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Connexive Negation

Abstract. Seen from the point of view of evaluation conditions, a usual way to obtain a connexive logic is to take a well-known negation, for example, Boolean negation or de Morgan negation, and then assign special properties to the conditional to validate Aristotle's and Boethius' Theses. Nonetheless, another theoretical possibility is to have the extensional or the material conditional and then assign special properties to the negation to validate the theses. In this paper we examine that possibility, not sufficiently explored in the connexive literature yet.We offer a characterization of connexive negation disentangled from the cancellation account of negation, a previous attempt to define connexivity on top of a distinctive negation. We also discuss an ancient view on connexive logics, according to which a valid implication is one where the negation of the consequent is incompatible with the antecedent, and discuss the role of our idea of connexive negation for this kind of view.

Keywords: Connexive negation, Negation as cancellation, Contradiction, Compatibility.

1. Introduction

According to "the now standard notion of connexive logic" (cf. [59]), a logic **L** is *connexive* if it validates the following schemas, where N and $>$ are a negation and an implication, respectively, in the underlying language of **L**:

$$N(A > NA) \qquad \text{(Aristotle's Thesis)}$$

$$N(NA > A) \qquad \text{(Variant of Aristotle's Thesis)}$$

$$(A > B) > N(A > NB) \qquad \text{(Boethius' Thesis)}$$

$$(A > NB) > N(A > B) \qquad \text{(Variant of Boethius' Thesis)}$$

and, moreover, it invalidates the following one:

$$(A > B) > (B > A). \qquad \text{(Symmetry of Implication)}$$

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Derivatively, any $>$ that satisfies these requirements is called 'connexive'. Although it can be disputed whether meeting all these conditions is either necessary or sufficient for connexivity,[1] we will assume that it is sufficient and, accordingly, we will call collectively the schemas above 'the connexive schemas'.

The usual procedure to obtain a connexive logic is to take a negation with standard truth and falsity conditions, for example, Boolean negation—as in Angell-McCall's **CC1**; see [31]—or de Morgan negation—as in Wansing-style connexive logics; see [41, 58]—, and then modify the (standard) truth or falsity conditions of a conditional —as the intuitionistic conditional (see [58]) or the material conditional (see [59])– to validate the schemas above. Nonetheless, another theoretical possibility is to take a conditional with standard truth and falsity conditions —like the extensional or the material conditional—and then modify the (standard) truth or falsity conditions of a negation to validate the schemas above. In this paper we examine this latter possibility, that has not been sufficiently explored in the connexive literature yet.

Thus, just like a $>$ that validates the connexive schemas (in the company of a 'standard' negation) is called a 'connexive conditional' or 'connexive implication', any N that validates the connexive schemas in the company of a 'standard' conditional will be called a 'connexive negation'. True, in many cases further discussion is needed to make sure that the connexive connective, whether a conditional or a negation, belongs in fact to the intended category, that is, that it in fact is a conditional or a negation. We will devote some time to that discussion in due course, in Section 4.

The closest antecedent to the idea that interests us has been the considerations about the account of negation as cancellation, explored in [57] and [46]. Nonetheless, such an account of negation does not leave other notions untouched. In particular, the conditional is required to be neither extensional nor material. Actually, what precludes contradictions to imply anything (different from a contradiction) in Priest's proposal is not a distinctive evaluation condition for negation, but either the modified notion of logical consequence or the intensional conditional proposed.

Our general motivation is lead by the validation of the connexive schemas by means of a suitable negation. What is going to count as a negation will not be delivered by a theory of negation occurring in a natural language, but

[1] For example, McCall [30] only required invalidating the Symmetry of Implication and validating at least one of Aristotle's or Boethius' Theses, not necessarily both.

rather by examining the evaluation conditions of the relevant unary connectives. In a way, our project can be seen as more general than the account of negation as cancellation, since it is also disentangled from the particularities of the properties of negation interacting with other connectives.[2]

The plan of the paper is as follows. In Section 2 we set the technical preliminaries for the rest of the paper. In Section 3, we introduce four families of negations and, in Section 4, we discuss whether our negations can achieve the status of connexive negations. In Section 5 we compare some properties of these connectives with those of negation under the cancellation account. Finally, in Section 6 we examine whether our negations can be used to define compatibility connectives that meet the requirements set out in [45].

2. Technical Preliminaries

Let NEG be a family of negations, with a negation denoted generically by N_i—or even without the subscript if there is no risk of ambiguity—, $COND$ a family of conditionals, with a conditional denoted generically by $>_j$—again, we might omit the subscript if the context allows so—and $CONJ$ a family of conjunctions, with a conjunction denoted generically by $\triangle_k$. Let $\mathcal{L}_{\{N_i, >_i\}}$ be a propositional language with a denumerable set Var of propositional variables, and a finite set $\{N_i, >_j\}$ of connectives. The set of formulas $FORM$ is defined on $\mathcal{L}_{\{N_i, >_i\}}$ in the usual way. We use upper case letters A, B, C and so on, as meta-variables ranging over arbitrary formulas, and lower case letters p, q, r and so on, as meta-variables ranging over arbitrary propositional variables.

We express an expansion of a language by indicating at subscripts the symbols added. For instance, we write $\mathcal{L}_{\triangle_k}$ to denote $\{N_i,\ >_j,\ \triangle_k\}$. The set of formulas is then defined on this expanded language as usual. In many cases below, the exact shape of a language, and therefore of a logic, will be left implicit and will be indicated by the *de facto* use of connectives.

Let $\sigma : Var \longrightarrow \{\{1\}, \{1, 0\}, \{\ \}, \{0\}\}$ be a valuation function from the set of propositional variables to the set $\{\{1\}, \{1, 0\}, \{\ \}, \{0\}\}$. (Thus, propositional variables are interpreted in terms of sets of truth values.) Valuation functions are extended to the set $FORM$ according to some fixed conditions, depending on the particular connectives one is dealing with. As usual,

[2] There have been attempts to make connexive other connectives using a different strategy to the one employed here. For further references see [18–21]. For a critical assessment of Francez's attempts see [16].

we write $\Sigma \models_{\mathbf{L}} A$ if and only if for all $B \in \Sigma$, if $1 \in \sigma(B)$ then $1 \in \sigma(A)$. For simplicity, in this paper we use classical logic in the meta-theory.

DEFINITION 2.1. A *Dunn atom* is an expression of the form $v_i \in \sigma(A)$ or $v_j \notin \sigma(A)$, with $v_i, v_j \in \{1, 0\}$. If it is of the first form, we will say that it is a *positive* Dunn atom; in the latter case, we will say that it is *negative*.

We observe that valuations can be rewritten in terms of Dunn atoms as follows:

- $\sigma(A) = \{1\}$ iff $1 \in \sigma(A)$ and $0 \notin \sigma(A)$
- $\sigma(A) = \{1, 0\}$ iff $1 \in \sigma(A)$ and $0 \in \sigma(A)$
- $\sigma(A) = \{\ \}$ iff $1 \notin \sigma(A)$ and $0 \notin \sigma(A)$
- $\sigma(A) = \{0\}$ iff $1 \notin \sigma(A)$ and $0 \in \sigma(A)$.

DEFINITION 2.2. Let $v_i \in \sigma(A)$ (resp. $v_j \notin \sigma(A)$) be a Dunn atom. We will say that $v_j \notin \sigma(A)$ (resp. $v_i \in \sigma(A)$), with $v_i, v_j \in \{1, 0\}$ and $v_i \neq v_j$, is its *Boolean counterpart*. (And we will assume that the relation of being a Boolean counterpart is symmetric.)

For instance, the following cases—horizontal-wise—are Boolean counterparts of each other:

$$\begin{array}{cc} 1 \in \sigma(N_x A) & 0 \notin \sigma(N_x A) \\ 0 \in \sigma(A \triangle_z B) & 1 \notin \sigma(A \triangle_z B) \\ 0 \notin \sigma(A >_y B) & 1 \in \sigma(A >_y B). \end{array}$$

DEFINITION 2.3. A *tweaking* is a modification in the evaluation conditions of a connective where the only changes are substituting at least one Dunn atom by its Boolean counterpart.

