Consistency and permission in deontic justification logic

FEDERICO L. G. FAROLDI, *Institute of Computer Science, University of Bern, Bern, Switzerland, and Research Foundation, Flanders (FWO)/Ghent University, Ghent, Belgium.*
E-mail: federico.faroldi@ugent.be

MEGHDAD GHARI, *Department of Philosophy, Faculty of Literature and Humanities, University of Isfahan, PO Box 8174673441, Isfahan, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), PO Box 19395-5746, Tehran, Iran.*
E-mail: m.ghari@ltr.ui.ac.ir

EVELINE LEHMANN, *Institute of Computer Science, University of Bern, Bern, Switzerland.*

THOMAS STUDER, *Institute of Computer Science, University of Bern, Bern, Switzerland.*

Abstract

Different notions of the consistency of obligations collapse in standard deontic logic. In justification logics, which feature explicit reasons for obligations, the situation is different. Their strength depends on a constant specification and on the available set of operations for combining different reasons. We present different consistency principles in justification logic and compare their logical strength. We propose a novel semantics for which justification logics with the explicit version of axiom **D**, **jd**, are complete for arbitrary constant specifications. Consistency is sometimes formulated in terms of permission. We therefore study permission in the context of justification logic, introducing a notion of free-choice permission for the first time. We then discuss the philosophical implications with regard to some deontic paradoxes.

Keywords: Justification logic, consistency of obligations, free-choice permission, completeness

1 Introduction

Deontic logic is the logic of obligations, permissions and sometimes other (primitive or derived) normative notions. What has emerged as the benchmark version, a system called standard deontic logic (SDL), is nothing more than KD, the smallest normal modal logic with the **D** axiom schema added. For an introduction and historical overview, see [20].

The **D** axiom is in place to ensure the consistency of obligations, but can take different formulations, for instance $\neg\mathcal{O}\bot$, or $\mathcal{O}A \to \neg\mathcal{O}\neg A$ or $\neg(\mathcal{O}A \land \mathcal{O}\neg A)$. In a normal modal logic, all these formulations are provably equivalent, and therefore it does not matter much which one is chosen. In a non-normal (but still classical) setting, for instance when an aggregation principle (such as $\mathcal{O}A \land \mathcal{O}B \to \mathcal{O}(A \land B)$) is missing, different versions of **D** are not interderivable, and it

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therefore matters which one is chosen, both for philosophical and for technical reasons (for deontic logic in a paraconsistent setting, see e.g. [10]). One might want to distinguish, conceptually, between an obligation for an impossible or logically contradictory state of affairs ($\mathcal{O}\bot$) on one hand, and multiple obligations for jointly inconsistent states of affairs ($\mathcal{O}A \wedge \mathcal{O}\neg A$) on the other, because the former might thought to be self-defeating or conceptually impossible, whereas the latter can derive from different background or contingent normative systems (e.g. ethics and the law) and are only practically unenforceable, but logically possible. Moreover, SDL and its variants in the standard modal language lack the power to distinguish the source of one situation (an obligation for the impossible) from the source of the other (inconsistent obligations), or to exclude one situation for logical reasons and admit the other for contingent reasons.

Justification logic [3, 31] is an explicit version of modal logic originally developed to provide a logic of proofs [2, 30]. Instead of formulas such as $\Box A$, the language of justification logic includes formulas such as $t : A$ saying, for instance, that *t justifies knowledge of A* or *A is obligatory because of reason t*, where t is a term representing the reason. Systems of justification logic are parameterized by a so-called *constant specification* that states which logical axioms do have a justification. Hence the constant specification can be used to calibrate the strength of a justification logic. Of particular interest are axiomatically appropriate constant specifications where every axiom has a justification. In that case the justification logic enjoys a constructive analogue of the modal necessitation rule. (See Sect. 3 for a formal definition of constant specification).

The explicit counterpart in justification logic of one version of axiom **D** (in standard modal logic) was first formulated by Brezhnev [6] as axiom **jd**, i.e. $\neg(t : \bot)$. This axiom turned out to be rather notorious. Usually one can establish completeness of a justification logic for an arbitrary constant specification. However, in the presence of **jd**, this is not the case. Systems that include **jd** usually need an axiomatically appropriate constant specification in order to be complete. Kuznets [24] defined M-models for justification logics with **jd** and Pacuit [40] presented F-models for **jd**. Modular models for **jd** have been studied in [28] and subset models for **jd** are introduced in [32]. For all these different semantics, an axiomatically appropriate constant specification is required in order to obtain a completeness result. Notable exceptions to this phenomenon are M-models (defined in [24]) and Fk-models (defined in [25]) for which completeness holds for arbitrary constant specifications.

The requirement of an axiomatically appropriate constant specification is often overlooked. In particular, this requirement is omitted in the completeness theorems given in [14] and [32] (although for the latter paper it seems that it has been corrected later [33]). In the case of [14] the appropriateness requirement is important since the solution to avoid some of the known paradoxes, such as Ross', is to restrict the constant specification. But then the resulting justification logic is not complete anymore.

In this paper we study in detail the **jd** axiom and related principles of consistency. The paper is split in two parts: the first focuses on consistency of obligations, the second on permission, and especially free choice permission. In the first part, we compare their logical strength and we investigate the role of the constant specification. After an informal discussion, we present the basic syntax and semantics of system JD (with axiom **jd**). We propose a novel semantics for which justification logics with **jd** are complete for arbitrary constant specifications. We consider system JNoC, which has a different version of the consistency axiom, **noc**. We establish that various formulations of consistency are equivalent only with an axiomatically appropriate constant specification. We put forward some philosophical remarks on the role of justification logic when it comes to the study of impossible and inconsistent obligations. In the second part, we introduce and study a modality of free choice permission, and show how it preserves consistency, respects desiderata and avoids well-known paradoxes. We discuss the issues of the dual of free-choice permission and weak obligation. We conclude with a brief discussion and a plan for future work.

2 Impossible vs inconsistent obligations: an overview

Standard (implicit) systems of deontic logic conflate impossible and conflicting obligations. One thing is to say that nothing logically impossible can be obligatory, i.e. $\neg\mathcal{O}\bot$, another to say that there are not (or there should not be) conflicting provisions that are obligatory, i.e. $\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$. Standard systems can derive $\mathcal{O}\bot$ from $\mathcal{O}A \wedge \mathcal{O}\neg A$ and vice versa, thus suffering a collapse. One way to see the difference is that the former might be argued to be unacceptable for conceptual or logical reasons (e.g. that such an obligation would be conceptually self-defeating), whereas the latter might be argued to be unacceptable for contingent reasons (e.g. that such obligations cannot be fulfilled in reality, although can potentially still arise in real-life situations). [8, 9] use minimal models, [41] uses multiple accessibility relations in the disjunctive truth condition of the ought operator: in such ways the authors avoid aggregation and therefore the collapse of impossible to inconsistent obligations (multi-relational semantics has also been used more recently, cf. e.g. [7]).

In justification logic we have the explicit counterparts **jd**: $\neg(t : \bot)$ and **noc**: $\neg(t : A \wedge t : \neg A)$, respectively, of the above implicit principles, giving rise to systems we call JD and JNoC (respectively). Corollary 8.2 establishes that the former implies the latter. Lemma 8.5 shows that the converse direction holds in the presence of an axiomatically appropriate constant specification. In this situation we have the same collapse as in the standard implicit systems. There are two options to avoid this consequence:

- In justification logic we can use the constant specification to adjust the power of the logical systems and thus avoid the collapse. Lemma 8.6 shows that $\neg(t : A \wedge t : \neg A)$ does not imply $\neg(t : \bot)$ if the constant specification is not axiomatically appropriate. Theorems 5.3 and 7.3 prove that JD and JNoC with an arbitrary constant specification are complete with regard to a novel semantics we develop.
- As explained in Remark 8.7, we can avoid the collapse even in the presence of an axiomatically appropriate CS. It suffices to consider a language without the $+$ operation. We denote this system JNoC^-.

Avoiding this collapse is important in situations with conflicting obligations. Let us look at Sartre's Dilemma [36] as presented in [39]:

(1) It is obligatory that I now meet Jones (say, as promised to Jones, my friend).
(2) It is obligatory that I now do not meet Jones (say, as promised to Smith, another friend).

