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Category theory and physical structuralism

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Abstract As a metaphysical theory, radical ontic structural realism (ROSR) is characterised mainly in terms of the ontological primacy it places on relations and structures, as opposed to the individual relata and objects that inhabit these relations/structures. The most popular criticism of ROSR is that its central thesis (that there can exist 'relations without relata') is incoherent. Bain (*Synthese*, 190, 1621–1635, 2013) attempts to address this criticism by arguing that the mathematical language of category theory allows for a coherent articulation of ROSR's key thesis. Subsequently, Wüthrich and Lam (2014) and Lal and Teh (2015) have criticised Bain's arguments and claimed that category theory fares no better than set theory in coherently articulating the main ideas of ROSR. In this paper, we defend Bain's main arguments against these critiques, and attempt to elaborate on the sense in which category theory can be seen as providing a coherent articulation of ROSR. We also consider the relationship between ROSR and Categorical Quantum Mechanics.

Keywords Structuralism · Category theory · Radical ontic structural realism · Categorical Quantum Mechanics

1 Introduction

The primary aim of this paper is to defend Bain's arguments for the coherence of a category theoretic articulation of ROSR against the critiques offered by Wüthrich and Lam (2014) and Lal and Teh (2015). Section 2 provides a concise overview of the

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key ideas of ROSR and the main argument against it (that it is incoherent). Section 3 recaps Bain's main argument for the coherence of a category theoretic articulation of ROSR. Sections 4, 5 and 6 then provide critical assessments of the recent critiques of Bain's argument. Section 7 then considers the philosophical relationship between Categorical Quantum Mechanics (see, for example Abramsky and Coecke 2008) and ROSR. Section 8 concludes.

2 ROSR: Object free ontology

ROSR is a metaphysical theory that was formulated mainly in response to philosophically problematic phenomena arising in modern physics (the supposed violation of Leibniz's law concerning the identity of indiscernibles by entangled particles in quantum theory[1] and the problem of individuating spacetime points in general relativity.) In particular, ROSR argues that classical metaphysics, with its ontology of individual objects and intrinsic properties, is unable to deal with the conceptual upheavals necessitated by modern physics. What is needed, according to the advocate of ROSR, is a metaphysics that treats structure and relations, not individual objects, as ontologically fundamental. This leads us to the key thesis of ROSR, that there can exist physical relations devoid of relata (for the remainder of this paper, we will refer to this thesis as 'STRUC').

Now, ROSR has been heavily criticised for its reliance on STRUC. Specifically, it has often been contended that STRUC is not even false, but incoherent. Thus, we read 'For the relations to be instantiated, there has to be something that instantiates them' (Esfeld and Lam 2008, p 31) and 'insofar as relations can be exemplified, they can only be exemplified by some relata. Given this conceptual dependence of relations upon relata, any contention that relations can exist floating freely from some objects that stand in those relations seems incoherent' (Wüthrich and Lam 2014, p 2). Clearly, the problem here is the apparent conceptual dependence of relations upon relata. We will now review Bain's attempt to break this dependence by employing the mathematical language of category theory.

3 Category theory and ROSR

The argument for the incoherence of ROSR can be made formally rigorous by appealing to the mathematical resources of set theory.[2] For, in the language of set theory,

[1]Consider, for example, Schrödinger's assertion that 'it is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of the sameness, of identity, really and truly has no meaning' (Schrödinger 1952, p18).

[2]In order to keep the paper at a reasonable length, we will assume that the reader is familiar with the basic concepts of category theory. Chapter 1 of (Mac Lane 1998) provides a thorough introduction.

a relation is nothing more than a set of ordered pairs[3] of elements from the relevant domain of first order quantification. These elements are the relata that are taken to instantiate the given relation. Thus, the language of set theory does not allow for any conceptual space between relations and the relata that instantiate them, since a relation is simply defined to be the set of all the order pairs of its relata. So of course it seems outright incoherent to claim that there can be relations with no relata.[4]

At this stage, the advocate of ROSR is placed in a difficult position. For, it seems that set theory, the language in which the vast majority of modern mathematics and physics is usually phrased, does not allow for the coherent articulation of its core thesis, STRUC. Thus, it seems that the only hope for ROSR is to find a new mathematical language that allows for the coherent formulation of STRUC and is of comparable expressive power to the language of set theory. This is exactly the strategy employed by Bain, who identifies category theory as a mathematical language of this kind.

Bain's first observation is that category theory allows us to reformulate (and generalise) many traditional set-theoretic notions that would normally require reference to elements and individual sets purely in terms of morphisms in the relevant category. In particular, we can replace all talk of elements of a set with descriptions of arrows from the terminal object of the ambient category.