Consider the evaluation conditions for the connectives in **FDE**, as they will be useful throughout the paper:

$1 \in \sigma({\sim} A)$ iff $0 \in \sigma(A)$
$0 \in \sigma({\sim} A)$ iff $1 \in \sigma(A)$
$1 \in \sigma(A \wedge_e B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$
$0 \in \sigma(A \wedge_e B)$ iff $0 \in \sigma(A)$ or $0 \in \sigma(B)$
$1 \in \sigma(A \vee B)$ iff $1 \in \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \vee B)$ iff $0 \in \sigma(A)$ and $0 \in \sigma(B)$
$1 \in \sigma(A \to B)$ iff $0 \in \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \to B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$.

As an illustration of tweakings, consider negation as evaluated in **FDE** in the upper left corner and three connectives obtained by changing at least part of its evaluation conditions:

$$\begin{array}{ll} 1 \in \sigma({\sim} A) \text{iff } 0 \in \sigma(A) & 1 \in \sigma({\sim_2} A) \text{iff } 1 \notin \sigma(A) \\ 0 \in \sigma({\sim} A) \text{iff } 1 \in \sigma(A) & 0 \in \sigma({\sim_2} A) \text{iff } 1 \in \sigma(A) \\ & \\ 1 \in \sigma({\sim_1} A) \text{iff } 0 \in \sigma(A) & 1 \in \sigma({\sim_3} A) \text{iff } 1 \notin \sigma(A) \\ 0 \in \sigma({\sim_1} A) \text{iff } 0 \notin \sigma(A) & 0 \in \sigma({\sim_3} A) \text{iff } 0 \notin \sigma(A). \end{array}$$

DEFINITION 2.4. Let e be an evaluation condition expressed in Dunn semantics that is classically equivalent to some evaluation condition e^c in classical logic. e is *(classically) redundant* if and only if there are Dunn atoms at the right of the 'iff' of e that could be eliminated, giving rise to an evaluation condition e^- and it is still classically equivalent to e^c. Otherwise, e is *(classically) non-redundant*.

For example, the conditional defined through the following evaluation conditions[3] is redundant from the classical logic perspective, for at the right of the 'iff' there are Dunn atoms that could be eliminated without loss (in classical logic):

- $1 \in \sigma(A \to_\varphi B)$ iff (1) $0 \in \sigma(A)$ and $1 \notin \sigma(A)$, or (2) $1 \in \sigma(B)$ and $0 \notin \sigma(B)$, or (3) both $1 \in \sigma(A)$ iff $1 \in \sigma(B)$ and $0 \in \sigma(A)$ iff $0 \in \sigma(B)$
- $0 \in \sigma(A \to_\varphi B)$ iff (1) $1 \in \sigma(A)$, $0 \notin \sigma(A)$ and $0 \in \sigma(B)$, or (2) $1 \in \sigma(A)$, and $1 \notin \sigma(B)$, or (3) $0 \notin \sigma(A)$, $1 \notin \sigma(B)$ and $0 \in \sigma(B)$.

Note that already the first clause in the truth condition is redundant from the classical point of view, as it states that one of the options for the $A \to_\varphi B$ to be true is that the antecedent is false but not true, but these two claims amount to the same thing in classical logic. We present here its truth table for self-containment:

$A \to_\varphi B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{0}	{0}	{0}
{1,0}	{1}	{1}	{0}	{0}
{ }	{1}	{0}	{1}	{0}
{0}	{1}	{1}	{1}	{1}

DEFINITION 2.5. We will say that a connective is a *classically clear case* of negation/conjunction/disjunction/conditional if

[3] This is the "Philonian conditional" discussed many times in the relevance logic tradition. See for example [56].

1. its evaluation conditions are those of negation/conjunction/disjunction/ conditional in the logic **FDE**; or
2. its evaluation conditions are obtained from these by tweaking them; or
3. its evaluation conditions are classically redundant.

In the first two cases, the evaluation conditions will be called *non-redundant*.[4]

For example, the four negations above are clear cases of negations. Note that $\sim$ and $\sim_3$ are de Morgan negation and Boolean negation, respectively.

For definiteness, in this paper we will employ the following conditionals:

$A \to B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

$A \to_m B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

$A \to_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{\ \}$
$\{\ \}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$

$A \to_W B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

We refer to $\to$ and $\to_m$, respectively, as the extensional and material conditional, and to $\to_{AM}$ and $\to_W$, respectively, as the AM-conditional (for *Angell-McCall*) and W-conditional (for *Wansing*). We call 'extensional' all the usual connectives definable through positive Dunn atoms only. The 'material' conditional is, in a sense, the weakest and most intuitive modification of the extensional conditional that delivers the usually expected properties of a conditional, such as Identity—$A > A$—or Detachment—If $A > B$ and A, therefore, B—. Note that the extensional conditional is definable as $\sim A \vee B$, and the material conditional is definable in a similar way but with another negation, $\neg_b$ below, which is not available in the language of **FDE**.

These conditionals have the following evaluation conditions[5]:

$$1 \in \sigma(A \to B) \text{ iff } 0 \in \sigma(A) \text{ or } 1 \in \sigma(B)$$
$$0 \in \sigma(A \to B) \text{ iff } 1 \in \sigma(A) \text{ and } 0 \in \sigma(B)$$

[4]Note that we have only suggested a sufficient condition for non-redundant classically clear cases. There are many other clear cases that are obtained in different ways, for example, enriching the semantic machinery by allowing multiple non-empty indexes of evaluation.

[5]We make heavy use of the method in [42] to transform many-valued tables to evaluation conditions and vice versa.

$$1 \in \sigma(A \rightarrow_m B) \text{ iff } 1 \notin \sigma(A) \text{ or } 1 \in \sigma(B)$$
$$0 \in \sigma(A \rightarrow_m B) \text{ iff } 1 \in \sigma(A) \text{ and } 0 \in \sigma(B)$$

$$1 \in \sigma(A \rightarrow_{AM} B) \text{ iff both } 1 \notin \sigma(A) \text{ or } 1 \in \sigma(B) \text{ and } 0 \in \sigma(A) \text{ iff } 0 \in \sigma(B)$$
$$0 \in \sigma(A \rightarrow_{AM} B) \text{ iff } (0 \in \sigma(A) \text{ iff } 0 \notin \sigma(B))$$

$$1 \in \sigma(A \rightarrow_W B) \text{ iff } 1 \notin \sigma(A) \text{ or } 1 \in \sigma(B)$$
$$0 \in \sigma(A \rightarrow_W B) \text{ iff } 1 \notin \sigma(A) \text{ or } 0 \in \sigma(B).$$

According to the definitions given above, only the first and second connectives are clear cases of conditionals because the evaluation conditions of $\rightarrow$ are those of the conditional of **FDE**, which is definable in that logic in terms of (de Morgan) negation and disjunction, and the evaluation conditions of $\rightarrow_m$ are obtained by tweaking the truth condition of the conditional of **FDE**. Thus, the evaluation conditions of $\rightarrow$ and $\rightarrow_m$ are classically non-redundant. We also consider the following negations:

A	$\sim A$	$\neg_b A$
$\{1\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{\ \}$
$\{\ \}$	$\{\ \}$	$\{1,0\}$
$\{0\}$	$\{1\}$	$\{1\}$

Their evaluation conditions correspond, from left to right, to the evaluation conditions of the negations $\sim A$ and $\sim_3 A$ presented above, but for reasons that will be given later, we will use the notation of these last tables. These connectives are clear cases of negation because the evaluation conditions of $\sim$ are those of the negation of **FDE**, and the evaluation conditions of $\neg_b$ are obtained by tweaking both the truth and falsity conditions of the negation of **FDE**. Thus, the evaluation conditions of $\sim$ and $\neg_b$ are classically non-redundant.