In implicit SDL featuring the principle $\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$, we immediately get a contradiction if we represent (1) and (2) as $\mathcal{O}A$ and $\mathcal{O}\neg A$, respectively. However, in a system such as JNoC^-, there is no conflict as there are two different reasons in (1) and (2). Hence (1) and (2) are represented by $s : A$ and $t : \neg A$ for two different terms s and t, which is consistent with axiom **noc**.

Moreover, in normal deontic logic, one can pass from *two* inconsistent obligations to *one* impossible obligation. This is dubious on philosophical grounds: we have pointed out that one may argue that one impossible obligation is conceptually self-defeating, whereas two inconsistent obligations may be in place for contingent reasons (e.g. different promises).

Justification logic gives us the means to not conflate the two, without loosing much reasoning power. Even more, one can do justice to the background philosophical intuitions to exclude impossible obligations for logical reasons, for instance by focusing on the system JNoC and calibrating the constant specification. Keeping track of the source of obligations, for instance through reasons, opens up the possibility to solve conflicts if one has a priority ordering on reasons (see for instance [21], and [13] for an implementation in justification logic).[1]

In the rest of the paper we present the formal results starting from the basic syntax and semantics of system JD (with axiom **jd**).

3 Syntax

Justification terms are built from countably many constants c_i and variables x_i according to the following grammar:

$$t ::= c_i \mid x_i \mid t \cdot t \mid (t + t) \mid\, !\, t.$$

The set of all terms is denoted by Tm.

Formulas are built from countably many atomic propositions P_i and the symbol $\bot$ according to the following grammar:

$$F ::= P_i \mid \ \bot \ \mid F \to F \mid t : F.$$

The set of all atomic propositions is denoted by Prop and the set of all formulas is denoted by $\mathcal{L}_J$. The other classical Boolean connectives $\neg, \top, \wedge, \vee, \leftrightarrow$ are defined as usual, in particular we have $\neg A := A \to \bot$ and $\top := \neg\bot$. Informally, $+$ mimics the aggregation of reasons, $\cdot$ embodies modus ponens reasoning and ! is positive introspection. We keep ! for ease of exposition, but it can be dispensed with. For a discussion on the interpretation of the operations in a deontic context, see [14].

The axioms of JD are the following[2] :

cl	all axioms of classical propositional logic;
j+	$s : A \vee t : A \to (s + t) : A$;
j	$s : (A \to B) \to (t : A \to s \cdot t : B)$;
jd	$\neg(t : \bot)$.

Note that since $\neg$ is a defined notion, **jd** actually stands for $t : \bot \to \bot$.

Justification logics are parameterized by a so-called constant specification, which is a set

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom of } \mathsf{JD}\}.$$

Our logic $\mathsf{JD}_{\mathsf{CS}}$ is now given by the axioms of JD and the rules modus ponens:

$$\frac{A \qquad A \to B}{B}\ (\text{MP})$$

[1]Normative reasoning with priority has, of course, a long history. [37], for instance, presents a framework to capture prioritized normative reasoning with conflicts and exceptions in an argumentation framework. While very important, we do not think we can consider it adequately in the context of the present paper, which is mostly concerned with explicit reasons and permission expressed in the framework of justification logic.

[2]The formulation of axioms **j+** and **j** as implications (instead of equivalences) has historical reasons in the semantics for the Logic of Proofs. The formulation as implication corresponds to a closure condition on the models whereas an equivalence would additionally require a leastness condition. That would be much stronger and is usually not needed. Equivalence formulations are studied in the context of dynamic epistemic justification logics, see, e.g. [29], and in the context intuitionistic justification logic [38].

and axiom necessitation

$$\frac{}{\underbrace{!\ldots!}_{n}\, c : \underbrace{!\ldots!}_{n-1}\, c : \ \ldots : !\,!\,c : !\,c : c : A} \text{(AN!)} \quad \forall n \in \mathbb{N}, \text{ where } (c, A) \in \mathsf{CS}$$

DEFINITION 3.1 (Axiomatically appropriate CS).
A constant specification CS is called *axiomatically appropriate* if for each axiom A, there is a constant c with $(c, A) \in \mathsf{CS}$.

Axiomatically appropriate constant specifications are important as they provide a form of necessitation. A proof of the following lemma can be found in [2, 3, 31].

LEMMA 3.2
Let CS be an axiomatically appropriate constant specification. For each formula A with

$$\mathsf{JD}_{\mathsf{CS}} \vdash A,$$

there exists a term t such that

$$\mathsf{JD}_{\mathsf{CS}} \vdash t : A.$$

It is not difficult to show that the Deduction Theorem holds for $\mathsf{JD}_{\mathsf{CS}}$ (and also holds for all logics introduced in the next sections).

4 Semantics

We recall the basic definitions and results about subset models for justification logic [32, 34, 35].

DEFINITION 4.1
[General subset model] Given some constant specification CS, then a general CS-subset model $\mathcal{M} = (W, W_0, V, E)$ is defined by:

- W is a non-empty set of objects called worlds.
- $W_0 \subseteq W$ and $W_0 \neq \emptyset$.
- $V : W \times \mathcal{L}_J \to \{0, 1\}$ such that for all $\omega \in W_0$, $t \in \mathsf{Tm}$, $F, G \in \mathcal{L}_J$:
  - $V(\omega, \bot) = 0$;
  - $V(\omega, F \to G) = 1$ iff $V(\omega, F) = 0$ or $V(\omega, G) = 1$;
  - $V(\omega, t : F) = 1$ iff $E(\omega, t) \subseteq \{\upsilon \in W | V(\upsilon, F) = 1\}$.
- $E : W \times \mathsf{Tm} \to \mathcal{P}(W)$ that meets the following conditions where we use

$$[A] := \{\omega \in W \mid V(\omega, A) = 1\}. \tag{1}$$

  For all $\omega \in W_0$, and for all $s, t \in \mathsf{Tm}$:
  - $E(\omega, s + t) \subseteq E(\omega, s) \cap E(\omega, t)$;
  - $E(\omega, s \cdot t) \subseteq \{\upsilon \in W \mid \forall F \in \mathsf{APP}_\omega(s, t)(\upsilon \in [F])\}$ where APP contains all formulas that can be justified by an application of s to t, see below;
  - $\exists \upsilon \in W_0$ with $\upsilon \in E(\omega, t)$;

- for all $n \in \mathbb{N}$ and for all $(c, A) \in \mathsf{CS} : E(\omega, c) \subseteq [A]$ and

$$E(\omega, \underbrace{! \ldots !}_{n}\, c) \subseteq [\underbrace{! \ldots !}_{n-1}\, c : \ldots .!\, c : c : A].$$

The set APP is formally defined as follows:

$$\mathsf{APP}_\omega(s,t) := \{F \in \mathcal{L}_J \mid \exists H \in \mathcal{L}_J \text{ s.t. } E(\omega, s) \subseteq [H \to F] \text{ and } E(\omega, t) \subseteq [H]\}.$$

$W_0$is the set of *normal* worlds. The set $W \setminus W_0$ consists of the *non-normal* worlds. Note that there are no conditions on functions V and E for non-normal worlds. Moreover, using the notation introduced by (1), we can read the condition on V for justification formulas $t : F$ as:

$$V(\omega, t : F) = 1 \quad \text{iff} \quad E(\omega, t) \subseteq [F].$$

In subset semantics terms are not treated only syntactically (as in most other semantics for justification logics), but they get assigned a set of worlds.

$E(\omega, t)$ tells us the states that are ideal according to t from ω's perspective. Then $t : F$ is true at ω just in case F is true at those ideal states. We have seen that a formula of the form $t : F$ is true at a world ω just in case the interpretation of t at ω (a set of worlds) is a subset of the truth set of F (the set of worlds where F is true). However, take two axioms A and B. They are true in all possible (or normal) worlds. Therefore, every term that is a reason for the former will also be a reason for the latter (if terms get assigned sets of *possible* worlds). But in this way, there is no control on the constant specification. Using *impossible* (or non-normal) worlds, however, let us solve this problem, because at impossible worlds classical logically equivalent propositions can differ in truth value, and a justification for one may not be a justification for the other. This makes the semantics able to capture hyperintensionality.

