CHARACTERIZING QUANTIFIER EXTENSIONS OF DEPENDENCE LOGIC

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Abstract. We characterize the expressive power of extensions of Dependence Logic and Independence Logic by monotone generalized quantifiers in terms of quantifier extensions of existential second-order logic.

1. Introduction

We study extensions of dependence logic D by monotone first-order generalized quantifiers. Dependence logic [12] extends first-order logic by dependence atomic formulas

$$D(t_1, \ldots, t_n)$$

the meaning of which is that the value of the term t_n is functionally determined by the values of $t_1, \ldots, t_{n-1}$. While in first-order logic the order of quantifiers solely determines the dependence relations between variables, in dependence logic more general dependencies between variables can be expressed. In fact, dependence logic is equivalent to existential second-order logic ESO in expressive power. Historically dependence logic was preceded by partially ordered quantifiers (Henkin quantifiers) of Henkin [7] and Independence-Friendly (IF) logic of Hintikka and Sandu [8].

The framework of dependence logic, so-called team semantics, has turned out be very flexible to allow interesting generalizations. For example, the extensions of dependence logic in terms of intuitionistic implication and linear implication was introduced in [1]. Also new variants of the dependence atoms was introduced in [4], [6] and [5], and generalized quantifiers in [4] and [2].

Engström, in [4], considered extensions of D in terms of first-order generalized quantifiers. The reason for doing so was partly to have a logical framework to analyze partially ordered generalized quantifier prefixes compositionally. The paper introduces a general schema to extend dependence logic with first-order generalized quantifiers. There are also alternative ways of extending dependence logic with generalized quantifiers, as in [2], where a version of the majority quantifier for dependence logic is studied. It is shown that dependence logic with that majority quantifier leads to a new descriptive complexity characterization of the counting hierarchy.

In this paper we continue the study of the logics $\mathrm{D}(Q)$ in the framework developed in [4]. Our main result shows that the logic $\mathrm{D}(Q)$ is equivalent, for sentences, to $\mathrm{ESO}(Q)$, i.e., existential second-order logic extended with Q. We also show analogous characterizations for extensions of Independence logic $\mathrm{I}(Q)$, a variant of

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dependence logic introduced in [6] and independently in [4], by generalized quantifiers. At the end of the paper, we characterize the open formulas of I(Q). For D(Q), finding a characterization of the open formulas remains open.

2. Preliminaries

2.1. **Dependence Logic.** In this section we give a brief introduction to dependence logic. For a detailed account see [12].

The syntax of dependence logic extends the syntax of first-order logic with new atomic formulas, the dependence atoms. There is one dependence atom for each arity. We write the atom expressing that the term t_n is uniquely determined by the values of the terms $t_1, \ldots, t_{n-1}$ as $D(t_1, \ldots, t_n)$.[1] We assume that all formulas of dependence logic are written in negation normal form, i.e., all negations in formulas occur in front of atomic formulas. For a vocabulary τ, D[τ] denotes the set of τ-formulas of dependence logic.

The set of free variables of a formula is defined as in first-order logic with the extra clause that all variables in a dependence atom are free. We denote the set of free variables of a formula ϕ by FV(ϕ).

To define a compositional semantics for dependence logic we use *sets of assignments*, called *teams* instead of single assignments as in first-order logic. An assignment is a function $s : V \to M$ where V is a finite set of variables and M is the universe under consideration. Given a universe M a team of M is a set of assignments for some fixed finite set of variables V. If $V = \emptyset$ there is only one assignment, the empty assignment, denoted by ϵ. Observe that the team of the empty assignment $\{ \epsilon \}$ is different from the empty team $\emptyset$.

Given an assignment $s : V \to M$ and $a \in M$ let $s[a/x] : V \cup \{ x \} \to M$ be the assignment:

$$s[a/x] : y \mapsto \begin{cases} s(y) & \text{if } y \in V \setminus \{ x \}, \text{ and} \\ a & \text{if } x = y. \end{cases}$$

Furthermore, let $X[M/y]$ be the team

$$\{ s[a/y] \mid s \in X, a \in M \},$$

and whenever $f : X \to M$, let $X[f/y]$ denote

$$\{ s[f(s)/y] \mid s \in X \}.$$

The domain of a non-empty team X, denoted dom(X), is the set of variables V. The interpretation of the term t in the model $\mathbb{M}$ under the assignment s is denoted by $t^{M,s}$.