Finally, we also consider the following connectives:

$A \wedge_e B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

$A \wedge_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$	$\{\ \}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$

These connectives have the following evaluation conditions:

$$1 \in \sigma(A \wedge_e B) \text{ iff } 1 \in \sigma(A) \text{ and } 1 \in \sigma(B)$$
$$0 \in \sigma(A \wedge_e B) \text{ iff } 0 \in \sigma(A) \text{ or } 0 \in \sigma(B)$$

$$1 \in \sigma(A \wedge_{AM} B) \text{ iff } 1 \in \sigma(A) \text{ and } 1 \in \sigma(B)$$

$$0 \in \sigma(A \wedge_{AM} B) \text{ iff } (0 \in \sigma(A) \text{ iff } 0 \notin \sigma(B)).$$

Only the first connective is a clear case of conjunction, as its evaluation conditions are classically non-redundant. Its evaluation conditions are those of the conjunction of **FDE**. Even though $\wedge_{AM}$ it is not a clear case of conjunction, one can make a case for its being a conjunction. Its truth condition is that of the extensional conjunction of **FDE**. On the other hand, $A \wedge_{AM} B$ is untrue if and only if A is untrue or B is untrue. This will be enough for us to treat '$\wedge_{AM}$' as a conjunction as well.[6]

As we mentioned in the introduction, the usual procedure to obtain a connexive logic is to take a well-known negation, Boolean negation or de Morgan negation, for example, and then change the evaluation conditions of a familiar conditional to validate the connexive schemas. Consider the approach by Angell [1] and McCall [31]. They aim at obtaining a (negation-) consistent connexive logic, and it can be easily seen that they keep Boolean negation untouched. Then, they work with the AM-conditional, but to further avoid contradictions, they also need a special conjunction. Thus, they work with the language $\mathcal{L}_{\{\neg_b, \wedge_{AM}, \rightarrow_{AM}\}}$. On the other hand, Wansing in [58] works on top of Nelson's **N4**, keeping de Morgan negation and as much as possible of the conditional. Else, working on top of material logic in [59], Wansing keeps de Morgan negation and as much as possible of the material conditional $\rightarrow_m$. In fact, the W-conditional presented above is like the material conditional, with the exception that the falsity condition of the latter, i.e.

- $0 \in v(A \rightarrow_m B)$ iff $1 \in v(A)$ and $0 \in v(B)$

is changed to the following one:

- $0 \in v(A \rightarrow_W B)$ iff $1 \notin v(A)$ or $0 \in v(B)$.

Moreover, in Wansing's approach, (negation-)inconsistency is not a problem, only non-triviality is aimed at.

[6] A more systematic analysis of the conditions to say that two connectives are of the same kind is needed, of course. We decided not to complicate things for conjunction here, but we will be more careful with negation below. For such a more systematic analysis and comparison of connectives, see [17].

3. Presenting Classical Negation, Anti-Classical Negation, Falsity Negation and Truth Negation

So much for the preliminaries. In this section, we discuss four families of negation-related connectives in order to approach the topic of negation-driven connexivity. According to [6] and [43], there is no unique unary connective one can expand the language of **FDE** with, and that can be properly called 'classical negation'. In this context, classical negation can be represented by any of the following sixteen connectives[7]:

A	$\neg_b A$	$\neg_e A$	$\neg_1 A$	$\neg_2 A$	$\neg_3 A$	$\neg_4 A$	$\neg_5 A$	$\neg_6 A$
$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$
$\{1,0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$

A	$\neg_7 A$	$\neg_8 A$	$\neg_9 A$	$\neg_{10} A$	$\neg_{11} A$	$\neg_{12} A$	$\neg_{13} A$	$\neg_{14} A$
$\{1\}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{\ \}$
$\{1,0\}$	$\{\ \}$	$\{0\}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$

For the sake of simplicity, let us suppose for the moment that the distinctiveness of classical negation, represented here by $\neg$, is the following truth condition:

$$1 \in v(\neg A) \text{ iff } 1 \notin v(A).$$

This is the truth condition of Boolean negation, the connective $\neg_b$ given in the preliminaries and in the tables above. It is is easy to see that all the unary connectives just introduced have this truth condition.

But one can overcome the bias towards truth, put forward a dual idea and say that the distinctiveness of classical negation is the following falsity condition:

$$0 \in v(\neg A) \text{ iff } 0 \notin v(A)$$

that is the falsity condition of Boolean negation. Then one can obtain sixteen new unary connectives. With the only exception of $\neg_b$, we distinguish these connectives from the previous ones by inverting the symbol for Boolean

[7] See [6] for criticisms on why some of these connectives cannot be considered classical negations.

negation and putting it backwards, i.e., $⌐ A$, attaching the corresponding subscripts[8]:

A	$⌐_b A$	$⌐_e A$	$⌐_1 A$	$⌐_2 A$	$⌐_3 A$	$⌐_4 A$	$⌐_5 A$	$⌐_6 A$
$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{1,0\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{1\}$

A	$⌐_7 A$	$⌐_8 A$	$⌐_9 A$	$⌐_{10} A$	$⌐_{11} A$	$⌐_{12} A$	$⌐_{13} A$	$⌐_{14} A$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$
$\{1,0\}$	$\{\ \}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$

Instead of the evaluation conditions of Boolean negation, one can inquire into the evaluation conditions of de Morgan negation. Recall its truth and falsity conditions:

$$1 \in v({\sim} A) \text{ iff } 0 \in v(A)$$
$$0 \in v({\sim} A) \text{ iff } 1 \in v(A).$$

Each condition defines again two groups of sixteen unary connectives. As in the previous cases, in what follows we use $\sim$ with appropriate subscripts to denote the connectives that have the truth condition of de Morgan negation and $\backsim$, with appropriate subscripts, to denote the connectives that have the falsity condition of de Morgan negation[9]:

Unary connectives with the truth condition of de Morgan negation

A	$\sim_m A$	$\sim_e A$	$\sim_1 A$	$\sim_2 A$	$\sim_3 A$	$\sim_4 A$	$\sim_5$	$\sim_6 A$
$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$

A	$\sim_7 A$	$\sim_8 A$	$\sim_9 A$	$\sim_{10} A$	$\sim_{11} A$	$\sim_{12} A$	$\sim_{13} A$	$\sim_{14} A$
$\{1\}$	$\{\ \}$	$\{0\}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1,0\}$

[8]Note that $\neg_b$ and $⌐_b$ are the same connective.

[9]Note that, like in the previous cases, $\backsim_m$ and $\sim_m$ are the same connective.

Unary connectives with the falsity condition of de Morgan negation

A	$\backsim_m A$	$\backsim_e A$	$\backsim_1 A$	$\backsim_2 A$	$\backsim_3 A$	$\backsim_4 A$	$\backsim_5 A$	$\backsim_6 A$
$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$

A	$\backsim_7 A$	$\backsim_8 A$	$\backsim_9 A$	$\backsim_{10} A$	$\backsim_{11} A$	$\backsim_{12} A$	$\backsim_{13} A$	$\backsim_{14} A$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{1\}$
$\{0\}$	$\{\ \}$	$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$

After the study of connectives with the same truth condition as Boolean negation presented in [6], it was just a matter of time to consider connectives with the truth condition of de Morgan negation, and then also to dualize the approach and focus on falsity conditions. This was done independently in [38, 39] and [44]. In the latter, and although they do not cover the connectives based on falsity conditions, plenty of results about the $\sim_i$'s added to **FDE**, **K3** and **LP** are presented.

According to Avron [2, p. 160], the truth condition of de Morgan negation "represents the idea of falsehood within the language", that is, the negation of A expresses that A is false. If this is correct, by parity of reasoning, the falsity condition of de Morgan negation would "represent the idea of truth within the language" in the sense that a formula A expresses its falsity, and when one negates it, one obtains its truth.[10]

Definition 3.1. We will say that a unary connective N is *negative* if

1. it has the truth or falsity condition of a clear case of a negation; or,
2. its evaluation conditions have the following structure:

$$v \in \sigma(NA) \text{ iff } \textit{condition}_1 \text{ and } \textit{condition}_2$$

where $\textit{condition}_1$ and $\textit{condition}_2$ are Dunn atoms, either $\textit{condition}_1$ or $\textit{condition}_2$ is the truth or falsity condition of a clear case of negation, and $\textit{condition}_1$ does not contradict $\textit{condition}_2$.

Thus, all the tables given above in this section are tables of negative connectives as they meet these requirements. Now, according to the terminology

[10]Studies on paraconsistency favoring falsity conditions over truth conditions can be found in [32, Ch. 11] [33] and then continued in [11] and [12].

introduced in the preliminaries, not all negative connectives are (classically) clear cases of negations. There are twelve clear cases of negations, given by $\neg_b$ (or $\ulcorner_b$), $\neg_e$ (or $\backsim_e$), $\neg_1$, $\neg_2$, $\ulcorner_e$ (or $\sim_e$), $\ulcorner_1$, $\ulcorner_2$, $\sim_m$ (or $\backsim_m$), $\sim_1$, $\sim_2$, $\backsim_1$ and $\backsim_2$. The connectives $\neg_b$, $\neg_e$ (or $\backsim_e$), $\ulcorner_e$ (or $\sim_e$) are obtained by tweakings, and the evaluation conditions of the other connectives are classically redundant.[11]

4. The Idea of a Connexive Negation

Remember that, under the now standard notion of connexive logic, there is a negative condition to be met to ensure that we are working with a conditional and not with a biconditional. However, that might not be enough, because there are other binary connectives that can be confused with a connexive conditional, in particular, conjunction. Omori has argued in [40] that the alleged conditional in $\mathbf{MRS}^P$, studied in [10] and [14], is in fact a conjunction. Therefore, some extra conditions are required in order to properly characterize a conditional. Likewise, there are a few extra negative conditions to be met to ensure that we are working with a negation and not with any other unary connective, especially a unary connective that delivers formulas that are always true.

