

BENJAMIN EVA

Modality and Contextuality in Topos Quantum Theory

Abstract. Topos quantum theory (TQT) represents a whole new approach to the formalization of non-relativistic quantum theory. It is well known that TQT replaces the orthomodular quantum logic of the traditional Hilbert space formalism with a new intuitionistic logic that arises naturally from the topos theoretic structure of the theory. However, it is less well known that TQT also has a dual logical structure that is paraconsistent. In this paper, we investigate the relationship between these two logical structures and study the implications of this relationship for the definition of modal operators in TQT.

Keywords: Quantum logic, Topos theory, Modal logic, Topos quantum theory.

1. Introduction

Traditionally, quantum logic is a field concerned primarily with the study of the orthomodular lattice of projection operators onto the Hilbert space of a quantum system. In this sense, quantum logic can be characterized as the study of the logical structure of the Hilbert space formalism of quantum mechanics. However, in recent years, new approaches to the formalization of quantum mechanics have proliferated, and some of these new formalizations come with in-built logical structures of their own that are not entirely well described by the lattice of projections on a Hilbert space.

In this paper, we will be examining some aspects of the logical structure that arises naturally from the topos-theoretic reformulation of quantum mechanics. Specifically, we will be concerned with the problem of providing a precise formal characterization of the relationship between the paraconsistent and intuitionistic aspects of this structure. We will subsequently go on to study the implications of this relationship for the problem of defining modal operators in TQT.^{1,2}

¹I use the name ‘topos quantum theory’ to refer to the topos theoretic approach to quantum theory developed primarily by Isham and Döring. In the literature, this is sometimes referred to as the ‘contravariant approach’, in contrast to the ‘covariant approach’ developed, for example, in [6].

²A quick caveat is needed here. Please note that this paper will be concerned purely with the *algebraic* aspects of the logical structure of TQT, as opposed to the topos theoretic aspects that utilize and emphasize the internal language of the ambient topos.

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In Section 2 we will give a short introductory account of some of the key concepts and results from the topos theoretic reformulation of quantum mechanics. In Section 3, we will describe the logical structure that arises naturally from this reformulation, paying special attention to the fact that the algebra of propositions associated with a quantum system is not an orthomodular lattice, but rather a complete bi-Heyting algebra. Section 4 contains the main formal results of the paper, and provides a characterization of the relationship between the paraconsistent and intuitionistic aspects of the logic of TQT. In Section 5, we use these results to show that the most intuitive way of defining modal operators in TQT trivializes.

2. Topos Quantum Theory

Over the past twenty years, the topos theoretic reformulation of quantum theory (TQT) has been developed by Isham, Butterfield and Döring (e.g [1,2,7]), amongst others.³ TQT was originally motivated largely by the Kochen-Specker theorem, which states the impossibility of simultaneously assigning classical truth values to all of the projections onto a Hilbert space (of dimension greater than 2) in a way that respects the functional relations between those projections. The motivating factor here is that the Kochen specker theorem tells us (modulo some assumptions about non-contextuality) that given a quantum system Q , there will always exist propositions⁴ that can be asserted of Q that have no determinate truth value, and that this fact renders any attempt at understanding quantum mechanics as describing how Q ‘really is’ impossible. Thus, we read

“Any attempt to construct a realist interpretation of quantum theory founders on the Kochen–Specker theorem” [7]

Topos quantum theory can, for present purposes, be seen as the project to provide a new formalism for quantum theory that circumvents the Kochen-Specker theorem, and so is amenable to a realist interpretation. We will now

³For a thorough book length introduction to TQT, see [3].

⁴Note that by a ‘proposition that can be asserted of a physical system’ or ‘physical proposition’, we will always mean a proposition of the form $A \in \Delta$, i.e. the value of the physical quantity A is in the Borel subset Δ of the reals. Of course, in quantum theory, such propositions are in bijective correspondence with the projection operators onto the Hilbert space of the system in question, by the spectral theorem.

sketch how this is achieved, working by analogy with classical physics, the archetypal realist physical theory.

In classical physics, the classical system S being studied is assigned a phase space P , representing the space of possible states of the system. A physical quantity A is then formalized as a real-valued function f_A on S , taking each possible state λ of S to the value that S would have for A if its state were λ . The proposition $A \in \Delta$ ('the value of A is in the Borel subset Δ ') can then be represented by the inverse image of Δ under f_A , i.e. we represent the proposition $A \in \Delta$ with the set of all possible states λ of S such that $f_A(\lambda) \in \Delta$. So the algebra of propositions that can be asserted of S is just the algebra of all sets $f_A^{-1}(\Delta)$, for a physical quantity A and a Borel subset Δ . This algebra is Boolean and will be denoted ' $Sub(P)$ '. In classical physics, possible states of S and Boolean algebra homomorphisms from $Sub(P)$ to $\{0, 1\}$ are in bijective correspondence. Specifically, given a state $\psi \in P$, define $h_\psi : Sub(P) \rightarrow \{0, 1\}$ by $h_\psi(f_A^{-1}(\Delta)) = 1$ if and only if $f_A(\psi) \in \Delta$, and 0 otherwise.

Note that a classical state, as described above, can also be characterized as an assignment of a definite real-number value to every physical quantity associated with the system being described. Thus, an equivalent statement of the Kochen–Specker theorem is that it is impossible to assign definite values to all of the physical quantities associated with a quantum system in a way that preserves the functional relationships between those quantities.

Now, mathematically, it is well known that the root cause of the Kochen–Specker theorem is the existence of non-commuting self-adjoint operators representing incompatible observables (specifically, non-commuting observables are necessary for the application of the Kochen–Specker theorem). Thus, if we let $B(H)$ represent the set of all bounded self-adjoint operators on a fixed Hilbert space H , we know that since $B(H)$ contains non-commuting operators, the Kochen–Specker theorem will (usually) apply and it will be impossible to assign definite values to all of the operators in $B(H)$ in a consistent way. However, if we take any Abelian sub-algebra of V of $B(H)$, the Kochen–Specker theorem won't apply to V since all of V 's operators are mutually commuting. Thus, if we were to consider a quantum system solely from the perspective of the physical quantities represented by operators in V , it would be possible to define a state for the system, in the sense that it would be possible to assign that system definite values for all of the observables in V in a satisfactory way.