The satisfaction relation for dependence logic $\mathbb{M}, X \vDash \phi$ is now defined as follows. Below, the notation $\mathbb{M}, s \vDash \phi$ refers to the ordinary satisfaction relation of first-order logic.

(1) For first-order atomic or negated atomic formulas ψ: $\mathbb{M}, X \vDash \psi$ iff $\forall s \in X : \mathbb{M}, s \vDash \psi$.
(2) $\mathbb{M}, X \vDash D(t_1, \ldots, t_{n+1})$ iff $\forall s, s' \in X \bigwedge_{1 \le i \le n} t_i^{\mathbb{M},s} = t_i^{\mathbb{M},s'} \to t_{n+1}^{\mathbb{M},s} = t_{n+1}^{\mathbb{M},s'}$
(3) $\mathbb{M}, X \vDash \neg D(t_1, \ldots, t_{n+1})$ iff $X = \emptyset$
(4) $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$
(5) $\mathbb{M}, X \vDash \phi \vee \psi$ iff $\exists Y, Z$ s.t. $X = Y \cup Z$, and both $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$

[1] The dependence atom is denoted by $=(t_1, \ldots, t_n)$ in the original exposition [12].

(6) $\mathbb{M}, X \models \exists y\phi$ iff $\exists f : X \to M,$ such that $\mathbb{M}, X[f/y] \models \phi$
(7) $\mathbb{M}, X \models \forall y\phi$ iff $\mathbb{M}, X[M/y] \models \phi$.

We define $M \models \sigma$ for a sentence σ to hold if $M, \{ \epsilon \} \models \sigma$.

Let us make some easy remarks. First, every formula is satisfied by the empty team. Second, satisfaction is preserved under taking subteams:

Proposition 2.1. *If* $\mathbb{M}, X \models \phi$ *and* $Y \subseteq X$ *then* $\mathbb{M}, Y \models \phi$.

And thirdly, the satisfaction relation is invariant of the values of the non-free variables of the formula:

Proposition 2.2. $\mathbb{M}, X \models \phi$ *iff* $\mathbb{M}, Y \models \phi$ *where* $Y = \{ s \restriction FV(\phi) \mid s \in X \}$.

The satisfaction relation for first-order formulas reduces to ordinary satisfaction in the following way.

Proposition 2.3. *For first-order formulas* ϕ *and teams* X*,* $\mathbb{M}, X \models \phi$ *iff for all* $s \in X : \mathbb{M}, s \models \phi$.

By formalizing the satisfaction relation of dependence logic in existential second order logic we get the following upper bound on the expressive power of dependence logic. For a team X with domain $\{ x_1, \dots, x_k \}$, let $\mathrm{rel}(X)$ be the k-ary relation $\{ \langle s(x_1), \dots, s(x_k)\rangle \mid s \in X \}$.

Proposition 2.4. *Let* τ *be a vocabulary and* ϕ *a* $\mathrm{D}[\tau]$*-formula with free variables* $x_1, \dots, x_k$*. Then there is a* $\tau \cup \{ R \}$*-sentence* ψ *of* ESO*, in which* R *appears only negatively, such that for all models* $\mathbb{M}$ *and teams* X *with domain* $\{ x_1, \dots, x_k \}$*:*

$$\mathbb{M}, X \models \phi \iff (\mathbb{M}, \mathit{rel}(X)) \models \psi.$$

For sentences the proposition gives that $\mathrm{D} \leq \mathrm{ESO}$ and in [12] the converse inequality was shown, hence $\mathrm{D} \equiv \mathrm{ESO}$. In [9] the following theorem was shown, which together with Proposition 2.4 characterizes open τ-formulas of dependence logic as the R-negative (downwards closed) fragment of $\mathrm{ESO}[\tau \cup \{ R \}]$.

