Gravitational redshift, inertia, and the role of charge

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I argue that the gravitational redshift effect cannot be explained purely by way of uniformly accelerated frames, as sometimes suggested in the literature. This is due to the fact that in terms of physical effects a uniformly accelerated frame is not exactly equivalent to a homogeneous gravitational field let alone to a gravitational field of a point mass. In other words, the equivalence principle only holds in the regime of certain approximations (even in the case of uniform acceleration). The concepts in need of clarification are spacetime curvature, inertia, and the weak equivalence principle with respect to our understanding of gravitational redshift. Furthermore, I briefly discuss gravitational redshift effects due to charge.

Contents

1 Introduction

In 1911 Einstein foresaw a phenomenon thereafter known as 'gravitational redshift' [Einstein, 1911]. His thought experiment initiated the revolutionary idea that mass warps space and time. There exist some misconceptions, as evidenced in the literature, regarding the nature of the gravitational field in Einstein's General Theory of Relativity (GR) and how it relates to redshift effects. My aim is to give a consistent analysis of the gravitational redshift effect, in the hope of thereby advancing in some small measure our understanding of GR. Moreover, I will show that when charge is taken into account, gravitational redshift is subject to further corrections.

In the first part of this paper I shall derive and discuss the gravitational redshift in the framework of GR, from the equivalence principle, and from energy conservation principles to then compare and relate the different results. In the second part of this paper I shall examine effects on the redshift due to charge with some remarks on the relationship between GR and electromagnetism.

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2 Gravitational redshift

It is a straightforward task to derive the relative shift in coordinate time of two clocks in a given gravitational field with metric $g_{\mu\nu}$. Since we will employ some alternative approximate approaches to derive the gravitational redshift in the following sections, we shall choose to present the exact and most general derivation from GR first, variants of which are standard fare (see for example, [Wald, 2010, p. 136]).

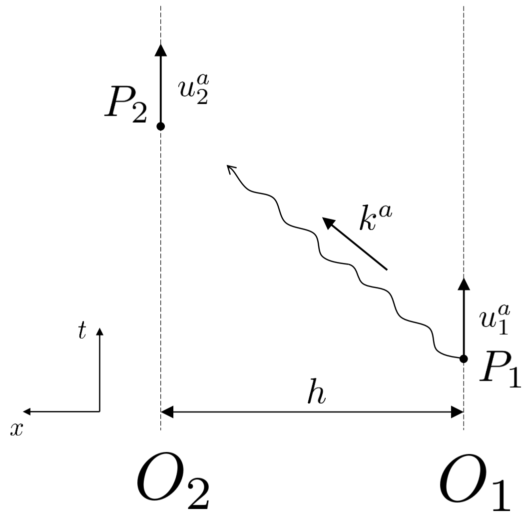

Figure 1: Two observers at different heights experience a time dilation effect in Earth's gravitational field. Emitter O_1 on the surface of the Earth sends a train of electromagnetic pulses from point P_1 with energy momentum 4-vector k^a to a receiver O_2, placed at point P_2, at height h above P_1. We assume O_1 and O_2 are static, i.e. their 4-velocities u_1^a and u_2^a are tangential to the Killing field $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$.

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Recall that for a given energy-momentum 4-vector $p^a = mu^a$ of a particle, with respect to a local inertial frame, the energy observed by an observer that moves with 4-velocity v^a is

$$E = -p^a v_a.^1 \tag{2.1}$$

Therefore, for the frequency ν_i of the photon observed by O_i, which moves with 4-velocity u_i^a, we find the relation $h\nu_i = E_k = -k_a u_i^a|_{P_i}$ (compare Equation 2.1), where E_k is the energy of the photon. By definition of the vector field ξ^a, we have $\xi_a\xi^a|_{P_i} = g_{00}|_{P_i}$ since ξ^a has vanishing spatial components. It would involve a fair amount of work to derive the gravitational redshift by finding the geodesic equation. However, this can be avoided by taking advantage of a useful proposition. Light travels on null geodesics (in the geometrical optics approximation, i.e. the spacetime scale of variation of the electromagnetic field is much smaller than that of the curvature), from which it follows that the inner product $k_a\xi^a$ is constant along geodesics, that is $k_a\xi^a|_{P_1} = k_a\xi^a|_{P_2}$.[2]