To be precise, what we do is consider the set $V(H)$ of all Abelian Von-Neumann⁵ sub-algebras of $B(H)$. Intuitively, each $V \in V(H)$ represents a ‘classical perspective’ on the quantum system, in the sense that the Kochen-Specker theorem does not rule out the existence of a state for the system, seen only from the perspective of V , i.e. we can always, in principle, simultaneously assign classical truth-values to all of the projection operators in V in a consistent manner. One way to see this is to note that the lattice $P(V)$ of all projection operators in any Abelian Von-Neumann algebra V will always be a Boolean algebra, and so there will exist a Boolean algebra homomorphism from $P(V)$ into $\{0, 1\}$.

The existence of states for any $V \in V(H)$ is expressed technically by the fact that any Abelian Von-Neumann algebra V comes equipped with a Gelfand spectrum, which consists of the set of all homomorphisms from V into the complex numbers, i.e. an element of the Gelfand spectrum of V is just a way of assigning complex-number values to all of the observables in V in a way that respects their functional and ordering relations. By analogy with classical physics, we can think of the Gelfand spectrum of each $V \in V(H)$ as a kind of ‘local state space’ of the system seen from the classical perspective represented by V .

The inclusion relation turns $V(H)$ into a partially ordered set. In what follows, we will treat $V(H)$ as a skeletal category.⁶ All this leads us to the following definition,

DEFINITION 2.1. The spectral presheaf⁷ $\underline{\Sigma}$ on $V(H)$ is defined by

Objects: Given $V \in V(H)$, the component $\underline{\Sigma}_V$ of $\underline{\Sigma}$ at V is the Gelfand spectrum of V .

Arrows: Given $i : V' \subseteq V$, $\underline{\Sigma}_{i_{V',V}} : \underline{\Sigma}_V \rightarrow \underline{\Sigma}_{V'}$
 $\lambda \mapsto \lambda|_{V'}$

Intuitively, the spectral presheaf takes each classical perspective on the quantum system to the local state space associated with that classical perspective. Also, given a classical perspective V whose states encode more information than those of another classical perspective V' (in this case,

⁵The technical details of Von-Neumann algebras are largely irrelevant for present purposes, and any important technical facts will be noted along the way.

⁶A category whose objects are just the elements of $V(H)$ and whose arrows are just the inclusion relations between the elements of $V(H)$, i.e. there exists an arrow $i : V' \rightarrow V$ if and only if $V' \subseteq V$.

⁷By a presheaf on $V(H)$, we will mean a contravariant set-valued functor on $V(H)$.

$V' \subseteq V$), the spectral presheaf takes a state λ for V and throws away any information that ‘can’t be seen’ by V' (i.e. it returns the restriction of λ to V').

Now, in topos quantum theory, the spectral presheaf is interpreted as the state space of the quantum system being described. The idea is that for each classical perspective V , $\underline{\Sigma}_V$ is the ‘local state space’ for V , and $\underline{\Sigma}$ is the result of ‘pasting’ all these local states spaces together into one global object. One of the most important technical results in TQT is the following,

THEOREM 2.2. (Isham and Butterfield [7]) *The Kochen-Specker theorem is equivalent to the fact that the spectral presheaf on $V(H)$ (for $\dim(H) \geq 3$) has no global elements.*

3. Topos Quantum Logic

In Section 2, we saw how topos quantum theory provides us with a new definition of the quantum state space. We will now explore the way in which this state space carries a natural logical structure that differs significantly from traditional quantum logic.

The first question that we need to ask is how we should go about representing a proposition of the form $A \in \Delta$, given our new definition of the quantum state space. Recall that in classical physics, a proposition corresponds to a special kind of subset of the system’s phase space (a measurable subset). So, extending the analogy with classical physics, we want to represent a proposition as a special kind of sub-object of the spectral presheaf. How do we go about doing this? The natural thing to do would be to go to each $V \in V(H)$ and find the part of V ’s Gelfand spectrum that makes the projection operator P corresponding to the proposition in question true. However, this will not work since there will generally be some $V \in V(H)$ such that $P \notin V$, and so the elements of V ’s Gelfand spectrum won’t be defined on P . So, we do the next best thing, i.e. we go to each $V \in V(H)$ and find V ’s best *approximation* to P . Specifically, we define

DEFINITION 3.1. The outer daseinisation of a projection operator P for a classical perspective $V \in V(H)$ is defined to be the projection operator given by

$$\delta^\circ(P)_V = \bigwedge \{Q \in P(V) \mid Q \geq P\}$$

where $P(V)$ is the set of all projection operators in V .

The idea is that the outer daseinisation of P at V is the strongest proposition that can be asserted from the perspective of V and is implied by P .

For example, it might be that P asserts that the momentum of the system is in a particular range, and V is a classical perspective from which we can't know the momentum to the degree of accuracy asserted by P (maybe V allows to talk about position to a relatively high degree of accuracy), and so the outer daseinisation of P at V is the most precise approximation that we can make to the momentum of the system without going beyond what we can know from the perspective represented by V .

Returning to the analogy with classical physics, if we think of $\delta^o(P)_V$ as a 'local proposition', then, as in classical physics, we can identify this local proposition with the part of V 's local state space that makes it true.

DEFINITION 3.2. Let $V \in V(H)$, $P \in P(H)$ (where $P(H)$ is the lattice of all projection operators on H). Then define

$$S_{\delta^o(P)_V} = \{\lambda \in \underline{\Sigma}_V \mid \lambda(\delta^o(P)_V) = 1\}$$

i.e. $S_{\delta^o(P)_V}$ is the set of all V 's possible states that make V 's approximation to P true.