According to some recent theories (see for instance [41], [47, Ch. 4]), a negation has to form contradictory pairs, i.e., A and NA have to be contradictories. Recall that A and B are *contradictories* iff $1 \in \sigma(A)$ iff $0 \in \sigma(B)$ and $0 \in \sigma(A)$ iff $1 \in \sigma(B)$.[12] In such an account, a negation has to meet the following negative conditions:

$$\text{(Non-verifier) There is an } A \text{ such that } A \not\models NA$$
$$\text{(Non-antiverifier) There is an } A \text{ such that } NA \not\models A$$

and the following positive conditions, for any formula A:

$$\text{(EDN) } NNA \models A$$
$$\text{(IDN) } A \models NNA.$$

(Marcos [28] considered the negative conditions as minimal requirements for a negation in any logic that is non-degenerate, such as empty or overfilled logics.)

[11] However, this does not imply that the other connectives not mentioned above are not negations, just that they are not classically clear cases of negations.

[12] We take this formulation using Dunn atoms from [41, p. 115].

As we said above, the negative conditions make sure that negation is not any other unary connective, similarly as the invalidation of Symmetry of Implication is meant to secure that the conditional is not a biconditional. Since the consequence relation is truth-preserving, the negative conditions ensure that we are not speaking of other disguised unary connectives $\mp$, $\pm$ and $\equiv$, with the following evaluation conditions:

(Taut) $1 \in \sigma(\mp A)$ iff $1 \in \sigma(A)$ or $1 \notin \sigma(A)$ (that is, $\mp A$ is always true, or equivalently it is true under any interpretation.)[13]

(Antaut) $1 \notin \sigma(\pm A)$ iff $1 \in \sigma(A)$ or $1 \notin \sigma(A)$ (that is, $\mp A$ is never true, or equivalently it is not true under any interpretation.)

(Equa) $v_i \in \sigma(\equiv A)$ if and only if $v_i \in \sigma(A)$

for any $i \in \{1, 0\}$. In other words, unary connectives that, when they are the main connective in a formula, make it true under any interpretation—this condition is obtained by (Non-verifier)—, or that make it untrue under any interpretation —this condition is obtained by (Non-antiverifier)—or that do not alter its evaluation—this condition is obtained by both (Non-verifier) and (Non-antiverifier).[14]

Now, we consider the case where the extensional conditional $\rightarrow$ appears uniformly in the connexive schemas, i.e.:

$$N(A \rightarrow NA) \tag{A1e}$$

$$N(NA \rightarrow A) \tag{A2e}$$

$$(A \rightarrow B) \rightarrow N(A \rightarrow NB) \tag{B1e}$$

$$(A \rightarrow NB) \rightarrow N(A \rightarrow B) \tag{B2e}$$

and the case where the material conditional $\rightarrow_m$ appears uniformly in the connexive schemas:

$$N(A \rightarrow_m NA) \tag{A1m}$$

$$N(NA \rightarrow_m A) \tag{A2m}$$

[13]It might be replied that a connective that is always true might be a negation. Such always-true connective has the falsity condition of a negation and satisfies some properties that one can consider as pertaining to any negation. What one would actually require is a condition that ensures that one is not speaking of a unary connective which is *just true* under any interpretation, but we will not press this point further and assume (Non-verifier) as a necessary condition for a negation.

[14]In fact, as we are concerned with validating the connexive schemas—and not antiverifying them, making them untrue in all interpretations—we are already assuming (Antaut).

$$(A \to_m B) \to_m N(A \to_m NB) \tag{B1m}$$

$$(A \to_m NB) \to_m N(A \to_m B) \tag{B2m}$$

Proposition 4.1. *No unary connective presented in Section 3 can validate all the connexive schemas in the company of the extensional conditional and also reject either (Taut), (Antaut) or (Equa).*

Proof. For the countermodels below we will ignore $\ulcorner_5$ and $\backsim_5$ since they do not reject (Non-verifier).

- For a countermodel to (A1e) with all the $\neg$'s, all the $\sim$'s and $\ulcorner_b$, $\ulcorner_e$, $\ulcorner_2$, $\ulcorner_4$, $\ulcorner_{10}$, $\ulcorner_{13}$, $\ulcorner_{14}$, $\backsim_m$, $\backsim_e$, $\backsim_2$, $\backsim_3$, $\backsim_4$ or $\backsim_8$, consider the case when $\sigma(A) = \{\ \}$.
- For a countermodel to (A1e) with $\ulcorner_1$, $\backsim_1$, $\backsim_9$ or $\backsim_{10}$, consider the case when $\sigma(A) = \{1, 0\}$.
- For a countermodel to (A1e) with $\ulcorner_3$, $\ulcorner_6$, $\ulcorner_7$, $\ulcorner_{11}$, $\ulcorner_{12}$, $\ulcorner_{13}$, $\backsim_6$, $\backsim_{11}$, $\backsim_{12}$, $\backsim_{13}$ or $\backsim_{14}$, consider the case when $\sigma(A) = \{1\}$.
- For a countermodel to (B2e) with $\backsim_7$ or $\ulcorner_8$ consider the case when $\sigma(A) = \{1\}$ and $\sigma(B) = \{0\}$.
- For a countermodel to (B2e) with $\ulcorner_9$ consider the case when $\sigma(A) = \{\ \}$ and $\sigma(B) = \{\ \}$.

To put it boldly, granting the extensional conditional, no negative connective is a connexive negation.

Things are different with the material conditional, as one can validate the connexive schemas using some negative connectives from Section 3 that also reject (Taut), (Antaut) or (Equa).

Proposition 4.2. *The only connectives from Section 3 that can stand in place of N to validate the connexive schemas in the company of the material conditional and also reject (Taut), (Antaut) or (Equa) are $\ulcorner_8$, $\ulcorner_9$, $\backsim_7$ and $\backsim_8$.*

Proof. We provide countermodels to all the connectives that are not $\ulcorner_8$, $\ulcorner_9$, $\backsim_7$ and $\backsim_8$; that these validate the connexive schemas can be easily verified through truth tables.

- For a countermodel to (A1m) with all the $\neg$'s, all the $\sim$'s, $\ulcorner_b$, $\ulcorner_e$, $\ulcorner_1$, $\ulcorner_2$, $\ulcorner_4$, $\ulcorner_6$, $\ulcorner_{10}$, $\ulcorner_{13}$, $\backsim_m$, $\backsim_e$, $\backsim_1$, $\backsim_2$, $\backsim_4$, $\backsim_6$, $\backsim_{11}$ or $\backsim_{12}$ consider the case when $\sigma(A) = \{0\}$.

- For a countermodel to (A1m) with $\neg_3$, $\neg_7$, $\neg_{11}$, $\neg_{12}$, $\backsim_9$, $\backsim_{10}$, $\backsim_{13}$ or $\backsim_{14}$, consider the case when $\sigma(A) = \{1\}$. Again, we do not consider $\neg_5$ and $\backsim_5$ as they do not reject (Non-verifier).

$\neg_8$, $\neg_9$, $\backsim_7$ and $\backsim_8$ fail to satisfy the positive requirement (EDN)—just consider the case where $A = \{0\}$—, yet all of them satisfy (IDN). However, if the problem is to keep both principles of double negation, one can verify that these connectives satisfy at least the following weak versions:

$$\text{If} \models NNA, \text{ then} \models A.$$
$$\text{If} \models A, \text{ then} \models NNA.$$

Thus, any of $\neg_8$, $\neg_9$, $\backsim_7$ and $\backsim_8$ can do the job of being a connexive negation with the material conditional. Note that, as we said in the previous section, these connectives are not clear cases of negations. Strictly speaking, they are non-(classically)-clear cases of connexive negations.

