Hindawi
Complexity
Volume 2018, Article ID 7257083, 15 pages
https://doi.org/10.1155/2018/7257083

Research Article

An *r*-Order Finite-Time State Observer for Reaction-Diffusion Genetic Regulatory Networks with Time-Varying Delays

Xiaofei Fan,[1,2] Yantao Wang,[1,3] Ligang Wu,[4] and Xian Zhang[1,3]

[1]*School of Mathematical Science, Heilongjiang University, Harbin 150080, China*
[2]*Institute of Systems Science, Northeastern University, Shenyang 110819, China*
[3]*Heilongjiang Provincial Key Laboratory of the Theory and Computation of Complex Systems, Heilongjiang University, Harbin 150080, China*
[4]*Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin 150001, China*

Correspondence should be addressed to Xian Zhang; xianzhang@ieee.org

Received 27 February 2018; Accepted 2 July 2018; Published 20 September 2018

Academic Editor: Mohamed Boutayeb

It will be settled out for the open problem of designing an r-order finite-time (F-T) state observer for reaction-diffusion genetic regulatory networks (RDGRNs) with time-varying delays. By assuming the Dirichlet boundary conditions, aiming to estimate the mRNA and protein concentrations via available network measurements. Firstly, sufficient F-T stability conditions for the filtering error system have been investigated via constructing an appropriate Lyapunov–Krasovskii functional (LKF) and using several integral inequalities and (reciprocally) convex technique simultaneously. These conditions are delay-dependent and reaction-diffusion-dependent and can be checked by MATLAB toolbox. Furthermore, a method is proposed to design an r -order F-T state observer, and the explicit expressions of observer gains are given. Finally, a numerical example is presented to illustrate the effectiveness of the proposed method.

1. Introduction

Recently, due to a great many applications in the real world, genetic regulatory networks have become one of the hot topics in many fields. Much significant results (see [1–10] and the references therein) have been obtained. Usually, owing to the highly complexity of genetic regulatory networks, spatial homogeneity (i.e., the mRNA and protein concentrations are independent on their space positions) is assumed in the process of modelling genetic regulatory networks. However, this assumption is sometimes unreasonable, for example, the concentrations of proteins in ribosomal gathering are higher than ones of other parts of the cell in the process of translation. So, bringing the diffusing phenomenon into the models of genetic regulatory networks is urgent and necessary, which results in RDGRNs. Generally, the models of genetic regulatory networks are divided into discrete-time models and continuous-time ones [11]. A continuous-time model has wide applications in studying the complex features and the nonlinear behaviors of genetic regulatory networks. Moreover, due to the slow processes of transcription and translation, time delays should be considered in the continuous-time models of RDGRNs. It should be emphasized that time delays may lead to poor network performance, even instability. To the best our knowledge, those works in [12–17] have researched the problem of stability analysis of delayed RDGRNs. The asymptotic stability analysis of delayed RDGRNs have been involved in [13–15] by constructing an appropriate LKF and applying some inequality techniques. In [16], a sufficient condition of F-T stability for delayed RDGRNs has been given by constructing an LKF including quad-slope integrations and applying the Gronwall inequality and Wirtinger-type integral inequality. Related research on uncertain stochastic time-delay RDGRNs and impulsive stochastic time-delay RDGRNs can be found in [12, 17], respectively.

Generally speaking, with the change of environment, not all mRNA and protein concentrations are measurable. So, it is important and necessary to estimate the mRNA and protein concentrations. Currently, one of effective approaches to estimate system states is to design observers based on the available measurement. As we all know, the problem of estimating the states of delayed RDGRNs is only studied in [18, 19], although some scholars have addressed the diffusion-free case (see [20–22] and the references therein). A full-order observer based on available measurement has been designed in [18] by introducing an LKF and employing several integral inequalities, convex approach and Green's second identity. The existence condition and design method of a full-order F-T observer have been given in [19]. It is worth emphasizing that all these results are about the full-order observers, and all approaches proposed in these literatures are not available for designing reduced-order observers. But, the design of a reduced-order observer is necessary, since it can be more easily realized than the full-order one in engineering practise.

The above discussion motivates us to design an r-order (i.e., reduced-order) F-T state observer for delayed RDGRNs. By constructing a novel LKF and employing several integral inequalities and (reciprocally) convex technique to estimate its derivative, a F-T stability criterion in the form of linear matrix inequalities (LMIs) is established for the resulting error system. In addition, we propose a method for designing an r-order F-T state observer for RDGRNs with time-varying delays, and the observer gains are parameterized by the solutions of these LMIs. Furthermore, the method proposed in this paper is explained by a numerical example.

It is worth emphasizing that the method proposed in this paper has the following advantages:

(i) The r-order observer is delay-dependent and reaction-diffusion-dependent, which is more practical.

(ii) Compared with the full-order observers, the designed r-order one can save the cost in the engineering.

(iii) For delayed genetic regulatory networks without reaction-diffusion items, the method is still keeping available by removing the corresponding parts of $\mathscr{V}_1$.

(iv) The method can also extend to some other time-delay models, including Markov jump neural networks [23–25] and stochastic delayed systems [26].

Notation 1. For given $n\times n$ matrices X and Y, we say $X>Y$ and $X\geq Y$, if $X-Y$ is real symmetric positive definite and semidefinite, respectively. The $n\times n$ identity matrix is defined by I_n, and the $m\times n$ zero matrix by $0_{m\times n}$. A^{T} and $\mathrm{Sym}(A)$ stand for the transpose matrix of A and the sum of A and its transpose, respectively. For an arbitrary but fixed positive integer l, we denote by $\langle l\rangle$ the set $\{1,2,\dots,l\}$. Let $\Omega=\{x\in\mathbb{R}^l : |x_k|\leq L_k, k\in\langle l\rangle\}$ with $L_i>0$, $i=1,2,\dots,l$. The set of all functions $f : X\rightarrow\mathbb{R}^n$ having the continuous second derivatives is defined by $C^2(X,\mathbb{R}^n)$. $\|\cdot\|$ and $\|\cdot\|_d$ represent the norms on $C^2([-d,0]\times\Omega,\mathbb{R}^n)$ and are defined by

$$\|y(t,x)\|=\left(\int_\Omega y^{\mathrm{T}}(t,x)y(t,x)\mathrm{d}x\right)^{1/2} \tag{1}$$

and

$$\begin{aligned}\|h(t,x)\|_d=\max\Bigg\{&\sup_{-d\leq t\leq 0}\|h(t,x)\|,\ \sup_{-d\leq t\leq 0}\left\|\frac{\partial h(t,x)}{\partial t}\right\|,\\ &\max_{1\leq k\leq n}\sup_{-d\leq t\leq 0}\left\|\frac{\partial h(t,x)}{\partial x_k}\right\|\Bigg\}.\end{aligned} \tag{2}$$

The symbol $\mathrm{col}(A_1,\dots,A_{\mathrm{m}})$ refers to $[A_1^{\mathrm{T}}\cdots A_m^{\mathrm{T}}]^{\mathrm{T}}$.

2. Problem Formulation

Consider the following delayed RDGRN [15]:

$$\begin{aligned}\frac{\partial\tilde{\mathbf{m}}(t,x)}{\partial t}&=\sum_{k=1}^{l}D_k\frac{\partial^2\tilde{\mathbf{m}}(t,x)}{\partial x_k^2}-A\tilde{\mathbf{m}}(t,x)\\ &\quad+\mathrm{W}g\left(\tilde{\mathbf{p}}(t-\kappa(t),x)\right)+q,\\ \frac{\partial\tilde{\mathbf{p}}(t,x)}{\partial t}&=\sum_{k=1}^{l}D_k^*\frac{\partial^2\tilde{\mathbf{p}}(t,x)}{\partial x_k^2}-C\tilde{\mathbf{p}}(t,x)\\ &\quad+\mathrm{B}\tilde{\mathbf{m}}(t-\rho(t),x),\end{aligned} \tag{3}$$

where

$$\begin{aligned}A&=\mathrm{diag}\,(a_1,a_2,\dots,a_n),\\ B&=\mathrm{diag}\,(b_1,b_2,\dots,b_n),\\ C&=\mathrm{diag}\,(c_1,c_2,\dots,c_n),\\ W&=\left[w_{ij}\right]\in\mathbb{R}^{n\times n},\\ q&=\mathrm{col}(q_1,q_2,\dots,q_n),\\ \tilde{\mathbf{m}}(t,x)&=\mathrm{col}(\tilde{\mathbf{m}}_1(t,x),\dots,\tilde{\mathbf{m}}_n(t,x)),\\ g\left(\tilde{\mathbf{p}}(t,x)\right)&=\mathrm{col}\left(g_1\left(\tilde{\mathbf{p}}_1(t,x)\right),g_2\left(\tilde{\mathbf{p}}_2(t,x)\right),\dots,g_n\left(\tilde{\mathbf{p}}_n(t,x)\right)\right),\\ \tilde{\mathbf{p}}(t,x)&=\mathrm{col}\left(\tilde{\mathbf{p}}_1(t,x),\tilde{\mathbf{p}}_2(t,x),\dots,\tilde{\mathbf{p}}_n(t,x)\right),\end{aligned} \tag{4}$$

$x=\mathrm{col}(x_1,x_2,\dots,x_l)\in\Omega\subset\mathbb{R}^l$, $\tilde{\mathbf{m}}_i(t,x)$, and $\tilde{\mathbf{p}}_i(t,x)$ stand for the concentrations of mRNAs and proteins, respectively; a_i, c_i, and b_i are the rate constants; $D_k>0$ and $D_k^*>0$ denote the diagonal diffusion rate matrices; W represents the coupling matrix with elements defined as in [15]; g_j : $s\rightarrow s^H/1+s^H$ is the Hill function, q_i is the sum of dimensionless transcriptional rates which repress gene i, $\kappa(\mathrm{t})$ and $\rho(\mathrm{t})$ are delays subject to

$$\begin{aligned}0&\leq\rho(t)\leq\bar{\rho},\ \dot{\rho}(t)\leq\mu_\rho,\\ 0&\leq\kappa(t)\leq\bar{\kappa},\ \dot{\kappa}(t)\leq\mu_\kappa,\end{aligned} \tag{5}$$

where $\bar{\rho}$, $\bar{\kappa}$, μ_ρ, and μ_κ are nonnegative real numbers.

The expression of g_i indicates that

$$0 \le \frac{g_i(y) - g_i(z)}{y - z} \le \xi_i, y, z \in \mathbb{R}, y \ne z \tag{6}$$

for some common scalar $\xi_i > 0$.

Assume that $(\mathfrak{m}^*(x), \mathfrak{p}^*(x))$ is the unique equilibrium solution of (3). Set

$$(\mathfrak{m}(t,x), \mathfrak{p}(t,x)) = \left(\tilde{\mathfrak{m}}(t,x) - \mathfrak{m}^*(x), \tilde{\mathfrak{p}}(t,x) - \mathfrak{p}^*(x)\right). \tag{7}$$

Then the delayed RDGRN (3) turns into

$$\begin{aligned}\frac{\partial \mathfrak{m}(t,x)}{\partial t} &= \sum_{k=1}^{l} D_k \frac{\partial^2 \mathfrak{m}(t,x)}{\partial x_k^2} - A\mathfrak{m}(t,x) + Wf(\mathfrak{p}(t-\kappa(t),x)),\\ \frac{\partial \mathfrak{p}(t,x)}{\partial t} &= \sum_{k=1}^{l} D_k^* \frac{\partial^2 \mathfrak{p}(t,x)}{\partial x_k^2} - C\mathfrak{p}(t,x) + B\mathfrak{m}(t-\rho(t),x),\end{aligned} \tag{8}$$

where

$$\begin{aligned}f(\mathfrak{p}(s,x)) &= \operatorname{col}(f_1(\mathfrak{p}_1(s,x)), \dots, f_n(\mathfrak{p}_n(s,x))),\\ f_i(\mathfrak{p}_i(s,x)) &= g_i(\mathfrak{p}_i(s,x) + p_i^*) - g_i(\mathfrak{p}_i^*), i \in \langle n \rangle.\end{aligned} \tag{9}$$

Next, it is assumed that the initial conditions and the Dirichlet boundary conditions of (8) are as follows:

$$\begin{aligned}&\mathfrak{m}(t,x) = \psi(t,x), \mathfrak{p}(t,x) = \psi^*(t,x), x \in \Omega, t \in [-d, 0],\\ &\mathfrak{m}(t,x) = 0, \mathfrak{p}(t,x) = 0, x \in \partial\Omega, t \in [-d, +\infty),\end{aligned} \tag{10}$$

where $d = \max\{\bar{\kappa}, \bar{\rho}\}$, and $\psi(t,x)$ and $\psi^*(t,x)$ are functions in $C^2([-d,0] \times \Omega, \mathbb{R}^n)$. Furthermore, let the network outputs be

$$z_{\mathfrak{p}}(t,x) = N_{\mathfrak{p}}\mathfrak{p}(t,x), z_{\mathfrak{m}}(t,x) = N_{\mathfrak{m}} m(t,x) \tag{11}$$

with the full-row-rank constant matrices $N_{\mathfrak{m}}$ and $N_{\mathfrak{p}}$. Here, $z_{\mathfrak{m}}(t,x)$ and $z_{\mathfrak{p}}(t,x)$ represent the expression levels of mRNAs and proteins at time t, respectively.