Furthermore, in category theory, when we are concerned with any particular type of structure (group structure, ring structure etc), we can generally just build a category in which the relevant kind of structured sets (groups, rings etc) appear as objects. Once we have moved to this category theoretic setting, we generally cannot 'look inside' the structured sets in order to talk about their elements. We can only talk about the category theoretic properties of the objects (for example, the arrows from the terminal object into that object). Generally, these properties involve arrows between objects. So it seems that category theory allows (indeed, requires) us to describe structural properties without referring to individual relata. In this sense, category theory looks like a much better formal setting for the claims of ROSR.

'... category theory lacks the resources for direct reference to "internal elements" of an object...This suggests that the definition of structure as an object in a category does not make ineliminable reference to relata in the set-theoretic sense." (Bain 2013, p 1624)

At this stage, category theory looks like a promising formal setting for ROSR, since it seems to be capable of providing a notion of structure that is not conceptually tied to individual relata. However, Bain immediately anticipates a possible objection to his proposal, namely the argument that 'category theory eliminates reference to relata only in name'. The idea here is that, by talking about arrows from the terminal object, we are really making reference to elements of the relevant structured sets

[3]For ease of exposition, we will talk mainly about binary relations. For n-ary relations, just replace 'ordered pair' with 'ordered n-tuple'.

[4]Of course, we do not need to worry about the trivial empty relation here.

(relata), and the category theoretic setting only really offers us new terminology. As Bain puts it,

'...any given set theoretic structure will have a category theoretic analog, and however many relata the former is associated with, so the latter will be associated with the same number of morphisms from the terminal object. The argument against the radical ontic structural realist then gets translated from the slogan "no relations without relata" to the slogan "no objects (of the relevant sort) without morphisms from the terminal object."' (Bain 2013, p 1625)

Bain replies to this argument by noting that there are many categories whose objects are not structured sets, and so the structures encoded by these objects can not depend completely on their elements. If it were the case that every category was a category of some kind of structured set, together with the structure-preserving morphisms, then Bain would presumably accept the criticism. However, since there are categories that do not fit this mold, he argues, it is possible to talk about structure in category theoretic terms without talking about relata.

Now, Bain gives several examples[5] of categories whose objects are not structured sets, but in which it is perfectly possible to talk about meaningful structures. However, in the interest of concision, we will only focus on one of the examples, namely the category **HILB** of Hilbert spaces and bounded linear operators.

Bain argues that the objects of **HILB** cannot simply be thought of as structured sets, since the morphisms of **HILB** are 'not simply functions that preserve the relevant set-theoretic notion of structure associated with them.' In particular, he notes that

*'Set-theoretically, the functions that preserve the structure of a Hilbert space are unitary operators that preserve the inner-product. The morphisms in **HILB** in contrast are general bounded linear operators that do not necessarily have to be unitary.'* (Bain 2013, p 1630)

The argument here is that since the morphisms of **HILB** do not preserve all of the relevant physical structure, the structure of **HILB** as a whole cannot be adequately expressed purely in terms of the elements of its objects. Bain also cites the fact that, unlike **SETS**, **HILB** is a monoidal dagger category, and notes that since **HILB** has a terminal object, it allows us to talk about elements of objects in the category. In particular, he notes that

'Insofar as the objects of these categories are not structured sets, their elements are not essential in articulating the relevant notions of structure. Again, because the objects of these categories are not structured sets, the "properties" of their elements are not what get preserved under the morphisms.'

So, according to Bain, **HILB** is an example of a category that allows us to talk about structure without referring to individual relata. Furthermore, since **HILB** cannot be construed as a category of structured sets, relata are not being eliminated 'only

[5]The most important example other than **HILB** is the category **nCOB** of $n-1$ dimensional compact oriented manifolds and n-dimensional oriented cobordisms, which plays an important role in general relativity and quantum field theory.

in name'. The structure that is embodied in the web of morphisms in **HILB** cannot be understood merely in terms of the elements of the category's objects.

At this stage, it is worth noting that we could easily have considered any category that lacks a terminal object in order to show the possibility of articulating a notion of structure that does not depend on individual relata. For, such a category would lack the resources for talking about individual relata in a way that is directly comparable to the set theoretic notion. However, Bain chose to focus on categories like **HILB** because as well as being categories that allow for the articulation of the desired notion of structure, they are also categories that play an important role in modern physics. This is important because ROSR is usually forwarded as a form of physical structuralism, concerned about the ontology of the physical world.

4 Throwing out the relations with the relata?

So far, we have seen how Bain attempts to argue for the coherence of ROSR by employing a new category theoretic notion of structure. This strategy relies on the existence of categories like **HILB** that cannot simply be thought of as categories of 'structured sets', and whose structural properties cannot be reduced to properties of individual relata.

Recall that the basic motivating problem behind Bain's use of category theory was to provide a coherent articulation of STRUC, the thesis that there can exist physical relations devoid of physical relata. However, we have not yet seen an explicit explanation of how category theory can provide a new formalisation of relations that does not make ineliminable reference to relata. Specifically, we have not been given a detailed account of how category theory allows for a coherent articulation of STRUC. This leads us to Wüthrich and Lam's first major criticism of Bain's strategy for rehabilitating ROSR.