Since the valuation function V is defined on worlds and formulas, the definition of truth is standard.

DEFINITION 4.2 (Truth).
Given a subset model

$$\mathcal{M} = (W, W_0, V, E)$$

and a world $\omega \in W$ and a formula F we define the relation $\Vdash$ as follows:

$$\mathcal{M}, \omega \Vdash F \quad \text{iff} \quad V(\omega, F) = 1.$$

Validity is defined with respect to the normal worlds.

DEFINITION 4.3 (Validity).
Let CS be a constant specification. We say that a formula F is *general* CS*-valid* if for each general CS-subset model

$$\mathcal{M} = (W, W_0, V, E)$$

and each $\omega \in W_0$, we have $\mathcal{M}, \omega \Vdash F$.

As expected, we have soundness [32].

THEOREM 4.4 (Soundness).
Let CS be an arbitrary constant specification. For each formula F we have that if $\mathsf{JD}_{\mathsf{CS}} \vdash F$, then F is general CS-valid.

However, completeness only holds if the constant specification is axiomatically appropriate [33].

THEOREM 4.5 (Completeness).
Let CS be an axiomatically appropriate constant specification. For each formula F we have that if F is general CS-valid, then $\mathsf{JD}_{\mathsf{CS}} \vdash F$.

One might need more control on the constant specification, e.g. by relinquishing the requirement that each axiom be justified. For instance, [14] argued that restricting the constant specification is one way to avoid certain deontic paradoxes, such as Ross'. In the next section, we prove soundness and completeness with regard to an arbitrary constant specification.

5 D-arbitrary subset models

We present a novel class of subset models for JD and establish soundness and completeness.

DEFINITION 5.1 (D-arbitrary subset model).
A *D-arbitrary* CS*-subset model* $\mathcal{M} = (W, W_0, V, E)$ is defined like a general CS-subset model with the condition

$$\exists \upsilon \in W_0 \text{ with } \upsilon \in E(\omega, t)$$

being replaced with

$$\exists \upsilon \in W_{\not\perp} \text{ with } \upsilon \in E(\omega, t)$$

where $W_{\not\perp} := \{\omega \in W \mid V(\omega, \perp) = 0\}$.

The notion of D-arbitrary CS-validity is now as expected.

DEFINITION 5.2 (D-arbitrary validity).
Let CS be a constant specification. We say that a formula F is *D-arbitrary* CS*-valid* if for each D-arbitrary CS-subset model $\mathcal{M} = (W, W_0, V, E)$ and each $\omega \in W_0$, we have $\mathcal{M}, \omega \Vdash F$.

We have soundness and completeness with respect to arbitrary constant specifications.

THEOREM 5.3 (Soundness and completeness).
Let CS be an arbitrary constant specification. For each formula F we have

$$\mathsf{JD}_{\mathsf{CS}} \vdash F \quad \text{iff} \quad F \text{ is D-arbitrary } \mathsf{CS}\text{-valid.}$$

The completeness proof is by a canonical model construction as in the case of general subset models [32]. We will only sketch main steps here. The canonical model is given as follows.

DEFINITION 5.4 (Canonical model).
Let CS be an arbitrary constant specification. We define the canonical model $\mathcal{M}^C = (W^C, W_0^C, V^C, E^C)$ by:

- $W^C = \mathcal{P}(\mathcal{L}_J)$.
- $W_0^C = \{\Gamma \in W^C | \Gamma$ is maximal $\mathsf{JD}_{\mathsf{CS}}$-consistent set of formulas$\}$.
- $V^C(\Gamma, F) = 1 \quad \text{iff} \quad F \in \Gamma$;
- $E^C(\Gamma, t) = \{\Delta \in W^C | \Delta \supseteq \Gamma/t\}$ where

$$\Gamma/t := \{F \in \mathcal{L}_J \mid t : F \in \Gamma\}.$$

The essential part of the completeness proof is to show that the canonical model is a D-arbitrary CS-subset model.

LEMMA 5.5
Let CS be an arbitrary constant specification. The canonical model $\mathcal{M}^C$ is a D-arbitrary CS-subset model.

PROOF. Let us only show the condition

$$\exists \upsilon \in W_{\not\perp}^C \text{ with } \upsilon \in E(\omega, t) \tag{2}$$

for all $\omega \in W_0$ and all terms t.

So let t be an arbitrary term and $\Gamma \in W_0^C$. Since Γ is a maximal $\mathsf{JD}_{\mathsf{CS}}$-consistent set of formulas, we find $\neg(t : \perp) \in \Gamma$ and thus $t : \perp \notin \Gamma$. Let $\Delta := \Gamma/t$. We find that $\perp \notin \Delta$ and by definition $V^C(\Delta, \perp) = 0$. Thus $\Delta \in W_{\not\perp}^C$. Moreover, again by definition, $\Delta \in E^C(\Gamma, t)$. Thus (2) is established. ∎

Now the Truth lemma and the completeness theorem follow easily as in [32].

6 Reducing application to sum

In subset models, it is possible to reduce application to sum by introducing a new term constant $\mathsf{c}^\star$, see [32]. We let JD be the background logic in this section. We define **$\mathsf{c}^\star$-term** inductively as follows:

- $\mathsf{c}^\star$ is a **$\mathsf{c}^\star$-term**;
- if s and t are terms and c is a **$\mathsf{c}^\star$-term** then $s + c$ and $c + t$ are **$\mathsf{c}^\star$-terms**.

So a **$\mathsf{c}^\star$-term** is either $\mathsf{c}^\star$ itself or a sum-term where $\mathsf{c}^\star$ occurs at least once. Instead of axiom **j**, we consider the axiom **jc*** given by

$$c : A \wedge c : (A \rightarrow B) \rightarrow c : B, \quad \text{for all } \mathsf{c}^\star\text{-terms } c,$$

and define $s \cdot t$ as $s + t + \mathsf{c}^\star$. Lemma 8 in [33] shows how then axiom **j** follows.

On the semantic side, D-arbitrary subset models are required to satisfy $E(\omega, \mathsf{c}^\star) \subseteq W_{MP}$ (for ω in W_0) where

$$W_{MP} := \{\omega \in W \mid \forall A, B \in \mathcal{L}_J \; ((V(\omega, A) = 1 \text{ and } V(\omega, A \rightarrow B) = 1) \text{ implies } V(\omega, B) = 1)\}.$$

Our completeness results also hold in the setting with $\mathsf{c}^\star$. However, the proof that the canonical model is well defined is substantially more complicated.

In the definition of the canonical model we use

$$E^C(\Gamma, t) = \{\Delta \in W^C_{MP} | \Delta \supseteq \Gamma/t\} \text{ if } t \text{ is a } \mathbf{c^\star\text{-term}}$$
$$E^C(\Gamma, t) = \{\Delta \in W^C | \Delta \supseteq \Gamma/t\} \text{ otherwise,}$$

with

$$W^C_{MP} := \{\Gamma \in W^C | \forall A, B \in \mathcal{L}_J : \text{ if } A \to B \in \Gamma \text{ and } A \in \Gamma \text{ then } B \in \Gamma\}.$$

To show that the canonical model indeed is a model we have to show the condition

$$\exists \upsilon \in W^C_{\not\perp} \text{ with } \upsilon \in E(\omega, t) \tag{3}$$

for all $\omega \in W_0$ and all terms t. For terms that are not $\mathsf{c}^\star$**-terms** this is done as in the proof of Lemma 5.5.

It remains to show the case when t is a $\mathsf{c}^\star$**-term**. Define $\Delta := \Gamma/t$. As in Lemma 5.5 we find $\Delta \in W^C_{\not\perp}$. To show that $\Delta \in E(\omega, t)$, it is enough to show $\Delta \in W^C_{MP}$. Assume $A \to B \in \Delta$ and $A \in \Delta$. By the definition of Δ, we obtain

$$t : (A \to B) \in \Gamma \quad \text{and} \quad t : A \in \Gamma.$$

Using axiom $\mathbf{jc}^\star$ and the fact that Γ is maximal consistent, we get $t : B \in \Gamma$ and thus $B \in \Delta$. Therefore $\Delta \in W^C_{MP}$, which establishes (3) for t being a $\mathsf{c}^\star$**-term**.