This paper aims at designing an r-order F-T state observer for the delayed RDGRN (8), which is described as:

$$\begin{aligned}\frac{\partial \hat{\mathfrak{m}}(t,x)}{\partial t} &= \hat{A}\hat{\mathfrak{m}}(t,x) + \hat{M}_{\mathfrak{m}} z_{\mathfrak{m}}(t,x),\\ \frac{\partial \hat{\mathfrak{p}}(t,x)}{\partial t} &= -\hat{C}\hat{\mathfrak{p}}(t,x) + \hat{M}_{\mathfrak{p}} z_{\mathfrak{p}}(t,x),\\ &\hat{\mathfrak{m}}(t,x) = \hat{\psi}(t,\mathrm{x}), \hat{\mathfrak{p}}(t,x) = \hat{\psi}^*(t,x), x \in \Omega, t \in [-d,0],\\ &\hat{\mathfrak{m}}(t,x) = 0, \hat{\mathfrak{p}}(t,x) = 0, x \in \partial\Omega, t \in [-d, +\infty).\end{aligned} \tag{12}$$

Here, $\hat{\mathfrak{m}}(t,x)$ and $\hat{\mathfrak{p}}(t,x)$ are the r-order observer states, and $\hat{A}$, $\hat{C}$, $\hat{M}_{\mathfrak{m}}$, and $\hat{M}_{\mathfrak{p}}$ are the observer gains.

Remark 1. Clearly, when $r = n$ the observer (12) is of full-order. So, our method is also available to establish full-order observers for the delayed RDGRN (8).

Define the augmented vectors

$$\begin{aligned}\mathfrak{e}_{\mathfrak{m}}(t,x) &= \begin{bmatrix} \mathfrak{m}(t,x) \\ \hat{\mathfrak{m}}(t,x) \end{bmatrix},\\ \mathfrak{e}_{\mathfrak{p}}(t,x) &= \begin{bmatrix} \mathfrak{p}(t,x) \\ \hat{\mathfrak{p}}(t,x) \end{bmatrix}.\end{aligned} \tag{13}$$

According to (8), (11), and (12), one can obtain the resulting error system as follows:

$$\begin{aligned}\frac{\partial \mathfrak{e}_{\mathfrak{m}}(t,x)}{\partial t} &= \sum_{k=1}^{l} \bar{D}_k \frac{\partial^2 \mathfrak{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2} - \bar{A}\mathfrak{e}_{\mathfrak{m}}(t,x)\\ &\quad + \bar{W}f(\mathfrak{p}(t-\kappa(t),x)), x \in \Omega, t \in [-d, +\infty),\\ \frac{\partial \mathfrak{e}_{\mathfrak{p}}(t,x)}{\partial t} &= \sum_{k=1}^{l} \bar{D}_k^* \frac{\partial^2 \mathfrak{e}_{\mathfrak{p}}(t,x)}{\partial x_k^2} - \bar{C}\mathfrak{e}_{\mathfrak{p}}(t,x)\\ &\quad + \bar{B}\mathfrak{m}(t-\rho(t),x), x \in \Omega, t \in [-d, +\infty),\\ &\mathfrak{e}_{\mathfrak{m}}(t,x) = \bar{\psi}(t,x), \mathfrak{e}_{\mathfrak{p}}(t,x) = \bar{\psi}^*(t,x), x \in \Omega, t \in [-d, 0],\\ &\mathfrak{e}_{\mathfrak{m}}(t,x) = 0, \mathfrak{e}_{\mathfrak{p}}(t,x) = 0, x \in \partial\Omega, t \in [-d, +\infty),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\bar{D}_k &= \operatorname{diag}(D_k, 0),\\ \bar{W} &= \operatorname{col}(W, 0),\\ \bar{D}_k^* &= \operatorname{diag}(D_k^*, 0),\\ \bar{B} &= \operatorname{col}(B, 0),\\ \bar{A} &= \begin{bmatrix} A & 0 \\ -\hat{M}_{\mathfrak{m}} N_{\mathfrak{m}} & -\hat{A} \end{bmatrix},\\ \bar{C} &= \begin{bmatrix} C & 0 \\ -\hat{M}_{\mathfrak{p}} N_{\mathfrak{p}} & -\hat{C} \end{bmatrix},\\ \bar{\psi}(t,x) &= \operatorname{col}\left(\psi(t,x), \hat{\psi}(t,x)\right),\\ \bar{\psi}^*(t,x) &= \operatorname{col}\left(\psi^*(t,x), \hat{\psi}^*(t,x)\right).\end{aligned} \tag{15}$$

Definition 1 (see [12]). The trivial solution of system (14) is called F-T stable with respect to positive scalars c_1, c_2, and T, if

$$\begin{aligned}\|\bar{\psi}(t,x)\|_d^2 + \|\bar{\psi}^*(t,x)\|_d^2 \le c_1 \Rightarrow \|\mathfrak{e}_{\mathfrak{m}}(t,x)\|^2 + \left\|\mathfrak{e}_{\mathfrak{p}}(t,x)\right\|^2\\ \le c_2, \forall t \in [0, T].\end{aligned} \tag{16}$$

We say that system (14) is F-T stable with respect to positive scalars c_1, c_2, and T, if so, it is its trivial solution.
To achieve our aim, one requires to seek r-order observer gains $\widehat{A}$, $\widehat{C}$, $\widehat{M}_{\mathrm{m}}$, and $\widehat{M}_{\mathrm{p}}$ such that system (14) is F-T stable with respect to positive scalars c_1, c_2, and T.

3. Preliminaries

The following two lemmas are needed to design an r-order F-T state observer.

Lemma 1 (Jensen's inequality) [27]. *For given scalars $a < b$, an integral function $\chi : [a, b] \to \mathbb{R}^n$ and a matrix $M^{\mathrm{T}} = M > 0$, there holds the following inequality:*

$$\begin{aligned} &\frac{b^3 - a^3}{6} \int_b^a \int_\theta^0 \int_\lambda^0 \chi^T(s) M \chi(s) \mathrm{d}s \mathrm{d}\lambda \mathrm{d}\theta \\ &\quad \geq \int_b^a \int_\theta^0 \int_\lambda^0 \chi^T(s) \mathrm{d}s \mathrm{d}\lambda \mathrm{d}\theta M \int_b^a \int_\theta^0 \int_\lambda^0 \chi(s) \mathrm{d}s \mathrm{d}\lambda \mathrm{d}\theta. \end{aligned} \tag{17}$$

Lemma 2 (Wirtinger-type integral inequalities) [28]. *For given scalars $a < b$, a function $\chi : [a, b] \to \mathbb{R}^n$ which is derivative and a matrix $M^{\mathrm{T}} = M > 0$, there hold the following inequalities*:

$$\begin{aligned} &(b-a) \int_a^b \chi^{\mathrm{T}}(s) M \chi(s) \mathrm{d}s \geq \left[\Theta_5^{\mathrm{T}} \Theta_6^{\mathrm{T}}\right] \bar{M} \left[\Theta_5^{\mathrm{T}} \Theta_6^{\mathrm{T}}\right]^T, \\ &\int_a^b \int_b^b \dot{\chi}^{\mathrm{T}}(s) M \dot{\chi}(s) \mathrm{d}s \mathrm{d}\alpha \geq \left[\Theta_3^{\mathrm{T}} \Theta_4^{\mathrm{T}}\right] \tilde{M} \left[\Theta_3^{\mathrm{T}} \Theta_4^{\mathrm{T}}\right]^{\mathrm{T}}, \\ &(b-a) \int_a^b \dot{\chi}^{\mathrm{T}}(s) M \dot{\chi}(s) \mathrm{d}s \geq \left[\Theta_0^{\mathrm{T}} \Theta_1^{\mathrm{T}} \Theta_2^{\mathrm{T}}\right] \widehat{M} \left[\Theta_0^{\mathrm{T}} \Theta_1^{\mathrm{T}} \Theta_2^{\mathrm{T}}\right]^{\mathrm{T}}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\bar{M} = \operatorname{diag}(M, 3M), \\ &\tilde{M} = \operatorname{diag}(2M, 4M), \\ &\widehat{M} = \operatorname{diag}(M, 3M, 5M), \\ &\Theta_0 = \chi(b) - \chi(a), \\ &\Theta_1 = \chi(b) + \chi(a) - 2(b-a)^{-1} \int_a^b \chi(s) \mathrm{d}s, \\ &\Theta_2 = \Theta_0 + \frac{6}{b-a} \int_a^b \chi(s) \mathrm{d}s - \frac{12}{(b-a)^2} \int_a^b \int_a^b \chi(s) \mathrm{d}s \mathrm{d}\alpha, \\ &\Theta_3 = \chi(b) - (b-a)^{-1} \int_a^b \chi(s) ds, \Theta_5 = \int_a^b \chi(s) ds, \\ &\Theta_4 = \chi(b) + \frac{2}{b-a} \int_a^b \chi(s) \mathrm{d}s - \frac{6}{(b-a)^2} \int_a^b \int_a^b \chi(s) \mathrm{d}s \mathrm{d}\alpha, \\ &\Theta_6 = \int_a^b \chi(s) \mathrm{d}s - 2(b-a)^{-1} \int_a^b \int_a^b \chi(s) \mathrm{d}s \mathrm{d}\alpha. \end{aligned} \tag{19}$$

4. Design Method of Observer

In this section, a method to design an r-order F-T state observer for the delayed RDGRN (8) is proposed, that is, determine the observer gains $\widehat{A}$, $\widehat{C}$, $\widehat{M}_{\mathrm{m}}$, and $\widehat{M}_{\mathrm{p}}$ such that the error system (14) is F-T stable. For this end, we define

$$\begin{aligned} &E_1 = \operatorname{col}\left(I_{n+r}, 0_{(17n+3r)\times(n+r)}\right), \\ &E_2 = \operatorname{col}\left(0_{(n+r)\times n}, I_n, 0_{(16n+3r)\times n}\right), \\ &E_3 = \operatorname{col}\left(0_{(2n+r)\times n}, I_n, 0_{(15n+3r)\times n}\right), \\ &E_4 = \operatorname{col}\left(0_{(3n+r)\times(n+r)}, I_{n+r}, 0_{(14n+2r)\times(n+r)}\right), \\ &E_{4+i} = \operatorname{col}\left(0_{(\mathrm{in}+3n+2r)\times n}, I_n, 0_{(14n-\mathrm{in}+2r)\times n}\right), i \in \langle 4 \rangle, \\ &E_9 = \operatorname{col}\left(0_{(8n+2r)\times(n+r)}, I_{n+r}, 0_{(9n+r)\times(n+r)}\right), \\ &E_{10} = \operatorname{col}\left(0_{(9n+3r)\times(n+r)}, I_{n+r}, 0_{8n\times(n+r)}\right), \\ &E_{10+i} = \operatorname{col}\left(0_{(9n+\mathrm{in}+4r)\times n}, I_n, 0_{(8-i)n\times n}\right), i \in \langle 8 \rangle, \\ &\Pi_1 = [E_3 - E_2E_3 + E_2 - 2E_{11}E_3 - E_2 + 6E_{11} - 12E_{12}], \\ &\Pi_2 = \Big[E_1\bar{K}^{\mathrm{T}} - E_3E_1\bar{K}^{\mathrm{T}} + E_3 - 2E_{13}E_1\bar{K}^{\mathrm{T}} \\ &\qquad - E_3 + 6E_{13} - 12E_{14}\Big], \\ &\Pi_3 = [E_6 - E_5E_6 + E_5 - 2E_{15}E_6 - E_5 + 6E_{15} - 12E_{16}], \\ &\Pi_4 = \Big[E_4\bar{K}^{\mathrm{T}} - E_6E_4\bar{K}^{\mathrm{T}} + E_6 - 2E_{17}E_4\bar{K}^{\mathrm{T}} \\ &\qquad - E_6 + 6E_{17} - 12E_{18}\Big], \\ &\Pi_5 = [E_{11}E_{11} - 2E_{12}], \\ &\Pi_6 = [E_{13}E_{13} - 2E_{14}], \\ &\Pi_7 = [E_{15}E_{15} - 2E_{16}], \\ &\Pi_8 = [E_{17}E_{17} - 2E_{18}], \\ &\Upsilon_1 = \Big[E_1\bar{K}^{\mathrm{T}} - E_{13}E_1\bar{K}^{\mathrm{T}} + 2E_{13} - 6E_{14}\Big], \\ &\Upsilon_2 = [E_3 - E_{11}E_3 + 2E_{11} - 6E_{12}], \\ &\Upsilon_3 = \Big[E_4\bar{K}^{\mathrm{T}} - E_{17}E_4\bar{K}^{\mathrm{T}} + 2E_{17} - 6E_{18}\Big], \\ &\Upsilon_4 = [E_6 - E_{15}E_6 + 2E_{15} - 6E_{16}], \\ &\Psi_1 = \Psi_{11} + \Psi_{12}, \\ &\Psi_{11} = \operatorname{Sym}\bigg(-\frac{\pi^2}{4} E_1 U_1 D_L E_1^{\mathrm{T}} - E_9 U_1 E_9^{\mathrm{T}} \\ &\qquad - \frac{\pi^2}{4} E_4 U_2 D_L^* E_4^{\mathrm{T}} - E_{10} U_2 E_{10}^{\mathrm{T}} + E_9 U_1 \bar{W} E_8^{\mathrm{T}} \\ &\qquad + E_4 U_2 \bar{B} E_3^{\mathrm{T}} + E_{10} U_2 \bar{B} E_3^{\mathrm{T}} + E_1 U_1 \bar{W} E_8^{\mathrm{T}}\bigg), \end{aligned}$$