Spacetime around Earth (if considered as generated by a point mass M at $r = 0$) can be modelled by the Schwarzschild metric

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu = &-\left(1-\frac{r_S}{r}\right)c^2dt^2 \\ &+\left(1-\frac{r_S}{r}\right)^{-1}dr^2 \\ &+r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2), \end{aligned} \tag{2.2}$$

where

$$r_S = \frac{2GM}{c^2} \tag{2.3}$$

[1]In particular, if $u^a = v^a$, i.e. the particle's 4-velocity aligns with the observer's, then $E = -mv^a v_a = mc^2$.

[2]For a detailed proof see for instance, [Wald, 2010, p. 442]

is the so-called Schwarzschild radius, r the distance from the Earth's centre, G the gravitational constant, c the speed of light, and M the mass of the Earth. This yields

$$\frac{\nu_1}{\nu_2} = \frac{\sqrt{-\xi^b\xi_b}\Big|_{P_2}}{\sqrt{-\xi^b\xi_b}\Big|_{P_1}} = \frac{\sqrt{1-\frac{2GM}{c^2 r_2}}}{\sqrt{1-\frac{2GM}{c^2 r_1}}}$$
$$\approx 1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \approx 1 + \frac{gh}{c^2}, \quad (2.4)$$

or

$$\frac{\Delta\nu}{\nu} \approx \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \quad (2.5)$$

with $g := \frac{GM}{c^2 r_1^2}$ the gravitational constant at r_1, $\nu = \nu_1$, $\Delta\nu = \nu_1 - \nu_2$, and $r_2 - r_1 = h$. For the last approximation in the second last line we have used $\frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_2 r_1} \approx \frac{h}{r_1^2}$ if $r_1 \approx r_2$ and $r_1, r_2 \gg h$. Moreover, we used the approximations

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$
$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x. \quad (2.6)$$

Experimental tests of the gravitational redshift were first conducted by Cranshaw, Schiffer and Whitehead in the UK in 1960 [Cranshaw et al., 1960]. It was not clear whether significant conclusions could be drawn from their results. In the same year, the experiments by Pound and Rebka in Harvard successfully verified the gravitational redshift effect [Pound and Rebka Jr, 1960].

3 Uniformly accelerated frames and the equivalence principle

Einstein's equivalence principle (also called the weak equivalence principle) assumes that any experiment in a uniform gravitational field yields the same results as the analogous experiment performed in a frame removed from any source of gravitational field but moving in uniform accelerated motion with respect to an inertial frame [Norton, 1985].[3]

However, it is clear that Einstein was well aware of the mere linearly approximate validity of the equivalence principle when he wrote:

> '*...we arrive at a principle* [the equivalence principle] *which, if it is really true, has great heuristic importance. For by theoretical consideration of processes which take place relative to a system of reference with uniform acceleration, we obtain information as to the behaviour of processes in a homogeneous gravitational field. ... It will be shown in a subsequent paper that the gravitational field considered here is homogeneous only to a first approximation.*' [Einstein, 1911, p. 900]

The principle, thus, only holds in a 'small neighbourhood' of a point-like observer. Nonetheless, a treatment of the redshift effect in a uniform static gravitational field proves instructive, insofar as it shows that certain consequences of GR can be explained by means of geometry without resorting to gravitational fields. Dealing with uniform accelerations to derive the gravitational redshift, however, is a delicate business, and we shall see that the field, resulting from uniform (proper) acceleration, is not *uniform* if we demand a constant (proper) distance between emitter and observer!

We consider a spaceship that is uniformly accelerated. An emitter E and receiver R inside the spaceship, separated by a height h, compare frequencies of signals ascending the spaceship. For an illustration, see Figure 2.