7 No conflicts

So far, we have considered the explicit version of $\neg\mathcal{O}\bot$. In normal modal logic, this is provably equivalent to $\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$.[3] In this section we study the explicit version of this principle, which we call NoC (*No Conflicts*), saying that reasons are self-consistent. That is A and $\neg A$ cannot be obligatory for one and the same reason. The axioms of JNoC are the axioms of JD where **jd** is replaced with:

noc $\neg(t : A \wedge t : \neg A)$.

Accordingly, a constant specification for JNoC is defined like a constant specification for JD except that the constants justify axioms of JNoC.

Given a constant specification CS for JNoC, the logic $\mathsf{JNoC}_{\mathsf{CS}}$ is defined by the axioms of JNoC and the rules modus ponens and axiom necessitation.

DEFINITION 7.1 (NoC subset model).
A *NoC CS-subset model*

$$\mathcal{M} = (W, W_0, V, E)$$

is defined like a general CS-subset model with the condition

$$\exists \upsilon \in W_0 \text{ with } \upsilon \in E(\omega, t)$$

[3] Via and aggregation and distribution axiom, such as $\mathcal{O}A \wedge \mathcal{O}B \leftrightarrow \mathcal{O}(A \wedge B)$.

being replaced with

$$\exists \upsilon \in W_{\mathsf{nc}} \text{ with } \upsilon \in E(\omega, t)$$

where $W_{\mathsf{nc}} := \{\omega \in W \mid \text{for all formulas A } (V(\omega, A) = 0 \text{ or } V(\omega, \neg A) = 0)\}$.

The notion of NoC CS-validity is now as expected.

DEFINITION 7.2 (NoC validity).
Let CS be a constant specification. We say that a formula F is *NoC CS-valid* if for each NoC CS-subset model $\mathcal{M} = (W, W_0, V, E)$ and each $\omega \in W_0$, we have $\mathcal{M}, \omega \Vdash F$.

THEOREM 7.3 (Soundness and completeness).
Let CS be an arbitrary constant specification. For each formula F we have

$$\mathsf{JNoC}_{\mathsf{CS}} \vdash F \quad \text{iff} \quad F \text{ is NoC } \mathsf{CS}\text{-valid.}$$

Again the completeness proof uses the canonical model construction from Definition 5.4 except that we set

- $W_0^C = \{\Gamma \in W^C | \Gamma \text{ is maximal } \mathsf{JNoC}_{\mathsf{CS}}\text{-consistent set of formulas}\}$.

Now we have to show that the defined structure is an NoC CS-subset model.

LEMMA 7.4
Let CS be an arbitrary constant specification. The canonical model $\mathcal{M}^C$ is an NoC CS-subset model.

PROOF. As before, we only show the condition

$$\exists \upsilon \in W_{\mathsf{nc}}^C \text{ with } \upsilon \in E(\omega, t) \tag{4}$$

for all $\omega \in W_0$ and all terms t.

So let t be an arbitrary term and $\Gamma \in W_0^C$. Let A be an arbitrary formula. Since Γ is a is maximal $\mathsf{JNoC}_{\mathsf{CS}}$-consistent set of formulas, we find

$$\neg(t : A \wedge t : \neg A) \in \Gamma$$

and thus $t : A \wedge t : \neg A \notin \Gamma$. Thus, again by maximal consistency,

$$t : A \notin \Gamma \text{ or } t : \neg A \notin \Gamma.$$

Let $\Delta := \Gamma / t$. We find that

$$A \notin \Delta \text{ or } \neg A \notin \Delta$$

and hence, by definition,

$$V^C(\Delta, A) = 0 \text{ or } V^C(\Delta, \neg A) = 0.$$

Thus $\Delta \in W_{\mathsf{nc}}^C$. Moreover, again by definition, $\Delta \in E^C(\Gamma, t)$. Thus (4) is established. □

Again the truth lemma and the completeness theorem follow easily.

8 Formal comparison

In the following we will study the exact relationship of JD and JNoC. The reasoning strength of a justification logic can be calibrated via the constant specification. Hence the relationship between various principles and formulas usually depends on the constant specification. We start with a simple example before we consider the case of JD and JNoC.

EXAMPLE 8.1
Let A and B be two formulas such that $A \to B$ is a logical validity, i.e. $A \to B$ is provable. If we work with a very weak (or even empty) constant specification, it may happen that there exists a justification term s such that $s : A$ is true but $t : B$ does not hold for any justification term t. Thus $s : A \to t : B$ will be false for each term t although $A \to B$ is a valid formula.

If we work with an axiomatically appropriate constant specification, such a situation is no longer possible. Since $A \to B$ is provable, we find by Lemma 3.2 a term r such that $r : (A \to B)$. Thus by axiom **j**, we obtain $s : A \to r \cdot s : B$.

Considering the case of our consistency principles, we will see, for instance, that $\neg(s : A \wedge t : \neg A)$ can be true for all terms s and t while at the same time $r : \bot$ holds for some term r. The reason is that even though $\bot \to A$ and $\bot \to \neg A$ are logical validities, it is not possible to derive $s : A$ or $t : \neg A$ for any terms s and t from a given $r : \bot$ if the constant specification is weak.

However, Lemma 3.2 is a key property for many applications of justification logic. Thus one often would like to work with an axiomatically appropriate constant specification. We will see that dropping the $+$ operation makes it possible to have an axiomatically appropriate constant specification, self-consistent reasons and reasons for contradicting obligations in one model.

The following lemmas establish the exact relationship between JD and JNoC. First we show that $\mathsf{JD}_{\mathsf{CS}}$ proves that reasons are consistent among them, i.e. that $\neg(s : A \wedge t : \neg A)$ holds for arbitrary terms s and t, which is the consistency principle used in [14].

LEMMA 8.2
Let CS be an arbitrary constant specification. Then $\mathsf{JD}_{\mathsf{CS}}$ proves $\neg(s : A \wedge t : \neg A)$ for all terms s, t and all formulas A.

PROOF. Suppose towards a contradiction that $s : A \wedge t : \neg A$. Thus we have $s : A$ and $t : \neg A$ where the latter is an abbreviation for $t : (A \to \bot)$ (by the definition of the symbol $\neg$). Thus using axiom **j**, we get $t \cdot s : \bot$ and by axiom **jd** we conclude $\bot$. Hence from the assumption $s : A \wedge t : \neg A$ we prove $\bot$. Therefore, by the Deduction Theorem, we get $(s : A \wedge t : \neg A) \to \bot$. Finally, by the definition of $\neg$, we obtain $\neg(s : A \wedge t : \neg A)$. □

COROLLARY 8.3
For any constant specification CS, $\mathsf{JD}_{\mathsf{CS}}$ proves every instance of **noc**.

REMARK 8.4
It is only by coincidence that Lemma 8.2, and thus also Corollary 8.3, hold for arbitrary constant specifications. If we base our propositional language on different connectives (say $\wedge$ and $\neg$ instead of $\to$ and $\bot$), then Lemma 8.2 and Corollary 8.3 only hold for axiomatically appropriate constant specifications.

The proof of Lemma 8.2 is as follows. Since CS is axiomatically appropriate, there exists a term r such that

$$r : (\neg A \to (A \to \bot)) \tag{5}$$

is provable where $\bot$ is defined as $P \wedge \neg P$ (for some fixed P) and $F \to G$ is defined as $\neg(F \wedge \neg G)$. From (5) and axiom **j** we get

$$t : \neg A \to r \cdot t : (A \to \bot).$$

Thus from $s : A \wedge t : \neg A$, we obtain $(r \cdot t) \cdot s : \bot$, which contradicts axiom **jd** as before.

Next we show that also $\mathsf{JNoC}_{\mathsf{CS}}$ proves that reasons are consistent among them.

LEMMA 8.5
Let CS be an arbitrary constant specification. Then $\mathsf{JNoC}_{\mathsf{CS}}$ proves $\neg(s : A \wedge t : \neg A)$ for all terms s, t and all formulas A.