We are now in a position to represent any quantum proposition as a sub-object of the spectral presheaf.

DEFINITION 3.3. Let $P \in P(H)$. Then we define P 's outer daseinisation presheaf $\underline{\delta^o(P)}$ on $V(H)$ by

Objects: Given $V \in V(H)$, the component $\underline{\delta^o(P)}_V$ of $\underline{\delta^o(P)}$ at V is given by $\underline{\delta^o(P)}_V = S_{\delta^o(P)_V}$

Arrows: Given $i : V' \subseteq V$, $\underline{\delta^o(P)}_{i_{V',V}} : S_{\delta^o(P)_V} \rightarrow S_{\delta^o(P)_{V'}}$

$$\lambda \mapsto \lambda|_{V'}$$

Note that this is well defined since if $\lambda \in S_{\delta^o(P)_V}$, then $\lambda(\delta^o(P)_V) = 1$, and since $V' \subseteq V$, we have

$$\delta^o(P)_V = \bigwedge \{Q \in P(V) \mid Q \geq P\} \leq \bigwedge \{Q \in P(V') \mid Q \geq P\} = \delta^o(P)_{V'}$$

i.e. $\delta^o(P)_V \leq \delta^o(P)_{V'}$.

So, since λ is a homomorphism and $\lambda(\delta^o(P)_V) = 1$, we have that $\lambda(\delta^o(P)_{V'}) = 1$, proving that the presheaf is well defined.

Note that we refer to the general fact that $\delta^o(P)_V \leq \delta^o(P)_{V'}$ for $V' \subseteq V$ as 'coarse graining'.

Thus, in analogy with classical physics, we represent a proposition as a collection of subsets of local state spaces, one for each $V \in V(H)$. Specifically, for $V \in V(H)$, we choose the subset of V 's local state space that

makes V 's approximation to the proposition true. The following definition will be useful,

DEFINITION 3.4. A sub-object (sub-presheaf) \underline{S} of the spectral presheaf $\underline{\Sigma}$ is a presheaf on $V(H)$ such that (i) $\forall V \in V(H) (\underline{S}_V \subseteq \underline{\Sigma}_V)$ (ii) Given $i : V' \subseteq V$, $\underline{\Sigma}_{i_{V',V}}(\underline{S}_V) \subseteq \underline{S}_{V'}$

Now, it should be noted that for any $P \in P(H)$ and for any $V \in V(H)$, the set $S_{\delta^o(P)_V} \subseteq \underline{\Sigma}_V$ is always a clopen subset of V 's Gelfand spectrum $\underline{\Sigma}_V$ (where $\underline{\Sigma}_V$ is given the weak $*$ -topology). Together with Def 3.4, this motivates the following definition,

DEFINITION 3.5. A clopen sub-object \underline{S} of the spectral presheaf is a subobject \underline{S} of $\underline{\Sigma}$ such that $\forall V \in V(H) \underline{S}_V$ is a clopen subset of $\underline{\Sigma}_V$

Thus, in topos quantum theory, we represent propositions as clopen sub-objects of the spectral presheaf. Note that for each $V \in V(H)$, the set $cl(\underline{\Sigma}_V)$ of clopen subsets of V 's Gelfand spectrum is in bijective correspondence with the set $P(V)$ of all projection operators in V (by Gelfand duality). Thus, any clopen subobject \underline{S} of the spectral presheaf defines a 'local proposition' $P_{\underline{S}_V} \in P(V)$ for each classical perspective V . It is easily seen that coarse graining will apply to these local propositions. For, if $V' \subseteq V$ then $\underline{\Sigma}_{i_{V',V}}(\underline{S}_V) \subseteq \underline{S}_{V'}$. So, given $\lambda \in \underline{\Sigma}_V$ such that $\lambda(P_{\underline{S}_V}) = 1$, we have that $\lambda \in \underline{S}_V$ and hence $\lambda|_{V'} \in \underline{S}_{V'}$, which implies $\lambda|_{V'}(P_{\underline{S}_{V'}}) = 1$, i.e. $\lambda(P_{\underline{S}_{V'}}) = 1$. This proves that $P_{\underline{S}_V} \leq P_{\underline{S}_{V'}}$.

So we can think of \underline{S} as a kind of 'global proposition' whose local components get more general as we lose information by moving to smaller classical perspectives. Note that it will not generally hold that $\underline{S} = \underline{\delta^o(P)}$ for some $P \in P(H)$. The outer daseinisation presheaves are only a special subclass of the clopen sub-objects of the spectral presheaf. By interpreting the set $Sub_{cl}(\underline{\Sigma})$ of all clopen sub-objects of the spectral presheaf as the set of all physical propositions that can be made about the quantum system, we are obtaining a logical structure that is strictly richer than that of traditional quantum logic. However, it should be noted that the problem of providing a physical interpretation for these new physical propositions that don't correspond to projection operators is a significant one that has not yet been addressed in the literature.

The natural question to ask now is "what is the algebraic structure of $Sub_{cl}(\underline{\Sigma})$?" To answer this, we begin by defining a lattice structure on $Sub_{cl}(\underline{\Sigma})$ in the following way,

$$\text{Given } \underline{S}, \underline{T} \in sub_{cl}(\underline{\Sigma}) : \underline{S} \leq \underline{T} \leftrightarrow (\forall V \in V(H) : \underline{S}_V \subseteq \underline{T}_V)$$

We define the meet and join operations component-wise,

For any family $(\underline{S}_i)_{i \in I}$, for any $V \in V(H)$,

$$\left(\bigwedge_{i \in I} \underline{S}_i \right)_V = \text{int} \left(\bigcap_{i \in I} \underline{S}_{i;V} \right), \left(\bigvee_{i \in I} \underline{S}_i \right)_V = \text{cls} \left(\bigcup_{i \in I} \underline{S}_{i;V} \right)$$

where *int* and *cls* are the topological interior and closure operations, respectively, which are required in order to ensure that the lattice is complete (since the arbitrary of intersection of clopen sets is not necessarily open, and the arbitrary union of clopen sets is not necessarily closed.)

