Models of Set Theory with Definable Ordinals*

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July 17, 2003

Abstract

A DO model (here also referred to a *Paris* model) is a model $\mathfrak{M}$ of set theory all of whose ordinals are first order definable in $\mathfrak{M}$. Jeffrey Paris (1973) initiated the study of DO models and showed that (1) every consistent extension T of ZF has a DO model, and (2) for complete extensions T, T has a *unique* DO model up to isomorphism iff T proves $\mathbf{V} = \mathbf{OD}$. Here we provide a comprehensive treatment of Paris models. Our results include the following:

1. *If T is a consistent completion of $ZF+\mathbf{V} \neq \mathbf{OD}$ then T has continuum-many countable nonisomorphic Paris models.*
2. *Every countable model of ZFC has a Paris generic extension.*
3. *If there is an uncountable well-founded model of ZFC, then for every infinite cardinal κ there is a Paris model of ZF of cardinality κ which has a nontrivial automorphism.*
4. *For a model $\mathfrak{M} \models ZF$, $\mathfrak{M}$ is a prime model $\Rightarrow \mathfrak{M}$ is a Paris model and satisfies $AC \Rightarrow \mathfrak{M}$ is a minimal model. Moreover, Neither implication reverses assuming* $\mathrm{Con}(ZF)$.

*2000 *Mathematics Subject Classification*: 03C62, 03C50; Secondary 03H99.

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1. INTRODUCTION

Some first order theories such as Peano Arithmetic PA, or the theory RCF of real closed fields, have the luxury of possessing a *pointwise definable model*, i.e., a model in which every element is definable by a first order formula. In the context of set theory, a distinguished example of a pointwise definable model is the Shepherdson-Cohen minimal model [C, III.6] which served as the ground model for Cohen's original forcing constructions. It is natural to wonder "When does an extension of Zermelo-Fraenkel set theory ZF possess a pointwise definable model?" The concept of *ordinal definability*[1] provides a satisfying answer to this question. Set theorists now know that the assertion "every set is definable from an ordinal parameter" can be expressed for models of ZF by a single first order sentence $\mathbf{V} = \mathbf{OD}$, a sentence which explains the existence of pointwise definable models by the following result [My-Sc].

Theorem 1.1. *Suppose T is a completion of ZF. The following are equivalent.*

(i) *T has a pointwise definable model.*

(ii) *There is a parameter-free formula $\varphi(x, y)$ such that T proves "φ well-orders the universe".*

(iii) *T proves* $\mathbf{V} = \mathbf{OD}$.

The point of departure for the work presented here is a fruitful generalization of the notion of pointwise definability introduced by Jeffrey Paris. Paris dubbed a model of set theory whose *ordinals* are first order definable a *DO model* (for **d**efinable **o**rdinals) and established the following elegant existence and uniqueness results via Henkin constructions.

Theorem 1.2. (Paris [P]) *Every consistent extension of ZF has a DO model. Moreover, a completion T of ZF has a unique DO model up to isomorphism iff* $T \vdash \mathbf{V} = \mathbf{OD}$.

In this paper we acknowledge Paris' pioneering work by referring to DO models as *Paris* models. Our aim is to provide a broad view of the fascinating territory of Paris models. After dealing with preliminaries in Part 1, we use the omitting types method of Model Theory, and the forcing technique of Set Theory, to construct a

[1] The concept of ordinal definability was independently discovered by various logicians, including Gödel, Post, Myhill-Scott [My-Sc], Takeuti, and Vopěnka-Balcar.

large variety of Paris models in Part 2. For instance, in Theorem 2.4 we fine-tune the second clause of Theorem 1.2 by showing that every completion of $ZF + \mathbf{V} \neq \mathbf{OD}$ has *continuum-many* countable nonisomorphic Paris models. In Theorem 2.8 we prove that every countable model of ZFC has a *generic* extension to a Paris model of ZFC. Aside from its intrinsic interest, Theorem 2.8 helps to answer a question of Mycielski by showing that under reasonable conditions the ordinal heights of *well-founded* Paris models are unbounded in ω_1 (see Corollary 2.9). Section 2.2 focuses on another question of Mycielski on the existence of Paris *elementary submodels*, while Section 2.3 concentrates on *uncountable* Paris models. For instance in Theorem 2.19 we show that if ZFC has an uncountable model, then there exist Paris models of ZF of *arbitrarily large cardinality.* Part 3 focuses on the *structural properties* of Paris models of ZF. Section 3.1 deals with well-foundedness, Section 3.2 addresses the existence of a pair of indiscernibles in Paris models, Section 3.3 discusses the *rigidity* of Paris models, and Section 3.4 probes the relation between Paris models, *prime* models, and *minimal* models of ZF.

I owe a debt of gratitude to Jan Mycielski, whose thought provoking questions prompted me to initiate this project, and whose moral support assisted me in bringing it to fruition. I am also indebted to Robert Solovay and Amir Togha for helpful comments on earlier drafts of this paper.

Preliminaries

Our notation is standard and follows the canonical texts of Jech [J] and Kunen [Ku], however we wish to review a list of definitions and results involving models of set theory which are central to this paper.

Suppose $\mathfrak{M} = (M, E)$ is a model of ZF, where $E = \in^{\mathfrak{M}}$.

- $\mathbf{Ord}^{\mathfrak{M}}$ denotes the ordered set of "ordinals" of $\mathfrak{M}$.
- $\mathfrak{M}$ is a *Paris model* if for every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$ there is a first order formula $\varphi(x)$ in the language $\{\in\}$ with only one free variable such that α is the unique element of $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi(\alpha)$.
- If $\mathfrak{M} \subseteq \mathfrak{N} = (N, F)$, then $\mathfrak{N}$ *end extends* $\mathfrak{M}$, written $\mathfrak{M} \subseteq_e \mathfrak{N}$, if $\mathfrak{N}$ fixes every element of $\mathfrak{M}$, i.e., for every $c \in M$, $\{x \in M : xEc\} = \{x \in N : xFc\}$.
- $\mathbf{OD}$ is the class of *ordinal definable* sets, and the submodel $\mathbf{HOD}^{\mathfrak{M}}$ of $\mathfrak{M}$ consisting of *hereditarily* ordinal definable elements of $\mathfrak{M}$ satisfies ZFC, but $\mathbf{HOD}^{\mathfrak{M}}$ is not necessarily a model of $\mathbf{V} = \mathbf{OD}$.

- L_α is the α-th level of Gödel's class of constructible sets $\mathbf{L}$. If $\mathfrak{N}$ is also model of ZF with $\mathfrak{M} \subseteq_e \mathfrak{N}$ and $\mathbf{Ord}^{\mathfrak{M}} = \mathbf{Ord}^{\mathfrak{N}}$, then $\mathbf{L}^{\mathfrak{M}} = \mathbf{L}^{\mathfrak{N}}$. Moreover, within ZF, $\mathbf{V} = \mathbf{L} \Rightarrow \mathbf{V} = \mathbf{OD} \Rightarrow AC$ (the axiom of choice), but neither implication in general reverses.

- A partial order $\mathbb{P}$ is said to be *weakly homogeneous* if for any two conditions p and q in $\mathbb{P}$, there is an automorphism f of $\mathbb{P}$ such that $f(p)$ and q are compatible. If $\mathbb{P}$ is a weakly homogeneous partial order in the sense of $\mathfrak{M}$, then (1) the first order theory of $\mathbb{P}$-generic extensions is independent of the choice of the $\mathbb{P}$-generic filter, (2) $\mathbf{V} \neq \mathbf{OD}$ holds in $\mathbb{P}$-generic extensions of $\mathfrak{M}$.

- A model is *standard* if it is isomorphic to some $(M, \in)$, where M is a transitive set. *When M is standard we shall use M instead of $(M, \in)$ or $\mathfrak{M}$.* By Mostowski's collapsing lemma, each well-founded model of *extensionality* is isomorphic to a *unique* standard model. $\mathfrak{M}$ is said to be *ω-standard* if $\mathfrak{M}$ has no nonstandard integers.

- ρ is the usual ordinal valued rank function on sets and V_α is the α-th level of the von Neumann cumulative hierarchy, so $V_\alpha = \{x : \rho(x) < \alpha\}$.

- Thanks to the Scott-Solovay *Boolean-valued* formulation of the method of forcing ([J, Ch.3], [Ku, Ch.VII, sec.7]), one can use forcing over countable models $\mathfrak{M}$ of ZF to build generic extensions even when $\mathfrak{M}$ is a *nonstandard* model.

- Our blanket assumption throughout the paper is that *ZF is a consistent first order theory.* However, we do not make a similar blanket assumption regarding stronger consistency assertions such as "ZF has a well-founded model", or "ZFC has an uncountable well-founded model". Also, a *completion* of a consistent theory T_0 refers to a *consistent* completion of T_0.

2. CONSTRUCTING PARIS MODELS

2.1. A Wealth of Paris Models

The following fundamental result of Paris was originally proved via a Henkin construction but, as shown below, it can also be conveniently obtained by a straightforward application of the Henkin-Orey omitting types theorem [CK, Theorem 2.2.15].

