 117–125, 2018.

[12] A. M. El-Sayed and M. M. Al-Fadel, "On the weak solutions of the Urysohn-Stieltjes functional integral equations," *International Journal of Applied Mathematics*, vol. 31, no. 2, pp. 263–277, 2018.

[13] A. M. A. El-Sayed and N. F. M. El-haddad, "Weak solutions of a functional integral equation in reflexive Banach space," *Nonlinear Analysis and Differential Equations*, vol. 4, no. 7, pp. 305–313, 2016.

[14] A. M. A. El-Sayed and H. H. G. Hashem, "Existence results for coupled systems of quadratic integral equations of fractional orders," *Optimization Letters*, vol. 7, no. 6, pp. 1251–1260, 2013.

[15] H. H. G. Hashem, "On successive approximation method for coupled systems of Chandrasekhar quadratic integral equations," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 1, pp. 108–112, 2015.

[16] N. Dunford and J. T. Schwartz, *Linear Operators*, Interscience, Wiley, New York, 1958.

[17] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, Providence, RI, USA, 1957.

[18] D. O'regan, "Fixed-point theory for weakly sequentially continuous mappings," *Mathematical and Computer Modelling*, vol. 27, no. 5, pp. 1–14, 1998.

[19] A. Szep, "Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces," *Studia Scientiarum Mathematicarum Hungarica*, vol. 6, pp. 197–203, 1971.