The most straightforward and popular way of defining relations in category theory is a direct generalisation of the set theoretic notion. Specifically, just as relations in set theory are subsets of the Cartesian product of the relevant domains of quantification, the category theoretic generalisation defines relations to be subobjects of the relevant Cartesian product in the ambient category.[6] So, it is natural to expect that we can define relations in this way in **HILB**, thereby introducing a notion of relation that allows for the coherence of STRUC and defends ROSR against its many critics. However, as Wüthrich and Lam point out, this strategy will not work. For, **HILB** is not what is known as a 'Cartesian Closed' category. This means that finite limits don't generally exist in **HILB** and, more to the point, Cartesian products do not generally exist in **HILB**. So we can't define relations in **HILB** to be subobjects of Cartesian products, since there may be no Cartesian products.

At this stage, it looks like Bain's strategy to defend ROSR against its critics has failed. For, his strategy relied on finding examples of physically significant categories

[6] By 'subobject of A' we mean 'monic arrow into A' and by 'Cartesian product' we of course mean the usual category theoretic notion of product that generalises the set theoretic notion.

that allow us to talk about their structure without (implicitly) referring to individual relata. He managed to find categories of this sort, but it turns out that these categories do not have the resources to define a meaningful notion of 'relation'. This being the case, it seems that these categories can't possibly provide any support to Bain's attempts to provide a coherent articulation of STRUC. As Wüthrich and Lam put it, it looks like Bain has 'thrown the relations out with the relata' (Wüthrich and Lam 2014, p 9).

The advocate of ROSR is unlikely to admit defeat just yet. For, there might yet be other ways of defining relations in category theory that will suit their purposes. Indeed, Wüthrich and Lam anticipate one other strategy that might be available to Bain. In particular, they consider the possibility of using 'concretisable categories' to define relations. We call a category **C** 'concretisable' if and only if there exists a faithful functor $F : \mathbf{C} \rightarrow \mathbf{SETS}$. In this case, we call the ordered pair $\langle \mathbf{C}, F \rangle$ a 'concrete category'. Intuitively, the idea is that a concretisable category is one that can be accurately 'reflected' in **SETS** (more specifically, in the image of the relevant faithful functor). Now, there is a natural way to define relations in a concretisable category **C**. Specifically, if we introduce one functor $F^n : \mathbf{C} \rightarrow \mathbf{SETS}$ for each $n \in \mathbb{N}$ satisfying the condition $F^n(A) = (F(A))^n$, then we can define an n-ary relation to be a subfunctor[7] of F^n.

Now, it turns out that **HILB** is a concretisable[8] category, which means that this approach to introducing relations might be suitable for Bain's purposes. However, Wüthrich and Lam argue that this approach is not philosophically satisfactory for the advocate of ROSR.

'Concretizing a category and studying its implicit structure via its underlying set-theoretical structures and, specifically, the structure of its objects via the relevant subfunctors may be an extremely powerful tool. It does, however, depend on the "exponentiability" of $U(A)$, as is evident in the definition of n-ary relations. Since **SETS** is Cartesian closed, $U(A)^n$ exists. The Cartesian closedness of **SETS**, however, entails that it is also possible to introduce elements of objects in a similar fashion. Thus, if it is legitimate to introduce relations in this way, then it is hard to see why it would be illegitimate to similarly introduce elements of objects.' (Wuthrich and Lam 2014, p 11)

The key idea here is that it is only by utilising the internal structure of **SETS** that we are able to define relations in **HILB** (via the concretisation). But this set theoretical structure is exactly what Bain was trying to avoid when he cited **HILB** as an example of the kind of category theoretic structure that proves the intelligibility of STRUC. For, by reintroducing this set theoretic structure, we also reintroduce the resources for talking about individual elements and relata. So, it is still not possible to talk about relations in **HILB** without relying, however implicitly, on individual relata.

[7] We call a functor $G : \mathbf{C} \rightarrow \mathbf{SETS}$ a 'subfunctor' of F if for every object A of **C**, $G(A) \subseteq F(A)$ and for any morphism $f : A \rightarrow B$, $G(f)$ is the restriction of $F(f)$ to the domain $G(A)$.

[8] For the remainder of this paper, we will use the terms 'concrete' and 'concretisable' interchangeably, since we will not need to consider any specific faithful functors.

However, this argument may not be as convincing as it seems at first blush. For, **HILB** already possesses the resources to talk about individual relata, even before we consider its concretisation. This is true because it has a terminal object (namely the zero-dimensional space $\{0\}$[9]), and so we can talk about arrows from $\{0\}$ into objects in **HILB** which, as we have seen, is the same as talking about elements of those objects. Bain's argument never relied on **HILB** lacking the resources to talk about individual relata. Rather, his point was that the structure of **HILB** cannot be articulated purely in terms of the individual elements of the objects of the category. This is reflected in the fact that the morphisms in **HILB** do not preserve all of the relevant set theoretic structure. So, Bain has nothing to lose by defining relations via the conretisation of **HILB**, since **HILB** is already capable of talking about elements of objects.