Theorem 2.1. (Paris [P]) *Every completion of ZF has a Paris model.*

Proof: Suppose T is a completion of ZF. Let $FORM(x)$ denote the set of first order formulas in the language of set theory $\{\in\}$ with one free variable x, and for each formula $\varphi(x) \in FORM(x)$ define

$$\overline{\varphi}(x) := \varphi(x) \to \exists y(\varphi(y) \wedge y \neq x).$$

Next consider the 1-type $\Sigma(x) := \{x \in \mathbf{Ord}\} \cup \{\overline{\varphi}(x) : \varphi(x) \in FORM(x)\}$. Note that Paris models are precisely the models omitting $\Sigma(x)$. We wish to use the omitting types theorem [CK, Theorems 2.2.9] to show that T has a model omitting $\Sigma(x)$, i.e., we wish to show that T *locally omits* $\Sigma(x)$. To see this, suppose $\psi(x) \in FORM(x)$ is a formula such that:

(1) $T \vdash \exists x \psi(x)$, and

(2) $T \vdash \psi(x) \to x \in \mathbf{Ord}$.

Consider the formula $\theta(x) = \psi(x) \wedge \forall y \in x \neg\psi(y)$. Clearly $T \vdash \theta(x) \to \neg\overline{\psi}(x)$. Moreover, since one can prove in ZF that every nonempty subset of ordinals has a least member, $T \vdash \exists x \theta(x)$. Therefore T locally omits $\Sigma(x)$, and our proof is complete. □

Theorem 2.1 allows us to obtain a straightforward proof of Myhill-Scott's extension [My-Sc] of the Montague-Vaught reflection theorem ([J, Theorem 29], [Ku, Theorem 7.5]):

Corollary 2.2. (The Extended Reflection Principle) *Given any unary formula* $\varphi(x)$, *we have* $ZF \vdash ERP_\varphi$, *where*

$$ERP_\varphi := [\forall \alpha \in \mathbf{Ord}\ \exists \gamma > \alpha \text{ such that } \alpha \text{ is first order definable in } (V_\gamma, \in), \text{ and } \forall x \in V_\gamma (\varphi(x) \leftrightarrow \varphi^{V_\gamma}(x))].$$

Proof: Thanks to Theorem 2.1, it is sufficient to observe that ERP_φ holds in every Paris model of ZF by the reflection theorem. □

It is natural to wonder about the *number* of nonisomorphic Paris models of ZF. The Gödel-Rosser incompleteness theorem implies that ZF has continuum-many completions. Coupled with Theorem 2.1, this implies that there are continuum-many nonisomorphic countable Paris models of ZF. This does not address the question of the number of countable Paris models of a *complete theory* T extending ZF. As we shall see, the answer to this question depends *only* on whether T contains the axiom $\mathbf{V} = \mathbf{OD}$ or not. If T includes the sentence $\mathbf{V} = \mathbf{OD}$, then there is exactly one Paris model of T up to isomorphism, thanks to the following observation:

Proposition 2.3. (Folklore) *If* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are elementarily equivalent Paris models of* ZF *then* $\mathbf{OD}^{\mathfrak{M}} \cong \mathbf{OD}^{\mathfrak{N}}$.

Proof: The desired isomorphism is given by the map $\tau^{\mathfrak{M}} \mapsto \tau^{N}$, where $\tau^{\mathfrak{M}}$ and $\tau^{\mathfrak{N}}$ respectively are the denotations of the ordinal definable term τ in $\mathfrak{M}$ and $\mathfrak{N}$. □

In contrast, in Theorem 2.4 below we fine-tune the second clause of Theorem 1.2 by showing that if a completion T of ZF includes the axiom $\mathbf{V} \neq \mathbf{OD}$, then the number of countable nonisomorphic Paris models of T is the maximum possible.

Theorem 2.4. *Every completion of* $ZF+\ \mathbf{V} \neq \mathbf{OD}$ *has* $2^{\aleph_0}$ *nonisomorphic countable Paris models.*

Proof: We shall first establish the existence of at least $\aleph_1$ nonisomorphic countable Paris models of a completion T of $ZF+\ \mathbf{V} \neq \mathbf{OD}$. To do so, use Theorem 2.1 to get hold of some countable Paris model $\mathfrak{M}$ of T to serve as the stepping stone to the construction of a sequence of nonisomorphic countable Paris models $(\mathfrak{M}_\gamma : \gamma < \omega_1)$ of T with $\mathfrak{M}_0 := \mathfrak{M}$. To construct $\mathfrak{M}_1$ we first observe:

$$\mathfrak{M}_0 \models\ \exists s_0 \subseteq \mathbf{Ord} \text{ such that } s_0 \notin \mathbf{OD}.$$

This is an immediate consequence of (1) and (2) below.

(1) By a result of Vopěnka and Balcar [J, Lemma 15.1], if M and N are transitive models of ZF of the same ordinal height which have the same *sets* of ordinals, and $\mathfrak{N}$ satisfies AC, then $\mathfrak{M} = \mathfrak{N}$ (we invoke this result *within* $\mathfrak{M}_0$, with $M = \mathbf{V}^{\mathfrak{M}_0}$ and $N = \mathbf{HOD}^{\mathfrak{M}_0}$).

(2) $\mathbf{HOD}^{\mathfrak{M}_0} \neq \mathfrak{M}_0$, and AC holds in $\mathbf{HOD}^{\mathfrak{M}_0}$.

Since $\mathfrak{M}_0$ is Paris, for each $\alpha \in \mathbf{Ord}^{\mathfrak{M}_0}$ there is a first order formula $\varphi_\alpha(x)$ which defines α in $\mathfrak{M}_0$. This allows us to construct the 1-types $\Pi(x)$ and $\Lambda_0(x)$ in the language $\{\in\}$, *with no parameters*, by

$$\Pi(x) := \{x \in \mathbf{Ord}\} \cup \{\neg\varphi_\alpha(x) : \alpha \in \mathbf{Ord}^{\mathfrak{M}_0}\},$$
$$\Lambda_0(x) := \{x \subseteq \mathbf{Ord}\} \cup \{\forall v(\varphi_\alpha(v) \rightarrow v \in x) : (\alpha \in s_0)^{\mathfrak{M}_0}\} \cup \{\forall v(\varphi_\alpha(v) \rightarrow v \notin x) : (\alpha \notin s_0)^{\mathfrak{M}_0}\}.$$

Clearly $\Pi(x)$ is locally omitted by T since $\mathfrak{M}_0$ omits $\Pi(x)$. To see that $\Lambda_0(x)$ is also locally omitted by T, suppose that for some formula $\psi(x)$, we have

$$T \vdash \exists x \psi(x),$$

and

$$\alpha \in s_0 \text{ iff } T \vdash \psi(x) \rightarrow \varphi_\alpha(x) \; .$$

This implies that the term "the unique x satisfying $\psi(x)$" serves as a definition of s_0 in $\mathfrak{M}_0$, contradicting the choice of $s_0 \notin \mathbf{OD}^{\mathfrak{M}_0}$. By the extended omitting types Theorem [CK, Theorem 2.2.15] there exists a countable model $\mathfrak{M}_1$ of T which omits both types Π and Λ_0. Moreover, since $\mathfrak{M}_0$ realizes Λ_0, $\mathfrak{M}_0$ is not isomorphic to $\mathfrak{M}_1$.

We can now easily repeat this process for each $\delta < \omega_1$. Suppose we have constructed a sequence of nonisomorphic countable models $(\mathfrak{M}_\gamma : \gamma < \delta)$ for some $\delta < \omega_1$ such that each $\mathfrak{M}_\gamma$ is a countable model of T omitting $\Pi(x)$. For each $\gamma < \delta$, choose $s_\gamma \subseteq \mathbf{Ord}^{\mathfrak{M}_\gamma}$ such that

$$\mathfrak{M}_\gamma \models \; s_\gamma \notin \mathbf{OD},$$

and let $\Lambda_\gamma(x)$ denote the type defined by

$$\Lambda_\gamma(x) := \{x \subseteq \mathbf{Ord}\} \cup \{\forall v(\varphi_\alpha(v) \to v \in x) : (\alpha \in s_\gamma)^{\mathfrak{M}_\gamma}\} \cup \{\forall v(\varphi_\alpha(v) \to v \notin x) : (\alpha \notin s_\gamma)^{\mathfrak{M}_\gamma}\}.$$

Since each of the types $\Lambda_\gamma(x)$ is locally omitted by T, by the extended omitting types theorem there is a model $\mathfrak{M}_\delta$ of T omitting the type $\Pi(x)$ as well as the *countable* set of types $\{\Lambda_\gamma(x) : \gamma < \delta\}$. Therefore $\mathfrak{M}_\delta$ is not isomorphic to $\mathfrak{M}_\gamma$ for $\gamma < \delta$ since for each $\gamma < \delta$, $\mathfrak{M}_\gamma$ realizes $\Lambda_\gamma(x)$ but $\mathfrak{M}_\delta$ omits $\Lambda_\gamma(x)$.