$$\Psi_{12} = \mathrm{Sym}\Bigg(-E_1\begin{bmatrix} U_{11}A & 0 \\ -M_{\mathfrak{m}}N_{\mathfrak{m}} & -A_{\mathfrak{m}}\end{bmatrix}E_1^{\mathrm{T}} - E_4\begin{bmatrix} U_{21}C & 0 \\ -M_{\mathfrak{p}}N_{\mathfrak{p}} & -C_{\mathfrak{p}}\end{bmatrix}E_4^{\mathrm{T}} - E_1\begin{bmatrix} A^{\mathrm{T}}U_{11} & -N_{\mathfrak{m}}^{\mathrm{T}}M_{\mathfrak{m}}^{\mathrm{T}} \\ 0 & -A_{\mathfrak{m}}^{\mathrm{T}}\end{bmatrix}E_9^{\mathrm{T}} - E_4\begin{bmatrix} C^{\mathrm{T}}U_{21} & -N_{\mathfrak{p}}^{\mathrm{T}}M_{\mathfrak{p}}^{\mathrm{T}} \\ 0 & -C_{\mathfrak{p}}^{\mathrm{T}}\end{bmatrix}E_{10}^{\mathrm{T}}\Bigg),$$

$$\Psi_2 = E_1\bar{K}^{\mathrm{T}}(V_1+V_2)\bar{K}E_1^{\mathrm{T}} - E_2V_2E_2^{\mathrm{T}} + \left(\mu_\rho - 1\right)E_3V_1E_3^{\mathrm{T}} + E_4\bar{K}^{\mathrm{T}}(V_3+V_4)\bar{K}E_4^{\mathrm{T}} - E_5V_4E_5^{\mathrm{T}} + (\mu_\kappa - 1)E_6V_3E_6^{\mathrm{T}},$$

$$\Psi_3 = (\mu_\kappa - 1)E_8V_5E_8^{\mathrm{T}} + E_7V_5E_7^{\mathrm{T}},$$

$$\Psi_4(\rho,\kappa) = \Psi_{40} + \Psi_{41} + \Psi_{42} + \Psi_{43}(\rho) + \Psi_{44}(\kappa),$$

$$\Psi_{40} = \bar{\rho}^2\left(E_9\bar{K}^{\mathrm{T}}W_1\bar{K}E_9^{\mathrm{T}} + E_1\bar{K}^{\mathrm{T}}W_3\bar{K}E_1^{\mathrm{T}}\right) + \bar{\kappa}^2\left(E_{10}\bar{K}^{\mathrm{T}}W_2\bar{\mathrm{K}}E_{10}^{\mathrm{T}} + E_4\bar{K}^{\mathrm{T}}W_4\bar{K}E_4^{\mathrm{T}}\right),$$

$$\Psi_{41} = -[\Pi_1 \quad \Pi_2]\begin{bmatrix}\widehat{W}_1 & \widehat{H}_1 \\ \widehat{H}_1^{\mathrm{T}} & \widehat{W}_1\end{bmatrix}[\Pi_1 \quad \Pi_2]^{\mathrm{T}},$$

$$\Psi_{42} = -[\Pi_3 \quad \Pi_4]\begin{bmatrix}\widehat{W}_2 & \widehat{H}_2 \\ \widehat{H}_2^{\mathrm{T}} & \widehat{W}_2\end{bmatrix}[\Pi_3 \quad \Pi_4]^{\mathrm{T}},$$

$$\Psi_{43}(\rho) = \bar{\rho}(\bar{\rho}-\rho)\Pi_5\tilde{W}_3\Pi_5^{\mathrm{T}} - \bar{\rho}\rho\Pi_6\tilde{W}_3\Pi_6^{\mathrm{T}},$$

$$\Psi_{44}(\kappa) = -\bar{\kappa}(\bar{\kappa}-\kappa)\Pi_7\tilde{W}_4\Pi_7^{\mathrm{T}} - \bar{\kappa}\kappa\Pi_8\tilde{W}_4\Pi_8^{\mathrm{T}},$$

$$\Psi_5(\rho,\kappa) = \Psi_{50} + \Psi_{51}(\rho) + \Psi_{52}(\kappa),$$

$$\Psi_{50} = \frac{\bar{\rho}^2}{2}E_9\bar{K}^{\mathrm{T}}X_1\bar{K}E_9^{\mathrm{T}} + \frac{\bar{\kappa}^2}{2}E_{10}\bar{K}^{\mathrm{T}}X_2\bar{K}E_{10}^{\mathrm{T}},$$

$$\Psi_{51}(\rho) = -\Upsilon_1\tilde{X}_1\Upsilon_1^{\mathrm{T}} - \Upsilon_2\tilde{X}_1\Upsilon_2^{\mathrm{T}} - \frac{\bar{\rho}-\rho}{\rho}\Pi_2\widehat{X}_1\Pi_2^{\mathrm{T}},$$

$$\Psi_{52}(\kappa) = -\Upsilon_3\tilde{X}_2\Upsilon_3^{\mathrm{T}} - \Upsilon_4\tilde{X}_2\Upsilon_4^{\mathrm{T}} - \frac{\bar{\kappa}-\kappa}{\kappa}\Pi_4\widehat{X}_2\Pi_4^{\mathrm{T}},$$

$$\Psi_6(\rho,\kappa) = \Psi_{60} + \Psi_{61}(\rho) + \Psi_{62}(\kappa) - (\bar{\rho}-\rho)\Upsilon_1\tilde{Y}_1\Upsilon_1^{\mathrm{T}} - (\bar{\kappa}-\kappa)\Upsilon_3\tilde{Y}_2\Upsilon_3^{\mathrm{T}},$$

$$\Psi_{60} = \frac{\bar{\rho}^3}{6}E_9\bar{K}^{\mathrm{T}}Y_1\bar{K}E_9^{\mathrm{T}} + \frac{\bar{\kappa}^3}{6}E_{10}\bar{K}^{\mathrm{T}}Y_2\bar{K}E_{10}^{\mathrm{T}},$$

$$\Psi_{61}(\rho) = -\frac{3}{2}\rho\left(E_1\bar{K}^{\mathrm{T}} - 2E_{14}\right)Y_1\left(E_1\bar{K}^{\mathrm{T}} - 2E_{14}\right)^{\mathrm{T}} - \frac{3}{2}(\bar{\rho}-\rho)(E_3 - 2E_{12})Y_1(E_3 - 2E_{12})^{\mathrm{T}},$$

$$\Psi_{62}(\kappa) = -\frac{3}{2}\kappa\left(E_4\bar{K}^{\mathrm{T}} - 2E_{18}\right)Y_2\left(E_4\bar{K}^{\mathrm{T}} - 2E_{18}\right)^{\mathrm{T}} - \frac{3}{2}(\bar{\kappa}-\kappa)(E_6 - 2E_{16})Y_2(E_6 - 2E_{16})^{\mathrm{T}},$$

$$\Psi_{01} = -2E_7Z_1E_7^{\mathrm{T}} + E_4\bar{K}^{\mathrm{T}}KZ_1E_7^{\mathrm{T}} + E_7Z_1K\bar{K}E_4^{\mathrm{T}},$$

$$\Psi_{02} = -2E_8Z_2E_8^{\mathrm{T}} + E_6KZ_2E_8^{\mathrm{T}} + E_8Z_2KE_6^{\mathrm{T}},$$

$$\tilde{X}_i = \mathrm{diag}\,(2X_i, 4X_i),$$

$$\widehat{W}_i = \mathrm{diag}\,(W_i, 3W_i, 5W_i),$$

$$\widehat{X}_i = \mathrm{diag}\,(X_i, 3X_i, 5X_i),$$

$$\tilde{Y}_i = \mathrm{diag}\,(2Y_i, 4Y_i)\; i = 1, 2,$$

$$\tilde{W}_j = \mathrm{diag}\left(W_j, 3W_j\right), j = 3, 4,$$

$$U = \mathrm{diag}\,(U_1, U_2),$$

$$U_k = \mathrm{diag}\,(U_{k1}, U_{k2}), k = 1, 2,$$

$$\lambda_{11} = \lambda_{\max}(U_1) + \bar{\rho}\lambda_{\max}(V_1) + \bar{\rho}\lambda_{\max}(V_2) + \frac{1}{6}\bar{\rho}^3{}_{\max}(X_1) + \sum_{k=1}^{l}\lambda_{\max}(U_1)\lambda_{\max}(\overline{D_k}) + \frac{1}{2}\bar{\rho}^3{}_{\max}(W_{1+}W_3) + \frac{1}{24}\bar{\rho}^4\lambda_{\max}(Y_1),$$

$$\lambda_{12} = \lambda_{\max}\,(U_2) + \bar{\kappa}\,\lambda_{\max}\,(V_3) + \bar{\kappa}\lambda_{\max}\,(V_4) + \bar{\kappa}\,\lambda_{\max}\,(V_5)\lambda_{\max}\left(K^{\mathrm{T}}K\right) + \frac{1}{6}\bar{\kappa}^3\lambda_{\max}(X_2) + \sum_{k=1}^{l}\lambda_{\max}(U_2)\lambda_{\max}(D_k^*) + \frac{1}{2}\bar{\kappa}^3\,\lambda_{\max}(W_2 + W_4) + \frac{1}{24}\bar{\kappa}^4\lambda_{\max}(Y_2),$$

$$\eta(t,x) = \mathrm{col}\Bigg(\mathfrak{e}_{\mathfrak{m}}(t,x), \mathfrak{m}(t-\bar{\rho},x), \mathfrak{m}(t-\rho(t),x), \mathfrak{e}_{\mathfrak{p}}(t,x), \mathfrak{p}(t-\bar{\kappa},x), \mathfrak{p}(t-\kappa(t),x), f(\mathfrak{p}(t,x)), f(\mathfrak{p}(t-\kappa(t),x)), \frac{\partial\mathfrak{e}_{\mathfrak{m}}(t,x)}{\partial t}, \frac{\partial\mathfrak{e}_{\mathfrak{p}}(t,x)}{\partial t}, \frac{1}{\bar{\rho}-\rho(t)}\int_{t-\bar{\rho}}^{t-\rho(t)}\mathfrak{m}(s,x)\mathrm{d}s, \frac{1}{(\bar{\rho}-\rho(t))^2}\int_{t-\bar{\rho}}^{t-\rho(t)}\int_{\alpha}^{t-\rho(t)}\mathfrak{m}(s,x)\mathrm{d}s\mathrm{d}\alpha,$$

$$\frac{1}{\rho(t)}\int_{t-\rho(t)}^{t}\mathfrak{m}(s,x)\mathrm{d}s,$$
$$\frac{1}{\rho^2(t)}\int_{t-\rho(t)}^{t}\int_{\alpha}^{t}\mathfrak{m}(s,x)\mathrm{d}s\mathrm{d}\alpha,$$
$$\frac{1}{\overline{\kappa}-\kappa(t)}\int_{t-\kappa}^{t-\kappa(t)}\mathfrak{p}(s,x)\mathrm{d}s,$$
$$\frac{1}{(\overline{\kappa}-\kappa(t))^2}\int_{t-\overline{\kappa}}^{t-\kappa(t)}\int_{\alpha}^{t-\kappa(t)}\mathfrak{p}(s,x)\mathrm{d}s\mathrm{d}\alpha,$$
$$\frac{1}{\kappa(t)}\int_{t-\kappa(t)}^{t}\mathfrak{p}(s,x)\mathrm{d}s,$$
$$\left.\frac{1}{\kappa^2(t)}\int_{t-\kappa(t)}^{t}\int_{\alpha}^{t}\mathfrak{p}(s,x)\mathrm{d}s\mathrm{d}\alpha\right),$$
$$D_L=\sum_{k=1}^{1}\frac{\bar{D}_k}{L_k^2},$$
$$D_L^*=\sum_{k=1}^{1}\frac{\bar{D}_k^*}{L_k^2},$$
$$\bar{K}=[I\quad 0], \tag{20}$$

where L_k, $\bar{D}_k$, and $\bar{D}_k^*$ are the same with previous ones.

Now we can provide an approach to design an r-order F-T state observer for the delayed RDGRN (8).

Theorem 1. *For given scalars $\overline{\rho},\overline{\kappa},\mu_\rho$, and μ_κ satisfying (5) and positive constants T, c_1, c_2, and α, system (14) is F-T stable with respect to c_1, c_2, and T, if there exist matrices $0<V_i^{\mathrm{T}}=V_i\in\mathbb{R}^{n\times n}$ $(i\in\langle 5\rangle)$, $0<W_j^{\mathrm{T}}=W_j\in\mathbb{R}^{n\times n}$ $(j\in\langle 4\rangle)$, $0<X_k^{\mathrm{T}}=X_k\in\mathbb{R}^{n\times n}$, and $0<Y_k^{\mathrm{T}}=Y_k\in\mathbb{R}^{n\times n}$, diagonal matrices $0<U_k\in\mathbb{R}^{(n+r)\times(n+r)}$ and $0<Z_k\in\mathbb{R}^{n\times n}$, and matrices $\widehat{H}_k\in\mathbb{R}^{3n\times 3n}$ $(k\in\langle 2\rangle)$, $A_{\mathrm{m}}\in\mathbb{R}^{r\times r}$, $C_{\mathrm{p}}\in\mathbb{R}^{r\times r}$, M_{m}, and M_{p} of appropriate dimensions, such that the following inequalities are feasible for $\rho\in\{0,\overline{\rho}\}$ and $\kappa\in\{0,\overline{\kappa}\}$:*

$$\begin{bmatrix}\widehat{W}_k & \widehat{H}_k\\ \widehat{H}_k^{\mathrm{T}} & \widehat{W}_k\end{bmatrix}\geq 0,\ k\in\langle 2\rangle, \tag{21}$$

$$\Psi(\rho,\kappa):=\sum_{i=1}^{3}\Psi_i+\sum_{i=4}^{6}\Psi_i(\rho,\kappa)+\sum_{i=1}^{2}\Psi_{0i}-\alpha E_1U_1E_1^{\mathrm{T}}-\alpha E_4U_2E_4^{\mathrm{T}}<0, \tag{22}$$

$$c_1e^{\alpha\mathrm{T}}(\lambda_{11}+\lambda_{12})-c_2\lambda_{\min}(U)\leq 0, \tag{23}$$

where $K=\mathrm{diag}\ (\xi_1,\xi_2,\ldots,\xi_n)>0$, and $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{W}$ are defined previously.