Having considered and dismissed these two strategies for defining relations in **HILB**, Wuthrich and Lam conclude that 'relations, not just relata, are conceptually intimately tied to a set- theoretical understanding of "structure" ' (Wuthrich and Lam 2014, p 11). Indeed, if Bain's only examples of categories in which it is possible to define a notion of 'structure' without reference to relata are categories that lack the resources to talk about relations, then it is natural to conclude that his strategy has done nothing to render STRUC coherent.

However, we will now argue that Wüthrich and Lam did not actually consider all of the possible ways of defining relations in **HILB**. Indeed, we will see that it is perfectly possible to define a notion of 'relation' in **HILB** without referring in any way to the conretisation of the category (even though it is not clear why this approach is not suitable) or introducing any reference to individual relata.

One standard way of defining relations in categories without finite products that is not considered by Wüthrich and Lam is the following.

Definition 4.1 Given objects A, B of a category **C**, an internal relation between A and B is a pair[10] of morphisms, $f : R \to A$, $g : R \to B$, with joint domain R, that are jointly monic.[11]

This is a popular definition in category theory that is often used to generalise the set theoretic concept of 'relation'. Clearly, it is a definition that, unlike those considered by Wüthrich and Lam, does not rely at all on any kind of set theoretic structure. It is justified partly by the fact that in the cases where **C** has binary (finite) products, the internal relations between A and B are in bijective correspondence with the subobjects of the product of A and B. So, in the case where **C** has products, Def 4.1 just gives the expected category theoretic reformulation of the set theoretic definition of relations. However, in the general case where **C** does not have products,

[9]Actually, $\{0\}$ is what is known as a 'zero object', i.e. it is both initial and terminal in **HILB**.

[10]This definition is for binary relations, but we can extend it to cover relations of arbitrary arity in the obvious way.

[11]This means that given any other morphisms h, h' with joint codomain R, the condition $(f \circ h = f \circ h') \wedge (g \circ h = g \circ h')$ implies that $h = h'$.

this definition provides us with a generalisation that goes beyond the set theoretic concept of ‘relation’.

Now, the critic of ROSR might respond that this definition of relations is not satisfactory because it lacks some of the standard properties of the usual set theoretic calculus of relations. In particular, we need to impose various stringent conditions[12] on **C** to obtain properties like, for example, the associative composition of internal relations. Unfortunately, **HILB** is not known to satisfy these conditions. So this definition of relations does not allow for associative composition in **HILB**. This will probably be enough for ROSR’s critics to conclude that this definition is unsatisfactory.

However, at this point, the advocate of ROSR might be justified in biting the bullet and accepting that it may be necessary to give up a significant part of the set theoretic calculus of relations in order to arrive at a new, more general category theoretic notion of ‘relation’ that renders STRUC coherent and fits in with the radical new metaphysical picture offered by the theory. Indeed, it could be argued that it is unfair to expect that categories like **HILB**, which are supposed to embody a radically new, category theoretic, conception of structure, should be able to recreate the old, set theoretic, calculus of relations in all its classical glory. To the advocate of ROSR, this kind of requirement is likely to sound like unjustified conservatism, motivated by exactly the kind of classical metaphysics they are trying to overturn. Compared to giving up an ontology of individual objects that satisfy Leibniz’s laws, surrendering the associative composition of relations (for example) looks like small change. The set theoretic calculus of relations, they might say, is just another relic of the classical metaphysics that has been refuted by modern physics.

It might still be objected that, in the cases where the ambient category does not have products, the notion of an internal relation does not capture any kind of genuine, physically meaningful concept of relation. Of course, the fact that internal relations have not yet played any important role in physics makes it difficult to refute this kind of allegation. However, these internal relations have several properties that mark them out as providing a meaningful notion of relation that could have potential applications to physics. Firstly, as we have already seen, internal relations directly generalise the set theoretic concept in a natural way (hence their popularity amongst category theorists). Secondly, they allow for the articulation of all of the usual properties of relations (reflexivity, transitivity, associativity etc) in purely category theoretic terms. So this definition allows us to talk about ordering relations, equivalence relations etc, as usual.