The above argument not only shows that there are at least $\aleph_1$ nonisomorphic Paris models of T, but also shows that the set $S_1(\overline{T})$ of complete first order 1-types which are realized in models of the the $L_{\omega_1,\omega}$ theory

$$\overline{T} := T \cup \{\forall x \in \mathbf{Ord} \bigvee_{\alpha \in \mathbf{Ord}^{\mathfrak{M}}} \varphi_\alpha(x)\}$$

is uncountable. But it is well-known that $S_1(\overline{T})$ can be viewed as an *analytic* subset of the Cantor set [Mo, Theorem 2.3], hence by the perfect set property of analytic sets [J-2, Theorem 94(c)],

$$\left|S_1(\overline{T})\right| = 2^{\aleph_0}.$$

So T has $2^{\aleph_0}$ nonisomorphic countable Paris models. $\square$

Remark 2.4.1. One can also prove Theorem 2.4 using a "purely set theoretic" argument at the cost of substantially increasing the complexity of the proof. Here is an outline. Suppose, first, that T is a completion of the theory ZF + "$\mathbf{V} \neq \mathbf{OD}$" + $\exists a \mathbf{V} = \mathbf{L}[a]$. By a classical result of Vopěnka [J, Theorem 65] there is a partial order $\mathbb{P}$ in $\mathbf{HOD}$ with the property that the universe $\mathbf{V}$ is a $\mathbb{P}$-generic extension of $\mathbf{HOD}$. Moreover, as noted by Grigorieff [G, Sec.5, Theorem 1] the partial order $\mathbb{P}$ can be arranged to be *weakly homogeneous*. Relativizing these to some Paris model $\mathfrak{M}$ of T, we conclude that $\mathfrak{M}$ is a generic extension of $\mathfrak{N} = \mathbf{HOD}^{\mathfrak{M}}$ via a weakly homogeneous notion of forcing. Now let $\{G_\alpha : \alpha < 2^{\aleph_0}\}$ denote a family of mutually generic $\mathbb{P}$-filters over $\mathfrak{M}$. The weak homogeneity of $\mathbb{P}$ can be used to show that for each $\alpha < 2^{\aleph_0}$, $\mathfrak{N}[G_\alpha]$ is a Paris model of T. Moreover, if $\alpha < \beta < 2^{\aleph_0}$, then by mutual genericity,

$$\mathfrak{N}[G_\alpha \times G_\beta] \models \mathfrak{N}[G_\alpha] \cap \mathfrak{N}[G_\beta] = \mathfrak{N}.$$

This implies that $\mathfrak{N}[G_\alpha]$ and $\mathfrak{N}[G_\beta]$ are not isomorphic since they realize different types (recall the types $\Lambda_\gamma(x)$ in the proof of Theorem 2.4.1). For theories containing $\forall a \mathbf{V} \neq \mathbf{L}(a)$ one has to work harder by constructing a weakly homogeneous *class notion of forcing* **P** in **HOD** such that **V** is a **P**-generic extension of **HOD**.

Remark 2.4.2. Let ψ denote the axiom "there is a real number which is not ordinal definable". The proof of Theorem 2.4 can be easily adapted to show that if T is a complete extension of $ZF+\psi$, then there are continuum many Paris models of T whose "real lines", viewed as *fields*, are pairwise nonisomorphic. Mycielski (private communication) considers this result to be of significant foundational interest since it demonstrates that even in the presence of an "ideal" complete set theory T, and a commitment to only definable ordinals, the structure of the real line remains dynamic and highly elusive.

Remark 2.4.3. Paris [P] observed that if T has a well-founded model then *all* Paris models of T must be well-founded (see Theorem 3.1). Coupled with Mostowski's collapsing lemma and Theorem 2.4, this shows if a complete extension T of $ZF + \mathbf{V} \neq \mathbf{OD}$ has at least one well-founded model, then the collection

$$\mathcal{M}_T := \{M : M \text{ is a countable transitive Paris model of } T\}$$

has size continuum. The following examples illustrate the fact that the structure of the partially ordered set $(\mathcal{M}_T, \subseteq)$ is heavily dependent on the theory T. In Example A, $\mathcal{M}_T$ turns out to be a trivial partial order, but in Example B, $\mathcal{M}_T$ contains a *directed* subfamily of size continuum.

Example A. Suppose L_α is a Paris model of $ZF + \mathbf{V} = \mathbf{L}$ and x is Sacks generic over L_α. For any distinct transitive Paris models M and M' of $Th(L_\alpha[x])$, $M \cap M' = L_\alpha$. This is a direct consequence of the fact that the Sacks reals are minimal, i.e., if x is Sacks generic over L_α and $y \in L_\alpha[x]$ then either $y \in L_\alpha$ or $x \in L_\alpha[y]$ [J, Theorem 64]. Note that $L_\alpha[x]$ satisfies $\mathbf{V} \neq \mathbf{OD}$ since the Sacks partial order is weakly homogeneous.

Example B. Suppose L_α is a Paris model of $ZF + \mathbf{V} = \mathbf{L}$, x is Cohen generic over L_α, and $T = Th(L_\alpha[x])$. $\mathcal{M}_T$ has a directed subfamily $\mathcal{F}$ of size continuum. To build $\mathcal{F}$, first build a family $\{G_\gamma : \gamma < 2^{\aleph_0}\}$ of mutually $\mathbb{P}$-generic filters over L_α (see, e.g., [Fr]), where $\mathbb{P}$ is the Cohen partial order. $\mathbb{P}$ is weakly homogenous, and moreover, forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{P}^n$ for any $n < \omega$. Therefore, all models in $\mathcal{F}$ satisfy the same first order theory T. To show that $\mathcal{F}$ is directed under inclusion is now easy, because if $M = L_\alpha[\prod_{\gamma \in S_1} G_\gamma]$, and

$M' = L_\alpha[\prod_{\gamma \in S_2} G_\gamma]$ are in $\mathcal{F}$ (where S_1 and S_2 are finite subsets of $2^{\aleph_0}$) then the desired model $\overline{M}$ of T containing $M \cup M'$ is $\overline{M} = L_\alpha[\prod_{\gamma \in S_1 \cup S_2} G_\gamma]$.

2.2. Paris Expansions and Extensions

We now turn to the construction of Paris expansions and Paris extensions of an arbitrary countable model $\mathfrak{M}$ of ZF. The following concept plays a key role in this section.

- Given a model $\mathfrak{M} = (M, E)$ of ZF, $X \subseteq M$ is said to be a *class* of $\mathfrak{M}$ if $(\mathfrak{M}, X) \models ZF(\widetilde{X})$, where $ZF(\widetilde{X})$ is the natural extension of ZF in the expanded language $\{\in, \widetilde{X}\}$ in which the unary predicate $\widetilde{X}$ is allowed to occur in the replacement schema.

Note that every parametrically first order definable subset of $\mathfrak{M}$ is a class of $\mathfrak{M}$, but the converse fails in general. For example, if $\mathfrak{M}$ is a countable model of ZF then $\mathfrak{M}$ has many undefinable classes (see, e.g., [E-2]). It is easy to see that every countable model $\mathfrak{A} = (A, \cdots)$ of *any* theory has an expansion $(\mathfrak{A}, R)$, where R is a binary relation on A, in which every element of A is definable without parameters (choose R to be a well-ordering of A of order type ω). If $\mathfrak{A}$ is a model of set theory then of course $(\mathfrak{A}, R)$ fails to satisfy $ZF(\widetilde{R})$ if R is a well-ordering of A of order type ω. However, by a forcing argument of Simpson [Si], every countable model $\mathfrak{M} \models ZFC$ has a class X such that $(\mathfrak{M}, X)$ is pointwise definable. Note that if $\mathfrak{M} \models ZF$ has a class X such that $(\mathfrak{M}, X)$ is pointwise definable then the axiom of choice holds in $\mathfrak{M}$ since $(\mathfrak{M}, X) \models \mathbf{V} = \mathbf{OD}(\widetilde{X})$. One can modify Simpson's argument to prove the following more general result.

Theorem 2.7. ([E-3, Lemma 4.1.1]) *Every countable model $\mathfrak{M}$ of ZF has a class $X \subseteq \mathbf{Ord}^{\mathfrak{M}}$ such that the expanded model $(\mathfrak{M}, X)$ is a Paris model.*

Aside from its intrinsic interest, Theorem 2.7 also plays a role in the proof of Theorem 2.8 .

Theorem 2.8. *Every countable model of ZFC has a generic Paris extension satisfying ZFC.*

Proof: Suppose $\mathfrak{M}$ is a countable model of ZFC. The desired extension $\mathfrak{N}$ is constructed in four stages:

- *Stage 1*: Use Theorem 2.7 to build a class $X \subseteq \mathbf{Ord}^{\mathfrak{M}}$ such that $(\mathfrak{M}, X)$ is Paris.