As in the derivation of the gravitational redshift from the Schwarzschild metric, we let the observers describe their world-lines. It suffices to consider only one spatial dimension x. Acceleration a is measured in an inertial frame S with momentary velocity v relative to the inertial frame S' outside the spaceship, inside of which

[3] Note that [Brown and Read, 2016] use 'Einstein equivalence principle' to refer to what is often called the 'strong equivalence principle'.

the acceleration is measured to be a'.[4] Relativistic transformation of 3-acceleration gives

$$a = \gamma^3 a', \tag{3.1}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.[5]

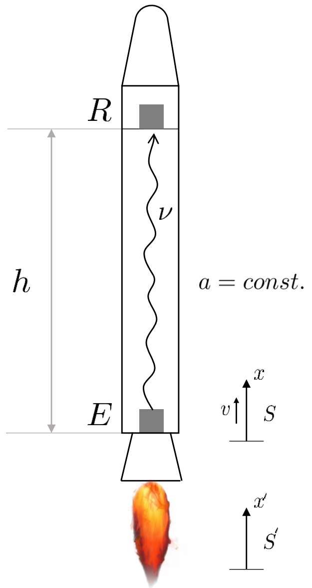

Figure 2: The gravitational redshift experiment in a uniformly accelerated spaceship. The redshift effect can be explained by the equivalence principle (to first order).

Note that the acceleration of the spaceship needs to be measured in the (momentary) inertial frame with instantaneous velocity v such that $a' = \frac{dv}{dt}$ (proper acceleration). With respect to the accelerated frame, sure enough, the ship's acceleration is zero. However, the principle of relativity — the requirement according to which the laws of physics take the same form in any inertial frame — no longer holds in accelerated, hence non-inertial, frames. Therefore, as expected, the two observers in the spaceship are going to feel a (pseudo)force $F = m_0 a$, where m_0 is the rest mass (invariant mass) of an object in the spaceship.

We want the (proper) acceleration a of the spaceship to be constant. The right hand side of Equation 3.1 is equal to $\frac{d}{dt}(\gamma v)$. Since a is constant we integrate Equation 3.1 twice to find the trajectory — so-called Rindler hyperbola — of a uniformly accelerated spacetime point as observed by the inertial frame S':

$$x(t) = \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2} + C, \tag{3.2}$$

with C a constant from integration. The second constant from the first integration was set to zero such that $v(0) = 0$. Without loss of generality we can also set $C = 0$. The result represents a hyperbolic path in Minkowski space, i.e.

$$x^2 - c^2 t^2 = \frac{c^4}{a^2}, \tag{3.3}$$

from which the term 'hyperbolic motion' is derived. We assume the back of the spaceship be subject to this motion. Note that $\dot{x} \overset{t\to\infty}{\to} c$, as expected.

We recover uniform acceleration in the Newtonian sense for $t \ll 1$. That is,

$$x(t) = x_0 + \frac{at^2}{2}, \tag{3.4}$$

with $x_0 = c^2/a$ the position at $t = 0$.

For an exact derivation, it would lead to inconsistencies to assume emitter and receiver be subject to the same Rindler hyperbola with only an additional spatial distance h in the coordinate x. For if we maintained a constant height between E and R relative to the inertial observing frame S', length contraction, as predicted by special relativity, would stretch the spaceship and eventually tear it apart (cf. also Bell's spaceship paradox in [Dewan and Beran, 1959] and [Bell, 1987, Chapter 9]). This is key. As was also pointed

[4] It is implicitly assumed that the proper time of co-moving clocks depends only on velocity and is independent of acceleration. This assumption is often called the Clock Hypothesis (see for example, [Brown and Read, 2016, Section 3]).

[5] To find the transformation of acceleration, one has to differentiate the spatial coordinates of the Lorentz transformation with respect to the time coordinates to first find the 3-velocity transformation (velocity-addition formula). Another differentiation of the velocities yields the transformation law for 3-acceleration.

out by [Alberici, 2006], assuming the gravitational acceleration to be the same for the top and bottom observers leads to all kinds of paradoxes. Most notably, it is not possible in this case to define a globally freely falling inertial frame because the corresponding metric would lead to a non-vanishing Riemann tensor, and hence curvature! The receiver R in the bow lying higher by height h with respect to the emitter E must follow the hyperbola

$$x^2 - c^2t^2 = \left(\frac{c^2}{a} + h\right)^2, \tag{3.5}$$

for the proper height (relative to S) to be constant. These are the two desiderata to simulate reasonably the gravitational redshift by uniform acceleration: First, the ship must have a constant acceleration; and second, the ship must have a constant proper height. The world-lines of emitter and receiver are denoted in Figure 3.