PROOF. Suppose towards a contradiction that $s : A \wedge t : \neg A$ holds. Using axiom **j+** we immediately obtain $s + t : A \wedge s + t : \neg A$. By axiom **noc** we conclude $\bot$, which by the Deduction Theorem establishes $\neg(s : A \wedge t : \neg A)$. □

Next we show that $\mathsf{JNoC}_{\mathsf{CS}}$ with an axiomatically appropriate constant specification proves $\neg(t : \bot)$.

LEMMA 8.6
Let CS be an axiomatically appropriate constant specification. Then $\mathsf{JNoC}_{\mathsf{CS}}$ proves $\neg(t : \bot)$ for each term t.

PROOF. Because CS is axiomatically appropriate, there are terms r and s such that

$$r : (\bot \to P) \quad \text{and} \quad s : (\bot \to \neg P).$$

Therefore, we get

$$t : \bot \to r \cdot t : P \quad \text{and} \quad t : \bot \to s \cdot t : \neg P.$$

Thus we have $t : \bot \to (r \cdot t : P \wedge s \cdot t : \neg P)$. Together with the previous lemma, this yields $t : \bot \to \bot$, which is $\neg(t : \bot)$. □

Here the requirement of an axiomatically appropriate constant specification is necessary.

LEMMA 8.7
There exists a NoC CS-subset model $\mathcal{M} = (W, W_0, V, E)$ with some $\omega \in W_0$ such that

$$\mathcal{M}, \omega \Vdash t : \bot$$

for some term t.

PROOF. Consider the empty CS and the following model:

(i) $W = \{\omega, \nu\}$ and $W_0 = \{\omega\}$
(ii) $V(\nu, \bot) = 1$ and $V(\nu, F) = 0$ for all other formulas F
(iii) $E(\omega, t) = \{\nu\}$ for all terms t.

We observe that $\nu \in W_{\mathsf{nc}}$. So the model is well defined. Further, we find $E(\omega, t) \subseteq [\bot]$. Since $\omega \in W_0$, we get $V(\omega, t : \bot) = 1$. We conclude

$$\mathcal{M}, \omega \Vdash t : \bot.$$

□

REMARK 8.8
For Lemmas 8.5 and 8.6, the presence of the + operation is essential. Consider a term language without + and the logic JNoC^- being JNoC without **j+**. Let CS be an axiomatically appropriate CS for JNoC^-. There is a NoC CS-subset model $\mathcal{M}$ for $\mathsf{JNoC}^-_{\mathsf{CS}}$ with a normal world ω such that

$$\mathcal{M}, \omega \Vdash s : P \wedge t : \neg P$$

for some terms s and t and some proposition P.

Hence if we drop the + operation, we can have self consistent reasons without getting reasons that are consistent among them even in the presence of an axiomatically appropriate constant specification.

Instead of using an axiomatically appropriate constant specification, we could also add the schema $s : \top$ to $\mathsf{JNoC}_{\mathsf{CS}}$ in order to derive **jd**.

LEMMA 8.9
Let CS be an arbitrary constant specification. Let $\mathsf{JNoC}^+_{\mathsf{CS}}$ be $\mathsf{JNoC}_{\mathsf{CS}}$ extended by the schema $s : \top$ for all terms s. We find that

$$\mathsf{JNoC}^+_{\mathsf{CS}} \vdash \neg(t : \bot) \quad \text{for each term } t.$$

PROOF. The following is an instance of axiom **noc**

$$\neg(t : \bot \wedge t : \neg\bot).$$

Using the definition $\top := \neg\bot$ and propositional reasoning, we obtain

$$t : \top \rightarrow \neg(t : \bot).$$

Using $t : \top$ and modus ponens, we conclude $\neg(t : \bot)$. □

9 Remarks on impossible and inconsistent obligations

There are two main advantages in using the justification logic framework to deal with deontic matters. First, one can explicitly track which reasons are reasons for what and perform operation on them, thus having a higher degree of accuracy in formal representations of normative reasoning: every obligation has a source. Puzzles and paradoxes such as Ross' are very easy to identify and, under a plausible set-up, disappear. In the present paper we have seen how justification logic provides a means to keep track of the source of impossible and inconsistent obligations, thus helping not to conflate the two.

Second, the framework allows for the hyperintensionality of obligation, namely that logically equivalent contents may not be normatively equivalent. In general it is not the case that if $t : F$ and $F \equiv G$, then $t : G$. This also ensures a finer-grained formal approach to everyday normative reasoning that is currently unavailable in more standard approaches.

When we come to the specific topic of the present paper, however, we have to remark that it is possible to distinguish between $\neg\mathcal{O}\bot$ and $\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$ also in some non-normal implicit modal systems, as we noted in Sect. 2, and in particular in Chellas' system D (cf. [8, 9]), which dispenses with axiom schema **M**: $\mathcal{O}(A \wedge B) \rightarrow \mathcal{O}A \wedge \mathcal{O}B$.

Chellas' minimal monadic deontic logic D builds as usual on propositional logic, adds $\neg\mathcal{O}\bot$ as an axiom, and has rule ROM: $A \rightarrow B/\mathcal{O}A \rightarrow \mathcal{O}B$. In Chellas' logic the collapse is indeed avoided, because $\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$ is not derivable from $\neg\mathcal{O}\bot$.

How does Chellas' approach compare to the one developed in the present paper? Given the apparent similarities, let's focus on the differences, both technical and philosophical. Rule ROM could be questioned in a deontic context: however, this rule is fundamental in Chellas' system, therefore one cannot ignore it (selectively or not); whereas in a justification logic context we can have a finer-grained control on which axioms get an 'automatic', as it were, normative justification, by fine-tuning the constant specification.

Philosophically, we can start from the semantic interpretation of the obligation operator. For Chellas, 'OA is true at a possible world just in case the world has a non-empty class of deontic alternatives throughout which A is true. The picture is one of possibly empty collections of non-empty classes of worlds functioning as moral standards: what ought to be true is what is entailed by one of these moral standards [8, p. 24]'. Chellas uses a neighborhood semantics. A standard, for him, is a collection of propositions. A term, in the context of the present paper, is instead interpreted as a set of worlds.

Moreover, Chellas' system is still an implicit modal logic, so it cannot keep track and reason with the sources of obligations. And indeed this reading is consistent with Chellas' intended interpretation of the obligation operator: what ought to be true is what is entailed by one of these moral standards. But which? In a justification logic context, for instance, if one wants to retain Chellas' ideas to interpret terms as moral standards, one can keep track of which moral standard requires what.

10 Consistency, permission and free choice

10.1 Weak permission

In the first part of the paper we studied different versions of consistency principles in justification logic with a deontic interpretation, and we showed how certain assumptions allow for finer-grained distinctions between impossible and inconsistent obligations. In SDL, as is well known, permission is introduced as the dual of obligation: $\mathcal{P}F := \neg\mathcal{O}\neg F$. Thus, the consistency principle **D** is sometimes expressed employing both obligation and permission, in the following way:

$$\mathcal{O}F \rightarrow \mathcal{P}F.$$

We call this the weak notion of permission. In the logics JD, JNoC and JNoC^{-}, a weak notion of permission can be defined as follows:

$$\langle t \rangle\, F := \neg t : \neg F.$$

This notion corresponds to the absence of a contrary obligation. Thus the formula $\langle t \rangle\, F$ can be read as 't tolerates F' (in the sense that it is not the case that t obliges one to $\neg F$. This is weaker than 'F is permitted because of reason t'.)[4]

[4] Let us justify this weaker reading of $\langle\cdot\rangle$ using an arithmetical interpretation. In the arithmetical interpretation of justification assertions, $t : F$ means 't is a proof of F'. Then, $\langle t \rangle\, F$, defined as $\neg t : \neg F$, means 't is not a proof of $\neg F$'. If $\neg F$ is defined as $F \rightarrow \bot$, then $\neg t : \neg F$ means 't is not a proof that assuming F leads to a contradiction'. But of course this does not mean that 't shows that F is consistent'. It only means that according to t, F could be consistent (but maybe there

Note that the axiom NoC in the logic JNoC is equivalent to

$$t : A \to \langle t \rangle A.$$

A good feature of subset models is that it provides a simple truth condition for this kind of permission:

$$V(\omega, \langle t \rangle F) = 1 \qquad \text{iff} \qquad E(\omega, t) \not\subseteq [\neg F],$$

where ω is a normal world. The above condition says that '$\neg F$ is not true in an ideal world according to reason t from ω's perspective', which is weaker than 'F is true in an ideal world according to reason t from ω's perspective (= F is permissible in w because of t)'.