PROPOSITION 3.6. (Isham and Döring [2]) *The operations defined above turn $Sub_{cl}(\underline{\Sigma})$ into a complete distributive lattice.*

The top and bottom elements, $\underline{1}$ and $\underline{0}$, of $Sub_{cl}(\underline{\Sigma})$ are just the presheaves that give $\underline{\Sigma}_V$ and \emptyset respectively, $\forall V \in V(H)$.

Thus, the logic that is obtained directly from the topos theoretic formulation of quantum mechanics is distributive, unlike the traditional quantum logic that arises from the Hilbert space formalism, which is only orthomodular. Distributivity allows us to define a canonical implication connective \Rightarrow on $Sub_{cl}(\underline{\Sigma})$ with the identifying property

$$\underline{R} \wedge \underline{S} \leq \underline{T} \leftrightarrow \underline{R} \leq (\underline{S} \Rightarrow \underline{T}).$$

So that

$$\underline{S} \Rightarrow \underline{T} = \bigvee \{ \underline{R} \in Sub_{cl}(\underline{\Sigma}) \mid \underline{R} \wedge \underline{S} \leq \underline{T} \}$$

Of course, \Rightarrow turns $Sub_{cl}(\underline{\Sigma})$ into a complete Heyting algebra. So the logic that arises naturally from the topos theoretic formalization of quantum theory is intuitionistic. The intuitionistic negation is obtained in the standard way, by defining

$$\neg \underline{S} = \underline{S} \Rightarrow \underline{0}$$

This implies that $\neg \underline{S}$ is the largest element of $Sub_{cl}(\underline{\Sigma})$ such that

$$\neg \underline{S} \wedge \underline{S} = \underline{0}.$$

So \neg has all the usual important properties of an intuitionistic negation operation, i.e.

$$\begin{aligned} \underline{S} \leq \underline{T} &\text{ implies } \neg \underline{T} \leq \neg \underline{S} \\ \neg \neg \underline{S} &\geq \underline{S} \end{aligned}$$

$$\neg\neg\neg\underline{S} = \neg\underline{S}$$

$$\neg\underline{S} \vee \underline{S} \leq \underline{1}$$

However, it is also possible to define another algebraic structure on the lattice $Sub_{cl}(\underline{\Sigma})$, dual to the Heyting algebraic structure defined above. First, we need the following definitions,

DEFINITION 3.7. A *co-Heyting algebra* is a distributive lattice L with top and bottom elements (1 and 0) that is equipped with a binary operation (called the ‘co-Heyting supplement operation’) \Leftarrow such that for any $A, B, C \in L$

$$(A \Leftarrow B) \leq C \leftrightarrow A \leq B \vee C$$

Given a co-Heyting algebra L , it is possible to define a corresponding *co-Heyting negation* operation \sim by setting

$$\sim A = 1 \Leftarrow A$$

It is easy to see that $\sim A$ has the defining property of being the smallest element of L such that $A \vee \sim A = 1$. Generally, \sim has the following important properties,

$$A \leq B \text{ implies } \sim B \leq \sim A$$

$$\sim\sim A \leq A$$

$$\sim\sim\sim A = \sim A$$

$$\sim A \wedge A \geq 0$$

Crucially, while the Heyting negation satisfies the law of non contradiction but generally violates the law of excluded middle, the co-Heyting negation satisfies excluded middle but generally violates non-contradiction. Thus, while Heyting algebras are naturally used to provide algebraic semantics for intuitionistic logics, co-Heyting algebras are naturally used to provide the algebraic semantics for various kinds of paraconsistent logic. Now, returning to the task at hand, it is possible to show (for details see [1]) that as well as being a complete Heyting algebra, $Sub_{cl}(\underline{\Sigma})$ is also a complete co-Heyting algebra.

Specifically, we can define the operation \Leftarrow on $Sub_{cl}(\underline{\Sigma})$ by

$$(\underline{S} \Leftarrow \underline{T}) = \bigwedge \{ \underline{R} \in Sub_{cl}(\underline{\Sigma}) \mid \underline{S} \leq \underline{T} \vee \underline{R} \}$$

The connective \Leftarrow has the identifying property

$$(\underline{S} \Leftarrow \underline{T}) \leq \underline{R} \leftrightarrow \underline{S} \leq \underline{T} \vee \underline{R}$$

This proves that \Leftarrow is a co-Heyting supplement operation and hence that $Sub_{cl}(\underline{\Sigma})$ is a complete co-Heyting algebra. Thus, we can define a co-Heyting negation \sim by

$$\sim \underline{S} = (\underline{\Sigma} \Leftarrow \underline{S})$$

So $\sim \underline{S}$ is the least element of $Sub_{cl}(\underline{\Sigma})$ such that

$$\sim \underline{S} \vee \underline{S} = \underline{\Sigma}$$

Thus, we have

THEOREM 3.8. (Döring [1]) *Sub_{cl}($\underline{\Sigma}$), equipped with the Heyting implication and co-Heyting supplement defined above, is both a complete Heyting algebra and a complete co-Heyting algebra, i.e. it is a complete bi-Heyting algebra.*

Theorem 3.8 raises some interesting question about the nature of the logic that has been obtained. One important question concerns the physical interpretation of the two new negations, and the relationship between them. Specifically, it would be interesting to know when, if ever, the two negations coincide, and whether the subset of $Sub_{cl}(\underline{\Sigma})$ on which the two negations agree corresponds to any physically significant property. This is the question to which we will turn in the following section.

4. Complemented Subobjects

DEFINITION 4.1. Given a complete bi-Heyting algebra B , equipped with intuitionistic and paraconsistent negations \neg and \sim , we call an element $b \in B$ ‘complemented’ if and only if $\neg b = \sim b$.