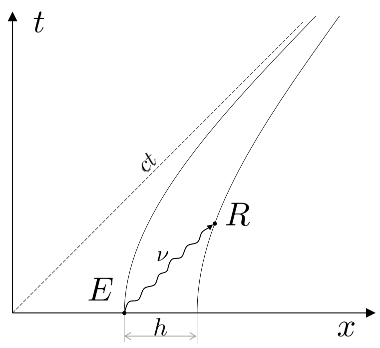

Figure 3: The world-lines of emitter E and receiver R are Rindler hyperbolae when experiencing constant proper acceleration.

Due to relativistic length contraction, the receiver's proper acceleration needs to be slightly greater. By comparing the two hyperbolae it immediately follows that the acceleration g_R of the receiver is related to the emitter's acceleration g_E by

$$g_R = \frac{g_E}{1 + \frac{g_E h}{c^2}}. \tag{3.6}$$

Compare also the treatment and related paradoxes in [Fabri, 1994]. Therefore, the gravitational field is not constant over the extended region of the spaceship. That is, however, not a surprise, for we would not expect the equivalence principle to hold globally in the first place. Further, it follows that proper time intervals along two different Rindler hyperbolae between two events having the same coordinate velocity are in a fixed proportion,

$$\frac{\tau_R}{\tau_E} = \frac{g_E}{g_R} = 1 + \frac{g_E h}{c^2}, \tag{3.7}$$

yielding the exact gravitational redshift formula for uniform acceleration. Alternatively, we can write

$$\nu_R = \frac{\nu_E}{1 + \frac{g_E h}{c^2}} = \nu_E \left(1 - \frac{g_R h}{c^2}\right) \tag{3.8}$$

for the corresponding observed frequencies, to highlight the dependence on the two different proper accelerations of emitter and receiver (cf. also the results in [Alberici, 2006]). From the preceding derivations we readily find for the (Rindler) metric of an accelerated frame

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{g_E x}{c^2}\right)^2 c^2 dt^2 - dx^2. \tag{3.9}$$

Thus, the gravitational redshift according to this metric reads

$$\frac{\nu_E}{\nu_R} = \frac{\Delta t_{x=h}}{\Delta t_{x=0}} = 1 + \frac{g_E h}{c^2}, \tag{3.10}$$

which is consistent with the first order approximation of the gravitational redshift from the Schwarzschild metric in Equation 2.4. Clocks at E and R, whose rates one wishes to compare, are permitted to describe their world-lines, i.e. Rindler hyperbolae, with respect to the inertial frame, and the value for the redshift is obtained by comparing the lengths of their world-lines in spacetime. Therefore, the treatment here is exact. The Rindler metric is, in fact, a solution to the vacuum Einstein field equations and has vanishing curvature ($R_{\mu\nu\rho\sigma} = 0$).

In the experiments of Pound-Rebka to confirm gravitational redshift, the emitter sends a signal at equal intervals on a clock at the surface of the Earth. The receiver measures the time interval between receipt of the signals on an identical clock at height h (see Figure 4).

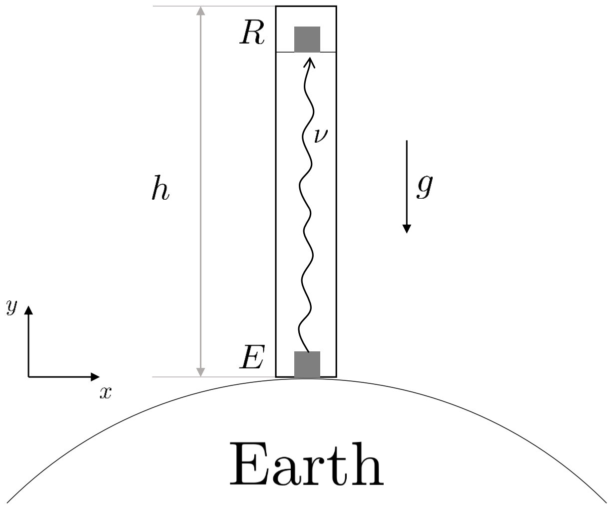

Figure 4: The Pound-Rebka experiment.