It is worth noting that, using the above truth condition, it is easy to show that $\langle t \rangle F \leftrightarrow \neg t : \neg F$ is valid, but it is not the case that $t : F \leftrightarrow \neg \langle t \rangle \neg F$ is valid. In fact, as the following result shows, in justification logics $t : \neg\neg F \leftrightarrow t : F$ is not valid.

LEMMA 10.1
There exists a NoC CS-subset model $\mathcal{M} = (W, W_0, V, E, P)$ with some $\omega \in W_0$ such that

$$\mathcal{M}, \omega \not\Vdash t : F \to t : \neg\neg F,$$

and

$$\mathcal{M}, \omega \not\Vdash t : \neg\neg F \to t : F,$$

for some term t and some formula F.

PROOF. Consider the empty CS and the following model $\mathcal{M}$:

(i) $W = \{\omega, \nu\}$ and $W_0 = \{\omega\}$,
(ii) $V(\nu, P) = 1$ and $V(\nu, F) = 0$ for all other formulas F,
(iii) $V(\omega, F) = 0$ for all other formulas F,
(iv) $E(\omega, t) = \{\nu\}$ and $E(\nu, t) = \emptyset$ for all terms t.

We observe that $\nu \in W_{\mathsf{nc}}$. So the model is a well-defined NoC $\emptyset$-subset model. Further, for every term t, we find $E(\omega, t) \subseteq [P]$ and $E(\omega, t) \not\subseteq [\neg\neg P]$. Since $\omega \in W_0$, we get

$$\mathcal{M}, \omega \not\Vdash t : P \to t : \neg\neg P.$$

Now consider the model $\mathcal{M}' = (W, W_0, V', E, P)$, defined similar to the above model $\mathcal{M}$, with the difference that:

- $V'(\nu, \neg\neg P) = 1$ and $V'(\nu, F) = 0$ for all other formulas F.

It is easy to show that $\mathcal{M}'$ is a well-defined NoC $\emptyset$-subset model. In addition, we have

$$\mathcal{M}', \omega \not\Vdash t : \neg\neg P \to t : P.$$

□

10.2 Free choice permission

In the context of justification logic, where there is a focus on explicit, rather than implicit, tools, there is another notion of permission that is worth considering. In this section, we add the notion of

is another term proving $\neg F$ and thus showing F's inconsistency). The same remark can be applied to the deontic reading of $\langle t \rangle F$. The fact that 'it is not the case that t obliges $\neg F$' does not mean that F is permitted because of reason t; it only means that t tolerates F.

free choice permission to deontic justification logics, which we write $\langle\!\langle t \rangle\!\rangle$. We discuss the resulting logic, together with a comparison with the different notions of permissions, in the next section.

Let us begin by defining the language upon which our justification logic with the free choice permission operator will be built. Justification terms are defined similar to that of Section 3. Formulas are built from a countably many atomic propositions from the set Prop according to the following grammar:

$$F ::= P_i \in \mathsf{Prop} \mid \bot \mid F \rightarrow F \mid t : F \mid \langle\!\langle t \rangle\!\rangle\, F.$$

The set of all formulas is denoted by $\mathcal{L}_{JF}$. Define $\langle t \rangle\, F := \neg t : \neg F$. Unlike the weak notion of permission expressed by the operator $\langle \cdot \rangle$, we suggest to read the strong notion of permission (i.e. the free choice permission) expressed by $\langle\!\langle t \rangle\!\rangle\, F$ as 'F is permitted because of reason t'.

Add the following axioms to JNoC:

fcp $\quad \langle\!\langle t \rangle\!\rangle\, (A \vee B) \rightarrow (\langle\!\langle t \rangle\!\rangle\, A \wedge \langle\!\langle t \rangle\!\rangle\, B)$;

con $\quad \langle\!\langle t \rangle\!\rangle\, A \rightarrow \langle t \rangle\, A$.

The resulting logic is denoted by $\mathsf{JNoC}^{\mathsf{FCP}}$. Note that a constant specification for $\mathsf{JNoC}^{\mathsf{FCP}}$ may contain instances of axiom **fcp** or axiom **con**.

Semantics for this language is based on subset models.

DEFINITION 10.2 (FCP subset model).
Given some constant specification CS for $\mathsf{JNoC}^{\mathsf{FCP}}$, then an FCP CS-subset model $\mathcal{M} = (W, W_0, V, E, P)$ is a NoC subset model (W, W_0, V, E) augmented by a function $P : W \times \mathsf{Tm} \rightarrow \mathcal{P}(\mathcal{L}_{JF})$ that meets the following conditions. For all $\omega \in W_0$, for all $t \in \mathsf{Tm}$, and for all $A, B \in \mathcal{L}_{JF}$:

- If $A \vee B \in P(\omega, t)$, then $A, B \in P(\omega, t)$.
- If $A \in P(\omega, t)$, then $E(\omega, t) \not\subseteq [\neg A]$.

In addition to the conditions of Definition 4.1, the valuation function V should satisfy the following condition:

$$V(\omega, \langle\!\langle t \rangle\!\rangle\, F) = 1 \quad \text{iff} \quad F \in P(\omega, t),$$

where $\omega \in W_0$.

Now the notion of FCP CS-validity can be defined similar to Definition 4.3.

Remember from the first part of the paper that the function $E(\omega, t)$ tells us the states that are ideal according to t from ω's perspective. Thus $t : F$ at ω was true just in case F is true at those ideal states. Similarly, the function P tells us which formulas t explicitly allows, from ω's perspective.[5] Thus, $\langle\!\langle t \rangle\!\rangle\, F$ is true at ω just in case F is explicitly allowed by t (from ω's perspective).[6]

Let us now turn to the completeness of $\mathsf{JNoC}^{\mathsf{FCP}}$.

[5] Note that the function P is similar to evidence functions used in Fitting models [16]. However, Fitting used the evidence function to present truth condition for justification formulas $t : F$, and the dual of $t : F$ was not mentioned in [16].

[6] We discuss the asymmetry between the semantic clauses of obligation (in terms of states) and permission (in terms of formulas) in the next section.

DEFINITION 10.3 (Canonical model).
Let CS be an arbitrary constant specification for $\mathsf{JNoC}^{\mathsf{FCP}}$. We define the canonical model $\mathcal{M}^C = (W^C, W_0^C, V^C, E^C, P^C)$ by:

- $W^C = \mathcal{P}(\mathcal{L}_{JF})$.
- $W_0^C = \{\Gamma \in W^C | \Gamma$ is maximal $\mathsf{JNoC}^{\mathsf{FCP}}_{\mathsf{CS}}$-consistent set of formulas$\}$.
- $V^C(\Gamma, F) = 1 \quad$ iff $\quad F \in \Gamma$.
- $E^C(\Gamma, t) = \{\Delta \in W^C | \Delta \supseteq \Gamma/t\}$, where $\Gamma/t := \{F \in \mathcal{L}_{JF} \mid t : F \in \Gamma\}$.
- $P^C(\Gamma, t) = \{F \mid \langle\!\langle t \rangle\!\rangle F \in \Gamma\}$.

LEMMA 10.4
Let CS be an arbitrary constant specification for $\mathsf{JNoC}^{\mathsf{FCP}}$. The canonical model $\mathcal{M}^C$ is an FCP CS-subset model.

PROOF. We first show the condition

$$A \vee B \in P^C(\Gamma, t) \Rightarrow A, B \in P^C(\Gamma, t),$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$, and $A, B \in \mathcal{L}_{JF}$.

From $A \vee B \in P^C(\Gamma, t)$, it follows that $\langle\!\langle t \rangle\!\rangle (A \vee B) \in \Gamma$. By axiom **fcp**, $\langle\!\langle t \rangle\!\rangle A \in \Gamma$ and $\langle\!\langle t \rangle\!\rangle B \in \Gamma$. Thus, $A, B \in P^C(\Gamma, t)$.

We now show the condition

$$A \in P^C(\Gamma, t) \Rightarrow E^C(\Gamma, t) \nsubseteq [\neg A],$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$ and $A \in \mathcal{L}_{JF}$.