- *Stage* 2: Build an expansion $(\mathfrak{M}, X, F) \models ZF(\widetilde{X}, \widetilde{F})$, where F is a global choice function. This is possible by a well-known forcing construction [Fe]. Using F we can define an ordering $<_F$ in $(\mathfrak{M}, F)$ such that every proper initial segment of $<_F$ is a set, and

$$(\mathfrak{M}, F) \models \text{“} <_F \text{ well-orders the universe”}.$$

Therefore there is a definable well-ordering $\{s_\alpha : \alpha \in \mathbf{Ord}^{\mathfrak{M}}\}$ in $(\mathfrak{M}, F)$ of all *subsets of ordinals* in the sense of $\mathfrak{M}$. Let $Y = \{(\alpha, \beta) : \alpha \in s_\beta\}$ and use the canonical pairing function $\Gamma : (\mathbf{Ord}^{\mathfrak{M}})^2 \to \mathbf{Ord}^{\mathfrak{M}}$ to code Y by $Z = \Gamma(Y) \subseteq \mathbf{Ord}^{\mathfrak{M}}$. Since in the presence of the axiom of choice every set is constructible from a set of ordinals [J, Exercise 15.4], $\mathfrak{M}$ is constructible from Z, in the following sense:

$$(\mathfrak{M}, Z) \models \mathbf{V} = \mathbf{L}[Z] = \bigcup_{\alpha \in \mathbf{Ord}} \mathbf{L}[Z \cap \alpha].$$

- *Stage* 3: Build a generic extension $(\mathfrak{M}[G], X, Z)$ of $(\mathfrak{M}, X, Z)$ in which GCH holds. This is possible by a theorem of Jensen [J, Exercise 20.8] which shows that for an appropriate partial order $\mathbf{P}$, which is a definable class of $\mathfrak{M}$,

$$\mathfrak{M} \models 1_{\mathbf{P}} \Vdash GCH.$$

Also, since G is generic over $(\mathfrak{M}[G], X, Z)$, X and Z remain classes of $\mathfrak{M}[G]$.

- *Stage* 4. Use the coding function Γ again to code X and Z into a single class $S \subseteq \mathbf{Ord}^{\mathfrak{M}}$. By Easton's Theorem [J, Theorem 46] there is a definable class partial order $\mathbf{Q}$ of $(\mathfrak{M}[G], S)$ such that

$$(\mathfrak{M}[G], S) \models 1_{\mathbf{Q}} \Vdash (2^{\aleph_{\alpha+1}} = \aleph_{\alpha+2}) \leftrightarrow \alpha \in S).$$

The desired Paris model is $\mathfrak{N} = \mathfrak{M}[G][H]$, where H is $\mathbf{Q}$-generic over $(\mathfrak{M}[H], X, Y)$. Clearly S is definable in $\mathfrak{N}$ as $\{\alpha \in \mathbf{Ord} : 2^{\aleph_{\alpha+1}} = \aleph_{\alpha+2}\}$. Therefore, by de-coding S via Γ^{-1}, both classes X and Z are also definable in $\mathfrak{N}$. Since $\mathfrak{M}$ is definable in $\mathfrak{N}$ as $\mathbf{L}[Z]$, and $(\mathfrak{M}, X)$ is Paris, $\mathfrak{N}$ is also Paris. □

Remark 2.8.1. In stage 4 of the proof above of Theorem 2.8, instead of using the partial order $\mathbf{Q}$ one can use McAloon's coding technique [Mc] to construct a definable class partial order $\mathbf{Q}^*$ of $(\mathfrak{M}[G], S)$ such that for every $\mathbf{Q}^*$-generic $H, \mathfrak{M}[G][H]$ not only codes the class Z, but also satisfies $\mathbf{V} = \mathbf{OD}$. *Therefore every model of ZFC can be generically extended to a pointwise definable model of ZFC.*

Mycielski (private communication) has asked whether the ordinal heights $\mathbf{Ord}^M$ of well-founded Paris models M of ZFC are unbounded in ω_1. Corollary 2.9 answers this question in the positive.

Corollary 2.9. *Assume ZFC has an uncountable well-founded model. The collection*

$$\{\alpha < \omega_1 : \alpha = \mathbf{Ord}^M \textit{for some transitive Paris model } M \models ZFC\}$$

is unbounded in ω_1.

Proof: Suppose N is an uncountable well-founded model of ZFC. By a Mostowski collapse, we may assume that N is transitive. Since the axiom of choice holds within N, $\omega_1 \subseteq \mathbf{Ord}^N$ (see Proposition 2.14). A routine Lowenheim-Skolem argument shows that $\{\mathbf{Ord}^M : M \prec N$ and M is countable$\}$ is unbounded in ω_1. We are now done, thanks to Theorem 2.8. □

2.3. Mycielski's Question

This section is concerned with another question of Mycielski:

- **Question** (Mycielski [My-1]) *Does every model $\mathfrak{M} \models ZF$ have a Paris elementary submodel?*

John Steel has recently answered Mycielski's question in the negative by constructing the following counterexample, which is presented below with his permission.

Theorem 2.10. (Steel). *If there is some $\alpha < \omega_1$ such that $L_\alpha \models ZF$ and some countable ordinal of L_α is undefinable in L_α, then there is a transitive model of ZFC which does not have an elementary Paris submodel.*

Proof: Suppose $L_\alpha \models ZF$ and some $\theta \in \aleph_1^{L_\alpha}$ is undefinable in L_α. Let $\mathbb{P} \in L_\alpha$ denote the usual notion of forcing for collapsing $\aleph_1$ to a countable ordinal, and let G be $\mathbb{P}$-generic over L_α. Since $\mathbb{P}$ is weakly homogeneous [J, Theorem 63], and is parameter-free definable in L_α, a standard truth-and-forcing argument shows that the definable ordinals of $L_\alpha[G]$ and L_α coincide.

We claim that $L_\alpha[G]$ has no Paris elementary submodel. To verify this claim, it suffices to show that if $M \prec L_\alpha[G]$ then $\aleph_1^{L_\alpha} \subseteq M$. To see this, we observe that $\beta = \aleph_1^{L_\alpha}$ is countable in the sense of $L_\gamma[G]$. Therefore, by elementarity, β is also countable in the sense of M. So there is some $f \in M$ such that f maps ω *onto* β. Hence, for each $n \in \omega$, $f(n) \in M$, and therefore $\beta \subseteq M$. $\square$

Remark 2.10.1. We can adapt the proof of Proposition 2.10 to produce a model of ZFC which has no Paris elementary submodel by just assuming Con(ZF). To do so, we need to replace the ground model L_α in the proof of Proposition 2.10 with a countable model $\mathfrak{M} \models ZF + \mathbf{V} = \mathbf{L}$ satisfying the following two conditions:

(1) Some $\theta \in \aleph_1^{\mathfrak{M}}$ is undefinable in $\mathfrak{M}$.
(2) Every "natural number" $c \in \omega^{\mathfrak{M}}$ is definable in $\mathfrak{M}$.

To build such an $\mathfrak{M}$, start with a countable Paris model $\mathfrak{M}_0$ of $ZF + \mathbf{V} = \mathbf{L}$, and use the regularity of $\aleph_1^{\mathfrak{M}_0}$ in $\mathfrak{M}_0$ to construct a nonprincipal ultrafilter $\mathcal{U}$ on $(\mathcal{P}(\aleph_1))^{\mathfrak{M}_0}$ such that $\mathcal{U}$ is $\aleph_1^{\mathfrak{M}_0}$-complete, i.e., $\mathcal{U} \cap p \neq \emptyset$ for all partitions p in $\mathfrak{M}_0$ of $\aleph_1^{\mathfrak{M}_0}$ which are countable in the sense of $\mathfrak{M}_0$. Then choose $\mathfrak{M}$ to be the ultrapower $\left(\mathbf{V}^{\aleph_1}\right)^{\mathfrak{M}_0}/\mathcal{U}$. $\mathfrak{M}$ satisfies condition (1) since $\mathfrak{M}$ introduces a new (hence undefinable) countable ordinal to $\mathfrak{M}_0$, but since $\mathcal{U}$ is $\aleph_1^{\mathfrak{M}_0}$-complete, $\mathfrak{M}$ does not introduce any new members to $\omega^{\mathfrak{M}_0}$, so $\mathfrak{M}$ also satisfies condition (2) since $\mathfrak{M}$ is an elementary extension of $\mathfrak{M}_0$.