Theorem 2.5. *Let* τ *be a signature and* R *a* k*-ary relation symbol such that* $R \notin \tau$*. Then for every* $\tau \cup \{ R \}$*-sentence* ψ *of* ESO*, in which* R *appears only negatively, there is a* τ*-formula* ϕ *of* D *with free variables* $x_1, \dots, x_k$ *such that, for all* $\mathbb{M}$ *and* X *with domain* $\{ x_1, \dots, x_k \}$*:*

$$\mathbb{M}, X \models \phi \iff (\mathbb{M}, \mathit{rel}(X)) \models \psi \vee \forall \bar{y} \neg R(\bar{y}).$$

2.2. **Independence logic.** Independence logic was introduced in [6] and independently in [4] as a variant of dependence logic in which the dependence atoms are replaced by independence atoms $\bar{x} \perp_{\bar{z}} \bar{y}$.[2] The semantics of these atoms are defined by:

$M, X \models \bar{y} \perp_{\bar{x}} \bar{z}$ iff

$$\forall s, s' {\in} X\Big(s(\bar{x}) = s'(\bar{x}) \to \exists s_0 {\in} X\big(s_0(\bar{x}, \bar{y}) = s(\bar{x}, \bar{y}) \wedge s_0(\bar{z}) = s'(\bar{z})\big)\Big).$$

The dependence atoms can easily be expressed using the independence atoms, implying that independence logic contains dependence logic, in fact this containment is proper, as seen from the lack of downwards closure.

[2]In [4] multivalued dependence atoms were introduced, denoted by $[\bar{z} \twoheadrightarrow \bar{x} | \bar{y}]$. The semantics are very similar to the independence atoms.

On the other hand the analogue of Proposition 2.4 holds for independence logic, if the restriction of R appearing only negatively is removed. Galliani, in [5], showed that the also the analogue of Theorem 2.5 holds with the same modification, i.e., the open τ-formulas of independence logic corresponds exactly to $\tau \cup \{ R \}$-sentences of ESO.

2.3. **D(Q).** The notion of a generalized quantifier goes back to Mostowski [11] and Lindström [10]. In a recent paper [4] Engström introduced semantics for generalized quantifiers in the framework of dependence logic. We will review the definitions here.

Let Q be a quantifier of type $\langle k \rangle$, meaning that Q is a class of τ-structures, where the signature τ has a single k-ary relational symbol. Also, assume that Q is monotone increasing, i.e., for every M and every $A \subseteq B \subseteq M^k$, if $A \in Q_M$ then also $B \in Q_M$. An assignment s satisfies a formula $Q\bar{x}\,\phi$ in the structure $\mathbb{M}$, written $\mathbb{M}, s \vDash Qx\,\phi$, if the set $\{ \bar{a} \in M^k \mid \mathbb{M}, s[\bar{a}/\bar{x}] \vDash \phi \}$ is in Q_M, where $Q_M = \{ R \subseteq M^k \mid (M, R) \in Q \}$.

In the context of teams we say that a team X satisfies a formula $Q\bar{x}\,\phi$,

(1) $\quad \mathbb{M}, X \vDash Q\bar{x}\,\phi$, if there exists $F : X \to Q_M$ such that $\mathbb{M}, X[F/\bar{x}] \vDash \phi$,

where $X[F/\bar{x}] = \{ s[\bar{a}/\bar{x}] \mid \bar{a} \in F(s) \}$. Note that this definition works well only with monotone (increasing) quantifiers, see [4] for details.

Let $D(Q)$ be dependence logic extended with the generalized quantifier Q with semantics as defined in (1).

The following easy proposition suggests that we indeed have the right truth condition for monotone quantifiers:

Proposition 2.6. *(i) $D(Q)$ is downwards closed.*
(ii) $D(Q)$ is local, in the sense that $\mathbb{M}, X \vDash \phi$ iff $\mathbb{M}, (X \restriction FV(\phi)) \vDash \phi$.
(iii) Viewing $\exists$ and $\forall$ as generalized quantifiers of type $\langle 1 \rangle$, the truth conditions in (1) *are equivalent to the truth conditions of dependence logic.*
(iv) For $\mathrm{FO}(Q)$*-formulas ϕ and teams X, $\mathbb{M}, X \vDash \phi$ iff for all $s \in X : \mathbb{M}, s \vDash \phi$.*
(v) For every $\mathrm{D}(Q)$ *formula ϕ we have $\mathbb{M}, \emptyset \vDash \phi$.*

The proofs of (i), (ii), (iv), and (v) are easy inductions on the construction of ϕ, and (iii) is proved by using (i).