In addition, based on a feasible solution of (21), (22), and (23), an r-order F-T state observer can be represented by (12) with the following gains:

$$\begin{bmatrix}\widehat{A} & \widehat{M}_{\mathrm{m}}\\ \widehat{C} & \widehat{M}_{\mathfrak{p}}\end{bmatrix}=\mathrm{diag}\ (U_{12},U_{22})^{-1}\begin{bmatrix}A_{\mathrm{m}} & M_{\mathrm{m}}\\ C_{\mathfrak{p}} & M_{\mathfrak{p}}\end{bmatrix}. \tag{24}$$

Proof 1. Choose the following LKF functional:

$$\mathscr{V}(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})=\sum_{i=1}^{6}\mathscr{V}_i(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}}), \tag{25}$$

where

$$\begin{aligned}\mathscr{V}_1(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})&=\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\mathbf{e}_{\mathfrak{m}}(t,x)\mathrm{d}x\\&+\int_{\Omega}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)U_2\mathbf{e}_{\mathfrak{p}}(t,x)\mathrm{d}x\\&+\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial x_k}U_1\bar{D}_k\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k}\mathrm{d}x\\&+\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)}{\partial x_k}U_2\bar{D}_k^*\frac{\partial\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial x_k}\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\mathscr{V}_2(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})&=\int_{\Omega}\int_{t-\rho(t)}^{t}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}V_1\bar{K}\mathbf{e}_{\mathfrak{m}}(s,x)\mathrm{d}s\mathrm{d}x\\&+\int_{\Omega}\int_{t-\bar{\rho}}^{t}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}V_2\bar{K}\mathbf{e}_{\mathfrak{m}}(s,x)\mathrm{d}s\mathrm{d}x\\&+\int_{\Omega}\int_{t-\kappa(t)}^{t}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}V_3\bar{K}\mathbf{e}_{\mathfrak{p}}(s,x)\mathrm{d}s\mathrm{d}x\\&+\int_{\Omega}\int_{t-\bar{\kappa}}^{t}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}V_4\bar{K}\mathbf{e}_{\mathfrak{p}}(s,x)\mathrm{d}s\mathrm{d}x,\end{aligned}$$

$$\mathscr{V}_3(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})=\int_{\Omega}\int_{t-\kappa(t)}^{t}f^{\mathrm{T}}(\mathfrak{p}(s,x))V_5f(\mathfrak{p}(s,x))\mathrm{d}s\mathrm{d}x,$$

$$\begin{aligned}\mathscr{V}_4(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})&=\bar{\rho}\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{t+\theta}^{t}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)}{\partial s}\bar{K}^{\mathrm{T}}W_1\bar{K}\\&\cdot\frac{\partial\mathbf{e}_{\mathfrak{m}}(s,x)}{\partial s}\mathrm{d}s\mathrm{d}\theta\mathrm{d}x+\bar{\kappa}\int_{\Omega}\int_{-\bar{\kappa}}^{0}\int_{t+\theta}^{t}\\&\cdot\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(s,x)}{\partial s}\bar{K}^{\mathrm{T}}W_2\bar{K}\frac{\partial\mathbf{e}_{\mathfrak{p}}(s,x)}{\partial s}\mathrm{d}s\mathrm{d}\theta\mathrm{d}x\\&+\bar{\rho}\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{t+\theta}^{t}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_3\bar{K}\mathbf{e}_{\mathfrak{m}}(s,x)\mathrm{d}s\mathrm{d}\theta\mathrm{d}x\\&+\bar{\kappa}\int_{\Omega}\int_{-\bar{\kappa}}^{0}\int_{t+\theta}^{t}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_4\bar{K}\mathbf{e}_{\mathfrak{p}}(s,x)\mathrm{d}s\mathrm{d}\theta\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\mathscr{V}_5(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}})&=\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{s}^{0}\int_{t+\theta}^{t}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\\&\cdot\frac{\partial\mathbf{e}_{\mathfrak{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}\theta\mathrm{d}s\mathrm{d}x+\int_{\Omega}\int_{-\bar{\kappa}}^{0}\int_{s}^{0}\int_{t+\theta}^{t}\\&\cdot\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_2\bar{K}\frac{\partial\mathbf{e}_{\mathfrak{p}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}\theta\mathrm{d}s\mathrm{d}x,\end{aligned}$$

$$\mathscr{V}_6(\mathrm{t}, \mathbf{e_m}, \mathbf{e_p}) = \int_\Omega \int_{-\bar{\rho}}^0 \int_s^0 \int_\alpha^0 \int_{t+\theta}^t \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} Y_1 \bar{K} \cdot \frac{\partial \mathbf{e_m}(u,x)}{\partial u} \mathrm{d}u\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}s\mathrm{d}x + \int_\Omega \int_{-\bar{\kappa}}^0 \int_s^0 \int_\alpha^0 \int_{t+\theta}^t \cdot \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} Y_2 \bar{K} \frac{\partial \mathbf{e_p}(u,x)}{\partial u} \mathrm{d}u\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}s\mathrm{d}x. \tag{26}$$

Then, calculating the derivatives of $\mathscr{V}_\mathrm{i}(\mathrm{t}, \mathbf{e_m}, \mathbf{e_p})$ $(\mathrm{i} \in \langle 6 \rangle)$ along on the solution of system (14), one can obtain that

$$\begin{aligned}\frac{\partial}{\partial \mathrm{t}}\mathscr{V}_1(\mathrm{t}, \mathbf{e_m}, \mathbf{e_p}) = {} & 2\int_\Omega \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x) U_1 \left[-\bar{A}\mathbf{e_m}(t,x) + \bar{W} f(p(t-\kappa(t),x)) + \sum_{k=1}^{l} \bar{D}_k \frac{\partial^2 \mathbf{e_m}(t,x)}{\partial x_k^2} \right] \mathrm{d}x + 2\int_\Omega \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x) U_2 \\ & \cdot \left[-\bar{C}\mathbf{e_p}(t,x) + \bar{B}\mathbf{m}(t-\rho(t),x) + \sum_{k=1}^{l} D_k^* \frac{\partial^2 \mathbf{e_p}(t,x)}{\partial x_k^2} \right] \mathrm{d}x + 2\sum_{k=1}^{l} \int_\Omega \\ & \cdot \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x)}{\partial x_k} U_1 \bar{D}_k \frac{\partial}{\partial x_k} \left(\frac{\partial \mathbf{e_m}(t,x)}{\partial t} \right) \mathrm{d}x + 2\sum_{k=1}^{l} \int_\Omega \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x)}{\partial x_k} U_2 \bar{D}_k^* \frac{\partial}{\partial x_k} \cdot \left(\frac{\partial \mathbf{e_p}(t,x)}{\partial t} \right) \mathrm{d}x,\end{aligned} \tag{27}$$

$$\begin{aligned}\frac{\partial}{\partial \mathrm{t}}\mathscr{V}_2(\mathrm{t}, \mathbf{e_m}, \mathbf{e_p}) = {} & \int_\Omega \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x) \bar{K}^\mathrm{T}(V_1+V_2)\bar{K}\mathbf{e_m}(t,x)\mathrm{d}x - (1-\dot{\rho}(t))\int_\Omega \mathbf{m}^\mathrm{T}(t-\rho(t),x) V_1 \mathbf{m} \\ & \cdot (t-\rho(t),x)\mathrm{d}x - \int_\Omega \mathbf{m}^\mathrm{T}(t-\bar{\rho},x) V_2 \mathbf{m} \cdot (t-\bar{\rho},x)\mathrm{d}x + \int_\Omega \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x)\bar{K}^\mathrm{T} \\ & \cdot (V_3+V_4)\bar{K}\mathbf{e_p}(t,x)\mathrm{d}x - (1-\dot{\kappa}(\mathrm{t}))\int_\Omega \mathfrak{p}^\mathrm{T} \cdot (t-\kappa(t),x)\mathrm{V}_3\mathfrak{p}(t-\kappa(t),x)\mathrm{d}x \\ & - \int_\Omega \mathfrak{p}^\mathrm{T}(t-\bar{\kappa},x)\mathrm{V}_4\mathfrak{p}(t-\bar{\kappa},x)\mathrm{d}x \\ \le {} & \int_\Omega \eta^\mathrm{T}(t,x)\Psi_2\eta(t,x)\mathrm{d}x,\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{V}_3(t, \mathbf{e_m}, \mathbf{e_p}) = {} & \int_\Omega f^\mathrm{T}(\mathfrak{p}(t,x)) V_5 f(\mathfrak{p}(t,x))\mathrm{d}x - (1-\dot{\kappa}(t))\int_\Omega f^\mathrm{T}(\mathfrak{p}(t-\kappa(t),x)) V_5 f \cdot (\mathfrak{p}(t-\kappa(t),x))\mathrm{d}x \\ \le {} & \int_\Omega \eta^\mathrm{T}(t,x)\Psi_3\eta(t,x)\mathrm{d}x,\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{V}_4(t, \mathbf{e_m}, \mathbf{e_p}) = {} & -\bar{\rho}\int_\Omega \int_{t-\bar{\rho}}^t \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(s,x)}{\partial s} \bar{K}^\mathrm{T} W_1 \bar{K} \frac{\partial \mathbf{e_m}(s,x)}{\partial s} \mathrm{d}s\mathrm{d}x + \bar{\rho}^2 \int_\Omega \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} W_1 \bar{K} \frac{\partial \mathbf{e_m}(t,x)}{\partial t} \mathrm{d}x \\ & - \bar{\kappa}\int_\Omega \int_{t-\bar{\kappa}}^t \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(s,x)}{\partial s} \bar{K}^\mathrm{T} W_2 \bar{K} \frac{\partial \mathbf{e_p}(s,x)}{\partial s} \mathrm{d}s\mathrm{d}x + \bar{\kappa}^2 \int_\Omega \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} W_2 \bar{K} \frac{\partial \mathbf{e_p}(t,x)}{\partial t} \mathrm{d}x \\ & - \bar{\rho}\int_\Omega \int_{t-\bar{\rho}}^t \mathbf{e}_\mathbf{m}^\mathrm{T}(s,x)\bar{K}^\mathrm{T} W_3 \bar{K}\mathbf{e_m}(s,x)\mathrm{d}s\mathrm{d}x + \bar{\rho}^2 \int_\Omega \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x)\bar{K}^\mathrm{T} W_3 \bar{K}\mathbf{e_m}(t,x)\mathrm{d}x \\ & - \bar{\kappa}\int_\Omega \int_{t-\bar{\kappa}}^t \mathbf{e}_\mathbf{p}^\mathrm{T}(s,x)\bar{K}^\mathrm{T} W_4 \bar{K}\mathbf{e_p}(s,x)\mathrm{d}s\mathrm{d}x + \bar{\kappa}^2 \int_\Omega \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x)\bar{K}^\mathrm{T} W_4 \bar{K}\mathbf{e_p}(t,x)\mathrm{d}x,\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{V}_5(t, e_m, e_p) = {} & \frac{\bar{\rho}^2}{2}\int_\Omega \frac{\partial e_m^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} X_1 \bar{K} \frac{\partial e_m(t,x)}{\partial t} \mathrm{d}x - \int_\Omega \int_{-\bar{\rho}}^0 \int_{t+s}^t \frac{\partial e_m^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} X_1 \bar{K} \frac{\partial e_m(u,x)}{\partial u} \mathrm{d}u\mathrm{d}s\mathrm{d}x \\ & + \frac{\bar{\kappa}^2}{2}\int_\Omega \frac{\partial e_p^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} X_2 \bar{K} \frac{\partial e_p(t,x)}{\partial t} dx - \int_\Omega \int_{-\bar{\kappa}}^0 \int_{t+s}^t \frac{\partial e_p^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} X_2 \bar{K} \frac{\partial e_p(u,x)}{\partial u} \mathrm{d}u\mathrm{d}s\mathrm{d}x,\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{V}_6(t, \mathbf{e_m}, \mathbf{e_p}) = {} & \frac{\bar{\rho}^3}{6}\int_\Omega \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} Y_1 \bar{K} \frac{\partial \mathbf{e_m}(t,x)}{\partial t} \mathrm{d}x - \int_\Omega \int_{-\bar{\rho}}^0 \int_\alpha^0 \int_{t+s}^t \frac{\partial \mathbf{e}_\mathbf{m}^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} Y_1 \bar{K} \frac{\partial \mathbf{e_m}(u,x)}{\partial u} \mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x \\ & + \frac{\bar{\kappa}^3}{6}\int_\Omega \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(t,x)}{\partial t} \bar{K}^\mathrm{T} Y_2 \bar{K} \frac{\partial \mathbf{e_p}(t,x)}{\partial t} \mathrm{d}x - \int_\Omega \int_{-\bar{\kappa}}^0 \int_\alpha^0 \int_{t+s}^t \frac{\partial \mathbf{e}_\mathbf{p}^\mathrm{T}(u,x)}{\partial u} \bar{K}^\mathrm{T} Y_2 \bar{K} \frac{\partial \mathbf{e_p}(u,x)}{\partial u} \mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x.\end{aligned} \tag{32}$$