The remaining results of this section deal with partial *positive* answers to Mycielski's question. Myhill and Scott [My-Sc] showed that if $\mathfrak{M} \models \mathbf{V} = \mathbf{OD}$, then the definable elements of $\mathfrak{M}$ form a Paris elementary submodel of $\mathfrak{M}$. The following result generalizes this fact:

Theorem 2.11. *Every model $\mathfrak{M}$ of ZF has a submodel $\mathfrak{N} \preceq \mathbf{HOD}^{\mathfrak{M}}$ whose ordinals are precisely the definable ordinals of $\mathfrak{M}$.*

Proof: Suppose $\mathfrak{M}$ is a model of ZF and let $D(\mathfrak{M})$ denote the set of *parameter-free definable* elements of $\mathfrak{M}$. Note that even though $\mathbf{OD}^{\mathfrak{M}}$ is a definable subset of $\mathfrak{M}$, $D(\mathfrak{M})$ itself is not a definable subset of $\mathfrak{M}$ in general, e.g., if $\mathfrak{M}$ has an undefinable ordinal, then $D(\mathfrak{M})$ cannot be a definable subset of $\mathfrak{M}$ (otherwise the least undefinable ordinal would have to be definable). Our desired submodel is $\mathfrak{N} = D(\mathfrak{M}) \cap \mathbf{HOD}^{\mathfrak{M}}$. To see that $\mathfrak{N} \preceq \mathbf{HOD}^{\mathfrak{M}}$, we shall use Tarski's test. Assume that for some $a \in N$

$$\mathbf{HOD}^{\mathfrak{M}} \models \exists x \varphi(x, a).$$

Since $a = \delta^{\mathfrak{M}}$ for some term δ, we can introduce a term α_0 by

$$\alpha_0 := \text{“the first ordinal } \alpha \text{ such that } \exists x \in V_\alpha \varphi(x, \delta)\text{”}.$$

Now we can employ the definable well-ordering $\lhd$ of the class of ordinal definable sets $\mathbf{OD}$ to introduce another term τ as:

$$\tau := \text{“the } \lhd \text{ - first member of } \{x \in V_{\alpha_0} : \varphi(x, a)\}\text{”}.$$

Clearly

$$\mathbf{HOD}^{\mathfrak{M}} \models \varphi(\tau^M, a).$$

□

Theorem 2.12. *If M is a well-founded model of ZF whose definable ordinals are not cofinal in $\mathbf{Ord}^M$, then M contains a submodel M^* as a transitive element which is a Paris model of $Th(M)$.*

Proof: Recall that within ZF, Σ_n-truth is Σ_n-definable for $n \geq 1$ [J, Section 14]. Therefore, by the reflection theorem, for each natural number n,

$$ZF \vdash \exists \alpha \in \mathbf{Ord}\ (V_\alpha \prec_n \mathbf{V}).$$

Here “$V_\alpha \prec_n \mathbf{V}$” is a single first order sentence of set theory expressing the statement “for all Σ_n formulae $\varphi(v_0, \cdots, v_k)$, and all $a_1, \cdots, a_k$ in $V_\alpha, \varphi(a_0, \cdots, a_k)$ holds iff its relativization $\varphi^{V_\alpha}(a_0, \cdots, a_k)$ to V_α holds”. So, given a well-founded model M of ZF, we can choose $\alpha_n \in \mathbf{Ord}^M$ to denote the *first* ordinal α in M such that

$$(V_\alpha \prec_n \mathbf{V})^M.$$

Since M is well-founded and the definable ordinals of M are not cofinal in $\mathbf{Ord}^M$, there is an ordinal $\delta \in \mathbf{Ord}^M$ such that $\delta = \sup_{n \in \omega} \alpha_n$. Note that $V_\delta^M \prec M$ since

$$V_{\alpha_1} \prec_1 V_{\alpha_2} \prec_2 \cdots \prec_{n-1} V_{\alpha_n} \prec_n V_{\alpha_{n+1}} \prec_{n+1} \cdots \ .$$

At this point we move *within* M and invoke Theorem 2.1 to get hold of a Paris model $\overline{M}$ of $Th(V_\delta)$ (note that the ordinals of $\overline{M}$ are also definable in the real world by routine absoluteness considerations). It is easy to see that $\overline{M}$ is also well-founded in the eyes of M, and therefore it is also well-founded in the real world. To see this, suppose on the contrary that

$$M \models \exists f : \omega \to \overline{M}\ \forall n \in \omega((f(n+1) \in f(n))^{\overline{M}},$$

then in M we can choose terms τ_n with $f(n) = \tau_n^{\overline{M}}$. This leads to the absurd conclusion

$$\forall n \in \omega\ M \models (\tau_{n+1} \in \tau_n)^{V_\delta}.$$

Hence the desired transitive M^* is the Mostowski collapse of $\overline{M}$. $\square$

Corollary 2.13. *If M is a well-founded model of ZF with $cf(\mathbf{Ord}^M) > \aleph_0$, then there is a Paris model $M^* \in M$ of Th(M).*

2.4. Uncountable Paris Models

This section focuses on the enigmatic class of *uncountable* Paris models which exhibit a dramatic tension between a gargantuan universe and a miniature class of ordinals. Clearly if $\mathfrak{M}$ is a Paris model then $\mathbf{Ord}^{\mathfrak{M}}$ is countable, but as we shall see, without at least some weak form of the axiom of choice this gives *no information whatsoever* on the cardinality of $\mathfrak{M}$ itself. We begin with a basic result which implies that Paris models of ZFC must be countable.

Proposition 2.14. *If $\mathfrak{M} \models ZFC$ then $\left|\mathbf{Ord}^{\mathfrak{M}}\right| = |M|$.*

Proof: This follows from (1), (2), and (3) below.

(1) $M = \bigcup_{\alpha \in \mathbf{Ord}^{\mathfrak{M}}} V_\alpha^{\mathfrak{M}}$ for any $\mathfrak{M}$ model of ZF.

(2) By basic cardinal arithmetic, $\left|\bigcup_{\alpha \in \mathbf{Ord}^{\mathfrak{M}}} V_\alpha^{\mathfrak{M}}\right| = \left|\mathbf{Ord}^{\mathfrak{M}}\right| . \sup_{\alpha \in \mathbf{Ord}^{\mathfrak{M}}} \left|V_\alpha^{\mathfrak{M}}\right|$.

(3) Since the axiom of choice holds in $\mathfrak{M}$, $\left|V_\alpha^{\mathfrak{M}}\right| \leq \left|\mathbf{Ord}^{\mathfrak{M}}\right|$ for every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$.$\square$

One does not need the full force of the axiom of choice in Proposition 2.14. To see this we need to look at a weaker form of the axiom of choice, introduced by Keisler.

Definition 2.15. (Keisler, [Ke, Ch. 25])

- $x \in \mathbf{WO}^0$ iff x is *well-orderable.*
- $x \in \mathbf{WO}^{\alpha+1}$ iff x can be written as a *well-ordered union* of sets in $\mathbf{WO}^{\alpha}$.
- For limit α, $\mathbf{WO}^{\alpha} = \bigcup\limits_{\beta<\alpha} \mathbf{WO}^{\beta}$.
- The axiom $\mathbf{V} = \mathbf{WO}^{\infty}$ denotes the assertion $\forall x \exists \alpha \in \mathbf{Ord}(x \in \mathbf{WO}^{\alpha})$.

Remark 2.15.1.

(a) As shown by Keisler [Ke, Ch. 25], the theory ZF + "every countable non-empty family has a choice function" proves "no infinite set which is Dedekind finite is in $\mathbf{WO}^{\infty}$". Coupled with the work of Halpern and Lévy [HL] on the consistency of the existence of infinite but Dedekind finite subsets of the real line, this implies that the theory $ZF + DC$ (dependent choice) + $\mathbf{V} \neq \mathbf{WO}^{\infty}$ is consistent.

(b) There is a model of $ZF + \ DC + \mathcal{P}(\omega) \notin \mathbf{WO}^{\infty}$. (Solovay [So] assuming Con($ZF + \exists$ an inaccessible cardinal), Shelah [Sh] without the inaccessible).

(c) The axiom of determinacy AD implies $\mathcal{P}(\omega) \notin \mathbf{WO}^{\infty}$(Mycielski-Swierczkowski [My-Sw]).

Theorem 2.16. (Keisler). *If M is a well-founded model of $ZF + \mathbf{V} = \mathbf{WO}^{\infty}$ then* $|M| = \left|\mathbf{Ord}^M\right|$.

Proof: Suppose M is a well-founded model of $ZF + \mathbf{V} = \mathbf{WO}^{\infty}$. An argument similar to that of the proof of Proposition 2.14 shows that $\left|(\mathbf{WO}^0)^M\right| = \left|\mathbf{Ord}^M\right|$. This helps to establish, via an *external* induction argument on $\alpha \in \mathbf{Ord}^M$, that $\left|(\mathbf{WO}^{\alpha})^M\right| = \left|\mathbf{Ord}^M\right|$, which shows that $|M| = \left|(\mathbf{WO}^{\infty})^M\right| = \left|\mathbf{Ord}^M\right|$. $\square$

We are now in a position to unveil examples of uncountable Paris models of ZF.

Theorem 2.17. *Every completion T of $ZF+$ $\mathbf{V} \neq \mathbf{WO}^{\infty}$ has a Paris model of cardinality $\aleph_1$.*

Proof: Suppose T is a completion T of $ZF+$ $\mathbf{V} \neq \mathbf{WO}^{\infty}$. By Theorem 2.1, T has a countable Paris model $\mathfrak{M}$. This model provides the stepping stone to the construction of an elementary *continuous* chain $(\mathfrak{M}_\alpha : \alpha < \omega_1)$ of countable models with $\mathfrak{M}_0 = \mathfrak{M}$, such that

$$\forall \alpha < \omega_1,\ \mathbf{Ord}^{\mathfrak{M}} = \mathbf{Ord}^{\mathfrak{M}_\alpha}.$$

The construction of $\mathfrak{M}_{\alpha+1}$ from $\mathfrak{M}_\alpha$ uses the Henkin-Orey omitting types theorem, and hinges on Keisler's key observation [Ke, Ch. 25] that $\mathfrak{M} \models \mathbf{V} \neq \mathbf{WO}^{\infty}$ ensures that the 1-type

$$\Sigma(x) := \{x \in \mathbf{Ord}\} \cup \{x \neq \widetilde{\alpha} : \alpha \in \mathbf{Ord}^{\mathfrak{M}}\}$$

is locally omitted by the following theory T formulated in the language $\{\in\}$ augmented with the constant c, and constants $\widetilde{m}$ for each element $m \in M$.

$$T := Th(\mathfrak{M}, m)_{m \in M} \cup \{c \notin \mathbf{WO}^{\infty}\} \cup \{c \neq \widetilde{m} : m \in M\}.$$

$\bigcup\limits_{\alpha<\omega_1} \mathfrak{M}_\alpha$ is the desired Paris model of T of power $\aleph_1$. $\square$

Theorems 2.16 and 2.17 together yield the following corollary.