2.4. ESO(Q)**.** We denote by ESO the existential fragment of second-order logic. The extension, ESO(Q), of ESO by a generalized quantifier Q is defined as follows.

Definition 2.7. The formulas of ESO(Q) are built up recursively from atomic and negated atomic formulas with conjuction, disjunction, first-order existential and universal quantification, Q quantification, and second-order existential relational and functional quantification.

A quantifier Q is *definable* in ESO if Q is the class of models of some ESO-sentence ϕ, i.e.,

$$Q = \mathrm{Mod}(\phi).$$

Note that if for every M, $\emptyset \in Q_M$ and $M \notin Q_M$ then we can use Q to simulate the classical negation, and thus full second-order logic is contained in ESO(Q). However, if we restrict to monotone (increasing) quantifiers we get the following result as in first-order logic:

Proposition 2.8. *Let Q be a monotone quantifier. Then Q is definable in* ESO *iff* $\mathrm{ESO}(Q) \equiv \mathrm{ESO}$.

Proof. Since the model class Q is trivially axiomatizable in $\mathrm{ESO}(Q)$, non-definability of Q in ESO implies that $\mathrm{ESO}(Q) > \mathrm{ESO}$. Assume then that Q is definable in ESO and let $\{ R \}$ be the vocabulary of Q, where R is k-ary. By the assumption, there is $\phi \in \mathrm{ESO}$ such that $\mathrm{Mod}(\phi) = Q$. The idea is now to use the sentence ϕ as a uniform definition of Q using substitution. The problem is that there might be negative occurrences of R in ϕ. By using the monotonicity of Q, this problem can be avoided. Define ψ as follows:

$$\exists P(\phi(P/R) \wedge \forall \bar{x}(P(\bar{x}) \to R(\bar{x}))).$$

By the monotonicity of Q, the sentence ψ also defines Q and it only has one positive occurrence of R. We can now compositionally translate formulas of $\mathrm{ESO}(Q)$ into the logic ESO, the clause for Q being the only non-trivial one:

$$Q\bar{x}\theta \rightsquigarrow \psi(\theta/R),$$

where $\psi(\theta/R)$ arises by substituting the unique subformula $R(\bar{x})$ of ψ by $\theta(\bar{x})$. □

The next example shows that it is easy to find monotone quantifiers which are not ESO-definable.

Example 1. Let $S \subseteq \mathbb{N}$. Then the following quantifiers of type $\langle 1 \rangle$ are monotone:

$$\begin{aligned} Q_1 &= \{ (M, X) : |M| \text{ finite, and } \emptyset \neq X \subseteq M \} \\ Q_S &= \{ (M, X) : |M| \in S \text{ and } X = M \} \cup \{ (M, X) : |M| \notin S \text{ and } X \neq \emptyset \} \end{aligned}$$

By, compactness of ESO, Q_1 is not definable in ESO. Furthermore, for only countably many S, the quantifier Q_S is ESO-definable.

3. The equivalence of D(Q) and ESO(Q)

In this section we consider monotone increasing quantifiers Q satisfying two non-triviality assumptions: $(M, \emptyset) \notin Q$ and $(M, M^k) \in Q$ for all M. We show that, for sentences, the logics $\mathrm{D}(Q)$ and $\mathrm{ESO}(Q)$ are equivalent.

3.1. A normal form for ESO(Q).

Definition 3.1. A formula of $\mathrm{ESO}(Q)$ is in *normal form* if it is of the form $\exists f_1 \ldots \exists f_k\, \phi$ and ϕ is a $\mathrm{FO}(Q)$-sentence in prenex normal form without existential quantifiers.

Thus an $\mathrm{ESO}(Q)$ formula is in normal form if it can be written as:

$$\exists f_1 \cdots f_n Q'_1 x_1 \cdots Q'_m x_m \psi,$$

where $Q'_i \in \{ Q, \forall \}$ and ψ is a quantifier-free formula. In order to show that every formula of $\mathrm{ESO}(Q)$ can be transformed into this normal form, we need the following lemma.