Finally, even if one is not satisfied with either of these attempts to define relations in **HILB** (via the concretisation or internal relations), there is still another response open to Bain. In particular, the problem posed by Wüthrich and Lam is that since **HILB** is not a Cartesian closed category, arbitrary products do not exist. So it is not generally possible to define relations in **HILB**, hence the charge of ‘throwing out the relations with the relata’. This is supposed to be a problem for Bain since it undermines the idea that **HILB** is an example of a category that allows for the articulation

[12] In particular, we generally need to assume that **C** is ‘locally regular’.

of a notion of structure that fits the aims of ROSR. However, we have not yet considered the distinct but closely related category **FHILB** of *finite dimensional* Hilbert spaces and linear maps. As in **HILB**, the morphisms in **FHILB** are linear but not generally unitary maps. So the morphisms do not preserve all of the relevant physical structure (as Bain would put it, **FHILB** is not a ‘category of structured sets’). So, like **HILB**, **FHILB** embodies a notion of structure that is suitable for ROSR. But there is a crucial difference between **HILB** and **FHILB**. In particular, **FHILB** is actually a regular category, which means that is possible to define relations in the usual way (using internal relations), giving a model of the usual calculus of relations.

Even if one accepts Wuthrich and Lam’s contention that **HILB** ‘throws out the relations with the relata’, it is still possible to cite **FHILB** as an example of a category theoretic codification of ROSR’s conception of relata-free structure. Furthermore, it is well known that **FHILB** is a very important category from a physical perspective (it is the setting for finite dimensional quantum theory). Indeed, it turns out that **FHILB** is particularly important from the perspective of categorical quantum mechanics (see Section 7 for more details).

So, the criticism that Bain’s strategy for defending ROSR from charges of incoherence ‘throws out the relations with the relata’ can be resisted on three fronts. Firstly, we have argued that Wüthrich and Lam are too quick to dismiss the possibility of defining relations via the concretisation of **HILB**. Since **HILB** is a category with a terminal object, it already possesses the resources to talk about individual elements, so (contra Wüthrich and Lam) introducing the conretisation makes no difference in this respect. Secondly, we have seen that even if the advocate of ROSR deemed this way of defining relations to be philosophically unsatisfactory, there is another way to introduce relations in **HILB**, via Def 4.1. Philosophically, this is a much bolder move as it essentially involves introducing a concept of ‘relation’ that is strictly broader than the concept enshrined in the set theoretic calculus of relations. But we have also argued that this need not prove problematic to the advocate of ROSR, who is already committed to a radical reshaping of the basic metaphysical concepts in terms of which we understand the world. Thirdly, we have seen that even if these two strategies are deemed unsatisfactory, one can just move from **HILB** to the regular category **FHILB** in which there is no problem in defining relations. Since, like **HILB**, **FHILB** has morphisms that do not preserve all the relevant structure, this makes no difference to Bain’s main arguments.

5 Concrete categories and physical structure

Lal and Teh (2015) also offer a number of criticisms of Bain’s main arguments. In this section, we will attempt to counter some of these criticisms. Unfortunately, most of the arguments offered in that paper go beyond the scope of the present analysis, so we will focus on a couple of crucial points in the dialectic.

Lal and Teh offer precise interpretations of some of Bain’s central claims. Firstly, they attempt to make more precise the sense in which Bain believes **nCOB** and **HILB** to instantiate a radically structuralist idea of ‘structure’.

'...the significance of these examples for Bain is their apparent status as purely category-theoretic formulations of physics which, in virtue of their generality, do not make any reference to O-objects[13] (represented in the standard way, i.e. as elements of sets)... Bain's key idea seems to be that this 'generality' consists of the fact that **nCOB** and **HILB**...have very different properties from **SET**...In fact, he claims that three such differences count in favor of (Objectless)[14]:

(i) **nCOB** and **HILB** are non-concrete categories, but **SET**[15] (and other categories based on it) are concrete
(ii) **nCOB** and **HILB** are monoidal categories, but **SET** is not
(iii) **nCOB** and **HILB** have a dagger functor, but **SET** does not ' (Lal and Teh 2015, p 15)

Having made this interpretation of Bain's key claims, Lal and Teh go on to note that (i)-(iii) are all either false, or insufficient for establishing Bain's intended conclusions.

Let's begin with (i). Lal and Teh note that (i) is straightforwardly false, sine although **nCOB** is not a concrete category, '**HILB** is certainly a concrete category, since the objects are Hilbert spaces, which are sets with extra conditions; and the morphisms are just functions with linearity conditions'. Furthermore, they also argue that non-concreteness of a category is not sufficient for establishing the claim that that category does not refer to O-objects. They use the example of **nCOB**, which is a 'non-concrete category that apparently contains O-objects...(viz. spacetime points)' to establish this.

However, their interpretation of Bain here is certainly contentious. Indeed, Lal and Teh themselves note that 'Bain does not himself use the language of 'non-concrete' category but this is the most reasonable-and indeed the most precise-interpretation of what he means'. But, as Bain notes, we are interested in more of the structure of **FHILB** than can be captured by an arbitrary faithful functor into **SET**. In particular, we are interested in the inner product structure of **FHILB**, which is not preserved by the morphisms of the category, because they are not generally unitary. This inner product structure is only captured by the dagger operation of **FHILB**, which is not generally preserved by faithful functors into **SET**. Thus, although, strictly speaking, **FHILB** is a concrete category, it is not the case that the relevant physical structure is always reproduced in the image of faithful functors into **SET**. Furthermore, Bain's claim that we cannot just think of **FHILB** as a category of structured sets still stands, since some of the vital category-theoretic structure (the inner product) will be lost if we think that way.