Intuitively, the complemented elements of a bi-Heyting algebra give us information about where the paraconsistent and intuitionistic logical structures ‘agree’. Following this intuition, we will now attempt to provide a characterization of complemented elements of $Sub_{cl}(\underline{\Sigma})$.

PROPOSITION 4.2. (Döring [1]) *Let $\underline{S} \in Sub_{cl}(\underline{\Sigma})$. Then, for any $V \in V(H)$, the projection operators $P_{\neg \underline{S}_V}$ and $P_{\sim \underline{S}_V}$ corresponding to $\neg \underline{S}$ and $\sim \underline{S}$ at V are given by*

$$P_{\neg \underline{S}_V} = \top - \bigvee_{V' \in m_V} P_{\underline{S}_{V'}}, P_{\sim \underline{S}_V} = \bigvee_{\bar{V} \in M_V} (\delta^\circ (\top - P_{\underline{S}_{\bar{V}}}))$$

where m_V represents the set of all minimal Abelian Von Neumann subalgebras of V , M_V represents the set of all maximal Abelian Von Neumann

superalgebras of V and \top denotes the top element of $P(H)$, i.e. the unit operator (we also let \perp represent the bottom element of $P(H)$, i.e. the zero operator).

So the proposition that $\neg \underline{S}$ corresponds to $\text{at } V$ is not the complement of the proposition that \underline{S} corresponds to $\text{at } V$. Rather, it is the complement of the disjunction of all of \underline{S} 's coarse grainings of the proposition that \underline{S} corresponds to $\text{at } V$. Thus, $P_{\neg \underline{S}_V}$ does not just say that $P_{\underline{S}_V}$ is false. It also says that there any way of generalizing $P_{\underline{S}_V}$ to a less informative classical perspective is also false. On the other hand, the proposition that corresponds to $\sim \underline{S}$ at V can be interpreted as saying, from the classical perspective represented by V , that at least one of the ‘precisifications’ or ‘fine grainings’ of the proposition that \underline{S} corresponds to $\text{at } V$ is false.

DEFINITION 4.3. A clopen subobject \underline{S} of $\underline{\Sigma}$ is uniform if and only if $\forall V', V \in V(H)$, if $V' \subseteq V$, then $P_{\underline{S}_{V'}} = P_{\underline{S}_V}$.

PROPOSITION 4.4. A clopen subobject \underline{S} of $\underline{\Sigma}$ is complemented if and only if it is uniform.

PROOF. Suppose that \underline{S} is complemented. Then, we have

$$(*) (\forall V \in V(H)) \top - \bigvee_{V' \in m_V} P_{\underline{S}_{V'}} = \bigvee_{\bar{V} \in M_V} \delta^\circ (\top - P_{\underline{S}_{\bar{V}}})_V$$

Fix $V \in V(H)$, and let $\bar{V} \in M_V$. Then, by coarse graining,

$$P_{\underline{S}_{\bar{V}}} \leq P_{\underline{S}_V} \leq P_{\underline{S}_{V'}}, \text{ for any } V' \in m_V. \text{ So, conversely}$$

$$\top - P_{\underline{S}_{\bar{V}}} \geq \top - P_{\underline{S}_V} \geq \top - P_{\underline{S}_{V'}}$$

This proves

$$(i) \delta^\circ (\top - P_{\underline{S}_{\bar{V}}})_V \geq \top - P_{\underline{S}_V} \geq \top - P_{\underline{S}_{V'}}$$

Note that, since \bar{V} and V' were arbitrary, (i) holds for any $\bar{V} \in M_V$ and any $V' \in m_V$. Conversely, the left hand side of (*) is just $\bigwedge_{V' \in m_V} \top - P_{\underline{S}_{V'}}$. So, by (*), we know that for any $V' \in m_V$ and for any $\bar{V} \in M_V$, the following holds,

(ii) $\top - P_{\underline{S}_{V'}} \geq \delta^o(\top - P_{\underline{S}_{\bar{V}}})_V$

Combining (i) and (ii), we have

(iii) $(\forall V \in V(H))(\forall V' \in m_V)(\forall \bar{V} \in M_V) \top - P_{\underline{S}_{V'}} = \delta^o(\top - P_{\underline{S}_{\bar{V}}})_V$

■

Now, suppose that we can find $V, V' \in V(H)$, $V' \subseteq V$, such that $P_{\underline{S}_V} \neq P_{\underline{S}_{V'}}$. Then we have $\top - P_{\underline{S}_V} \neq \top - P_{\underline{S}_{V'}}$. Thus, we can find $V_m \in m_V$ and $V_M \in M_V$ satisfying $V_m \subseteq V' \subseteq V \subseteq V_M$ such that

$$\delta^o(\top - P_{\underline{S}_{V_M}})_V \geq \top - P_{\underline{S}_{V_M}} \geq \top - P_{\underline{S}_V} \neq \top - P_{\underline{S}_{V'}} \geq \top - P_{\underline{S}_{V_m}}$$

But this contradicts (iii). So there can't be $V, V' \in V(H)$ such that $V' \subseteq V$ and $\underline{S}_V \neq \underline{S}_{V'}$. So \underline{S} is uniform.

Conversely, let \underline{S} be uniform. Then, of course, for any $V \in V(H), V' \in m_V, V^\sim \in M_V$, $\top - P_{\underline{S}_{V'}} = \top - P_{\underline{S}_V} = \top - P_{\underline{S}_{V^\sim}}$. So

$$\bigvee_{V^\sim \in M_V} \delta^o(\top - P_{\underline{S}_{V^\sim}})_V = \bigvee_{V^\sim \in M_V} \top - P_{\underline{S}_{V^\sim}} = \top - P_{\underline{S}_V} = \top - \bigvee_{V' \in m_V} P_{\underline{S}_{V'}}$$

But this proves that for any $V \in V(H)$, $P_{\neg \underline{S}_V} = P_{\sim \underline{S}_V}$. So \underline{S} is complemented. This completes the proof.