Firstly, it follows from Green formula that

$$2\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\sum_{k=1}^{l}\bar{D}_k\frac{\partial^2\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2}\mathrm{d}x = 2\sum_{k=1}^{l}\int_{\Omega}\frac{\partial}{\partial x_k}\left(\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\bar{D}_k\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k}\right)\mathrm{d}x - 2\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial x_k}U_1\bar{D}_k\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k}\mathrm{d}x. \quad (33)$$

Using Dirichlet boundary conditions, one can derive that

$$2\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\sum_{k=1}^{l}\bar{D}_k\frac{\partial^2\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2}\mathrm{d}x = -2\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial x_k}U_1\bar{D}_k\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k}\mathrm{d}x. \quad (34)$$

This, together with the so-called Wirtinger's inequality [29], implies that

$$2\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\sum_{k=1}^{l}\bar{D}_k\frac{\partial^2\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2}\mathrm{d}x \le -\frac{\pi^2}{2}\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1D_L\mathbf{e}_{\mathfrak{m}}(t,x)\mathrm{d}x. \quad (35)$$

In a similar way,

$$2\int_{\Omega}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)U_2\sum_{k=1}^{l}\bar{D}_k^{*}\frac{\partial^2\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial x_k^2}\mathrm{d}x \le -\frac{\pi^2}{2}\int_{\Omega}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)U_2D_L^{*}\mathbf{e}_{\mathfrak{p}}(t,x)\mathrm{d}x. \quad (36)$$

From (14) we get

$$2\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial t}U_1\left[-\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial t}+\sum_{k=1}^{l}\bar{D}_k\frac{\partial^2\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2} - \bar{A}\mathbf{e}_{\mathfrak{m}}(t,x)+\bar{W}f(p(t-\kappa(t),x))\right]\mathrm{d}x = 0 \quad (37)$$

and

$$2\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)}{\partial t}U_2\left[-\frac{\partial\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial \mathrm{t}}+\sum_{k=1}^{l}\bar{D}_k^{*}\frac{\partial^2\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial x_k^2} - \bar{C}\mathbf{e}_{\mathfrak{p}}(t,x)+\bar{B}\mathfrak{m}(t-\rho(t),x)\right]\mathrm{d}x = 0. \quad (38)$$

By means of Dirichlet boundary conditions, Green formula and [15], Lemma 4, it yields that

$$2\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial t}U_1\sum_{k=1}^{l}\bar{D}_k\frac{\partial^2\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial x_k^2}\mathrm{d}x = 2\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left[D_k\frac{\partial}{\partial x_k}\left(\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial t}\right)\right]\mathrm{d}x = -2\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial x_k}U_1\bar{D}_k\frac{\partial}{\partial x_k}\left(\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial t}\right)\mathrm{d}x. \quad (39)$$

Similarly,

$$2\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)}{\partial t}U_2\sum_{k=1}^{l}\bar{D}_k^{*}\frac{\partial^2\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial x_k^2}\mathrm{d}x = -2\sum_{k=1}^{l}\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)}{\partial x_k}U_2\bar{D}_k^{*}\frac{\partial}{\partial x_k}\left(\frac{\partial\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial t}\right)\mathrm{d}x. \quad (40)$$

Combining (27) and (35), (36), (37), (38), (39), and (40), we get

$$\begin{aligned}\frac{\partial}{\partial \mathrm{t}}\mathscr{V}_1(t,\mathbf{e}_{\mathfrak{m}},\mathbf{e}_{\mathfrak{p}}) \le{}& 2\int_{\Omega}\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)U_1\left[-\bar{A}\mathbf{e}_{\mathfrak{m}}(t,x)-\frac{\pi^2}{4}D_L\mathbf{e}_{\mathfrak{m}}(t,x) + \bar{W}f(p(t-\kappa(t),x))\right]\mathrm{d}x + 2\int_{\Omega}\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)U_2\\ &\cdot\left[-\bar{C}\mathbf{e}_{\mathfrak{p}}(t,x)-\frac{\pi^2}{4}D_{\mathrm{L}}^{*}\mathbf{e}_{\mathfrak{p}}(t,x) + \bar{B}\mathfrak{m}(t-\rho(t),x)\right]\mathrm{d}x + 2\int_{\Omega}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(t,x)}{\partial t}U_1\\ &\cdot\left[-\frac{\partial\mathbf{e}_{\mathfrak{m}}(t,x)}{\partial t}-\bar{A}\mathbf{e}_{\mathfrak{m}}(t,x) + \bar{W}f(p(t-\kappa(t),x))\right]\mathrm{d}x + 2\int_{\Omega}\\ &\cdot\frac{\partial\mathbf{e}_{\mathfrak{p}}^{\mathrm{T}}(t,x)}{\partial t}U_2\left[-\frac{\partial\mathbf{e}_{\mathfrak{p}}(t,x)}{\partial t}-\bar{C}\mathbf{e}_{\mathfrak{p}}(t,x) + \bar{B}\mathfrak{m}(\mathrm{t}-\mathrm{\rho}(t),x)\right]\mathrm{d}x\\ ={}& \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_1\eta(t,x)\mathrm{d}x.\end{aligned} \quad (41)$$

Secondly, in view of (21), the reciprocally convex technique [30] and Lemma 2, it follows that

$$\begin{aligned}-\bar{\rho}\int_{\Omega}\int_{t-\bar{\rho}}^{t}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)}{\partial s}\bar{K}^{\mathrm{T}}W_1\bar{K}\frac{\partial\mathbf{e}_{\mathfrak{m}}(s,x)}{\partial s}\mathrm{d}s\mathrm{d}x ={}& -\bar{\rho}\int_{\Omega}\int_{t-\rho(t)}^{t}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)}{\partial \mathrm{s}}\bar{K}^{\mathrm{T}}W_1\bar{K}\frac{\partial\mathbf{e}_{\mathfrak{m}}(s,x)}{\partial \mathrm{s}}\mathrm{d}s\mathrm{d}x\\ &-\bar{\rho}\int_{\Omega}\int_{t-\bar{\rho}}^{t-\rho(t)}\frac{\partial\mathbf{e}_{\mathfrak{m}}^{\mathrm{T}}(s,x)}{\partial \mathrm{s}}\bar{K}^{\mathrm{T}}W_1\bar{K}\frac{\partial\mathbf{e}_{\mathfrak{m}}(s,x)}{\partial s}\mathrm{d}s\mathrm{d}x\\ \le{}& -\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{41}\eta(t,x)\mathrm{d}x.\end{aligned} \quad (42)$$

Similarly,

$$-\bar{\kappa}\int_{\Omega}\int_{t-\bar{\kappa}}^{t}\frac{\partial \mathbf{e}_{\mathbf{p}}^{\mathrm{T}}(s,x)}{\partial s}\bar{K}^{\mathrm{T}}W_2\bar{K}\frac{\partial \mathbf{e}_{\mathbf{p}}(s,x)}{\partial s}\mathrm{d}s\mathrm{d}x \leq -\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{42}\eta(t,x)\mathrm{d}x. \quad (43)$$

Again using Lemma 2, one can obtain

$$\begin{aligned}
-\bar{\rho}\int_{\Omega}\int_{t-\bar{\rho}}^{t}\mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_3\bar{K}\mathbf{e}_{\mathbf{m}}(s,x)\mathrm{d}s\mathrm{d}x
&= -\bar{\rho}\int_{\Omega}\int_{t-\rho(t)}^{t}\mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_3\bar{K}\mathbf{e}_{\mathbf{m}}(s,x)\mathrm{d}s\mathrm{d}x \\
&\quad -\bar{\rho}\int_{\Omega}\int_{t-\bar{\rho}}^{t-\rho(t)}\mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_3\bar{K}\mathbf{e}_{\mathbf{m}}(s,x)\mathrm{d}s\mathrm{d}x \\
&\leq -\bar{\rho}\rho(t)\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Pi_6\tilde{W}_3\Pi_6^{\mathrm{T}}\eta(t,x)\mathrm{d}x \\
&\quad -\bar{\rho}(\bar{\rho}-\rho(t))\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Pi_5\tilde{W}_3\Pi_5^{\mathrm{T}}\eta(t,x)\mathrm{d}x \\
&= \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{43}(\rho(t))\eta(t,x)\mathrm{d}x
\end{aligned} \quad (44)$$

and

$$\begin{aligned}
-\bar{\kappa}\int_{\Omega}\int_{t-\bar{\rho}}^{t}\mathbf{e}_{\mathbf{p}}^{\mathrm{T}}(s,x)\bar{K}^{\mathrm{T}}W_4\bar{K}\mathbf{e}_{\mathbf{p}}(s,x)\mathrm{d}s\mathrm{d}x
&\leq -\bar{\kappa}\kappa(t)\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Pi_8\tilde{W}_4\Pi_8^{\mathrm{T}}\eta(t,x)\mathrm{d}x \\
&\quad -\bar{\kappa}(\bar{\kappa}-\kappa(t))\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Pi_7\tilde{W}_4\Pi_7^{\mathrm{T}}\eta(t,x)\mathrm{d}x \\
&= \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{44}(\kappa(t))\eta(t,x)\mathrm{d}x.
\end{aligned} \quad (45)$$

The combination of (30) and (42), (43), (44), and (45) gives

$$\frac{\partial}{\partial t}\mathscr{V}_4(t,\mathbf{e}_{\mathbf{m}},\mathbf{e}_{\mathbf{p}}) \leq \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_4(\rho(t),\kappa(t))\eta(t,x)\mathrm{d}x. \quad (46)$$

Thirdly, it is clear that

$$\begin{aligned}
-\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x
&= -\int_{\Omega}\int_{-\bar{\rho}}^{-\rho(t)}\int_{t+s}^{t-\rho(t)}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial \mathrm{u}}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial \mathrm{u}}\mathrm{d}u\mathrm{d}s\mathrm{d}x \\
&\quad -\int_{\Omega}\int_{-\rho(t)}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x \\
&\quad -(\bar{\rho}-\rho(t))\int_{\Omega}\int_{t-\rho(t)}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}x.
\end{aligned} \quad (47)$$

It follows from Lemma 2 that

$$\begin{aligned}
-\int_{\Omega}\int_{-\rho(t)}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x
&\leq -\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Upsilon_1\tilde{X}_1\Upsilon_1^{\mathrm{T}}\eta(t,x)\mathrm{d}x, \\
-\int_{\Omega}\int_{-\bar{\rho}}^{-\rho(t)}\int_{t+s}^{t-\rho(t)}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x
&\leq -\int_{\Omega}\eta^{\mathrm{T}}(t,x)\Upsilon_2\tilde{X}_1\Upsilon_2^{\mathrm{T}}\eta(t,x)\mathrm{d}x
\end{aligned} \quad (48)$$

and

$$-(\bar{\rho}-\rho(t))\int_{\Omega}\int_{t-\rho(t)}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial \mathrm{u}}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}x \leq -\int_{\Omega}\eta^{\mathrm{T}}(t,x)\frac{\bar{\rho}-\rho(t)}{\bar{\rho}}\Pi_2\hat{X}_1\Pi_2^{\mathrm{T}}\eta(t,x)\mathrm{d}x. \quad (49)$$

By (47), it implies that

$$-\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial \mathrm{u}}\bar{K}^{\mathrm{T}}X_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial \mathrm{u}}\mathrm{d}u\mathrm{d}s\mathrm{d}x \leq \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{51}(\rho(t))\eta(t,x)\mathrm{d}x. \quad (50)$$

Similarly,

$$-\int_{\Omega}\int_{-\bar{\kappa}}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{p}}^{\mathrm{T}}(u,x)}{\partial \mathrm{u}}\bar{K}^{\mathrm{T}}X_2\bar{K}\frac{\partial \mathbf{e}_{\mathbf{p}}(u,x)}{\partial \mathrm{u}}\mathrm{d}u\mathrm{d}s\mathrm{d}x \leq \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{52}(\kappa(t))\eta(t,x)\mathrm{d}x. \quad (51)$$

Combining (31), (50), and (51), we have

$$\frac{\partial}{\partial t}\mathscr{V}_5(t,e_m,e_p) \leq \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_5(\rho(t),\kappa(t))\eta(t,x)\mathrm{d}x. \quad (52)$$

Fourthly, it is obvious that

$$\begin{aligned}
-\int_{\Omega}\int_{-\bar{\rho}}^{0}\int_{\alpha}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x
&\leq -\int_{\Omega}\int_{-\rho(t)}^{0}\int_{\alpha}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x \\
&\quad -\int_{\Omega}\int_{-\bar{\rho}}^{-\rho(t)}\int_{\alpha}^{-\rho(t)}\int_{t+s}^{t-\rho(t)}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x \\
&\quad -(\bar{\rho}-\rho(t))\int_{\Omega}\int_{-\rho(t)}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathbf{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathbf{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x.
\end{aligned} \quad (53)$$