As it is known at least since the works by Dunn [7] —see also [8,49,53]—, the double negation principles are relatively strong, so demanding them for a connective to be a negation might be too much. We agree, the negative requirements suggested by Marcos or even Contraposition in Dunn's kite:

$$\text{If } A \models B \text{ then } NB \models NA$$

would be enough. Nonetheless, several of our unary connectives satisfy the stronger requirements, and those that do not, satisfy the weaker versions that we have just mentioned—even Contraposition—so we do not think it is necessary to consider weaker principles here. Of course, that would be interesting in other contexts where the expressible unary connectives are weaker than those presented here. Finally, it goes without saying that Sylvan (see [57, p. 312]) considered the possibility of requiring that a negation satisfied other principles, among them the connexive schemas; our negations would meet this requirement by default.

Let us make clear our position about the previous result. That certain unary connectives are not (classically)-clear cases of negations is not a negative result in itself. We have shown that the previous connectives meet the minimal requirements for being a negation. If we wanted to say something more about the nature of such connectives, especially by comparing them to other connectives, we would need to be more cautious about the underlying notion of logical consequence. Commonly, the standard properties of a connective depend heavily on the consequence relation adopted; this is particularly important to determine whether there are certain relations holding among connectives and to evaluate whether a connective is similar to another. For example, it has recently been argued—see [9], [15]—that certain connectives, like transplication[15], are closer to its intended connective—in this case, an implication—when the entailment relation is not Tarskian, in particular, when it is non-transitive, because with a Tarskian consequence relation it seems more a conjunction.

In general terms, the idea is that, in order to clarify the nature of a connective by comparing it with others, we probably would need to verify whether a clear case of a given connective—say, conjunction—entails or is entailed by a non-clear case of that connective. Until now, we assumed the usual Tarskian consequence relation. But this is not necessary. Let us suppose that the underlying consequence relation is p-consequence:
A is a p-logical consequence of Σ, in symbols, $\Sigma \models^p_{\mathbf{L}} A$, if and only if for all $B \in \Sigma$, if $1 \in \sigma(B)$ or $0 \notin \sigma(B)$ then $1 \in \sigma(A)$ or $0 \notin \sigma(A)$.
Then, one can make sense more easily of some of the previous unary connectives as negations, as some of them entail or are entailed by a clear case of negation. A more detailed and systematic description of the interaction of different consequence relations, connectives, and evaluation conditions would be needed, though.[16] We leave this point aside and continue using the usual Tarskian consequence relation.

Let us note in passing that these unary connectives do not exclude a connexive conditional, in the sense that there are languages containing any of these four negative connectives and either Wansing's conditional or Angell-McCall's conditional that validate the connexive schemas. Here we summarize the combinations that deliver the validity of the connexive schemas:

N	$\to_{AM}$	$\to_W$
$\urcorner_8$	×	✓
$\urcorner_9$	✓	✓
$\backsim_7$	×	✓
$\backsim_8$	×	✓

[15] A connective introduced by that name has been introduced in [5] through a table structurally identical to the following one:

$A \to_\varphi B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$

Nonetheless, this connective appeared earlier, in [48], and it is nowadays more well-known as the de Finetti conditional, see [9] and the references therein.

[16] A detailed discussion of this issue can be found in [17].

Definition 4.1. A negation N is *connexively stable* with respect to a family of connexive conditionals $COND_{cnnxv}$ iff
$\models N(A >_i N\ A)$
$\models N(N\ A >_i A)$
$\models (A >_i B) >_i N\ (A >_i N\ B)$
$\models (A >_i N\ B) >_i N(A >_i B)$

for any $>_i \in COND_{cnnxv}$.

Since it is already assumed that $>_i$ is a connexive conditional, there is no need to mention the Non-Symmetry condition.[17]

With a toy but still illustrative example, consider the case where the family of connexive conditionals has only two members: Wansing's and Angell-McCall's. Then, only $\neg_9$ is connexively stable with respect to that family of connexive conditionals. For this reason we express our preference for the negation $\neg_9$.

The results in this section suggest the following picture about connexivity: it depends on both negation and the conditional, not only on one of them. This is not news. What is new is the right emphasis on *both*. In the usual approaches, the conditional plays the most important role; it almost seems that what is at the heart of connexivity is the conditional. And in this context, that view seems to be supported by **Proposition 1**. The extensional conditional is not regarded by many as a real conditional, and it would be no wonder that, as per **Proposition 1**, none of the negations produce connexive validities because we did not have a conditional to begin with, it was merely the extensional disjunction in disguise.

Nonetheless, we have also shown that, if some specific negations do not accompany a conditional better than the extensional conditional, it might not validate the connexive schemas, either. The example in this case is the conditional by Angell-McCall, that validates the connexive schemas only with some negations. The reply here is that the Angell-McCall conditional is rather a biconditional. Thus, in the end it would not be better than the extensional conditional, so the proof that our negations do not deliver the connexive schemas when combined with the Angell-McCall conditional carries little weight. We do not agree with the claim that the Angell-McCall conditional is not a conditional, but let that pass.

Someone could say that the Wansing conditional in fact shows that what is at the heart of connexivity is the conditional, because it is stable under

[17] One can also define a *connexively stable conditional* $>$ with respect to some class of negations, as in [13]. In fact, we borrow the terminology from there.

changes of negation. This is a moot point, though. The Wansing conditional internalizes, so to speak, a particular conception of negation—at least, of the negation of a conditional—in its falsity condition. Thus, it is not entirely clear that in this case the conditional is more important than negation. What would be in need of explanation is why such a conception of negation internalized in the falsity condition is connexively stable under changes of negation. We admit not having an answer to that.

5. Our Connexive Negations and the Negations as Cancellation Account

A rather different approach to obtaining a connexive logic by looking at the negation, instead of the conditional, has been associated with the cancellation account of negation. The *cancellation account of negation* is the view according to which the pair A, NA has no content, since the contents of A and NA "cancel" or "erase" each other. (See [50–52].) In all fairness, nobody knows what 'content' amounts to here, but let us assume that the contents of A and NA cancel each other. If one couples the cancellation account of negation with the idea that a valid implication is one in which the content of the consequent is included in the content of the antecedent, and makes explicit that all propositions have some content—as the ancient and medieval thinkers might have assumed (see [55])—then neither A can imply NA nor vice versa, since the consequents in those implications cancel the contents of their antecedents. For the same reason, if A implies B, NB cannot be part of the content of A. Thus, one arrives at the connexive schemas.

Among the distinctiveness of the cancellation account of negation is the invalidation of Explosion $(A \triangle NA) > B$, and Simplification, $(A \triangle NA) > A$ or $(A \triangle NA) > NA$, because, as explained above, the antecedents cancel themselves, so there is nothing that implies the consequent. Additionally, the invalidation of Simplification is motivated by the fact that in the presence of (B2e), Simplification leads to inconsistency. Wansing and Skurt [60, p. 482] also suggest that the cancellation account of negation can motivate Abelard's First Principle, $N((A > B) \triangle (A > NB))$ and Aristotle's Second Thesis, $N((A > B) \triangle (NA > B))$. That is so because the former involves the negation of instances of Simplification, $((B \triangle NB) > B) \triangle ((B \triangle NB) > NB))$, and dually, the latter involves the negation of instances of Addition, both of which are rejected in the cancellation account, as we have remarked above.[18]

[18]The Routleys' proposal has some other distinctive features. Let $\oplus$ be a disjunction. Accepting Contraposition $(A > B) > (NB > NA)$ and all the de Morgan principles also

However, in [60] the authors pointed out several conceptual problems with the cancellation account of negation in the Routleys' version, and have argued that it is better not to consider it a conceptual basis of connexive logic. They observe that if the cancellation account of negation were correct, $A \triangle NA$ would have no content, but then the external N in $N(A \triangle NA)$ would be operating upon no content at all. However, this goes against the assumption that all negations operate over some content. Moreover, $N(A \triangle NA)$ could not cancel $A \triangle NA$ because $A \triangle NA$ would cancel itself, so there would not be anything to cancel to begin with.