Proposition 4.4 tells us that the two negations on $Sub_{cl}(\underline{\Sigma})$ coincide on a subobject \underline{S} if and only if \underline{S} is constant in the sense that \underline{S} corresponds to the same local proposition for any pair of classical perspectives that are related by inclusion. In this sense, complemented subobjects are precisely those subobjects that ‘don’t need to approximate’.

LEMMA 4.5. *If $\underline{S} \in Sub_{cl}(\underline{\Sigma})$ is complemented, then $\forall V \in V(H)$ ($P_{\underline{S}_V} = \top$ or $P_{\underline{S}_V} = \perp$).*

PROOF. Let \underline{S} be complemented. Then, by the previous proposition, it is uniform. So, given $V' \subseteq V \in V(H)$, $P_{\underline{S}_V} = P_{\underline{S}_{V'}}$. Now, let $V \in V(H)$ be such that $P(V)$ contains two elements, P and Q , satisfying $P \neq Q$, $P \neq \top - Q$, $Q \neq \top - P$. Then, V has minimal Abelian subalgebras V_P and V_Q , where V_P is the minimal Abelian subalgebra generated by P (and similarly for Q). Then $P(V_P) \cap P(V_Q) = \{\top, \perp\}$. Now, since \underline{S} is uniform, we have $P_{\underline{S}_{V_P}} = P_{\underline{S}_V} = P_{\underline{S}_{V_Q}}$. So $P_{\underline{S}_V} \in P(V_P) \cap P(V_Q) = \{\top, \perp\}$. So $P_{\underline{S}_V} = \perp$ or $P_{\underline{S}_V} = \top$. If $P(V)$ does not contain any elements like P and Q above, then V is the minimal Abelian subalgebra generated by some $P \in P(H)$. But then V will be contained in some maximal superalgebra \bar{V} which contains some other projection $Q \in P(\bar{V})$ satisfying $P \neq Q$, $P \neq \top - Q$, $Q \neq \top - P$. So the minimal Abelian algebra V_Q generated by Q is a subalgebra of \bar{V} .

So, since \underline{S} is uniform, $P_{\underline{S}_V} = P_{\underline{S}_{\bar{V}}} = P_{\underline{S}_{V_Q}}$. Again this proves that either $P_{\underline{S}_V} = \perp$ or $P_{\underline{S}_V} = \top$. ■

So, if \underline{S} is complemented, then at any given classical perspective V , the physical proposition to which \underline{S} corresponds will be either a contradiction or a tautology. \underline{S} will never correspond to a physically significant and non-trivial proposition. We are now ready to prove the key theorem that makes the relationship between the Heyting and co-Heyting negations of $Sub_{cl}(\underline{\Sigma})$ clear.

THEOREM 4.6. *The only complemented elements of $Sub_{cl}(\underline{\Sigma})$ are the top and bottom elements $\underline{1} = \underline{\delta^o(\top)}$ and $\underline{0} = \underline{\delta^o(\perp)}$, respectively.*

PROOF. Let $\underline{S} \in Sub_{cl}(\underline{\Sigma})$ be complemented. Then, by lemma 4.5, we know that for any $V \in V(H)$, either $P_{\underline{S}_V} = \top$ or $P_{\underline{S}_V} = \perp$. Now, suppose towards contradiction that there exist V and V' such that $P_{\underline{S}_V} = \top$ and $P_{\underline{S}_{V'}} = \perp$. Then, let V_1 be a minimal subalgebra of V , generated by a single projection P_1 and the identity, and let V_2 be a minimal subalgebra of V' generated by a single projection P_2 and the identity. Then, since S is complemented, it is also uniform, which guarantees that $P_{\underline{S}_{V_1}} = \top$ and $P_{\underline{S}_{V_2}} = \perp$. ■

Now, if P_1 and P_2 commute, then the algebra W generated by P_1, P_2 and the identity is a joint super-algebra of V_1 and V_2 , which means that $P_{\underline{S}_W} = \perp$ and $P_{\underline{S}_W} = \top$, by uniformity of \underline{S} , which is a contradiction. So P_1 and P_2 do not commute.

Since P_1 and P_2 do not commute, find a non-zero P_3 that is orthogonal to both of them. Let V_3 be generated by P_1, P_3 and \top . Then, since $V_1 \subseteq V_3$, we have $P_{\underline{S}_{V_3}} = \top$. Let V_4 be the algebra generated by P_3 and \top . Then $V_4 \subseteq V_3$, so $P_{\underline{S}_{V_4}} = \top$. Let V_5 be generated by P_2, P_3 and \top . Then this $V_4 \subseteq V_5$, so $P_{\underline{S}_{V_5}} = \top$. But $V_2 \subseteq V_5$, so $P_{\underline{S}_{V_5}} = \perp$. Contradiction. So there cannot exist V, V' such that $P_{\underline{S}_V} = \top$ and $P_{\underline{S}_{V'}} = \perp$. So either $\forall V \in V(H)(P_{\underline{S}_V} = \top)$ or $\forall V \in V(H)(P_{\underline{S}_V} = \perp)$, i.e. either $\underline{S} = \underline{1}$ or $\underline{S} = \underline{0}$.

Theorem 4.6 provides a definite characterization of the relationship between the paraconsistent and intuitionistic aspects of the logic of TQT. In particular, we now know that these two aspects ‘only agree at the limits’. In the following section, we will study the implications of this result for the definition of modal operators on $Sub_{cl}(\underline{\Sigma})$.