By Lemmas 1 and 2, one can obtain

$$-\int_{\Omega}\int_{-\bar{\rho}}^{-\rho(t)}\int_{\alpha}^{-\rho(t)}\int_{t+s}^{t-\rho(t)}\frac{\partial \mathbf{e}_{\mathrm{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathrm{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x - \int_{\Omega}\int_{-\rho(t)}^{0}\int_{\alpha}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathrm{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathrm{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x \le \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{61}(\rho(t))\eta(t,x)\mathrm{d}x \quad (54)$$

and

$$-(\bar{\rho}-\rho(t))\int_{\Omega}\int_{-\rho(t)}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathrm{m}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_1\bar{K}\frac{\partial \mathbf{e}_{\mathrm{m}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}x \le -\int_{\Omega}\eta^{\mathrm{T}}(t,x)(\bar{\rho}-\rho(t))\Upsilon_1\tilde{Y}_1\Upsilon_1^{\mathrm{T}}\eta(t,x)\mathrm{d}x, \quad (55)$$

respectively. Similarly,

$$-\int_{\Omega}\int_{-\kappa(t)}^{0}\int_{\alpha}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathrm{p}}^{\mathrm{T}}(u,x)}{\partial \mathrm{u}}\bar{K}^{\mathrm{T}}Y_2\bar{K}\frac{\partial \mathbf{e}_{\mathrm{p}}(u,x)}{\partial \mathrm{u}}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x - \int_{\Omega}\int_{-\bar{\kappa}}^{-\kappa(t)}\int_{\alpha}^{-\kappa(t)}\int_{t+s}^{t-\kappa(t)}\frac{\partial \mathbf{e}_{\mathrm{p}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_2\bar{K}\frac{\partial \mathbf{e}_{\mathrm{p}}(u,x)}{\partial u}\mathrm{d}u\mathrm{d}s\mathrm{d}\alpha\mathrm{d}x \le \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_{62}(\kappa(t))\eta(t,x)\mathrm{d}x \quad (56)$$

and

$$-(\bar{\kappa}-\kappa(t))\int_{\Omega}\int_{-\kappa(t)}^{0}\int_{t+s}^{t}\frac{\partial \mathbf{e}_{\mathrm{p}}^{\mathrm{T}}(u,x)}{\partial u}\bar{K}^{\mathrm{T}}Y_2\bar{K}\frac{\partial \mathbf{e}_{\mathrm{p}}(u,x)}{\partial \mathrm{u}}\mathrm{d}u\mathrm{d}s\mathrm{d}x \le -\int_{\Omega}\eta^{\mathrm{T}}(t,x)(\bar{\kappa}-\kappa(t))\Upsilon_3\tilde{Y}_2\Upsilon_3^{\mathrm{T}}\eta(t,x)\mathrm{d}x. \quad (57)$$

By (32) and (53), (54), (55), (56), and (57), it is obtained that

$$\frac{\partial}{\partial \mathrm{t}}\mathscr{V}_6(t,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \le \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi_6(\rho(t),\kappa(t))\eta(t,x)\mathrm{d}x. \quad (58)$$

Finally, from (6) and the relationship between f_i and g_i, it is clear to say that $f(0)=0$ and

$$(f(z)-Kz)^{\mathrm{T}}Z_i f(z) \le 0, \forall z \in \mathbb{R}^n, i=1,2, \quad (59)$$

that is, for given diagonal matrices $Z_1>0$ and $Z_2>0$, we get

$$\eta^{\mathrm{T}}(t,x)\Psi_{0i}\eta(t,x) \ge 0, i=1,2. \quad (60)$$

The combination of (22), (28), (29), (41), (46), (52), (39), and (60) yields

$$\begin{aligned}\frac{\partial}{\partial \mathrm{t}}\mathscr{V}(\mathrm{t},\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) &= \sum_{i=1}^{6}\frac{\partial}{\partial t}\mathscr{V}_{\mathrm{i}}(t,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \\ &\le \alpha\int_{\Omega}\left[\mathbf{e}_{\mathrm{m}}^{\mathrm{T}}(t,x)U_1\mathbf{e}_{\mathrm{m}}(t,x)+\mathbf{e}_{\mathrm{p}}^{\mathrm{T}}(t,x)U_2\mathbf{e}_{\mathrm{p}}(t,x)\right]\mathrm{d}x \\ &\quad + \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi(\rho(t),\kappa(t))\eta(t,x)\mathrm{d}x \\ &\le \int_{\Omega}\eta^{\mathrm{T}}(t,x)\Psi(\rho(t),\kappa(t))\eta(t,x)\mathrm{d}x + \alpha\mathscr{V}(t,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}).\end{aligned} \quad (61)$$

Since $\Psi(\rho(t),\kappa(t))$ depends affinely upon $\rho(t)$ and $\kappa(t)$, one can derive from (22) that $\Psi(\rho(t),\kappa(t))<0$ for any $0\le\kappa(t)\le\bar{\kappa}$ and $0\le\rho(t)\le\bar{\rho}$. Utilizing (61), we can get

$$\frac{\partial}{\partial \mathrm{t}}\mathscr{V}(\mathrm{t},\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \le \alpha\mathscr{V}(\mathrm{t},\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}). \quad (62)$$

For any $t\in[0,T]$, integrating the two sides of (62) from 0 to t, we derive

$$\mathscr{V}(\mathrm{t},\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \le \mathscr{V}(0,\mathbf{e}_{\mathrm{m}}(0,x),\mathbf{e}_{\mathrm{p}}(0,x)) + \int_0^t \alpha\mathscr{V}(s,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}})\mathrm{d}s. \quad (63)$$

By means of the so-called Gronwall inequality, one can obtain

$$\mathscr{V}(T,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \le \mathrm{e}^{\alpha T}\mathscr{V}(0,\mathbf{e}_{\mathrm{m}}(0,x),\mathbf{e}_{\mathrm{p}}(0,x)). \quad (64)$$

According to the expression of $\mathscr{V}(t,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}})$, it apparently will come

$$\begin{aligned}&\mathscr{V}(0,\mathbf{e}_{\mathrm{m}}(0,x),\mathbf{e}_{\mathrm{p}}(0,x)) \\ &\quad \le (\lambda_{11}+\lambda_{12})\left(\|\bar{\psi}(\mathrm{t},\mathrm{x})\|_d^2 + \|\bar{\psi}^*(t,x)\|_d^2\right).\end{aligned} \quad (65)$$

Combination of (64) and (65) derives

$$\mathscr{V}(T,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \le \mathrm{e}^{\alpha T}(\lambda_{11}+\lambda_{12})\left(\|\overline{\psi}(t,x)\|_d^2+\|\overline{\psi}^*(t,x)\|_d^2\right). \quad (66)$$

In addition, when $t\in[0,T]$, one can easily derive that

$$\mathscr{V}(T,\mathbf{e}_{\mathrm{m}},\mathbf{e}_{\mathrm{p}}) \ge \lambda_{\min}(U)\left(\|\mathbf{e}_{\mathrm{m}}(t,x)\|^2 + \|\mathbf{e}_{\mathrm{p}}(t,x)\|^2\right), \quad (67)$$

where $\lambda_{\min}(U)$ is the minimum eigenvalue of diag (U_1, U_2). It can be seen from (66) and (67) that

$$\|\mathbf{e}_{\mathrm{m}}(t,x)\|^2 + \|\mathbf{e}_{\mathrm{p}}(t,x)\|^2 \le \frac{\mathrm{e}^{\alpha \mathrm{T}}(\lambda_{11}+\lambda_{12})\left(\|\overline{\psi}(t,x)\|_d^2 + \|\overline{\psi}^*(\mathrm{t},\mathrm{x})\|_d^2\right)}{\lambda_{\min}(U)} \quad (68)$$

By Definition 1 and (23), under Dirichlet boundary conditions, system (14) is F-T stable with respect to c_1, c_2, and T. The proof is completed.

Following the proof of Theorem 1, one can easily derive the following conclusion which gives a method to design an r-order observer for the delayed RDGRN (8).

Corollary 1. *For given scalars $\bar{\rho}$, $\bar{\kappa}$, μ_ρ, and μ_κ subject to (5), system (14) is asymptotically stable under Dirichlet boundary conditions, if there are matrices $0 < V_i^{\mathrm{T}} = V_i \in \mathbb{R}^{n\times n}$ $(i \in \langle 5\rangle)$, $0 < W_j^{\mathrm{T}} = W_j \in \mathbb{R}^{n\times n}$ $(j \in \langle 4\rangle)$, $0 < X_k^{\mathrm{T}} = X_k \in \mathbb{R}^{n\times n}$, and $0 < Y_k^{\mathrm{T}} = Y_k \in \mathbb{R}^{n\times n}$, diagonal matrices $0 < U_k \in \mathbb{R}^{(n+r)\times(n+r)}$ and $0 < Z_k \in \mathbb{R}^{n\times n}$, and matrices $\widehat{H}_k \in \mathbb{R}^{3n\times 3n}$ $(k \in \langle 2\rangle)$, $A_{\mathbf{m}} \in \mathbb{R}^{r\times r}$, $C_{\mathbf{p}} \in \mathbb{R}^{r\times r}$, $M_{\mathbf{m}}$, and M_p of appropriate dimensions, such that LMIs (21) and (22) with $\alpha = 0$ are feasible for $\rho \in \{0, \bar{\rho}\}$ and $\kappa \in \{0, \bar{\kappa}\}$. In addition, the desired r-order state observer is described by (12) and (24).*

Since inequality (23) is not an LMI, the Toolbox YALMIP of MATLAB is not applicable. Now we will do the following transformation:

Theorem 2. *For given scalars $\bar{\rho}$, $\bar{\kappa}$, μ_ρ, and μ_κ satisfying (5) and positive constants T, c_1, c_2, and α, system (14) is F-T stable with respect to c_1, c_2, and T under Dirichlet boundary conditions, if there are real numbers $\lambda_{\mathrm{v}i} > 0 (i \in \langle 5\rangle)$, $\lambda_{\mathrm{w}j} > 0$ $(j \in \langle 4\rangle)$, $\lambda_{xk} > 0$, $\lambda_{yk} > 0$, $\lambda_{uk} > 0$ $(k \in \langle 2\rangle)$, and $\lambda_u > 0$, matrices $0 < V_i^{\mathrm{T}} = V_i \in \mathbb{R}^{n\times n}$ $(i \in \langle 5\rangle)$, $0 < W_j^{\mathrm{T}} = W_j \in \mathbb{R}^{n\times n}$, $(j \in \langle 4\rangle)$, $0 < X_k^{\mathrm{T}} = X_k \in \mathbb{R}^{n\times n}$, and $0 < Y_k^{\mathrm{T}} = Y_k \in \mathbb{R}^{n\times n}$, diagonal matrices $0 < U_k \in \mathbb{R}^{(n+r)\times(n+r)}$ and $0 < Z_k \in \mathbb{R}^{n\times n}$, and matrices $\widehat{H}_k \in \mathbb{R}^{3n\times 3n}$ $(k \in \langle 2\rangle)$, $A_{\mathbf{m}} \in \mathbb{R}^{r\times r}$, $C_{\mathbf{p}} \in \mathbb{R}^{r\times r}$, $M_{\mathbf{m}}$, and M_p of appropriate dimensions, such that LMIs (21), (22) and the following (69), (70), (71), (72), (73), (74), and (75) are feasible for $\rho \in \{0, \bar{\rho}\}$ and $\kappa \in \{0, \bar{\kappa}\}$:*

$$0 \le V_i \le \lambda_{vi} I, i \in \langle 5\rangle, \tag{69}$$

$$0 \le W_j \le \lambda_{wj} I, j \in \langle 4\rangle, \tag{70}$$

$$0 \le \mathrm{U}_k \le \lambda_{uk} I, k \in \langle 2\rangle, \tag{71}$$

$$0 \le X_k \le \lambda_{xk} I, k \in \langle 2\rangle, \tag{72}$$

$$0 \le Y_k \le \lambda_{yk} I, k \in \langle 2\rangle, \tag{73}$$

$$\lambda_u I \le U, \tag{74}$$

$$\begin{aligned} c_1 e^{\alpha T} \Bigg(& \lambda_{u1} + \bar{\rho}\lambda_{v1} + \bar{\rho}\lambda_{v2} + \frac{1}{6}\bar{\rho}^3\lambda_{x1} + \frac{1}{2}\bar{\rho}^3(\lambda_{w1} + \lambda_{w3}) \\ & + \lambda_{u1}\sum_{k=1}^{1}\lambda_{\max}(\bar{D}_k) + \lambda_{u2} + \bar{\kappa}\lambda_{v3} + \bar{\kappa}\lambda_{v4} + \frac{1}{24}\bar{\rho}^4\lambda_{y1} \\ & + \bar{\kappa}\lambda_{v5}\lambda_{\max}(K^{\mathrm{T}}K) + \frac{1}{6}\bar{\kappa}^3\lambda_{x2} + \frac{1}{2}\bar{\kappa}^3(\lambda_{w2} + \lambda_{w4}) \\ & + \frac{1}{24}\bar{\kappa}^4\lambda_{y2} + \lambda_{u2}\sum_{k=1}^{1}\lambda_{\max}(\bar{D}_k^*)\Bigg) \le \mathrm{c}_2\lambda_{\mathrm{u}}. \end{aligned} \tag{75}$$

In addition, the desired r-order F-T state observer is described by (12) and (24).