Merely when the experiment is taken to be at rest in a Rindler field, the equivalence principle implies that the relation between the clock times of emitter and receiver must be the same as if a spaceship were to accelerate vertically upwards in free space, as shown in Figure 2. The signals at the back are received at longer intervals than they are emitted because they are catching up with the accelerated bow of the spaceship and thus exhibit a Doppler shift. Note that the equivalence principle is a local law. Thus, in a field like that of the Earth it holds only approximately (to first order) for a small spacetime region.

4 Equivalence and gravitational redshift

Although GR is a well-established framework, it often occurs that its application amounts to an analysis that renders conclusions equivocal. This, in particular, happens to be the case for gravitational redshift. For instance, Brown and Read comment on the gravitational redshift effect as follows:

> *'The second possible misconception relates to the notion that gravitational redshift experiments provide evidence for spacetime curvature. They do, but contrary to what is claimed in some important modern textbooks on GR, a single gravitational redshift experiment does not require an explanation in terms of curvature. Rather, it is only multiple such experiments, performed at appropriately different locations in spacetime, that suggest curvature, via the notion that inertial frames are only defined locally, ...This "redshift" effect follows directly from the claim that the emitter and absorber are accelerating vertically at a rate of g m/s^2 relative to the (freely falling) inertial frames.'* [Brown and Read, 2016, p. 327, 329]

Here, Brown and Read assume the 'redshift' effect to be independent of 'tidal effects' (which is what they refer to as curvature). We have in fact already shown such a derivation is limited and does not fully account for gravitational redshift. There are indeed tidal effects in a single redshift experiment as outlined above in the most general derivation. Moreover, as we have seen, assuming both emitter and absorber to accelerate at the same rate is impossible given the two desiderata mentioned. However, they acknowledge there is nonetheless a connection between spacetime curvature and redshift experiments. This connection, to Brown and Read, amounts to the fact that redshift experiments carried out at different places on Earth reveal 'geodesic deviation' due to the spherical shape of the planet. That is, relative to a global freely falling frame at the site of one redshift experiment, a freely falling frame at another site is not moving inertially. Multiple gravitational redshift experiments thus require for their joint explanation the rejection of the global nature of inertial frames. Brown and Read think it is only geodesic deviation that reveals curvature. However, one experiment is sufficient to detect tidal effects of Earth's gravitational field. After all, an extended body will experience a stronger attraction on its nearest side than on its furthest side.

What Brown and Read deem to be a misconception, that is

> *'An explanation for the results of a single gravitational redshift experiment of Pound–Rebka type will appeal to a notion of spacetime curvature.'* [Brown and Read, 2016, p. 330],

is in fact one. However, this results not from an absence of curvature. Rather, since the Pound-Rebka experiment was solely designed to verify the first order effects predicted by GR, in this case a derivation via accelerated frames gives the desired result.

Brown and Read's proposal holds if the gravitational field of the Earth was assumed to be uniform, that is, independent of the radial distance from the centre of the earth, and also if $\frac{gh}{c^2} \ll 1$. In experiments involving larger spatial separations or stronger gravitational field variations, it is necessary to use the exact Schwarzschild solution of GR. By means of fully formed GR, of course, all approximations are bound to disappear. Incidentally, the second and higher order contributions to the redshift effect amounts to a correction of a magnitude of 10^{-9} relative to the first order result. Clearly, in the case of the Earth's gravitational field, a freely falling observer does not accelerate uniformly relative to a static frame on Earth, but rather her acceleration increases as she gets closer to the surface since the gravitational field strength increases.

I conclude with two clarifications. First, if the equivalence principle is to be used to explain the gravitational redshift, then it is important to realise that this can only be done to first order. Second, the quantitative results of Pound-Rebka can indeed be justified without appealing to spacetime curvature, but one should be aware that a complete theoretic description has to take into account the inhomogeneous gravitational field of Earth. After all, more sophisticated experiments with higher accuracy than those used by Pound and Rebka are in fact able to measure effects due to curvature in a single redshift experiment.[6] Although my considerations do not inhibit the successful comparison of the results of the Pound-Rebka experiment with first order calculation because higher order effects are beyond their measurement accuracy, they show that the qualitative explanation of the result requires to invoke spacetime curvature and an exact treatment of accelerations.