Lemma 3.2. *Then the following equivalences hold*

- $Q\bar{x}(\psi \vee \phi) \equiv Q\bar{x}\psi \vee \phi$,
- $Q\bar{x}(\psi \wedge \phi) \equiv Q\bar{x}\psi \wedge \phi$,

where the variables $\bar{x}$ do not appear free in ϕ.

Proposition 3.3. *Every sentence of* ESO(Q) *can be written in the normal form of Definition 3.1.*

Proof. The claim is proved using induction on ϕ. The proof is analogous to the corresponding proof for ESO (see e.g., Lemma 6.12 in [12]). The cases of conjunction and disjunction are proved using Lemma 3.2. The case corresponding to Q is analogous to the case of the universal quantifier using the observation that a formula of the form $Q\bar{x}\exists f\phi$ is equivalent to $\exists g Q\bar{x}\psi$, where ψ arises from ϕ by replacing terms $f(t_1,\ldots,t_k)$ by $g(\bar{x},t_1,\ldots,t_k)$. □

3.2. **The main result.** We will first show a compositional translation mapping formulas of D(Q) into sentences of ESO(Q). This is analogous to the translation from D into ESO of Proposition 2.4.

Proposition 3.4. *Let τ be a vocabulary and ϕ a* D(Q)[τ]*-formula with free variables $x_1,\ldots,x_k$. Then there is a $\tau\cup\{R\}$-sentence ψ of* ESO(Q)*, in which R appears only negatively, such that for all models $\mathbb{M}$ and teams X with domain $\{x_1,\ldots,x_k\}$:*

$$\mathbb{M},X\vDash\phi \iff (\mathbb{M},rel(X))\vDash\psi(R).$$

Proof. The claim is proved using induction on ϕ. It suffices to define a translation for $Q\bar{y}\,\theta(\bar{x},\bar{y})$, since the other cases are translated analogously to Proposition 2.4:

$$Q\bar{y}\,\theta \rightsquigarrow \exists P\big(\theta^*(P)\wedge\forall\bar{x}(R(\bar{x})\to Q\bar{y}\,P(\bar{x},\bar{y}))\big),$$

where θ^* is the translation for θ given by the induction assumption. □

Next we show that, for sentences, Proposition 3.4 can be reversed, and thus the following holds.

Theorem 3.5. ESO(Q) $\equiv$ D(Q).

Proof. Let ϕ be a ESO(Q)-sentence. We show that there is a logically equivalent sentence $\psi\in$ D(Q). By Proposition 3.3 we may assume that ϕ is of the form:

$$\exists f_1\cdots f_n Q'_1x_1\cdots Q'_mx_m\psi, \tag{2}$$

where $Q'_i\in\{\forall,Q\}$ and ψ is quantifier free. Before translating this sentence into D(Q), we apply certain reductions to it. We transform the quantifier-free part ψ of ϕ to satisfy the condition that for each of the function symbols f_i there is a unique tuple $\bar{x}^i$ of pairwise distinct variables such that all occurrences of f_i in ψ are of the form $f_i(\bar{x}^i)$. In order to achieve this, we might have to introduce new existentially quantified functions and also universal first-order quantifiers (as in the proof of Theorem 3.3 in [3]), but the quantifier structure of the sentence (2) does not change. We will now assume that the sentence (2) has this property.

We will next show how the sentence (2) can be translated into D(Q). We claim that the following sentence of D(Q) is a correct translation for (2):

$$Q'_1x_1\cdots Q'_mx_m\exists y_1\cdots\exists y_n\big(\bigwedge_{1\le j\le n}D(\bar{x}^i,y_i)\wedge\theta\big), \tag{3}$$

where θ is obtained from ψ by replacing all occurrences of the term $f_i(\bar{x}^i)$ by y_i.