Proof 2. It follows from (69), (70), (71) and (72) that

$$\begin{aligned} n_1 := c_1 e^{\alpha T} \Bigg(& \lambda_{\max}(U_1) + \bar{\rho}\lambda_{\max}(V_1) + \bar{\rho}\lambda_{\max}(V_2) \\ & + \frac{1}{6}\bar{\rho}^3\lambda_{\max}(X_1) + \sum_{k=1}^{l}\lambda_{\max}(U_1)\lambda_{\max}(\bar{D}_{\mathrm{k}}) \\ & + \frac{1}{2}\bar{\rho}^3\lambda_{\max}(W_1) + \frac{1}{2}\bar{\rho}^3\lambda_{\max}(W_3) + \frac{1}{24}\bar{\rho}^4\lambda_{\max}(Y_1)\Bigg) \\ \le c_1 e^{\alpha T} \Bigg(& \lambda_{u1} + \bar{\mathrm{\rho}}\lambda_{v1} + \bar{\mathrm{\rho}}\lambda_{v2} + \frac{1}{6}\bar{\rho}^3\lambda_{x1} + \frac{1}{2}\bar{\rho}^3\lambda_{w1} \\ & + \frac{1}{2}\bar{\mathrm{\rho}}^3\lambda_{\mathrm{w}3} + \frac{1}{24}\bar{\mathrm{\rho}}^4\lambda_{\mathrm{y}1} + \lambda_{\mathrm{u}1}\sum_{\mathrm{k}=1}^{1}\lambda_{\max}(\bar{\mathrm{D}}_{\mathrm{k}})\Bigg) \end{aligned} \tag{76}$$

and

$$\begin{aligned} \mathrm{n}_2 := c_1 e^{\alpha T} \Bigg(& \lambda_{\max}(U_2) + \bar{\kappa}\lambda_{\max}(V_3) + \bar{\kappa}\lambda_{\max}(V_4) \\ & + \bar{\kappa}\lambda_{\max}(V_5)\lambda_{\max}(K^{\mathrm{T}}K) + \frac{1}{2}\bar{\kappa}^3\lambda_{\max}(W_4) \\ & + \frac{1}{6}\bar{\kappa}^3\lambda_{\max}(X_2) + \sum_{k=1}^{l}\lambda_{\max}(U_2)\lambda_{\max}(\bar{D}_k^*) \\ & + \frac{1}{2}\bar{\kappa}^3\lambda_{\max}(W_2) + \frac{1}{24}\bar{\kappa}^4\lambda_{\max}(Y_2)\Bigg) \\ \le c_1 e^{\alpha T} \Bigg(& \lambda_{u2} + \bar{\kappa}\lambda_{\mathrm{v}3} + \bar{\kappa}\lambda_{\mathrm{v}4} + \bar{\kappa}\lambda_{\mathrm{v}5}\lambda_{\max}(K^{\mathrm{T}}K) \\ & + \frac{1}{6}\bar{\kappa}^3\lambda_{\mathrm{x}2} + \frac{1}{2}\bar{\kappa}^3\lambda_{w2} + \frac{1}{2}\bar{\kappa}^3\lambda_{w4} + \frac{1}{24}\bar{\kappa}^4\lambda_{\mathrm{y}2} \\ & + \lambda_{u2}\sum_{k=1}^{l}\lambda_{\max}(\bar{D}_k^*)\Bigg). \end{aligned} \tag{77}$$

This, together with (74) and (75), implies that

$$n_1 + n_2 \le \lambda_u c_2 \le c_2\lambda_{\min}(U), \tag{78}$$

which shows that (23) holds. By Theorem 1, we complete the proof.

Finally, we make several remarks on the method proposed in this paper.

Remark 2. Different from [18, 19], this paper gives an r -order F-T state observer (12) for the delayed RDGRN (8). It should be mentioned that, compared with the full-order observer, the reduced-order one is more practical. Particularly, a reduced-order observer can save the cost in the engineering applications.

Remark 3. In the proof of Theorem 1, we employ the so-called Wirtinger's inequality to obtain the equalities (18, 19), which claims the necessity of Dirichlet boundary conditions. However, by employing the technique used in [14, 15], one can deal with the cases of Robin boundary conditions and Neumann boundary conditions, which will make the LMI conditions corresponding to ones in Theorems 1 and 2 more conservative.

Remark 4. For delayed genetic regulatory networks without reaction-diffusion items, the method proposed in Theorems 1 and 2 is still keeping available by removing the corresponding parts of $\mathscr{V}_1(t, \mathbf{e_m}, \mathbf{e_p})$.

Remark 5. For delayed genetic regulatory networks without reaction-diffusion items, Zhang et al. [22] proposed a method to design full- and reduced-order state observers. It should be pointed that this method cannot be applied to delayed RDGRNs, since the equivalent decompositions of output matrices are required.

Remark 6. Several techniques used in this paper may be available for some other time-delay models:

(1) Wirtinger-type integral inequality, instead of Jensen's inequality, is applied to estimate some integral items in the derivative of LKFs;

(2) The convex technique and reciprocally convex technique are organically combined;

(3) The coefficients, $1/\rho(t)$ and $1/\bar{\rho} - \rho(t)$, are introduced into the augmented vector $\eta(t, x)$.

5. An Illustrative Example

In this section, we will give a numerical example to verify the availability of the proposed method to design the r-order F-T state observer.

Example 1. Consider the delayed RDGRN (8), where $l = L_1 = 1$, $f_i(x) = x^2/1 + x^2, i \in \langle 3 \rangle$, $D_1 = 0.1I_3$, $D_1^* = 0.2I_3$,and

$$\begin{aligned} A &= \text{diag}\,(0.2,1.1,1.2), \\ C &= \text{diag}\,(0.3,0.7,1.3), \\ B &= \text{diag}\,(1.0,0.4,0.7), \\ W &= \begin{bmatrix} 0 & 0 & -0.5 \\ -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \end{bmatrix}, \\ N_m &= \begin{bmatrix} 0.5 & -0.6 & 0 \\ 0.3 & 0.8 & -0.2 \end{bmatrix}, \\ N_p &= \begin{bmatrix} 0.7 & -0.25 & 0.3 \\ 0.4 & 0.2 & -0.3 \end{bmatrix}. \end{aligned} \tag{79}$$

In order to save space, we only design the 1-order F-T state observer. Let $\mu_\rho = \mu_\kappa = 1.5$, $\bar{\rho} = \bar{\kappa} = 1$, $c_1 = 1.2$, $c_2 = 5$, $\alpha = 0.002$, and $\mathrm{T} = 10$. The LMIs in Theorem 2 is solved by means of the MATLAB's toolbox. The solution matrices are listed as follows:

$$\begin{aligned} U_1 &= \text{diag}\,(0.0024,0.0023,0.0023,0.0023), \\ U_2 &= \text{diag}\,(0.0021,0.0022,0.0022,0.0022), \\ V_1 &= 10^{-4} * \begin{bmatrix} 0.1045 & 0.0025 & 0.0006 \\ 0.0025 & 0.4262 & 0.0008 \\ 0.0006 & 0.0008 & 0.3188 \end{bmatrix}, \\ V_2 &= 10^{-3} * \begin{bmatrix} 0.8349 & 0.0014 & -0.0001 \\ 0.0014 & 0.6863 & 0.0001 \\ -0.0001 & 0.0001 & 0.7273 \end{bmatrix}, \\ V_3 &= 10^{-4} * \begin{bmatrix} 0.2375 & 0.0007 & 0.0177 \\ 0.0007 & 0.3705 & 0.0004 \\ 0.0177 & 0.0004 & 0.2184 \end{bmatrix}, \\ V_4 &= 10^{-3} * \begin{bmatrix} 0.2459 & 0.0004 & -0.0026 \\ 0.0004 & 0.3530 & 0.0001 \\ -0.0026 & 0.0001 & 0.3982 \end{bmatrix}, \\ V_5 &= 10^{-3} * \begin{bmatrix} 0.0724 & 0.0001 & 0.0044 \\ 0.0001 & 0.1008 & 0.0001 \\ 0.0044 & 0.0001 & 0.0504 \end{bmatrix}, \\ W_1 &= 10^{-3} * \begin{bmatrix} 0.4365 & -0.0003 & -0.0010 \\ -0.0003 & 0.0602 & -0.0002 \\ -0.0010 & -0.0002 & 0.0979 \end{bmatrix}, \\ W_2 &= 10^{-3} * \begin{bmatrix} 0.0650 & -0.0000 & -0.0001 \\ -0.0000 & 0.0676 & -0.0000 \\ -0.0001 & -0.0000 & 0.1329 \end{bmatrix}, \\ W_3 &= \begin{bmatrix} 0.0019 & 0.0000 & -0.0000 \\ 0.0000 & 0.0013 & -0.0000 \\ -0.0000 & -0.0000 & 0.0017 \end{bmatrix}, \\ W_4 &= 10^{-3} * \begin{bmatrix} 0.3859 & 0.0007 & -0.0657 \\ 0.0007 & 0.7248 & 0.0003 \\ -0.0657 & 0.0003 & 0.8332 \end{bmatrix}, \\ X_1 &= 10^{-4} * \begin{bmatrix} 0.1836 & -0.0002 & -0.0002 \\ -0.0002 & 0.2028 & 0.0000 \\ -0.0002 & 0.0000 & 0.1890 \end{bmatrix}, \end{aligned}$$

$$X_2 = 10^{-4} * \begin{bmatrix} 0.2099 & -0.0000 & 0.0037 \\ -0.0000 & 0.2086 & -0.0000 \\ 0.0037 & -0.0000 & 0.1899 \end{bmatrix},$$

$$Y_1 = 10^{-3} * \begin{bmatrix} 0.5525 & 0.0028 & -0.0085 \\ 0.0028 & 0.2215 & -0.0002 \\ -0.0085 & -0.0002 & 0.2354 \end{bmatrix},$$

$$Y_2 = 10^{-3} * \begin{bmatrix} 0.1749 & 0.0000 & 0.0291 \\ 0.0000 & 0.2371 & -0.0003 \\ 0.0291 & -0.0003 & 0.2511 \end{bmatrix},$$

$$Z_1 = \text{diag}\,(0.0012, 0.0083, 0.0083),$$

$$Z_2 = \text{diag}\,(0.0007, 0.0010, 0.0015),$$

$$\lambda_{v1} = 1.0785\text{e} - 004,$$

$$\lambda_{v2} = 8.6624\text{e} - 004,$$

$$\lambda_{v3} = 1.0586\text{e} - 004,$$

$$\lambda_{v4} = 4.3530\text{e} - 004,$$

$$\lambda_{v3} = 1.0586\text{e} - 004,$$

$$\lambda_{v4} = 4.3530\text{e} - 004,$$

$$\lambda_{v5} = 2.5342\text{e} - 004,$$

$$\lambda_{w1} = 4.9465\text{e} - 004,$$

$$\lambda_{w2} = 2.4219\text{e} - 004,$$

$$\lambda_{w3} = 0.0019,$$

$$\lambda_{w4} = 9.0637\text{e} - 004,$$

$$\lambda_{x1} = 4.8617\text{e} - 004,$$

$$\lambda_{x2} = 4.8702\text{e} - 004,$$

$$\lambda_{u1} = 0.0024,$$

$$\lambda_{u2} = 0.0022,$$

$$\lambda_{y1} = 0.0020,$$

$$\lambda_{y2} = 0.0019,$$

$$\lambda_{u} = 0.0021,$$

$$A_{\mathfrak{m}} = -0.0037,$$

$$M_{\mathfrak{m}} = [0.0008 - 0.0011],$$

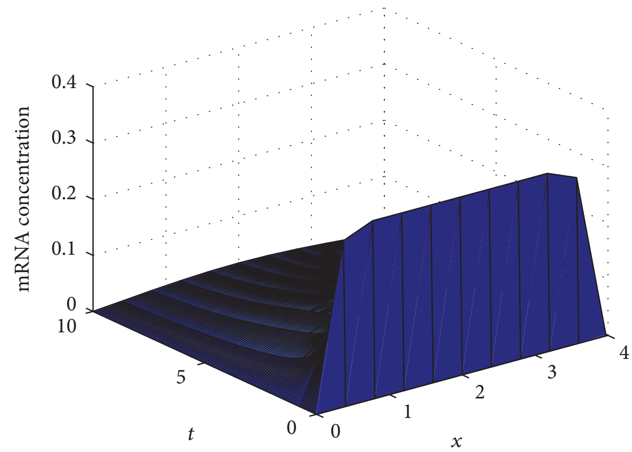

Figure 1: The real trajectories of mRNA concentration ($\|\mathfrak{m}(t,x)\|_2$).

$$C_{\mathfrak{p}} = -0036,$$

$$M_{\mathfrak{p}} = [0.0016 - 0.0024]. \tag{80}$$

Furthermore, we can obtain the corresponding observer gains as follows:

$$\begin{aligned} \widehat{A} &= -1.6087, \\ \widehat{M}_{\mathfrak{m}} &= [0.3478 - 0.4783], \\ \widehat{C} &= -1.6364, \\ \widehat{M}_{\mathfrak{p}} &= [0.7273 - 1.0909]. \end{aligned} \tag{81}$$

When the initial function $\psi(t) = [0.20.20.2]^T$ and $\psi^*(t) = [0.20.20.2]^T$ for $t \in [-1, 0]$, the state responses of RDGRN (8), 1-order observer (12), and error system (14) are presented in Figures 1–6. From which, it is seen that our approach is effective.