Our proposal is that one can obtain connexive logics by modifying the evaluation conditions of negations in such a way that, coupled with the evaluation conditions of a standard conditional, would deliver the validation of the connexive schemas, and that one can do so without assuming that such connexive negations represent some cancellationist ideas. Thus, in what follows we examine to what extent our connexive negations give similar results to the Routleys' idea of connexive negation. In particular, we want to know whether we can detach the idea of connexive negation from the failure of Simplification and the validation of Abelard's First Principle and Aristotle's Second Thesis.[19]

Remark 5.1. Simplification is valid with either the material or the extensional conditional, Angell-McCall's conjunction $\wedge_{AM}$ or extensional conjunction $\wedge_e$, and $\urcorner_8$ or $\urcorner_9$:[20]

- $\models (A \triangle_{\{\wedge_{AM}, \wedge_e\}} N_{\{\urcorner_8, \urcorner_9\}} A) >_{\{\rightarrow, \rightarrow_m\}} A$
- $\models (A \triangle_{\{\wedge_{AM}, \wedge_e\}} N_{\{\urcorner_8, \urcorner_9\}} A) >_{\{\rightarrow, \rightarrow_m\}} N_{\{\urcorner_8, \urcorner_9\}} A$.

However, by using the connexive negations $\backsim_7$ and $\backsim_8$ one obtains different results. All instances of Simplification are valid just with the material conditional:

lead to the rejection of Addition $A > (A \oplus B)$ and $B > (A \oplus B)$ (see [60, p. 480]). Nowadays, it is widely agreed that connexivity has nothing to do with the failure of Simplification, much less of Addition, connexive logic being characterized as we reported at the beginning of Section 1.

[19] These principles tend to divide connexivists. While in Angell-McCall's **CC1** Simplification and Addition are invalid but Abelard's First Principle and Aristotle's Second Thesis are valid, in Wansing's **C**, Simplification and Addition are valid, but Abelard's First Principle and Aristotle's Second Thesis are not.

[20] In the following we use the notation $(A \triangle_{\{\wedge_{AM} \wedge_e\}} N_{\{\urcorner_8, \urcorner_9\}} A) >_{\{\rightarrow, \rightarrow_m\}} N_{\{\urcorner_8, \urcorner_9\}} A$ to indicate that from the connectives that appear between the formulas one can uniformly choose any of them. For instance, if one chooses $\urcorner_8$ to appear in the antecedent, it also needs to appear in the consequent.

- $\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow_m A$
- $\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow_m N_{\{\backsim_7,\backsim_8\}}A$

The same schemas, but with the extensional conditional, are invalid, though. Just take A to be neither true nor false, $\{\ \}$:

- $\not\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow A$
- $\not\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow N_{\{\backsim_7,\backsim_8\}}A$.

Finally, the problem of inconsistency mentioned above for Simplification in the light of (B2e) arises for certain combinations of conditionals, conjunctions and negations.

Proposition 5.1.

$$(A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}A \rightarrow_m N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}A) \rightarrow_m N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}(A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}A \rightarrow_m A)$$

and

$$(A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}A) \rightarrow_m N_{\{\backsim_7,\backsim_8,\urcorner_8,\urcorner_9\}}A$$

leads to inconsistency—following the standard proof by the use of modus ponens. However, something similar does not happen by using the extensional conditional, since it is possible to find countermodels to $(A\triangle_{\{\wedge_{AM},\wedge_e\}} N_{\{\backsim_7,\backsim_8\}}A) \rightarrow A$*—just consider the case when* $\sigma(A) = \{\ \}$*. The extensional conditional in the company of either* $\urcorner_8$ *or* $\urcorner_9$*, and* $\wedge_{AM}$ *or* $\wedge_e$ *in the schemas above leads to inconsistency, notwithstanding.*

Proposition 5.2. *Consider the following argument:*

1. $((A\triangle NA) > A) > N((A\triangle NA) > NA)$	*Boethius*
2. $(A\triangle NA) > A$	*Simplification*
3. $(A\triangle NA) > NA$	*Simplification*
4. $N((A\triangle NA) > NA)$	*1, 2, Detachment*
5. $((A\triangle NA) > NA)\triangle N((A\triangle NA) > NA)$	*3, 4, Adjunction*

The previous argument is valid using $\rightarrow_m$*, in company of* $\urcorner_8$*,* $\urcorner_9$*,* $\backsim_7$ *or* $\backsim_8$*, and* $\wedge_e$ *or* $\wedge_{AM}$*. With respect to the conditional* $\rightarrow$*, the argument is valid in company of the negations* $\urcorner_8$ *or* $\urcorner_9$*, and* $\wedge_{AM}$*. However, it is not valid using* $\backsim_7$ *or* $\backsim_8$*, and* $\wedge_e$ *or* $\wedge_{AM}$*—just consider the case when* $\sigma(A) = \{\ \}$*.*

Remark 5.2. With respect to Abelard's First Principle and Aristotle's Second Thesis, only the former is valid with either Angell-McCall's or extensional conjunction, material conditional, and the negation $\urcorner_9$.

$\models_{\ulcorner_9} ((A \rightarrow_m B)\triangle_{\{\wedge_{AM},\wedge_e\}}(A \rightarrow_{m\ulcorner_9} B))$.
All other cases are rendered invalid:

- $\not\models N_{\{\ulcorner_8,\ulcorner_9,\backsim_7,\backsim_8\}}((A >_{\{\rightarrow_m,\rightarrow\}} B)\triangle_{\{\wedge_{AM},\wedge_e\}}(N_{\{\ulcorner_8,\ulcorner_9,\backsim_7,\backsim_8\}}A >_{\{\rightarrow_m,\rightarrow\}} B))$
- $\not\models N_{\{\ulcorner_8,\backsim_7,\backsim_8\}}((A >_{\{\rightarrow_m,\rightarrow\}} B)\triangle_{\{\wedge_{AM},\wedge_e\}}(A >_{\{\rightarrow_m,\rightarrow\}} N_{\{\ulcorner_8,\backsim_7,\backsim_8\}}B))$

For a countermodel to these schemas consider either the case when $\sigma(A) = \{1\}$ and $\sigma(B) = \{\ \}$, or the case when $\sigma(A) = \{1\}$ and $\sigma(B) = \{0\}$, or else the case when $\sigma(A) = \{\ \}$ and $\sigma(B) = \{0\}$.

Remark 5.3. Finally, for any of our negations, the truth of NA does not exclude the truth of A, nor vice versa, since NA is true in the same conditions under which A is true, i.e., $\{1\}$ or $\{1, 0\}$. This also has consequences on the validity of Explosion. Explosion is valid with either Angell-McCall's conjunction or extensional conjunction, the extensional conditional and the negations $\ulcorner_8$ and $\ulcorner_9$:

- $\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\ulcorner_8,\ulcorner_9\}}A) \rightarrow B$

In all other cases, it is invalid:

- $\not\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\ulcorner_8,\ulcorner_9\}}A) \rightarrow_m B$
- $\not\models (A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) >_{\{\rightarrow_m,\rightarrow\}} B$

For a countermodel to $(A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow B$ consider the case when $\sigma(A) = \sigma(B) = \{\ \}$. For a countermodel to the formulas $(A\triangle_{\{\wedge_{AM},\wedge_e\}} N_{\{\ulcorner_8,\ulcorner_9\}}A) \rightarrow_m B$ and $(A\triangle_{\{\wedge_{AM},\wedge_e\}}N_{\{\backsim_7,\backsim_8\}}A) \rightarrow_m B$ consider the case when $\sigma(A) = \{1\}$ and $\sigma(B) = \{\ \}$.