The problem here is that Bain is not concerned simply with whether or not **FHILB** is concrete. Rather, he is concerned with whether all of the physically important structure of the category will be captured by a faithful functor into **SET**. Although

[13] Lal and Teh introduce some useful terminology here. A 'C-object' is an object in a category, while an 'O-object' is an actual physical object in the world.

[14] For '(Objectless)', read 'STRUC'.

[15] For the remainder of the paper, we write '**SET**' instead of '**SETS**', in line with Lal and Teh's terminology.

FHILB is concrete, there are physically significant parts of its structure that are lost when we translate the objects and morphisms of **FHILB** into the set-theoretic setting. This is all that Bain needs in order to justify his claim that category theory allows for a more general approach to formalising physical theories, and in doing so provide us with a new notion of structure that is not inherently dependent on individual relata.

Moving onto (ii), Lal and Teh note that **SET**, like **nCOB** and **HILB**, is a monoidal category (where the monoidal product is just the Cartesian product). So this property cannot be used to support the claim that **nCOB** and **HILB** instantiate a different kind of structure to **SET**. Of course, this observation is technically correct. However, one thing that has not been taken into account here is that the properties of the monoidal product in **HILB/FHILB** in particular are fundamentally different to those of the monoidal product in **SET** in physically significant ways. In particular, the monoidal structure of **HILB** models a generalised category-theoretic definition of entanglement (see Section 7 for more details), whereas **SET** does not. Thus, although **SET** does have a monoidal structure, it is of a fundamentally different kind[16] to the monoidal structure of **HILB**. So it can legitimately be argued that the monoidal structure of **HILB** marks it out as instantiating a different kind of physical structure to **SET**.

Lal and Teh admit that (iii) is technically true, but argue that this is not sufficient for establishing the claim that **nCOB** and **HILB** instantiate a new kind of structure that is different from the usual set-theoretic notion. In particular, they provide a toy example (Lal and Teh 2015, p 16) of a category that can be equipped with a dagger functor, but can be thought of as making ineliminable reference to O-objects. However, it is not clear that Bain is arguing that the existence of a dagger functor is sufficient for a category to instantiate structure that can't be reduced to O-objects, and if he is, then he shouldn't be (as Lal and Teh's toy example shows). Rather, Bain can reasonably argue that the dagger functor is an essential part of the internal structure of **HILB**, and the fact that **SET** does not have such a functor is indicative of a fundamental difference between the categories (and the kinds of structure they instantiate). This does not commit him to the claim that the existence of a dagger functor for a category is sufficient for that category be a setting for ROSR's notion of 'structure'.

Lal and Teh go on to note that the category **REL** of sets and relations (see Section 7 for more details) also has a dagger functor. They argue that **REL** is an 'easy extension' of **SET** (they have the same objects and the morphisms of **SET** are a subset of the morphisms of **REL**) that 'is akin to **nCOB/HILB** in having a dagger functor, but which also appears to be at least as good a candidate as **SET** for codifying the standard notion of physical structure, since its morphisms are n-ary relations and they can be used to encode structure'. However, it could be argued that in terms of its internal structure, **REL** is actually a lot closer to **HILB** than it is to **SET**. For, just as the morphisms of **HILB** fail to preserve an important part of the structure of their objects

[16]One useful way to characterise this difference is to note that the monoidal product in **SET**, unlike the monoidal product in **HILB/FHILB** coincides with the Cartesian product. Indeed, this is precisely the reason that **SET** does not model the generalised category-theoretic definition of entanglement.

(the inner product structure), the morphisms in **REL** also fail to preserve a crucial aspect of the structure of their objects (they are one to many and so do not allow us to track the basic set theoretic structure of the objects). So it could be argued that **REL**, like **HILB**, cannot be adequately understood as a category of structured sets. In this case, **REL** could be seen as another exemplar of the kind of category theoretic structure required by ROSR. Furthermore, apart from **FHILB**, **REL** is probably the most important category theoretic model of categorical quantum mechanics (see Section 7 for more details). This is precisely because **REL** shares many key structural features with **FHILB** (it is a compact monoidal dagger category that allows for the articulation of a category theoretic definition of entanglement), all of which are missing from **SETS**.