6. Conclusions

The design problem of r-order F-T state observer of RDGRNs with time-varying delays has been researched under Dirichlet boundary conditions. Utilizing available measurement outputs, we proposed a method to design r-order F-T observer which can be used to estimate the mRNA and protein concentrations. Sufficient F-T stability conditions for error system have been investigated by constructing an appropriate LKF and employing several integral inequalities and (reciprocally) convex technique. Thereby, the concrete expression of r-order F-T state observer is given. A numerical example is presented to illustrate the validity of the proposed method. It is worth emphasizing that the reduced-order observer problem of delayed RDGRNs is studied at the first time.

In literature, the problem of full-order state estimation for complex systems have been addressed (see, for example, [23–25]). However, all approaches proposed in these

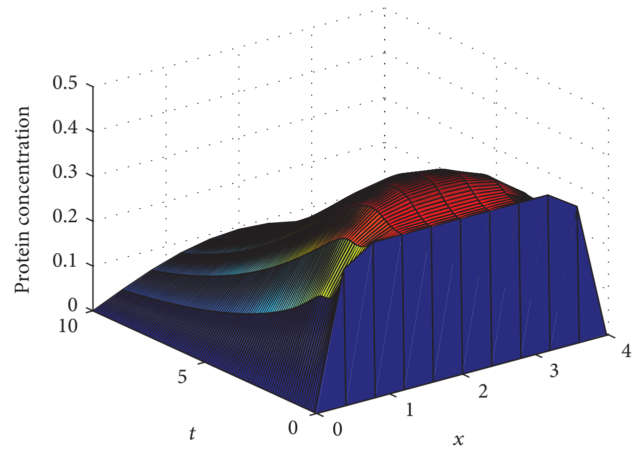

Figure 2: The real trajectories of protein concentration ($\|\mathfrak{p}(t,x)\|_2$).

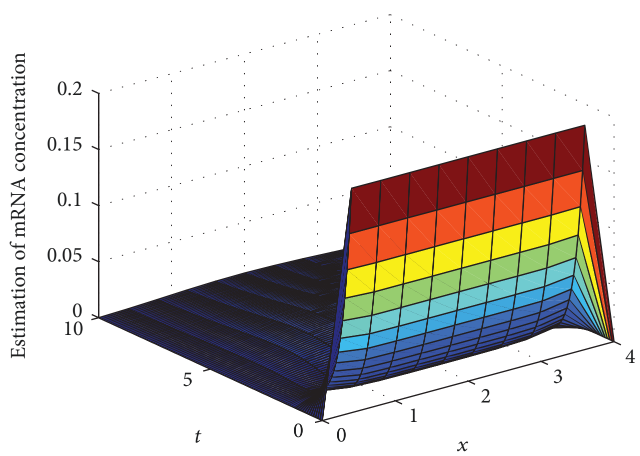

Figure 3: The estimation mRNA by the 1-order observer ($\|\widehat{\mathfrak{m}}(t,x)\|_2$).

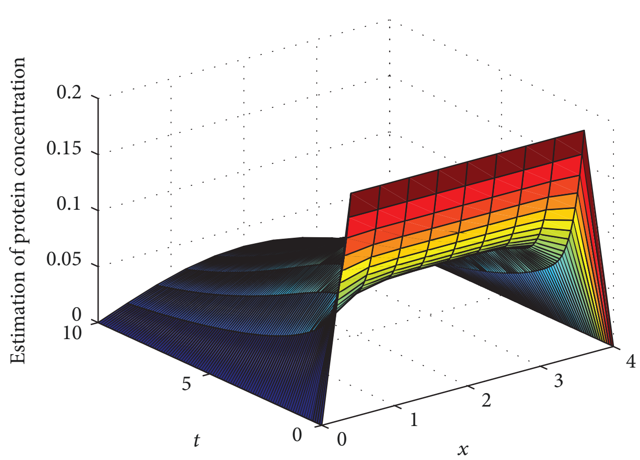

Figure 4: The estimation protein concentrations by the 1-order observer ($\|\widehat{\mathfrak{p}}(t,x)\|_2$).

literature are not available for designing reduced-order observers. Therefore, extending the method presented in this paper to the other system models will be left for future work.

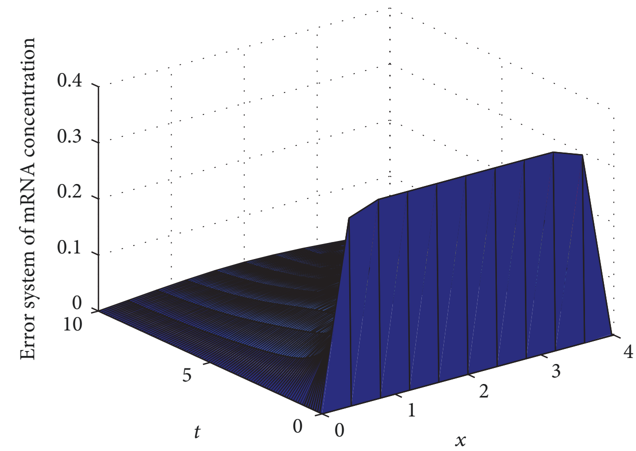

Figure 5: The mRNA concentrations of error system ($\|\mathfrak{e}_{\mathfrak{m}}(t,x)\|_2$).

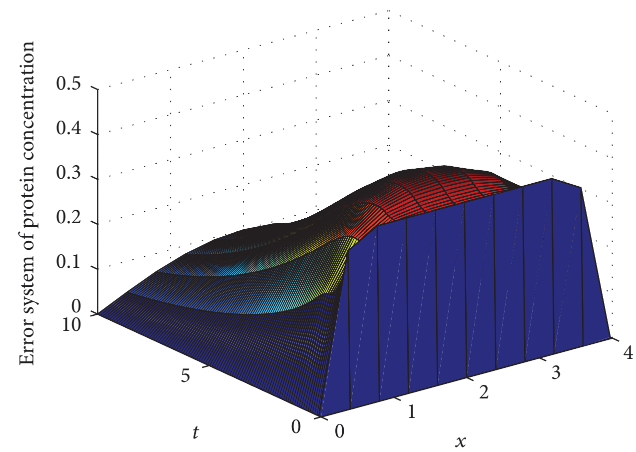

Figure 6: The protein concentrations of error system ($\|\mathfrak{e}_{\mathfrak{p}}(t,x)\|_2$).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request. All data are included within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant nos. 61174126, 11371006, 61222301, and 11501182), and the Fundamental Research Funds for the Central Universities (Grant no. HIT.BRETIV.201303).

References

[1] X. Zhang, L. Wu, and J. Zou, "Globally asymptotic stability analysis for genetic regulatory networks with mixed delays: an M-matrix-based approach," *IEEE/ACM Transactions on*

Computational Biology and Bioinformatics, vol. 13, no. 1, pp. 135–147, 2016.

[2] Z. Li, D. Chen, Y. Liu, and Y. Zhao, "New delay-dependent stability criteria of genetic regulatory networks subject to time-varying delays," *Neurocomputing*, vol. 207, pp. 763–771, 2016.

[3] W. Wang, Y. Wang, S. K. Nguang, S. Zhong, and F. Liu, "Delay partition method for the robust stability of uncertain genetic regulatory networks with time-varying delays," *Neurocomputing*, vol. 173, pp. 899–911, 2016.

[4] W. Wang and S. Zhong, "Delay-dependent stability criteria for genetic regulatory networks with time-varying delays and nonlinear disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 9, pp. 3597–3611, 2012.

[5] Y. He, J. Zeng, M. Wu, and C. K. Zhang, "Robust stabilization and H_∞ controllers design for stochastic genetic regulatory networks with time-varying delays and structured uncertainties," *Mathematical Biosciences*, vol. 236, no. 1, pp. 53–63, 2012.

[6] Y. He, L. Y. Fu, J. Zeng, and M. Wu, "Stability of genetic regulatory networks with interval time-varying delays and stochastic perturbation," *Asian Journal of Control*, vol. 13, no. 5, pp. 625–634, 2011.

[7] Q. Li, B. Shen, Y. Liu, and F. E. Alsaadi, "Event-triggered H_∞ state estimation for discrete-time stochastic genetic regulatory networks with Markovian jumping parameters and time-varying delays," *Neurocomputing*, vol. 174, pp. 912–920, 2016.

[8] M. Xiao, W. X. Zheng, G. Jiang, and J. Cao, "Stability and bifurcation analysis of arbitrarily high-dimensional genetic regulatory networks with hub structure and bidirectional coupling," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 8, pp. 1243–1254, 2016.

[9] H. Jiao, M. Shi, Q. Shen, J. Zhu, and P. Shi, "Filter design with adaptation to time-delay parameters for genetic regulatory networks," *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, vol. 15, no. 1, pp. 323–329, 2018.

[10] M. S. Ali, N. Gunasekaran, C. K. Ahn, and P. Shi, "Sampled-data stabilization for fuzzy genetic regulatory networks with leakage delays," *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, vol. 15, no. 1, pp. 271–285, 2018.

[11] J. Cao and F. Ren, "Exponential stability of discrete-time genetic regulatory networks with delays," *IEEE Transactions on Neural Networks*, vol. 19, no. 3, pp. 520–523, 2008.

[12] J. Zhou, S. Xu, and H. Shen, "Finite-time robust stochastic stability of uncertain stochastic delayed reaction-diffusion genetic regulatory networks," *Neurocomputing*, vol. 74, no. 17, pp. 2790–2796, 2011.

[13] Q. Ma, G. Shi, S. Xu, and Y. Zou, "Stability analysis for delayed genetic regulatory networks with reaction–diffusion terms," *Neural Computing Applications*, vol. 20, no. 4, pp. 507–516, 2011.

[14] Y. Y. Han and X. Zhang, "Stability analysis for delayed regulatory networks with reaction-diffusion terms (in Chinese)," *Journal of Natural Science of Heilongjiang University*, vol. 31, no. 1, pp. 32–40, 2014.

[15] Y. Han, X. Zhang, and Y. Wang, "Asymptotic stability criteria for genetic regulatory networks with time-varying delays and reaction–diffusion terms," *Circuits Systems and Signal Processing*, vol. 34, no. 10, pp. 3161–3190, 2015.

[16] X. Fan, X. Zhang, L. Wu, and M. Shi, "Finite-time stability analysis of reaction-diffusion genetic regulatory networks with time-varying delays," *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, vol. 14, no. 4, pp. 868–879, 2017.

[17] B. Cao, Q. Zhang, and M. Ye, "Exponential stability of impulsive stochastic genetic regulatory networks with time-varying delays and reaction-diffusion," *Advances in Difference Equations*, vol. 2016, no. 1, 2016.

[18] X. Zhang, Y. Han, L. Wu, and Y. Wang, "State estimation for delayed genetic regulatory networks with reaction-diffusion terms," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 2, pp. 299–309, 2018.

[19] X. Fan, Y. Xue, X. Zhang, and J. Ma, "Finite-time state observer for delayed reaction-diffusion genetic regulatory networks," *Neurocomputing*, vol. 227, pp. 18–28, 2017.

[20] R. Anbuvithya, K. Mathiyalagan, R. Sakthivel, and P. Prakash, "Sampled-data state estimation for genetic regulatory networks with time-varying delays," *Neurocomputing*, vol. 151, pp. 737–744, 2015.

[21] Y. T. Wang, X. Zhang, and Z. R. Hu, "Delay-dependent robust H_∞ filtering of uncertain stochastic genetic regulatory networks with mixed time-varying delays," *Neurocomputing*, vol. 166, pp. 346–356, 2015.

[22] X. Zhang, X. Fan, and L. Wu, "Reduced- and full-order observers for delayed genetic regulatory networks," *IEEE Transactions on Cybernetics*, vol. 48, no. 7, pp. 1989–2000, 2018.

[23] H. Shen, S. Huo, J. Cao, and T. Huang, "Generalized state estimation for Markovian coupled networks under round-robin protocol and redundant channels," *IEEE Transactions on Cybernetics*, pp. 1–10, 2018.

[24] H. Shen, M. Xing, S. Huo, Z. G. Wu, and J. H. Park, "Finite-time H_∞ asynchronous state estimation for discrete-time fuzzy Markov jump neural networks with uncertain measurements," *Fuzzy Sets and Systems*, 2018.

[25] J. Tao, Z.-G. Wu, H. Su, Y. Wu, and D. Zhang, "Asynchronous and resilient filtering for Markovian jump neural networks subject to extended dissipativity," *IEEE Transactions on Cybernetics*, pp. 1–10, 2018.

[26] X. Wu, Y. Tang, and W. Zhang, "Input-to-state stability of impulsive stochastic delayed systems under linear assumptions," *Automatica*, vol. 66, pp. 195–204, 2016.

[27] M. Fang and J. H. Park, "A multiple integral approach to stability of neutral time-delay systems," *Applied Mathematics and Computation*, vol. 224, pp. 714–718, 2013.

[28] P. Park, W. I. Lee, and S. Y. Lee, "Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems," *Journal of the Franklin Institute*, vol. 352, no. 4, pp. 1378–1396, 2015.

[29] H. Shen, X. Huang, J. Zhou, and Z. Wang, "Global exponential estimates for uncertain Markovian jump neural networks with reaction-diffusion terms," *Nonlinear Dynamics*, vol. 69, no. 1-2, pp. 473–486, 2012.

[30] P. G. Park, J. W. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, 2011.