Corollary 2.18. *Suppose T is a completion of ZF which has a well-founded model. T has an uncountable Paris model iff $T \vdash \mathbf{V} \neq \mathbf{WO}^{\infty}$.*

We can push the cardinality of Paris models even higher than $\aleph_1$, as witnessed by Theorem 2.19.

Theorem 2.19. *Suppose the statement "ZFC has an uncountable well-founded model" holds in the constructible universe. For every infinite cardinal κ there exists a Paris model of ZF of cardinality κ.*

Proof: Let $FORM(x)$ denote the set of first order formulas in the language of set theory $\{\in\}$ with one free variable x, and for each $\varphi \in FORM(x)$ define

$$\widehat{\varphi}(x) := \varphi(x) \wedge (\forall y \in x\ \neg\varphi(y)).$$

Consider the following countable $L_{\omega_1,\omega}$ theory T_0 consisting of ZF plus the infinitary formula

$$\forall v\ (\mathbf{Ord}(v) \rightarrow \bigvee_{\varphi \in FORM(x)} \widehat{\varphi}(v)).$$

Clearly models of T_0 are precisely the Paris models of ZF. By invoking *Morley's two cardinal theorem* ([Ke, chapter 17], [B, Section VII.4]) within the constructible universe $\mathbf{L}$, we observe that T_0 has a model of prescribed infinite cardinality κ as soon as

(∗) $\mathbf{L} \models$ "For unboundedly many ordinals α in ω_1, T_0 has a model of cardinality $\beth_\alpha$".

Happily, (∗) follows from the following known results (1) and (2) below.

(1) (Friedman [Fr]). *Every countable transitive model M of ZF has a generic extension which satisfies ZF and has cardinality $\beth_\alpha$, where $\alpha = \mathbf{Ord}^M$.*

(2) [E-3, Theorem 3.7] *If ZFC has an uncountable well-founded model then the collection of countable ordinals α such that L_α is a Paris model of ZF is unbounded in $\omega_1^{\mathbf{L}}$.*

□

3. THE STRUCTURE OF PARIS MODELS

We now turn our attention to the structural properties of Paris models. It is easy to see that if $\mathfrak{M}$ is a Paris model of $ZF + \mathbf{V} = \mathbf{OD}$ then $\mathfrak{M}$ is pointwise definable. In particular $\mathfrak{M}$ is rigid, prime, minimal, and does not contain a pair of indiscernibles. But as we shall see the situation is far more complex for other extensions of ZF, even when the axiom of choice holds in $\mathfrak{M}$. We begin with a brief discussion of well-foundedness of Paris models.

3.1. Well-foundedness

Theorem 3.1. (Paris [P]) *If a completion T of ZF has a well-founded model, then every Paris model of T is well-founded.*

Proof: Suppose that a Paris model $\mathfrak{M}$ of T contains an infinite decreasing "$\in$" sequence of elements. This implies that there exists a sequence of definable terms $(\tau_n : n \in \omega)$ such that

$$\forall n \in \omega,\ T \vdash \tau_{n+1} \in \tau_n.$$

Therefore,

$$\forall n \in \omega,\ T \vdash \rho(\tau_{n+1}) \in \rho(\tau_n),$$

where $\rho(x)$ = the ordinal rank of x. This contradicts the existence of a well-founded model of T. □

Remark 3.1.1.

(a) By Gödel's second incompleteness theorem, ZF has continuum many completions with no well-founded models since any model of ZF satisfying $\neg\mathrm{Con}(ZF)$ cannot be ω-standard.

(b) A similar argument to the proof of Theorem 3.1 shows that if T has an ω-standard model, then every Paris model of T is ω-standard.

3.2. Existence of Indiscernibles

Can a Paris model $\mathfrak{M}$ contain a pair of indiscernibles? Recall that a pair of indiscernibles for a model $\mathfrak{M}$ is a pair of distinct elements a and b of $\mathfrak{M}$ such that for every first order formula $\varphi(x)$ with one free variable x,

$$\mathfrak{M} \models \varphi(a) \leftrightarrow \varphi(b).$$

To answer this question we first need to briefly discuss the *Leibniz-Mycielski axiom LM* introduced by Mycielski[2]:

$$LM: \qquad \forall x \forall y\ [x \neq y \rightarrow \exists \alpha \exists \varphi(\{x, y\} \subseteq V_\alpha\ \wedge (\varphi(x) \wedge \neg\varphi(y))^{V_\alpha}).$$

By a general result of Ehrenfeucht and Mostowski [CK, Theorem 3.3.10] every first order theory which has an infinite model possesses a model with a pair of indiscernibles. In contrast, Mycielski [My-1] showed that LM captures the spirit of Leibniz's dictum on the identity of indiscernibles by proving that a complete theory T extending ZF possesses a model *without* a pair of indiscernible elements iff T proves LM. His *proof* uses Paris models, and also shows the following result[3].

[2] Mycielski refers to this axiom as A_2' in [My-1], and L in [My-2]. The Leibniz-Mycieskli appellation LM was proposed in [E-4].

[3] See [E-5] for more on models of set theory without indiscernibles.

Theorem 3.2. (Mycielski) *A Paris model $\mathfrak{M}$ of ZF contains a pair of indiscernibles iff LM fails in $\mathfrak{M}$.*

Note that ZF is consistent with LM since in the presence of ZF, LM is a consequence of $\mathbf{V} = \mathbf{OD}$. As shown in [E-4], LM and AC are mutually independent (the independence of AC from LM is due to Solovay). Moreover, LM turns out to be equivalent to certain global "choice-like" principles within ZF. This fact plays a key role in the proof of Theorem 3.5(2).

Theorem 3.3. [E-4] *The following are equivalent for a model $\mathfrak{M} \models ZF$.*

(i) *There is a parameter free definable map* $\mathbf{F}$ *in* $\mathfrak{M}$ *such that*

$$\mathfrak{M} \models \forall x(|x| > 1 \rightarrow (\emptyset \neq \mathbf{F}(x) \subsetneq x).$$

(ii) *There is a parameter free definable map* $\mathbf{G}$ *in* $\mathfrak{M}$ *such that*

$$M \models \text{"}\mathbf{G} \textit{ injects } \mathbf{V} \textit{ into the class of subsets of } \mathbf{Ord}\text{"}.$$

(iii) $\mathfrak{M} \models LM$.

3.3. Rigidity

Recall that a model $\mathfrak{M}$ is said to be *rigid* if the only automorphism of $\mathfrak{M}$ is the trivial one. The rigidity of a countable model $\mathfrak{M}$ also gives information about the definability of elements of $\mathfrak{M}$ by infinitary formulas, thanks to the following classical theorem.

Theorem 3.4. (Scott, [B, Ch.VII]). *The following are equivalent for a countable model $\mathfrak{M}$ in a countable language.*

(i) $\mathfrak{M}$ *is rigid.*

(ii) *Every element of $\mathfrak{M}$ is $L_{\omega_1,\omega}$-definable in $\mathfrak{M}$.*

One would expect Paris models to be rigid. Theorem 3.5 shows that this is true in many cases, but not always.

Theorem 3.5. *Assume that T is a complete extension of ZF.*

(1) *If T has a well-founded model, then every Paris model of T is rigid.*

(2) *If T includes AC or LM, then every Paris model of T is rigid.*

(3) *Paris models of ZF with nontrivial automorphisms exist.*

Proof:

(1) If f is an automorphism of a *transitive* model M, then

$$\forall x \in M \quad f(x) = \{f(y) : y \in x\}.$$

This immediately implies - via an induction argument on the ordinal rank of x - that f fixes every $x \in M$. In light of Theorem 3.1 and Mostowski's collapsing lemma, this completes the proof of (1).

□ (part 1)

(2) Suppose $\mathfrak{M}$ is a Paris model of T and f is an automorphism of $\mathfrak{M}$. f must fix every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$ since f fixes every definable element of $\mathfrak{M}$. Thanks to extensionality, this implies that f fixes every *subset* of ordinals in $\mathfrak{M}$. Suppose T includes AC, and let $a \in M$. In the presence of AC, a is constructible from a subset s of ordinals in $\mathfrak{M}$ [J, Exercise 15.4]. In particular, there is an ordinal definable term $\tau(x)$ such that $a = \tau^{\mathfrak{M}}(s)$. Therefore,

$$f(a) = f(\tau^{\mathfrak{M}}(s)) = \tau^{\mathfrak{M}}(f(s)) = \tau^{\mathfrak{M}}(s) = a.$$

Hence every Paris model of T is rigid.