6 Structure in set theory and category theory

Following on from the criticisms outlined in Section 4, Wüthrich and Lam offer a second challenge to Bain's general strategy for defending ROSR. They begin by outlining how the category theoretic notion of structure differs from its traditional set-theoretic counterpart,

'In fact, category theory offers a rather different approach to structure from the usual set-theoretical one. Given a category **C**, the structure of a **C**-object is given by the **C**-morphisms to and from this object. In category theory, it is by conditions on the web of morphisms of a category that internal structure is imposed on the objects of the category. Whatever these internal structures, they are determined by these conditions only up to isomorphism...An advocate of ROSR might be tempted to feel vindicated by the fact that the web of morphisms thus captures the internal structure of the objects of a category, combined with the assumptions that morphisms can be equated to relations, and that the objects -the relata- are in some sense, precisified above, eliminable. But such a feeling would rest on a confusion: if relational at all, the morphisms are relations between objects, not elements of objects...ROSR is a thesis emphasizing the fundamentality of relations among what is termed elements in this context- and not the objects-, at the ontological expense of these elements. Thus, this move cannot save the radical' (Wüthrich and Lam 2014, p 12).

This characterisation of category theoretic structure seems fair and useful. In category theory, the fundamental structure of a category **C** is encoded in the web of morphisms associated with the objects of **C**, rather than by the internal set theoretic constitution of those objects. As Wüthrich and Lam note, this conception of structure looks attractive to the advocate of ROSR. However, they go on to contend that ROSR requires that the morphisms of **C** should be understood as relations between the elements of the objects of **C**. But, in light of the arguments offered in the previous section, it is not clear why this requirement is necessary. There are at least two category theoretic ways to define relations that are suitable for the purposes of ROSR, and these methods do not require that we think of the morphisms of **C** as being relational with respect to the elements of the objects of **C**. Thus, the conception of structure outlined above seems to be a perfectly intelligible and useful notion that suits the philosophical purposes of ROSR.

Furthermore, even if this requirement is accepted, there are special cases where it will be satisfied. Specifically, the category **REL** whose objects are sets and whose morphisms are relations is a category in which the morphisms can be understood as relations between the elements of the objects. Thus, since the argument concerns the intelligibility of ROSR, it could be argued that the existence of a category such as **REL** proves the intelligibility of the relevant notions, given Wüthrich and Lam's requirement. And as we have already noted (and will see again in the next section), **REL** is a category that is actually quite interesting from a physical perspective, as it closely simulates many of the key structural properties of **FHILB** and so is capable of reproducing the mathematical structure of several key quantum protocols. In the previous section, we also argued that the fact that the morphisms in **REL** are relations rather than functions suggests that Bain's arguments concerning **HILB** might also be applicable to **REL**. For, the morphisms in **REL**, like the morphisms in **HILB**, do not preserve the internal set theoretic structure of the objects. So it could be argued that **REL**, like **HILB**, cannot be adequately understood as a category of structured sets. In this case, **REL** could be seen as another exemplar of the kind of category theoretic structure required by ROSR, and furthermore it satisfies Wüthrich and Lam's requirement concerning relational morphisms.

7 ROSR and categorical quantum mechanics

Having defended Bain's strategy for a category theoretic articulation of ROSR from Wüthrich and Lam's main arguments, we will now argue that categorical quantum mechanics (CQM)[17] provides a formulation of quantum theory that is particularly well suited to the philosophical sensibilities of ROSR.

The fundamental motivating aim behind CQM is to reformulate quantum theory in purely category theoretic language, by abstracting from the key category theoretic properties of **FHILB**.[18] The idea is that this will provide a more general and conceptually perspicuous formulation of the theory that is not 'cluttered' by the accidental particularities of the usual Hilbert space formalism. In particular, CQM takes the monoidal structure of **FHILB** very seriously and attempts to provide a new quantum formalism that begins by assuming that the ambient category has a monoidal product structure. Intuitively, this means that CQM assumes a tensor product structure as one of its 'primitives'. Clearly, this means that entanglement emerges as one of the basic notions of the formalism. Coecke, one of the founding developers of CQM, describes the philosophical implications of the project in the following way,

[17]For technical and conceptual introductions to CQM, see for example Abramsky and Coecke (2008) and Coecke (2005, 2009).

[18]It turns out that, from a category theoretic perspective, **FHILB** has more useful properties than **HILB**. In particular, **FHILB** is a 'compact category', and this property is very important in providing abstract category theoretic derivations of famous quantum protocols like teleportation. This is the main reason that **FHILB** is used more than **HILB** in CQM.

'(i) It shifts the conceptual focus from "material carriers" such as particles, fields or other "material stuff", to "logical flows of information", by mainly encoding how things stand in relation to each other.
(ii) Consequently, it privileges processes over states. The chief structural ingredient of the framework is the interaction structure on processes.
(iii) In contrast to other operational approaches,... we do not take probabilities, nor properties, nor experiments as a-priori, nor as generators of structure, but everything is encoded within the interaction of processes.
(iv) In contrast to other structural approaches, we do not start from a notion of system, systems now being "plugs" within a web of interacting processes. Hence systems are organized within a structure for which compoundness is a player and not the structure of the system itself: a system is implicitly defined in terms of its relation(ship)/interaction with other systems

Clearly, the philosophical vision forwarded by Coecke is deeply appealing to advocates of ROSR. In particular, CQM emphasises the primacy of relational and structural properties over isolated systems with intrinsic properties, and in doing so respects the basic sensibilities of ROSR.