5 Redshift due to charge

5.1 The weight of photons

What Pound and Rebka call the 'weight of photons' in their experiments, in fact, aptly describes how Einstein originally had thought of gravitational redshift and what he had termed the inertia of energy.

5.1.1 Einstein's thought experiment

Let us go back to the thought experiment alluded to in the introduction. Einstein foresaw the gravitational redshift on the basis of a thought experiment using the 'inertia of energy' he had discovered in 1905 [Einstein, 1905], six years before his famous paper on relativity [Einstein, 1911]. A variant of this I shall spell out here.

Consider a test body of mass m_0 at rest at a height h, with a total energy $m_0c^2 + m_0gh$. The mass subsequently is dropped, and when it reaches the ground the total energy $\gamma m_0 c^2$ is obtained, where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and v the velocity of the mass at the ground (such that $m_0c^2 + m_0gh = \gamma m_0 c^2$). The mass is then transformed into a packet of radiation of energy $h\nu_1$, which is then sent from the ground back to height h, where the mass m_0 initially had been. There, the packet is transformed back into a mass m. By energy conservation, m must equal the mass m_0, which amounts to saying that $h\nu_2 = m_0c^2$, where ν_2 is the frequency of the packet at height h. See steps 1–4 in Figure 5.

From this we regain the first order approxima-

[6] Note that for the Schwarzschild metric $R = 0$ and $R_{\mu\nu} = 0$, but not all entries of the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ vanish.

tion in Equation 2.4;

$$\frac{\nu_1}{\nu_2} = \frac{m_0c^2 + m_0gh}{m_0c^2} = 1 + \frac{gh}{c^2}. \tag{5.1}$$

Equation 5.1 again involves an approximation of the exact redshift formula, for we assume a uniform gravitational field. Hence we use m_0gh for the energy of the test body. If we were to take into account the $\frac{1}{r}$-dependence of the gravitational potential, we would obtain

$$\begin{aligned}\frac{\nu_1}{\nu_2} &= \frac{m_0c^2 + \int_{r_2}^{r_1} F_N dr}{m_0c^2} \\ &= 1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \\ &\approx 1 + \frac{gh}{c^2},\end{aligned} \tag{5.2}$$

with F_N Newton's gravitational force of a massive central body.

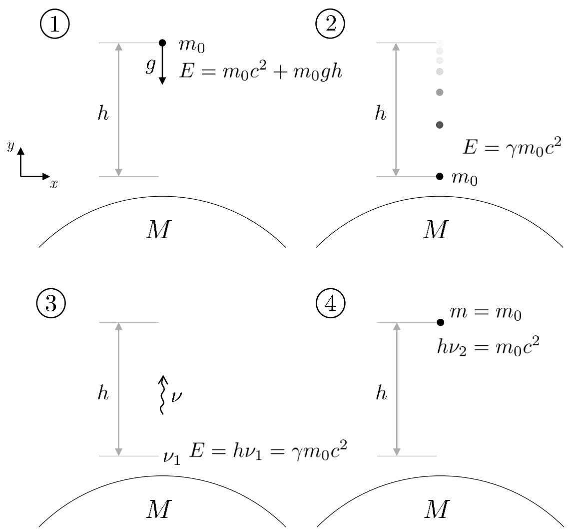

Figure 5: Gravitational redshift as a consequence of energy conservation. A test body of mass m_0 at rest at a height h is dropped. When it reaches the ground the total energy γm_0c^2 is obtained. The mass subsequently is transformed into a photon of energy $h\nu_1$, which is then sent from the ground back to height h. There, the photon is transformed back into a mass m. By energy conservation, m must equal the mass m_0, from which it follows that the photon's frequency must have decreased at its ascent.

Bear in mind that neither the derivation by means of uniformly accelerated frames nor the derivation by means of energy conservation yield the correct value for the gravitational redshift in the first line of Equation 2.4. The former holds in virtue of the inhomogeneity of Earth's gravitational field and the merely local validity of the equivalence principle. The latter is true because the Newtonian central body force law is an approximate limit of GR.

5.1.2 Inertia of energy

The approach of describing the redshift effect as a result of energy conservation suggests the following idea:

Any 'source' of energy causes clocks at different distances from the 'source' to exhibit time dilation effects.