6. Compatibility and Connexivity

According to a traditional view that can be traced back to Chrysippus (see [54] and the references therein), the connexive schemas arise from certain (in)compatibility relations between the antecedent and (the negation of) the consequent. In such a view, an implication is valid if and only if the negation of the consequent is incompatible with the antecedent. Let $\circ$ denote a (binary) compatibility connective, such that $(A \circ B)$ is read, for example, as "A is compatible with B", "A does not exclude B", "the realizability of A does not exclude the realizability of B" or "the truth of A does not exclude the truth of B". Then, according to the above,

$$(A > B) =_{def} N(A \circ NB)$$

We started with conditionals and negations instead, and we would like to know which compatibility connectives can be defined in terms of those. Our strategy is using the following usual definition derivable from the previous one:

$$(A \circ B) =_{def} N(A > NB)$$

For example, in the Angell-McCall approach, the evaluation conditions of $A \circ_{AM} B$ are as follows:

$A \circ_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$

Accordingly, in Wansing's approach, the evaluation conditions of $A \circ_W B$ are as follows:

$A \circ_W B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

Here, we examine the compatibility connectives arising not from standard negations and connexive conditionals, but again using standard conditionals and using our four connexive negations. We define those compatibility connectives as follows:

$A \circ_{\ulcorner_8} B =_{def} \ulcorner_8 (A \rightarrow_m \ulcorner_8 B)$

$A \circ_{\ulcorner_9} B =_{def} \ulcorner_9 (A \rightarrow_m \ulcorner_9 B)$

$A \circ_{\backsim_7} B =_{def} \backsim_7 (A \rightarrow_m \backsim_7 B)$

$A \circ_{\backsim_8} B =_{def} \backsim_8 (A \rightarrow_m \backsim_8 B)$

$A \circ_{\ulcorner_8} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

$A \circ_{\ulcorner_9} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

$A \circ_{\backsim_7} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

$A \circ_{\backsim_8} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1,0\}$	$\{1,0\}$	$\{\ \}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{\ \}$	$\{1,0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

Perhaps the more unusual connectives are $\circ_{\ulcorner_8}$, $\circ_{\ulcorner_9}$, $\circ_{\backsim_7}$, because they are true in all interpretations. In particular, this implies that $A \circ_{\ulcorner_8} N\ A$, $A \circ_{\ulcorner_9} N\ A$ and $A \circ_{\backsim_7} N\ A$ are true in all interpretations.

Can we do better in comparing these different compatibility connectives? Yes, we can. Pizzi [45] put forward some criteria for a compatibility connective.[21] According to him, a compatibility connective must satisfy the following properties:

$$\text{For all } A, \models A \circ A \qquad \text{(Reflexivity)}$$

$$\text{For all } A \text{ and } B, \models ((A > B) \rightarrow_m (A \circ B)) \qquad \text{(Subalternation)}$$

$$\text{For all } A \text{ and } B, \not\models (A \circ B) \rightarrow_m (A > B) \qquad \text{(Non-collapse)}$$

$$\text{For all } A, \models \Diamond A \circ A \qquad \text{(Modal compatibility)}$$

where $\Diamond$ is a unary connective for possibility. These requirements express, respectively, that the truth of A does not exclude the truth of A; that if A implies B, the truth of A does not exclude the truth of B, but the fact that the truth of A does not exclude the truth of B does not mean that A implies B; finally, that the possibility of A's truth does not exclude that A is (actually) true.

Pizzi also set forth the following intuitive requirement on a compatibility connective:

$$\text{For all } A, \models N(A \circ NA) \qquad \text{(Consistency)}$$

Nonetheless, as it makes $>$ reflexive, Pizzi considers (Consistency) a superfluous feature of a compatibility connective, derivable from the properties of a conditional.[22]

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[21] These properties have been originally set forth with respect to a consistent logic, which contains classical zero-order logic and is closed under replacement of material equivalents and uniform substitution. Nonetheless, it is clear that these requirements can be investigated for any kind of logic.

[22] $A > A$ certainly holds for the conditionals we are working with. Nonetheless, $A > A$ has failed in connexive settings. When the arrow is understood in certain ways, there might be implicative logical truths without $A > A$ being among them. For example, in Goddard and Sylvan's works on reason-giving conditionals and non-ponible reasoning—see for example [22]—, $A \ni B$, "A is a reason for B", is analyzed as "A (relevantly) implies B and A is prior enhancing information to B". This delivers Aristotle's Thesis, for A is not a reason against itself, but not $A \ni A$ (in general, a proposition is not a reason for itself). In that framework, (Consistency) might cause some trouble.

It is easy to check whether the compatibility connectives defined in this section meet Pizzi's requirements. Whereas the property of (Modal compatibility) had some purpose for Pizzi's discussion on consequential implication—an implication $\rightarrow_{ci}$ interpreted in modal terms that validates so-called '(Weak) Boethius' Thesis' $((A \rightarrow_{ci} B) \rightarrow_m \neg_b(A \rightarrow_{ci} \neg_b B))$—we leave it out in the consideration of our demodalized languages. The results are summarized in the following table:

	$\circ_{\neg_8}$	$\circ_{\neg_9}$	$\circ_{\backsim_7}$	$\circ_{\backsim_8}$
(Reflexivity)	✓	✓	✓	✓
(Subalternation)	✓	✓	✓	✓
(Non-collapse)	✓	✓	✓	✓
(Consistency)	✓	✓	✓	✓

Thus, all the previous connectives pass the Pizzi test for being compatibility connectives.

7. Conclusions

In this paper we considered a common view of connexive logic according to which the validation of the so called "connexive schemas" (Aristotle's Thesis and its variant, and Boethius' Thesis and its variant) and the invalidation of Symmetry of Implication constitute sufficient conditions for a connexive logic and, derivatively, to have a connexive conditional.

The standard way to obtain a connexive conditional is by taking a known negation and then modifying the conditional to validate the connexive schemas. We explored the converse situation: obtain a connexive negation by taking a known conditional—the extensional conditional or material conditional in this case—and then modifying the negation in such a way as to validate the connexive schemas. Our exploration led us to consider the cancellation account of negation, and compare its properties with our connexive negations. To round our analysis of the connexive negations, we connected the connexive schemas with the notion of (in)compatibility; we defined some connectives of compatibility by means of our connexive negations and discussed its adequacy with respect to some criteria given by Pizzi.

Finally, this sort of investigation opens some avenues for further research. One of the most obvious is whether the satisfaction of the connexive schemas is a sufficient condition for a non-classical negation be taken as connexive, or whether the shift of focus to negation makes necessary to consider some other conditions, even detached from conditionals. Another topic for future work is approaching the issue from the modal theories of negation; for example, using those in [3, 4, 29, 34–37]. Finally, one could employ other semantic frameworks already used to study connexivity, for example, relating semantics (cf. [23–27]), and try to adapt them to focus connexivity on negation. Negation in general remains understudied in the relating framework, and this could be a good topic for further testing of the machinery.

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Angell, R.B., A propositional logic with subjunctive conditionals, *Journal of Symbolic Logic* 27(3):327–343, 1962.

[2] Avron, A., A non-deterministic view on non-classical negations, *Studia Logica* 80:159–194, 2005.

[3] Berto, F., A modality called 'negation', *Mind* 124(495):761–793, 2015.

[4] Berto, F., and G. Restall, Negation on the Australian Plan, *Journal of Philosophical Logic* 48(6):1119–1144, 2018.

[5] Blamey, S., Partial logic, in D. Gabbay, and F. Guenthner, (eds.), *Handbook of Philosophical Logic. Vol. III, Alternatives to classical logic*, Kluwer, Dordrecht, 1986, pp. 1–70.

[6] De, M., and H. Omori, Classical negation and expansions of Belnap-Dunn logic, *Studia Logica* 103(4):825–851, 2015.

[7] Dunn, J.M., Star and perp: Two treatments of negation, *Philosophical Perspectives* 7:331–357, 1993.

[8] Dunn, J.M., and C. Zhou, Negation in the context of gaggle theory, *Studia Logica* 80:235–264, 2005.

[9] Egré, P., L. Rossi, and J. Sprenger, De Finettian logics of indicative conditionals Part I: Trivalent semantics and validity of indicative conditionals Part I: Trivalent semantics and validity, *Journal of Philosophical Logic* 50:187–213, 2021.

[10] Estrada-González, L., Weakened semantics and the traditional square of opposition, *Logica Universalis* 2(1):155–165, 2008.

[11] Estrada-González, L., Complement-topoi and dual intuitionistic logic, *Australasian Journal of Logic* 9:26–44, 2010.

[12] Estrada-González, L., The evil twin: The basics of complement-toposes, in J.Y. Beziau, M. Chakraborty, and S. Dutta, (eds.), *New Directions in Paraconsistent Logic*, Springer India, New Delhi, 2015, pp. 375–425

[13] Estrada-González, L., and F. Cano-Jorge, Mortensen logics, in A. Indrzejczak, and M. Zawidzki, (eds.), *Proceedings of the 10th International Conference on* Non-Classical Logics. Theory and Applications, Łódź, Poland, 14–18 Mar 2022, vol 358 of *Electronic Proceedings in Theoretical Computer Science*, Open Publishing Association, 2022, pp. 189–201.

[14] Estrada-González, L., and E. Ramírez-Cámara, A comparison of connexive logics, *IFCoLog Journal of Logics and their Applications* 3(3):341–355, 2016.