5. Modal Operators in TQT

In [8], it was shown that any complete bi-Heyting algebra comes equipped with a canonical modal structure that is obtained through the iteration of

the Heyting and co-Heyting negations of the algebra. We will now give a brief overview of this modal structure. First, we need the following definitions,

DEFINITION 5.1. An *interior operator* \Box on a complete lattice L with an implication connective \rightarrow is an operator satisfying

$$(i) (\forall x \in L) \Box x \leq x$$

$$(ii) \Box 1 = 1$$

$$(iii) (\forall x, y \in L) \Box(x \rightarrow y) = (\Box x \rightarrow \Box y).$$

$$(iv) (\forall x \in L) \Box \Box x = \Box x$$

$$(v) \text{ If } x \leq y \text{ then } \Box x \leq \Box y$$

DEFINITION 5.2. A *closure operator* \Diamond on a complete lattice L with an implication connective \rightarrow is an operator satisfying

$$(i) (\forall x \in L) \Diamond x \geq x$$

$$(ii) \Diamond 0 = 0$$

$$(iii) (\forall x, y \in L) \Diamond(x \rightarrow y) = (\Diamond x \rightarrow \Diamond y).$$

$$(iv) (\forall x \in L) \Diamond \Diamond x = \Diamond x$$

$$(v) \text{ If } x \leq y \text{ then } \Diamond x \leq \Diamond y$$

Of course, the notions of ‘interior’ and ‘closure’ operators, as defined above, are the lattice-theoretic translations of the necessity and possibility operators of modal logic ($S4$ in particular). So, in order to study modal notions in the context of TQT, we will attempt to find a canonical procedure for defining interior and closure operators on $Sub_{cl}(\underline{\Sigma})$. First, we need to see how all complete bi-Heyting algebras come equipped with canonically defined interior and closure operators.

Fix a complete bi-Heyting algebra B with Heyting negation \neg and co-Heyting negation \sim . Then, we can make the following inductive definition,

DEFINITION 5.3. We define the operators $\square_n, \diamond_n : B \rightarrow B$ ($n \in \mathbb{N}$) by

$$\begin{aligned} \square_0 &= \diamond_0 = id_B \\ \square_{n+1} &= \neg \sim \square_n, \diamond_{n+1} = \sim \neg \diamond_n \end{aligned}$$

So \square_n and \diamond_n are essentially the operations of iterating $\neg \sim$ and $\sim \neg$ n times, respectively. Now, we obtain the desired modal operators by taking the limit over n , i.e.

DEFINITION 5.4. We define the operators $\square, \diamond : B \rightarrow B$ by

$$\begin{aligned} \square a &= \bigwedge_{n \in \mathbb{N}} \square_n a \\ \diamond a &= \bigvee_{n \in \mathbb{N}} \diamond_n a \end{aligned}$$

for all $a \in B$.

It is possible to show that \square and \diamond , as defined above, satisfy the defining features of interior and closure operators, respectively. Specifically, we have

PROPOSITION 5.5. (Reyes and Zolfaghari [8]) *The operators \square and \diamond satisfy the following properties⁸*

- (i) \square and \diamond are order preserving
- (ii) $\square a \leq a \leq \diamond a$ ($\forall a \in B$)
- (iii) $\square \square a = \square a$ and $\diamond \diamond a = \diamond a$ ($\forall a \in B$).

PROPOSITION 5.6. (Reyes and Zolfaghari [8]) *Let $a \in B$. Then $\square a$ is the greatest complemented $x \in B$ such that $x \leq a$ and $\diamond a$ is the least complemented $x \in B$ such that $x \geq a$.*

Combined with Theorem 4.6, Proposition 5.6 immediately implies that this way of defining modal operators will trivialize for the bi-Heyting algebra $Sub_{cl}(\underline{\Sigma})$ (the image of the \square and \diamond operators will just be $\{0, 1\}$). However, there may yet be a way around this triviality. Specifically, rather than considering the usual case where we build the spectral presheaf over the poset of Abelian subalgebras of the set of $B(H)$ of all bounded operators on the Hilbert space H , let's suppose that we begin with a Von Neumann algebra W with a non-trivial center, $C(W)$, and let $V(W)$ represent the poset of all Abelian Von-Neumann subalgebras of W . Then, as before, we can

⁸We have not included condition (iii) from Def 4.1/Def 4.2 since we are using both the Heyting and co-Heyting algebraic structure of B , and so have not chosen a single privileged implication connective.

build the spectral presheaf over $V(W)$ and $Sub_{cl}(\underline{\Sigma})$ will still be a complete bi-Heyting algebra, and we obtain the following characterization of complemented clopen-subobjects,

THEOREM 5.7. *Let $\underline{S} \in Sub_{cl}(\underline{\Sigma})$. Then \underline{S} is complemented if and only if there exists some $P \in C(W)$ such that $\forall V \in V(H), P_{\underline{S}_V} = P$.*

PROOF. First, let $P \in C(W)$ be such that $\forall V \in V(W)(P = P_{\underline{S}_V})$. Then, using the formulae for the Heyting and bi-Heyting negations, we obtain

$$P_{\neg \underline{S}_V} = \top - \bigvee_{V' \in m_V} P_{\underline{S}_{V'}} = \top - P$$

$$P_{\sim \underline{S}_V} = \bigvee_{\bar{V} \in M_V} \delta^o(\top - P_{\underline{S}_{\bar{V}}})_V = \top - P$$

for arbitrary $V \in V(W)$, which proves that \underline{S} is complemented, as desired. Conversely, suppose that \underline{S} is complemented. It is easily checked that the proof of proposition 5.3 also applies in the new generalized setting. In particular, we can conclude that \underline{S} is uniform. We then apply the same argument used in the proof of lemma 5.4 to show that $P_{\underline{S}_V} \in C(W)$ will always hold. Finally, we use the argument from the proof of theorem 5.5 to complete the proof. ■

Theorem 5.7 gives us a new perspective on the philosophical significance of Theorem 4.6. Specifically, it tells us that the only reason that the modal structure of $Sub_{cl}(\underline{\Sigma})$ is generally trivial is that $B(H)$ generally has a trivial center. In the special cases where we build our spectral presheaf over an algebra with a non-trivial center, we obtain a non-trivial modal structure that encodes all the information about when the Heyting and co-Heyting negations coincide.