Thanks to the equivalence of LM with the existence of a parameter free definable injection of the universe into the class of subsets of ordinals (see Theorem 3.3), an identical argument shows that Paris models of $ZF + LM$ are also rigid.

□ (part 2)

(3) By Gödel, the theory $T_0 = ZF + \mathbf{V} = \mathbf{L} + \neg\mathrm{Con}(ZF)$ is consistent. Moreover, no model of T_0 can be ω-standard. Let $\mathfrak{M}$ be a Paris model of T_0. The desired Paris model with a nontrivial automorphism can be built as a generic extension of $\mathfrak{M}$ using the following surprising theorem of Cohen.

- (Cohen, [C-2]). *Every countable nonstandard model $\mathfrak{M} \models ZF$ has a generic extension $\mathfrak{M}[G]$ which possesses a nontrivial automorphism f of order 2, that is $f(f(x)) = x$ for every $x \in \mathfrak{M}[G]$).*

By Cohen's result above there is a generic extension $\mathfrak{M}[G]$ which has a nontrivial automorphism. $\mathfrak{M}[G]$ remains Paris because $\mathbf{L}^{\mathfrak{M}[G]}$ is definable in $\mathfrak{M}$, and equals $\mathfrak{M}$ itself.

$\square$ (part 3)

Remark 3.5.1. With more work, and invoking stronger consistency hypotheses, one can even construct *ω-standard* Paris models which possess nontrivial automorphisms. This can be done in at least two ways:

Method 1. Apply Cohen's aforementioned theorem to a nonstandard but ω-standard Paris model $\mathfrak{M}$. If ZF has an ω-standard model then such an $\mathfrak{M}$ exists, thanks to the fact, established by Suzuki and Wilmers [SW], that if ZF has an ω-standard model then (1) the theory $T_1 := ZF +$ "ZF has no ω-standard model" has an ω-model, and (2) every model of T_1 is nonstandard.

Method 2. An examination of the *proof* of Morley's two cardinal theorem reveals that the argument used to prove Theorem 2.19 produces an *Ehrenfeucht-Mostowski* Paris model of ZF which is generated from a prescribed ordered set $(X, <)$ of indiscernibles of cardinality κ. Since the property of being an ω-standard model can be expressed by a single $L_{\omega_1,\omega}$ sentence, the conclusion of Theorem 2.19 can be strengthened to "there is a Paris model of cardinality κ with a nontrivial automorphism".

Note that the first method produces a countable ω-standard Paris model with an automorphism of order 2, while the second method produces an arbitrarily large ω-standard Paris model with an automorphism of infinite order.

3.4. Relation with Prime and Minimal Models

Recall the following definitions from Model Theory.

- A model $\mathfrak{M}$ is *prime* if $\mathfrak{M}$ can be elementarily embedded into every model of $Th(\mathfrak{M})$.
- A model $\mathfrak{M}$ is *minimal* if $\mathfrak{M}$ has no proper elementary submodels.

It is known that arbitrary first order theories need not have either prime or minimal models. Also, in general, prime models need not be minimal, and minimal models need not be prime. However, the results of this section show that for $\mathfrak{M} \models ZF$,

$$\mathfrak{M} \text{ is prime} \Rightarrow \mathfrak{M} \text{ is Paris and satisfies } AC \Rightarrow \mathfrak{M} \text{ is minimal.}$$

It is easy to see that if $\mathfrak{M}$ satisfies $\mathbf{V} = \mathbf{OD}$ then both implications above reverse, but we shall see that in general, neither implication reverses (see Remark 3.9.1).

We first examine prime models of ZF.

Theorem 3.6. *Suppose T is a complete extension of ZF. The following are equivalent:*

(**i**) $T \vdash \mathbf{V} = \mathbf{OD}$.

(**ii**) *T has a unique Paris model up to isomorphism.*

(**iii**) *T has a prime model.*

Proof: (**i**) $\Rightarrow$ (**ii**) follows from Proposition 2.3, and (**ii**) $\Rightarrow$ (**i**) follows from Theorem 2.4. To see that (**i**) $\Rightarrow$ (**iii**), recall that if $T \vdash \mathbf{V} = \mathbf{OD}$ then the definability of a global well-ordering implies that T has definable Skolem functions. Consequently there is a pointwise definable model $\mathfrak{M}_0$ of T (take the Skolem hull of the empty set inside any model of T). Given any $\mathfrak{M} \models T$, the desired embedding given by:

$$\tau^{\mathfrak{M}_0} \mapsto \tau^{\mathfrak{M}},$$

where $\tau^{\mathfrak{M}_0}$ and $\tau^{\mathfrak{M}}$ respectively are the denotations of the definable term τ in $\mathfrak{M}$ and $\mathfrak{M}_0$.

We now complete the proof by showing (**iii**) $\Rightarrow$ (**i**). Suppose $\mathfrak{M}$ is a prime model, and suppose on the contrary that $\mathfrak{M}$ satisfies $\mathbf{V} \neq \mathbf{OD}$. The *proof* of Theorem 2.4 shows that there exists a model $\mathfrak{M}_1$ of T such that $\mathfrak{M}_1$ omits a 1-type which is realized in $\mathfrak{M}$. So $\mathfrak{M}$ cannot be elementarily embedded into $\mathfrak{M}_1$, which contradicts the primality of $\mathfrak{M}$. □

We now turn to the discussion of the relation between Paris and minimal models of ZF. The following proposition provides us with useful sufficient conditions under which Paris models are minimal.

Proposition 3.7. *Assume* $\mathfrak{N} \preceq \mathfrak{M} \models ZF$ *and* $\mathbf{Ord}^{\mathfrak{M}} = \mathbf{Ord}^{\mathfrak{N}}$.

(1) *If* $\mathfrak{M} \models AC$ *then* $\mathfrak{M} = \mathfrak{N}$.

(2) [Ke, Ch. 25] *If* $\mathfrak{M} \models \mathbf{V} = \mathbf{WO}^{\infty}$ *and* $\mathfrak{M}$ *is well-founded, then* $\mathfrak{M} = \mathfrak{N}$.

Proof of (1): It suffices to prove that for every $\alpha \in \mathbf{Ord}^{\mathfrak{M}}$, $V_\alpha^{\mathfrak{M}}$ and $V_\alpha^{\mathfrak{N}}$ have the same elements. Invoking the axiom of choice in $\mathfrak{N}$, there are f in $\mathfrak{M}$ and $\theta \in \mathbf{Ord}^{\mathfrak{M}}$ such that

$$\mathfrak{N} \models f : \theta \to V_\alpha \text{ is a surjection.}$$

Coupled with $\mathfrak{N} \preceq \mathfrak{M}$ this implies

$$\mathfrak{M} \models f : \theta \to V_\alpha \text{ is a surjection.}$$

Therefore, $m \in V_\alpha^{\mathfrak{M}}$ iff $\exists \beta \in \mathbf{Ord}^{\mathfrak{M}}(m = f(\beta))$ iff $\exists \beta \in \mathbf{Ord}^{\mathfrak{N}}(m = f(\beta))$ iff $m \in V_\alpha^{\mathfrak{N}}$.

□

Corollary 3.8. *Suppose* T *is a theory extending* ZF.

(1) *If* $T \vdash AC$, *then every Paris model of* T *is minimal.*

(2) *If* $T \vdash \mathbf{V} = \mathbf{WO}^{\infty}$ *and* T *has a well-founded model, then every Paris model of* T *is minimal.*

Theorem 3.6 and Corollary 3.8 together clarify the relation between prime, Paris, and minimal models of ZF:

Corollary 3.9. *Suppose* $\mathfrak{M} \models ZF$.

$$\mathfrak{M} \textit{ is prime} \Rightarrow \mathfrak{M} \textit{ is Paris and satisfies } AC \Rightarrow \mathfrak{M} \textit{ is minimal.}$$

Remark 3.8.1. Some completions of ZF lack minimal models. For example, let T denote the first order theory of Cohen's model N of the negation of AC obtained by adjoining an infinite set S of mutually generic Cohen reals to a model $\mathfrak{M}$ of $ZF + \mathbf{V} = \mathbf{L}$ without adding an enumeration of S itself [J-2, Ex.1, p.203]. If $\mathfrak{M} \models T$ then

$$\mathfrak{M} \models \text{“}\mathbf{V} = \mathbf{L}(S) \text{ for some infinite collection } S \text{ of mutually Cohen generic reals”}.$$

Choose some $s \in S$ and let $\overline{S} = S \backslash \{s\}$. A standard symmetry argument using the homogenous character of Cohen forcing reveals that $\mathbf{L}^{\mathfrak{M}}(\overline{S}) \prec \mathfrak{M}$, thereby showing that $\mathfrak{M}$ is not minimal. Also, as shown by Lévy, T proves that S is an infinite Dedekind finite set [J, Sec.21], so T also proves $\mathbf{V} \neq \mathbf{WO}^{\infty}$ (see Remark 2.15.1(a)). Therefore, AC cannot be deleted from part (1) of Corollary 3.8, and if ZF has a well-founded model, then $\mathbf{V} = \mathbf{WO}^{\infty}$ cannot be eliminated from part (2).