It is worth noting that CQM manages to provide category theoretic reformulations of several key notions from the Hilbert space formalism for quantum theory. For example, CQM allows for a purely category theoretic definition of entanglement that refers only to the web of morphisms in the ambient monoidal category. Unsurprisingly, in the case where this ambient category is **FHILB**, this definition is equivalent to the usual one. However, in categories that share a sufficient amount of structural similarity with **FHILB**, the definition still applies and we gain the opportunity to study entanglement in a new formal setting. We have already been acquainted with one such category, namely the category **REL** of sets and relations.[19] However, in categories that lack some of the key structural properties of **FHILB**, the new definition cannot be applied. One such category is **SET**. So, it is only by moving beyond set theory that we are able to apply this rich new approach to entanglement.

Given that one of the main motivations for ROSR is the apparent violation of Leibniz's law of the identity of indiscernibles by quantum particles (see e.g French and Krause 2010), which is a direct consequence of quantum entanglement, and the fact that CQM is itself a formalism that starts out by 'taking entanglement seriously', advocates of ROSR are likely to claim that it is only natural that the two projects should share such a close philosophical affinity. CQM, being a formalism based on the axiomatisation of quantum entanglement, was always bound to reject the traditional metaphysics of individual systems with intrinsic properties that is apparently refuted by the strange behaviour of entangled quanta.

This brings us to Wüthrich and Lam's final criticism of Bain's category theoretic defense of ROSR. Referring to Bain's purported 'concrete examples' of modern

[19]This is just one example of how the category **REL** shares important structural properties with **FHILB**. Other examples include the fact that, like **FHILB** and unlike **SETS**, **REL** is a compact monoidal dagger category.

physical theories whose category theoretic formulations support the ontology of ROSR, they write

‘In particular, the fact that standard fundamental physical objects or relata (such as spacetime points) might not be part of the ontology of these candidate fundamental physical theories in their categorial formulation does not imply that there are no physical objects or relata at all.’ (Wüthrich and Lam 2014, pg 19)

Indeed, this seems like a sound observation. Advocates of ROSR cannot use the absence of fundamental physical objects and relata from the category theoretic formulations of physical theories to support their ontologies any more than advocates of classical metaphysics can use the presence of fundamental physical objects and relata in the set theoretic formulations of those same theories to support theirs. However, this is not an argument that the advocate of ROSR should be making in the first place. For, the significance of category theory for ROSR is that is can be seen as a language in which the theory’s key principles can be coherently expressed. The simple fact that our best physical theories can be expressed in category theoretic terms should be enough to establish that ROSR is a viable candidate for describing the ontology of the physical world. The arguments supporting the claim that ROSR is the *right* description of that ontology are an entirely separate matter. However, this does not mean that advocate of ROSR cannot legitimately prefer formalisms like CQM to their set theoretic counterparts on the grounds that they provide more perspicuous representations of what they (for independent reasons) believe to be the metaphysical structure of physical reality.

8 Conclusion

In conclusion, we have seen that Bain’s arguments for the coherence of a category theoretic formulation of ROSR survive largely intact in the face of Wüthrich and Lam/Lal and Teh’s critiques. Specifically, we saw that, contra Wüthrich and Lam, the examples of categories that support Bain’s arguments do not necessarily lack the resources to define relations. However, we did see that it might be necessary to surrender some parts of the usual set theoretic calculus of relations in order to provide a definition of relations that does not rely on reintroducing some philosophically unseemly set theoretic structure. We argued that, for the advocate of ROSR, this is a relatively small price to pay given the size of the metaphysical upheavals already engendered by the theory. We also noted that even if one accepts Wüthrich and Lam’s arguments in the context of **HILB**, the problem can be avoided by moving to the equally natural setting of **FHILB**, which is a regular category.

We also countered some of the criticisms that Lal and Teh forwarded against Bain’s arguments. In particular, we argued that their interpretation of some of Bain’s central claims was unfair to the advocate of ROSR, since those claims can also be interpreted in other, less problematic ways. We also argued that the category **REL** can actually be seen as another example of a category that instantiates the kind of structure posited by ROSR.

Furthermore, we have also seen that ROSR can naturally be construed as the philosophical vision best suited to CQM. The fact that category theoretic formalisms like

CQM embody the kind of category theoretic structure advocated by Bain is sufficient to prove the intelligibility and *coherence* of ROSR as a metaphysical theory about the physical world. However, Wüthrich and Lam are correct to contend that the existence of such formalisms do not count as arguments for the *truth* of ROSR. But insofar as we are concerned only with the coherence of ROSR, this is beside the point.

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