From $A \in P^C(\Gamma, t)$, it follows that $\langle\!\langle t \rangle\!\rangle A \in \Gamma$. By axiom **con**, $\langle t \rangle A \in \Gamma$. Thus, $\neg t : \neg A \in \Gamma$, and hence $t : \neg A \notin \Gamma$. Therefore, $\neg A \notin \Gamma/t$. Let $\Delta := \Gamma/t$. Then, $\Delta \in E^C(\omega, t)$. From $\neg A \notin \Delta$, it follows that $\Delta \notin [\neg A]$. Hence, $E^C(\omega, t) \nsubseteq [\neg A]$.

We finally show that

$$V^C(\Gamma, \langle\!\langle t \rangle\!\rangle A) = 1 \quad \text{iff} \quad A \in P^C(\Gamma, t),$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$, and $A \in \mathcal{L}_{JF}$.

The proof is as follows:

$$V^C(\Gamma, \langle\!\langle t \rangle\!\rangle A) = 1 \quad \text{iff} \quad \langle\!\langle t \rangle\!\rangle A \in \Gamma \quad \text{iff} \quad A \in P^C(\Gamma, t).$$

□

Again the truth lemma and the completeness theorem follow easily.

THEOREM 10.5 (Soundness and completeness).
Let CS be an arbitrary constant specification for $\mathsf{JNoC}^{\mathsf{FCP}}$. For each formula $F \in \mathcal{L}_{JF}$ we have

$$\mathsf{JNoC}^{\mathsf{FCP}}_{\mathsf{CS}} \vdash F \quad \text{iff} \quad F \text{ is FCP CS-valid.}$$

11 Remarks on permission

11.1 On free choice permission

In the last section, we added a modality for free-choice permission, which is not simply the dual of obligation. The literature on free-choice permission is very large, and we cannot hope to account for it in a satisfying way in this context.[7]

However, we follow von Wright in calling something

> permitted in the weak sense if it is not forbidden; and [...] permitted in the strong sense if it is not forbidden, but subject to norm. Acts which are strongly permitted are thus weakly permitted but not necessarily vice versa ([44, p. 86]).

And in fact, von Wright identifies strong permission with an express permission: 'that which is in the strong sense permitted is, somehow, expressly permitted ([45])'.

In the language of justification logic, we can point to the reason why something is permitted, which we indicate with a term, and therefore we call this strong or explicit permission.

In the previous section we have logically studied a notion of 'explicit' permission as free-choice permission. It is tempting to endorse the possibility that strong permission and free-choice permission coincide, as [4] argues: the paper offers strong philosophical arguments to support an approach to free-choice permission as strong permission that is similar to ours, although it develops a different formal model of it (based on default conditionals).[8]

The peculiar nature of free-choice permission as explicit is reflected in the choice of our semantic clauses: while the clause for obligation uses the function E, which tells us the states that are ideal according to t from ω's perspective, the clause for permission uses the function P tells us which formulas t explicitly allows, from ω's perspective. This apparent asymmetry is explained by the nature of free-choice permission, which is very sensitive to the options offered. Such a level of fine-grainedness is well captured by focusing on the formulas that reasons directly allows, rather than states.

We now see how our account fares when it comes to a few known problems from the literature. It is well known that adding the free-choice permission principle **FCP**: $(\mathcal{P}(A \vee B) \rightarrow \mathcal{P}A \wedge \mathcal{P}B)$ to standard modal logic leads to disaster.

For instance, assuming a standard version of modal logic, that you may sit on the sofa implies that you may destroy it:

(1) $\mathcal{P}(A)$, assumption
(2) $\mathcal{P}(A \vee B)$, from 1 and reasoning in standard modal logic
(3) $\mathcal{P}A \wedge \mathcal{P}B$, from 2 and **FCP**

[7] A good starting point is the chapter on the varieties of permission in [19]. A recent logic contribution is [18]. For a different point of view, see [11], which suggests a way to capture the logic of free choice permission while avoiding known paradoxes by using a family of substructural logics augmented with a non-monotonic reasoning principle.

[8] For more on this point, consider [1], which starts from the 'open reading' of permissions: an action is permitted if it meets sufficient conditions to be ok in a given normative system. In turn, within this framework, obligations can be understood as the strongest action that is necessary for it being ok, or as the weakest permission. The open reading of permissions is in fact the same as strong permissions, at least in one interpretation. In fact, the open reading licenses the free-choice principle. In this reading, permission and obligation are (i) non-normal modalities, interpreted through a neighborhood semantics, (ii) not dual, but (iii) are nonetheless connected. In particular, obligation implied permission, and something is obligatory only in case no logically weaker action is both permitted and possible. Note that in this framework, both obligation and permission remain non-hyperintensional.

(4) $\mathcal{P}B$, from 3 by reasoning in standard modal logic

where B is completely arbitrary.

Within our framework, something equivalent to (2) cannot already be derived from the equivalent version of (1), for we would need another term that explicitly permits $A \vee B$.

Moreover, hyperintensionality matters. In a non-hyperintensional modal logic, thanks to the principle of the substitution of equivalents (RE: $\vdash A \leftrightarrow B / \vdash \mathcal{P}A \leftrightarrow \mathcal{P}B$), one can get: $\mathcal{P}A \rightarrow \mathcal{P}(A \wedge B)$.

(1) $\mathcal{P}(A)$, assumption
(2) $A \leftrightarrow (A \wedge B) \vee (A \wedge \neg B)$, by reasoning in standard modal logic
(3) $\mathcal{P}A \leftrightarrow \mathcal{P}((A \wedge B) \vee (A \wedge \neg B))$, from 1 and 2 by RE
(4) $\mathcal{P}A \rightarrow \mathcal{P}(A \wedge B)$, from 3 and **FCP**, by reasoning in standard modal logic.

Again, in our framework, something equivalent to RE is not available, so our notion of free choice permission expressed by $\langle\langle \cdot \rangle\rangle$ is safe.

11.2 Duality

In this section we discuss different notions of permission and obligations that can be expressed in our framework. On top of the standard obligation, expressed as $t : F$, which we read as F is obligatory for reason t, we have its dual, what we called weak permission, written as $\langle t \rangle F$. This is much weaker than that F is permitted for reason t, which in our framework we identify with the reading of $\langle\langle t \rangle\rangle F$. The formula $\langle t \rangle F$ therefore just denotes the absence of an obligation to the contrary. A similar pattern emerges with the (potential) dual of free-choice permission, $\langle\langle t \rangle\rangle F$, which we write $[\![t]\!]F$. Let

$$[\![t]\!]F := \neg\langle\langle t \rangle\rangle \neg F.$$

Since the operator $[\![\cdot]\!]$ is the dual of the free choice permission, it is expected to express some kind of obligation. However, $[\![\cdot]\!]$ does not obey the same properties of obligation as in JNoC. For example, the formula $[\![t]\!]A \rightarrow \langle\langle t \rangle\rangle A$ is equivalent to $\langle\langle t \rangle\rangle A \vee \langle\langle t \rangle\rangle \neg A$, which says that for every reason t and for every formula A either A is permitted because of t or $\neg A$ is permitted because of t. Obviously, this is not plausible, and thus $[\![t]\!]A \rightarrow \langle\langle t \rangle\rangle A$ is not an acceptable property of the operator $[\![\cdot]\!]$.

Let us add some other remarks on this notion. First, philosophically speaking, if we accept the equation of free choice permission with explicit, or strong, permission, which by the way has a certain affinity with the attitude of justification logic as an explicit modal logic, then it makes sense to identify its dual $[\![\cdot]\!]$ as weak obligation. Weak obligation can be approximated, in English, by the use of weak deontic modals, such as 'ought' and 'should', when contrasted to strong deontic (semi)modals, like 'must' and 'have to'. 'Ought' and 'should' are weaker than 'must' and 'have to', as shown by patterns such as:
OK: You should do the dishes, but you don't have to. NO: You must save his life, but you shouldn't.