Remark 3.9.1. Neither implication in Corollary 3.9 reverses, even if $\mathfrak{M}$ satisfies AC. For the first implication, this follows from Theorem 3.6 and the independence of $\mathbf{V} = \mathbf{OD}$ from ZFC. To see that the second implication does not reverse, start with a Paris model $\mathfrak{M}_0$ of $ZF + \mathbf{V} = \mathbf{L}$ and build a nonprincipal $\aleph_1^{\mathfrak{M}_0}$-complete ultrafilter $\mathcal{U}$ over $(\mathcal{P}(\aleph_1))^{\mathfrak{M}_0}$ such that whenever $(f : \aleph_1 \to \aleph_1)^{\mathfrak{M}_0}$ there is some $X \in \mathcal{U}$ such that the restriction of f to X is either constant or is 1-1 [E-1, Theorem 2.12]. Let $\mathfrak{M}$ be the ultrapower $\left(\mathbf{V}^{\aleph_1}\right)^{\mathfrak{M}_0}/\mathcal{U}$. Standard arguments show that $\mathfrak{M}$ is an elementary extension of $\mathfrak{M}_0$ which does not introduce any new members to $\omega^{\mathfrak{M}_0}$, and moreover, there is no proper intermediate elementary submodel between $\mathfrak{M}$ and $\mathfrak{M}_0$, i.e.,

(1) If $\mathfrak{M}_0 \preceq \overline{\mathfrak{M}} \preceq \mathfrak{M}$, then either $\mathfrak{M}_0 = \overline{\mathfrak{M}}$ or $\overline{\mathfrak{M}} = \mathfrak{M}$.

Note that $\mathfrak{M}$ is not a Paris model since it has new and therefore undefinable countable ordinals. Now, collapse $\aleph_1^{\mathfrak{M}}$ to $\aleph_0$ in the usual way to obtain the generic extension $\mathfrak{M}[G]$. *$\mathfrak{M}[G]$ is our desired example of a minimal model of ZFC which is not a Paris model.* $\mathfrak{M}[G]$ is not Paris since by the homogeneity of the collapsing partial order no new definable ordinals are introduced in the transition from $\mathfrak{M}$ to $\mathfrak{M}[G]$. To verify minimality, suppose $\mathfrak{N} \preceq \mathfrak{M}[G]$. Clearly

(2) $\mathfrak{M}_0 \preceq \mathbf{L}^{\mathfrak{N}} \preceq \mathbf{L}^{\mathfrak{M}[G]} = \mathfrak{M}$.

Since $\omega^{\mathfrak{M}_0}$ and $\omega^{\mathfrak{M}}$ have the same members, and $\mathbf{Ord}^{\mathfrak{M}[G]} = \mathbf{Ord}^{\mathfrak{M}}$, every $c \in \omega^{\mathfrak{M}[G]}$ is definable in $\mathfrak{M}[G]$. So we can repeat the argument establishing Proposition 2.10 to show that every member of $\aleph_1^{\mathfrak{M}}$ is in $\mathfrak{N}$ and therefore $\mathbf{Ord}^{\mathfrak{N}}$ properly contains $\mathbf{Ord}^{\mathfrak{M}_0}$. Coupled with (1) and (2) this shows that $\mathbf{L}^{\mathfrak{N}} = \mathfrak{M}$ and, *a fortiori*

(3) $\mathbf{Ord}^{\mathfrak{N}} = \mathbf{Ord}^{\mathfrak{M}} = \mathbf{Ord}^{\mathfrak{M}[G]}$.

Putting (3), Proposition 3.7(1), and the fact that $\mathfrak{M}[G] \vDash ZFC$ together, we obtain $\mathfrak{N} = \mathfrak{M}[G]$.

3.5. Rank Extensions

Suppose $\mathfrak{M}$ is a model of the fragment KP (Kripke-Platek set theory) of ZF. A model $\mathfrak{N}$ of KP *rank extends* $\mathfrak{M}$, written $\mathfrak{M} \subseteq_r \mathfrak{N}$, if the ordinal rank of each member of $N \setminus M$ (as computed within $\mathfrak{N}$) is greater than the ordinal rank of each member of M. It is easy to see that $\mathfrak{M} \subseteq_r \mathfrak{N} \Rightarrow \mathfrak{M} \subseteq_e \mathfrak{N}$, but in general end extensions are not rank extensions, e.g., generic extensions of models of ZF are always end extensions and never rank extensions. However, it is well-known that for models $\mathfrak{M}$ of ZF,

$$\mathfrak{M} \preceq_e \mathfrak{N} \Rightarrow \mathfrak{M} \preceq_r \mathfrak{N}.$$

Moreover, by a classical theorem of Keisler and Morley [CK, Theorem 2.2.18] every countable model of ZF has an elementary end extension. But as shown below, a rank extension of a Paris model of ZF can never have a least new ordinal. In particular, no Paris model of ZF can have a *well-founded* elementary end extension.

Theorem 3.10. *If $\mathfrak{M}$ is a Paris model of ZF which is rank extended to a model $\mathfrak{N}$ of KP, then $\mathbf{Ord}^{\mathfrak{N}} \setminus \mathbf{Ord}^{\mathfrak{M}}$ has no least element.*

Proof: Suppose, on the contrary, that $\mathfrak{M}$ is a Paris model of ZF which is rank extended by a model $\mathfrak{N}$ of KP with $\alpha := \min(\mathbf{Ord}^{\mathfrak{N}} \setminus \mathbf{Ord}^{\mathfrak{M}})$. It is easy to see that if $\mathfrak{M} \subseteq_r \mathfrak{N}$ then $\aleph_1^{\mathfrak{M}} = \aleph_1^{\mathfrak{N}} < \alpha$. Moreover, since Tarski's truth predicate $x \models y$ can be implemented in models of KP [B, III.1], for any first order formula $\varphi(x)$, and any $a \in M$,

$$\mathfrak{M} \models \varphi(a) \text{ iff } \mathfrak{N} \models \text{“}V_\alpha \text{ exists and } V_\alpha \models \varphi(a)\text{”}.$$

This immediately implies that $\mathfrak{N} \models$ "V_α is a Paris model". So α is a countable ordinal in the sense of $\mathfrak{N}$, which contradicts $\aleph_1^{\mathfrak{N}} < \alpha$. $\square$

In light of the fact that elementary end extensions of models of ZF are rank extensions, we obtain the following corollary.

Corollary 3.11. *No elementary end extension of a Paris model of ZF has a first new ordinal.*

4. OPEN QUESTIONS

1. *Let* $ZF_n := \{\varphi \in ZF : \varphi$ *is a* Σ_n*-formula*$\}$, *where* n *is a fixed natural number. Is there a consistent extension* T *of* ZF_n *which has no Paris model?*

   - This is motivated by the proof of Theorem 2.1, and the fact that some instance of the induction scheme is unprovable in ZF_n (by a general result of Kreisel and Wang [KW], coupled with the finite axiomatizability of ZF_n for $n \geq 1$).

2. (Paris [P]). *Can an* ω*-standard Paris model of* ZF *contain its own first order theory?*

- It is easy to see that if $\mathfrak{M}$ is a Paris model of ZF such that either (1) $\omega_1^{\mathfrak{M}}$ is well-founded, or (2) $(\mathcal{P}(\omega) \subseteq \mathbf{OD})^{\mathfrak{M}}$, then $\mathrm{Th}(\mathfrak{M}) \notin \mathfrak{M}$. In particular, Question 2 has a negative answer if $\mathfrak{M}$ satisfies $\mathbf{V} = \mathbf{OD}$, or $\mathfrak{M}$ is well-founded. Moreover, if T is an extension of $ZF + \mathbf{V} \neq \mathbf{OD}$ which has an ω-standard model, then the proof of Theorem 2.4 can be slightly modified to show that $T \notin \mathfrak{M}$ for continuum-many ω-standard Paris models $\mathfrak{M}$ of T.

- As observed by Paris [P], if $\mathfrak{M}$ is an ω-standard Paris model whose elementary diagram is recursive in its own theory T, then $T \notin \mathfrak{M}$ (such a Paris model exists by the proof of Theorem 2.1 since the Henkin proof of the omitting types theorem can be conveniently effectivized).

3. *Does every countable model of* ZF *have a generic extension to a Paris model of* ZF*?*

- This is inspired by Theorems 2.7 and 2.8.

4. *Is there a set of first order sentences* Σ *such that* $\mathfrak{M} \models ZF$ *has a Paris elementary submodel iff* $\mathfrak{M} \models \Sigma$*?*

- This is motivated by the results in Section 2.3.

5. *Do arbitrarily large Paris models of* ZF *with automorphisms of order* 2 *exist?*

- This is inspired by Theorem 3.5(3) and Remark 3.5.1.

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