[15] Estrada-González, L., and E. Ramírez-Cámara, Non-conditional contracting connectives, in S. Rahman, M. Mojtahedi, and M. S. Zarepour, (eds.), *Mathematics, Logics and their Philosophies. Essays in Honor of Mohammad Ardeshir*, Springer, 2021, pp. 349–364.

[16] Estrada-González, L., An analysis of poly-connexivity, *Studia Logica* 110:925-947, 2022.

[17] Estrada-González, L., and R. A. Nicolás-Francisco, Contra-classicality and the meaning of the connectives in the Bochum plan, *Unpublished typescript*, 202x.

[18] Francez, N., Natural deduction for two connexive logics, *IFCoLog Journal of Logics and their Applications* 3(3):479–504, 2016.

[19] Francez, N., A poly-connexive logic, *Logic and Logical Philosophy* 29(1):143–157, 2020.

[20] Francez, N., *A View of Connexive Logics*, College Publications, London, 2021.

[21] Francez, N., Poly-connexivity: connexive conjunction and disjunction, *Notre Dame Journal of Formal Logic* 63(3):343–355, 2022.

[22] Goddard, L., and R. Sylvan, On reasoning: (ponible) reason for (and also against), and relevance, in L.G.R. Sylvan, and N. da Costa, (eds.), *Reason, Cause and Relevant Containment with and application to Frame Problems*, Chapter 1. Australian National University, Canberra, 1989; reprint with minor modifications in: D. Hyde, and G. Priest, (eds.), *Sociative Logics and Their Applications: Essays by the Late Richard Sylvan*, Ashgate, 2000, pp. 141–173.

[23] Jarmużek, T., and J. Malinowski, Boolean connexive logics: Semantics and tableau approach, *Logic and Logical Philosophy* 28(3):427–448, 2019.

[24] Jarmużek, T., and J. Malinowski, Modal boolean connexive logics: semantics and tableau approach, *Bulletin of the Section of Logic* 48(3):213–243, 2019.

[25] Klonowski, M., Axiomatization of some basic and modal boolean connexive logics, *Logica Universalis* 15(4):517–536, 2021.

[26] Malinowski J. and R. A. Nicolás-Francisco, Relating Semantics for Hyper-Connexive and Totally Connexive Logics, *Logic and Logical Philosophy*, pp. 1–14, (2023).

[27] Malinowski, J., and R. Palczewski, Relating semantics for connexive logic, in A. Giordani, and J. Malinowski, (eds.), *Logic in High Definition*, Springer, Switzerland, 2021, pp. 49–65.

[28] Marcos, J., Nearly every normal modal logic is paranormal, *Logique Et Analyse* 48(189-192):279–300, 2005.

[29] Marcos, J., On negation: pure local rules, *Journal of Applied Logic* 11:185–219, 2005.

[30] McCall, S., *Non-Classical Propositional Calculi*, PhD thesis, Oxford University, 1963.

[31] McCall, S., Connexive implication, *Journal of Symbolic Logic* 31(3):415–433, 1966.

[32] Mortensen, C., *Inconsistent Mathematics*, Kluwer Mathematics and Its Applications Series. Kluwer Academic Publishers, 1995.

[33] Mortensen, C., Closed set logic, in R. T. Brady, (ed.), *Relevant Logics and Their Rivals*, Ashgate Publishing, Aldershot, 2003, pp. 254–262.

[34] Mruczek-Nasieniewska, K., and M. Nasieniewski, Syntactical and semantical characterization of a class of paraconsistent logics, *Bulletin of the Section of Logic* 34(4):229–248, 2005.

[35] Mruczek-Nasieniewska, K., and M. Nasieniewski, Logics with impossibility as the negation and regular extensions of the deontic logic D2, *Bulletin of the Section of Logic* 46(3/4):263-282, 2017.

[36] Mruczek-Nasieniewska, K., and M. Nasieniewski, A characterization of some Z-like logics, *Logica Universalis* 12:207–219, 2018.

[37] Mruczek-Nasieniewska, K., and M. Nasieniewski, On correspondence of standard modalities and negative ones on the basis of regular and quasi-regular logics, *Studia Logica* 105(5):1087-1123, 2020.

[38] Nicolás-Francisco, R. A., A classification of negative connectives, in O.M. Grigoriev, D.V. Zaitsev, Yu. V. Ivlev, V. I. Shalak, and N.E. Tomova, (eds.), *Proceedings of the XII Smirnov Readings in Logic*, Izdatel'stvo "Russkoe Obsestvo Istorii i Filosofii Nauki", 2021, pp. 143–144.

[39] Nicolás-Francisco, R. A., Las conectivas negativas (Negative connectives). Chapter 1 of the (ongoing) PhD Dissertation *Un estudio lógico y filosófico acerca de la semi-negación* (*A logical and philosophical study on demi-negation*), UNAM, Graduate Program in Philosophy of Science, 2021.

[40] Omori, H., MRS^p is classical logic with two conjunctions, *Unpublished typescript.*

[41] Omori, H., From paraconsistent logic to dialetheic logic, in A. Holger, and P. Verdée, (eds.), *Logical Studies of Paraconsistent Reasoning in Science and Mathematics*, Springer, Berlin, 2016, pp. 111–134.

[42] Omori, H., and K. Sano, Generalizing functional completeness in Belnap-Dunn logic, *Studia Logica* 103(5):883–917, 2015.

[43] Omori, H., and H. Wansing, 40 years of FDE: An introductory overview, *Studia Logica* 105(6):1021–1049, 2017.

[44] Omori, H., and H. Wansing, Varieties of negation and contra-classicality in view of Dunn semantics, in K. Bimbó, (ed.), *Relevance Logics and other Tools for Reasoning. Essays in Honour of J. Michael Dunn*, College Publications, London, 2022, pp. 309–337.

[45] Pizzi, C., Cotenability and the logic of consequential implication, *Logic Journal of the IGPL* 12(6):561–579, 2004.

[46] Priest, G., Negation as cancellation, and connexive logic, *Topoi* 18(2):141–148, 1999.

[47] Priest, G., *Doubt Truth to be a Liar*, Oxford University Press, 2006.

[48] Reichenbach, H., *Wahrscheinlichkeitslehre*, Sijthoff, Leiden, 1935.

[49] Ripley, D. W., *Negation in Natural Language*, PhD thesis, Chapel Hill, 2009.

[50] Routley, R., Semantics for connexive logics. I, *Studia Logica* 37(4):393–412, 1978.

[51] Routley, R., R. K. Meyer, V. Plumwood, and R. Brady, *Relevant Logics and Their Rivals*, Ridgeview Publishing Company Ohio, 1982.

[52] Routley, R., and V. Routley, Negation and contradiction, *Revista Colombiana de Matemáticas* 19(1–2):201–230, 1985.

[53] Shramko, Y., Dual intuitionistic logic and a variety of negations: The logic of scientific research, *Studia Logica* 80(2/3):347–367, 2005.

[54] Sylvan, R., An orientational survey of sociative logics, in *Bystanders' Guide to Sociative Logics*, Australian National University, Canberra, 1989; reprint with minor modifications in D. Hyde, and G. Priest, (eds.), *Sociative Logics and Their Applications: Essays by the Late Richard Sylvan*, Ashgate, 2000, pp. 29–51.

[55] Sylvan, R., A preliminary Western history of sociative logics, in *Bystanders' Guide to Sociative Logics*. Australian National University, Canberra, 1989; reprint with minor modifications in D. Hyde, and G. Priest, (eds.), *Sociative Logics and Their Applications: Essays by the Late Richard Sylvan*, Ashgate, 2000, pp. 53–138.

[56] Sylvan, R., On interpreting truth tables and relevant truth table logic, *Notre Dame Journal of Formal Logic* 33(2):207–215, 1992.

[57] Sylvan, R., What is that item designated negation?, in M.D. Gabbay, and H. Wansing, (eds.), *What is Negation?*, Springer Netherlands, Dordrecht, 1999, pp. 299–324.

[58] Wansing, H., Connexive modal logic, in R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, (eds.), *Advances in Modal Logic*, vol. 5, College Publications, London, 2005, pp. 367–383.

[59] Wansing, H., Connexive logic, in E.N. Zalta, (ed.), *The Stanford Encyclopedia of Philosophy*. CSLI Stanford, Spring 2020 edition, 2020.
[60] Wansing, H., and D. Skurt, Negation as cancellation, connexive logic, and qLPm, *Australasian Journal of Logic* 15(2):476–488, 2018.

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