Let us show that the sentences (2) and (3) are logically equivalent. Let $\mathbb{M}$ be a structure and let $\mathbf{f}_1,\ldots,\mathbf{f}_n$ interpret the function symbols f_i. We first show the following auxiliary result: for all teams X with domain $\{x_1,\ldots,x_m\}$ the following equivalence holds:

$$(\mathbb{M},\bar{\mathbf{f}}),X\vDash\psi \iff \mathbb{M},X^*\vDash\theta, \tag{4}$$

where $X^* = X(g_1/y_1)\cdots(g_n/y_n)$, and the functions g_i are defined as follows:

$$g_i(s) \quad = \quad \mathbf{f}_i(s(\bar{x}^i)),$$

and where $s(\bar{x}^i)$ is the tuple obtained by pointwise application of s. Since ψ and θ are first-order, by Proposition 2.3, (4) follows from the fact that for each $s \in X^*$ it holds that

$$(\mathbb{M}, \bar{\mathbf{f}}), s_i \vDash \psi \iff \mathbb{M}, s \vDash \theta, \tag{5}$$

where $s' = s \upharpoonright \{ x_1, \ldots, x_m \}$. The claim is proved using induction on the structure of the quantifier-free formula ψ.

Let us then show that ϕ (see (2)) and sentence (3) are logically equivalent. Suppose that $\mathbb{M} \vDash \phi$. Then there are $\mathbf{f}_1, \ldots, \mathbf{f}_n$ such that

$$(\mathbb{M}, \bar{\mathbf{f}}) \vDash Q'_1 x_1 \cdots Q'_m x_m \psi. \tag{6}$$

Now, by (6), there is a team X arising by evaluating the quantifiers Q'_i such that

$$(\mathbb{M}, \bar{\mathbf{f}}), X \vDash \psi. \tag{7}$$

By (4), and the way the functions g_i are defined, we get that

$$\mathbb{M}, X^* \vDash \bigwedge_{1 \le j \le n} D(\bar{x}^i, y_i) \wedge \theta,$$

and that

$$\mathbb{M}, X \vDash \exists y_1 \cdots \exists y_n \big(\bigwedge_{1 \le j \le n} D(\bar{x}^i, y_i) \wedge \theta \big). \tag{8}$$

Finally, (8) implies that

$$\mathbb{M} \vDash Q'_1 x_1 \cdots Q'_m x_m \exists y_1 \cdots \exists y_n (\bigwedge_{1 \le j \le n} D(\bar{x}^i, y_i) \wedge \theta).$$

The converse implication is proved by reversing the steps above. Note that there is some freedom when choosing the functions $\mathbf{f}_1, \ldots, \mathbf{f}_n$, since it is enough to satisfy the equivalence in (4). □

We remark that the theorem also holds for quantifiers satisfying only the assumptions that for all M, $(M, \emptyset) \notin Q$. This is achieved by a small trick: Let $\phi \in \mathrm{ESO}(Q)$ be a sentence. Suppose M is such that $(M, M^k) \in Q$ then the sentence (3), denoted ϕ^* in the following, is equivalent to ϕ on structures over M. However, if M is such that $(M, M^k) \notin Q$, then Q is trivially false in structures over M and hence ϕ is equivalent to $\phi_0 \in \mathrm{ESO}$, acquired by replacing subformulas headed by Q with $\bot$, in structures over M.

It is easy to show, by induction on ϕ, that

$$\phi_0 \Rightarrow \phi. \tag{9}$$

Let $\phi_0^* \in \mathrm{D}$ be a sentence equivalent to ϕ_0. Let θ be the following $\mathrm{D}(Q)$ sentence:

$$(Q\bar{x}\top \wedge \phi^*) \vee \phi_0^*.$$

Now, assume that $(M, M^k) \in Q$, then θ is equivalent, over M, to $\phi^* \vee \phi_0^*$. By using the fact that ϕ is equivalent to ϕ^* we can see that whenever ϕ_0^* is true ϕ^* is also true and thus θ is equivalent, again over M, to ϕ. On the other hand if $(M, M^k) \notin Q$ then θ is equivalent, over M, to ϕ_0 which in turn is equivalent to ϕ.

If we assume Q only to be monotone (i.e., it may be trivial on some universes), we can, by a similar trick as above and using the obvious generalization of Proposition 3.3 to $\mathrm{ESO}(Q_1, \ldots, Q_k)$, prove that

$$\mathrm{ESO}(Q, Q^d) \leq \mathrm{D}(Q, Q^d),$$

where Q^d is the dual of Q, i.e, $Q^d = \{ (M, A^c) \mid (M, A) \notin Q \}$. This in turn gives us that for any monotone Q:

$$\mathrm{ESO}(Q, Q^d) \equiv \mathrm{D}(Q, Q^d).$$

The logic $\mathrm{D}(Q, Q^d)$ might be considered more natural than $\mathrm{D}(Q)$ since $\mathrm{FO}(Q) \leq \mathrm{D}(Q, Q^d)$.