As one example, charged particles attracted by a charged source should likewise be expected to give rise to redshift effects. We can, however, not follow the procedure from above and play the same game with charged bodies, replacing the Newtonian potential with the Coulomb potential. Consider a charged source Q and a test particle of charge q and mass m_0. We assume the mass of the source to be negligible. The charged particle falls under the attraction of the source according to the Coulomb force. When it reaches height r_1, a photon is created out of it and sent back to the particle's initial position, where it is transformed back into a mass m with charge q. For this process to happen, we can imagine annihilating the descending charge by an anti-charge of size $-q$ to create a photon (or actually at least two photons, which we can think of a single photon for the discussion). The photon is sent back, and when it reaches the top, the initial charge q plus its anti-charge $-q$ is created via pair-production. We assume the two particles have the same mass m. The anti-charge $-q$ subsequently is brought back to the bottom to restore the initial situation. It is precisely the energy contribution of this last step that cancels a redshift effect in the calculations, which is not further analysed here.

5.2 Reissner-Nordström metric

In fact, charge does give rise to redshift effects — and consequently time dilation — in the standard formalism of GR, though not, in a way analogous to how mass curves spacetime.

The Einstein equations for a charged point-like (non-rotating) mass with stress-energy tensor $T_{\mu\nu}$ reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{5.3}$$

We neglect the cosmological constant Λ. From this we obtain the Reissner-Nordström metric (cf. [Reissner, 1916])

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}\right) c^2 dt^2 \\ &+ \left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}\right)^{-1} dr^2 \\ &+ r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \end{aligned} \tag{5.4}$$

from which we recover the Schwarzschild metric in the limit $Q = 0$. It is worth mentioning that the charge term in the Reissner-Nordström metric affects geodesics of particles even though they may be uncharged. For $Q \neq 0$, this metric gives rise to an additional gravitational redshift. Similar to the derivation of gravitational redshift due to mass we obtain

$$\begin{aligned} \frac{\nu_1}{\nu_2} &= \frac{\sqrt{\left(1 - \frac{2GM}{c^2 r_2} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r_2^2}\right)}}{\sqrt{\left(1 - \frac{2GM}{c^2 r_1} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r_1^2}\right)}} \\ &\approx 1 + \frac{gh}{c^2} - \frac{g_{C_2}h}{c^2}, \end{aligned} \tag{5.5}$$

where g defined as before, and $g_{C_2} := \frac{GQ^2}{4\pi\varepsilon_0 c^2 r^3}$. The approximations are as in the case without charge (first order terms in h and large radii r_1, r_2).

The effect is quadratic in the charge Q, and, in fact, leads to a *blueshift* of the photon. Thus, it partly compensates the gravitational redshift due to mass. Note that gravity is fully 'geometrised' by GR. That is, geodesics of the metric fully describe the motion of test particles. Whereas for charged sources, the usual force terms from elecrodynamics need to be considered additionally in the geodesic equation.

6 Conclusion

There are several lessons to be drawn. Alternative derivations of gravitational redshift, as well as Einstein's historical ones, hold approximately only under special circumstances. Brown and Read used the equivalence principle, in conjunction with accelerated frames, and claimed the gravitational redshift to derive from this. Moreover, they supposed only multiple Pound-Rebka experiments to reveal curvature. If Brown and Read's statements about the inertial nature of gravitational redshift are valid in some sense, then only to first order and under certain approximations. And if valid in this sense, then only in Rindler spacetime, for it is the only framework that consistently treats uniform acceleration (and uniform gravitational fields, respectively), in which the Riemann curvature tensor vanishes globally. Arguments of equivalence can be used to infer a general relationship between redshift and acceleration, but the details are subtle. Thus, the conclusion that a single gravitational redshift effect on the Earth's surface can be derived from the equivalence principle and uniformly accelerated frames should be taken with a grain of salt. In experiments involving strong gravitational field variations only the exact Schwarzschild solution is accurate.

If the field generating source is charged, a blueshift effect arises, which is captured by the Reissner-Nordström metric. It seems not difficult to convince oneself that any 'source of energy' may likewise lead to further redshift effects. Instances of these are the strong and weak force, angular momentum, magnetic fields, and all other terms that contribute to the stress-energy tensor in Einstein's field equations.

Acknowledgements

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