6. Modality and Contextuality in TQT and Hilbert Space

What Theorem 5.7 tells us is that the complemented clopen subobjects are precisely those that correspond to physical propositions that are not effected by quantum contextuality, i.e. they have the same meaning in every classical perspective, and can always be meaningfully asserted, regardless of the measurement context. Thus, the modal operators act by taking a clopen subobject to its best non-contextual approximations. Essentially, the modal operators in TQT can be thought of as ‘contextuality annihilators’ that serve to replace physical propositions with their best possible non-contextual

substitutes. At this point, it will be useful to provide a quick overview of some relevant results concerning the definition of modal operators on the orthomodular projection lattices of traditional quantum logic.

The problem of defining interior and closure operators on an orthomodular lattice $P(H)$ of projection operators on a Hilbert space H was thoroughly studied by Herman and Piziak [5]. They managed to prove the following important result,

THEOREM 6.1. (Herman and Piziak [5]) *Let L be an orthomodular lattice. Then the following objects are all in bijective correspondence,*

- (i) *Interior operators on L*
- (ii) *Closure operators on L*
- (iii) *Strong implication connectives on L*

Where ‘strong implications’ are lattice theoretic implication connectives on L satisfying some intuitive properties. For example, the Sasaki hook is an instance of a strong implication connective on an orthomodular lattice.

This theorem raises a significant problem for formalizing modal notions in the context of traditional quantum logic. Specifically, it tells us that there are just as many ways to formalize quantum necessity and possibility as there are ways to formalize the notion of implication in quantum logic. But it is well known that there is very little consensus in the literature concerning the choice of quantum implication connectives. Thus, it seems that, in the context of traditional quantum logic, the problem of finding a philosophically and technically adequate formalization of modality can only be solved simultaneously with the much more deeply entrenched and hotly contested issue of choosing a privileged implication connective.

In the literature, the problem of choosing a single formalization of possibility and necessity in the context of traditional quantum logic has largely been ignored. However, in recent years, there has been one significant attempt to address the issue, due to De Ronde, Freytes and Domenech [4].

Specifically, they identify a particular choice of closure operator as being physically and philosophically important. In particular, they choose to define the closure operator \diamond on the orthomodular projection lattice $P(H)$ in the following way,

$$\diamond x = \bigwedge \{z \in Z(P(H)) \mid x \leq z\}$$

where $Z(P(H))$ is the center of $P(H)$, i.e. the set of all elements of $P(H)$ which commute with all other elements. It is well known that the center of

$P(H)$ will be a complete Boolean algebra, and so $\diamond x \in Z(P(H))$ will always hold.

Philosophically, the justification of this choice of closure operator is justified by the idea that in quantum contexts, modal notions should be used to mitigate the philosophically problematic consequences of quantum contextuality. The key idea is that the Kochen Specker theorem tells us that we cannot simultaneously talk about all the propositions represented by elements of $P(H)$, but we may at least be able to talk about whether or not those propositions are *possible* (possibly true). Thus, we read

...one may predicate truth or falsity of all possibilities at the same time, i.e., possibilities allow an interpretation in a Boolean algebra. [4]

The hope is that we may be able to avoid quantum contextuality by ‘modalising’ our discourse about quantum systems, and prefixing our physical propositions with modal operators. Given this motivation, it is very natural to formalize modal propositions as elements of the center of $P(H)$, since the central elements are exactly those elements that correspond to propositions which are immune to the effects of contextuality, i.e. they are the propositions that can always be meaningfully asserted from any measurement context.

Now, it seems that this approach to defining modal operators in Hilbert space quantum theory has exactly the same kind of philosophical interpretation as that which has been given to the modal operators we defined for TQT. In both cases, modal operators serve to eliminate the contextual aspects of the propositions on which they act. What Theorem 4.6 tells us is that it is not generally possible to do this in a non-trivial way in TQT, i.e. the physical propositions of TQT are inherently contextual in a way that is not the case for the propositions of the Hilbert space formalism. But this should not come as a surprise, given the centrality of the notion of quantum-contextuality in the philosophical motivation for TQT. Indeed, the representation of physical propositions in TQT makes explicit use of variation over measurement contexts, whereas the Hilbert space formalism simply represents physical propositions as individual projection operators, with no reference to varying classical perspectives. So the apparent triviality of the modal structure of TQT is philosophically significant. It is symptomatic of the inherently contextual nature of the physical propositions of the formalism. We can roughly summarize the situation with the slogan ‘in TQT, modality = contextuality’.

It turns out that formalizing modal notions in TQT is a much neater and philosophically less contentious task than it is in the context of the Hilbert space formalism. The reason for this is that in TQT, the logical structure of the formalism provides us with a canonical modal structure that is completely determined by the interplay between the Heyting and co-Heyting algebraic structures of the lattice of physical propositions. As we have seen, the Hilbert space formalism has no such canonical structure. The reason for this is that unlike TQT, the Hilbert space formalism does not provide us with an obvious choice of implication connective, and, as we have seen, this means that the modal structure is not canonically determined. In this sense, modality is philosophically much simpler in TQT than it is in the Hilbert space formalism, where there are as many choices of modal structure as there are choices of implication connective.

However, it is interesting that in both cases, modality appears to have a natural connection to quantum contextuality. Specifically, the formalization of modality advocated by De Ronde, Freytes and Domenech, like the formalization of modality in TQT, can also be seen as the formalization of operators that serve to destroy the contextual elements of the propositions on which they act. This suggests that the connection between modality and quantum contextuality might be more than just a side effect of the way in which quantum theory is formalized. It might actually be a deep and important feature of the theory.

Finally, it should be noted that the definition of modal operators in TQT provides an example of how the bi-Heyting algebraic structure of $Sub_{cl}(\underline{\Sigma})$ can be philosophically and technically significant. Thus far, the literature has tended to focus entirely on the Heyting algebraic/intuitionistic aspect of TQT's logical structure. The author believes that the bi-Heyting algebraic structure discovered by Döring warrants further investigation.

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B. EVA
University of Bristol
Bristol, UK
be0367@bristol.ac.uk