In [6] it is shown that $\mathrm{I} \equiv \mathrm{ESO}$, and hence analogously to Proposition 3.4 it follows that $\mathrm{I}(Q) \leq \mathrm{ESO}(Q)$. On the other hand, since $\mathrm{D}(Q) \leq \mathrm{I}(Q)$ Theorem 3.5 implies the following.

Theorem 3.6. $\mathrm{I}(Q) \equiv \mathrm{ESO}(Q)$.

4. Characterizing the open formulas

In this section we note that Theorem 3.5 can be generalized to open formulas. We assume that the generalized quantifiers are monotone and satisfy the same non-triviality conditions as in the previous section.

Theorem 4.1. *Let τ be a signature and R a k-ary relation symbol such that $R \notin \tau$. Then for every $\tau \cup \{ R \}$-sentence ψ of $\mathrm{ESO}(Q)$ there is a τ-formula ϕ of $\mathrm{I}(Q)$ with free variables $\bar{z} = z_1, \ldots, z_k$ such that, for all $\mathbb{M}$ and X with domain $\{ \bar{z} \}$:*

$$\mathbb{M}, X \vDash \phi \iff (\mathbb{M}, rel(X)) \vDash \psi \vee \forall \bar{y} \neg R(\bar{y}). \tag{10}$$

Proof. The proof follows the proof of Theorem 3.5 closely with some additional tweaks. First we translate the formula ϕ into the form

$$\exists f_1 \cdots f_n Q'_1 x_1 \cdots Q'_m x_m \big(\forall \bar{w} \big(R(\bar{w}) \leftrightarrow f_1(\bar{w}) = f_2(\bar{w})\big) \wedge \psi\big), \tag{11}$$

where $Q'_i \in \{ \forall, Q \}$ and ψ is a quantifier free formula with no occurrence of R and such that all occurrences of f_i is of the form $f_i(\bar{x}^i)$. This is done by using the techniques of Proposition 3.3 and Theorem 6.1 in [5].

Instead of translating the formula (11) into (3) we need to assure that the sets chosen by the quantifier prefix $Q'_1 x_1 \ldots Q'_m x_m$ are chosen *uniformly* and not depending on the assignments in the team X. In $\mathrm{I}(Q)$ we can do this by adding independence atoms in the following way:

$$Q'_1 x_1 \cdots Q'_m x_m \exists y_1 \cdots \exists y_n \big(\bigwedge_{1 \leq l \leq m} x_l \bot_{\{ x_1, \ldots, x_{l-1} \}} \bar{z} \ \wedge \bigwedge_{1 \leq i \leq n} y_i \bot_{\bar{x}^i} y_i) \ \wedge \theta\big). \tag{12}$$

Here θ corresponds to the quantifier free formula in the proof of Theorem 6.1 in [5]. Observe that $y \bot_{\bar{x}} y$. is equivalent to the dependence atom $D(\bar{x}, y)$.

The rest of the proof goes through as in Theorem 3.6. □

The same proof cannot prove that $\mathrm{ESO}(Q) \leq \mathrm{D}(Q)$. This, and the closely related question of whatever we can express slashed and backslashed quantifiers in $\mathrm{D}(Q)$ remains open.

5. Conclusion

Our results show that the correspondence between dependence logic and independence logic on one hand and ESO on the other is robust in the sense that adding generalized quantifiers will not break the correspondences.

As discussed in section 3, even if we drop the non-triviality conditions, we can prove that for any monotone Q:

$$\mathrm{ESO}(Q, Q^d) \equiv \mathrm{D}(Q, Q^d).$$

The dual is used only to express that $\neg Qx\bot$, which is equivalent to $Q^d x\top$. We leave the question of whether $\mathrm{ESO}(Q) \leq \mathrm{D}(Q)$ open for arbitrary monotone quantifiers.

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