However, it is plausible that there are readings of 'ought' and 'should' that would make the second sentence acceptable, and in particular prudential or advisory readings.[9]

A similar remark applies to $\langle t \rangle$, i.e. weak permission as the dual of strong, or explicit, obligation, which could be expressed by 'might' and 'could' (in English), as conjectured in part of the literature (e.g. [12]). While historically 'might' and 'could', in English, were weaker forms of

[9] However, note that a few extant theories on weak deontic modals suggest they are weaker e.g. because of a counterfactual element (e.g. [15]). But see also [22].

'may' and 'can', nowadays they only seem to have retained pragmatic differences (e.g. in terms of politeness), contrary to many other e.g. European languages where the equivalents of 'must/have to' vs 'ought/should' patterns in the same way as the equivalents of 'may/can' vs 'might/could'.[10]

Second, if we interpret $[\![t]\!]$ as the dual of $\langle\!\langle t\rangle\!\rangle$, we are saying that t is not a reason for the free choice permission of the opposite action. Whether this makes t enough to generate a weak obligation is unclear. However, we can easily read $[\![t]\!]$ as an implicit modal operator, i.e. there is a weak obligation that, without t being a specific reason for it. In fact, a similar remark can be made for the dual of explicit obligation, i.e. $t : A$, which we wrote $\langle t\rangle$, it too can be interpreted as there being a weak implicit permission of an implicit kind, without t being a specific reason for it.[11]

Inferential patterns between weak obligation and explicit permission are unclear. On one hand, natural language data do not seem to suggest that explicit permission implies weak obligation in any straightforward sense, if we stick to the modal analysis suggested, nor vice versa. However, on the other hand, conceptual analysis might provide opposite evidence. Imagine a situation where person A welcomes person B in their library, and A utters 'you may sit on the reading chair or on the sofa'. It seems plausible that person B should sit on the chair or on the sofa—while clearly B does not have to sit, but could say she prefers to stand, there is some expectation that one of the options should be taken, barring some extenuating circumstance. This implication from free choice permission to weak obligation seems to be absent in the case of strong permission not offering a choice context. That you are now explicitly allowed to pay taxes with apples does not seem to imply a weak obligation to do so in the same way.

This account might be relatively simple and symmetric, but ultimately prove unrealistic. It is plausible to imagine that strong permission and free choice permission come apart in some cases. Our account is flexible enough to deal with this scenario. In fact, in addition to the simple logic of free-choice permission of the previous section, we present in the appendix A a further logic of explicit permission that combines elements of free choice and of strong permission (such as axiom **fcp**, and the existence of a dual, i.e. weak obligation, that is implied by strong permission). This logic has a novel semantics.

11.3 Some open problems

We now put forward a few problems and ideas for discussion. First, to what extent it makes sense to have non-disjunctive free-choice permission, e.g. $\langle\!\langle t\rangle\!\rangle A$? In this case, there is no explicit choice between different options. Second, there is the intuition that strong obligation should imply weak obligation. But if strong permission implies weak obligation, it is easy to see problems of triviality arising. Third, if there is such a thing as weak obligation as the dual of strong permission, should $\langle\!\langle\cdot\rangle\!\rangle$ have special modal axioms? Fourth, in our approach we assumed that reasons can be both requiring (i.e. justifying obligations) and permissive (i.e. justifying permissions). But it is also conceivable that we should conceptualize reasons as different sets of terms.

[10] Note that [23, p. 184] claims, without giving evidence, that English and German lack duals of weak necessity modals. [5] proposes to interpret the dual of weak deontic modals as 'faultlessness', expressed by 'justified'. For the purposes of this paper, it is interesting to report the semantics Beddor suggests for weak necessity modals and for their dual (in our parlance, strong permission). This semantics is couched in Kratzerian terms, but it does not matter: 'Weak necessity modals are universal quantifiers over the optimal worlds in the modal base [...]. Strong necessity modals quantify over all of the acceptable worlds in the modal base, where a world is acceptable as long as it's good enough [...]. [T]ake expressions of permission to be existential quantifiers over the acceptable worlds in the modal base. [...] it's natural to understand expressions of faultlessness as existential quantifiers over the very best worlds'.

[11] A different interpretation of the dual of $t : A$, although not in a deontic context, is offered by [27], where $\langle t\rangle A$ is indeed an explicit piece of evidence, although at the meta-level, consisting in a model that validates A.

12 Conclusion

We provided a novel semantics for justification logics with axiom **D** that does *not* require an axiomatically appropriate constant specification, i.e. not every axiom needs to be justified by a constant. This can be crucial to have more control on the logic and solve some traditional puzzles such as Ross'. Axiom **D** can be formulated in at least two equivalent ways in normal modal logic, either with inconsistent obligations ($\neg(\mathcal{O}A \wedge \mathcal{O}\neg A)$) or with one impossible obligation ($\neg\mathcal{O}\bot$). We proved that their explicit versions are interderivable in JD and JNoC only when the constant specification is axiomatically appropriate. In particular, our technical results are:

(i) $\mathsf{JD}_{\mathsf{CS}}$ proves **noc** for axiomatically appropriate CS and vice versa.
(ii) $\mathsf{JNoC}_{\mathsf{CS}}$ proves **jd** for axiomatically appropriate CS.
(iii) $\mathsf{JD}_{\mathsf{CS}}$ proves **noc** for arbitrary CS only if the language is based on the Boolean connectives $\rightarrow$ and $\bot$.
(iv) $\mathsf{JNoC}_{\mathsf{CS}}$ does not prove **jd** for arbitrary CS.
(v) $\mathsf{JNoC}^{-}_{\mathsf{CS}}$ does not prove **jd** for axiomatically appropriate CS.

Having more control not only on how to formulate axiom **D**, but also on how to specify the constant specification is philosophically perspicuous: it avoids conflating impossible and conflicting obligations and can encode why this is the case, e.g. for conceptual (logical) or contingent reasons.

Recently, it was shown that principle **noc** is also very useful for analysing epistemic situations in the context of quantum physics [43].

When it comes to permission, we added a free-choice permission operator to deontic justification logic which avoids known paradoxes from the literature. We proved that new logic is sound and complete. We discussed duality between different notions of permission. The semantics we adopted presents an asymmetry between obligation (which uses a subset-style clause) and free-choice permission (which uses a basic model-style clause). While we think this choice is justified for philosophical reasons, it will be interesting to investigate a subset-style clause for free-choice permission in future work.

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A On weak obligation

In this section, we develop a new semantics for the explicit free choice permission based on neighborhood semantics. This semantics enables us to define a proper truth condition for the dual of the explicit free choice permission.

We formulate our logic in the same language $\mathcal{L}_{JF}$ as $\mathsf{JNoC}^{\mathsf{FCP}}$. Define $\langle\cdot\rangle$ and $[\![\cdot]\!]$ as follows:

$$\langle t\rangle F := \neg t : \neg F.$$

$$[\![t]\!]F := \neg\langle\!\langle t\rangle\!\rangle \neg F.$$

Add the following axioms to JNoC:

fcp	$\langle\!\langle t\rangle\!\rangle (A \vee B) \rightarrow (\langle\!\langle t\rangle\!\rangle A \wedge \langle\!\langle t\rangle\!\rangle B)$;
con	$\langle\!\langle t\rangle\!\rangle A \rightarrow \langle t\rangle A$;
fcp-wo	$\langle\!\langle t\rangle\!\rangle A \rightarrow [\![t]\!]A$.

The resulting logic is denoted by $\mathsf{WoJNoC}^{\mathsf{FCP}}$.

DEFINITION A.1 (FCP subset-neighborhood model).
Given some constant specification CS for $\mathsf{WoJNoC}^{\mathsf{FCP}}$, an FCP CS-subset-neighborhood model $\mathcal{M} = (W, W_0, V, E, N)$ is a NoC subset model (W, W_0, V, E) augmented by a function $N : W \times \mathsf{Tm} \rightarrow \mathcal{P}(\mathcal{P}(W))$ that meets the following conditions. For all $\omega \in W_0$, for all $t \in \mathsf{Tm}$ and for all $A, B \in \mathcal{L}_{JF}$:

- If $[A \vee B] \in N(\omega, t)$, then $[A], [B] \in N(\omega, t)$.
- If $[A] \in N(\omega, t)$, then $E(\omega, t) \not\subseteq [\neg A]$.
- If $[A] \in N(\omega, t)$, then $[\neg A] \notin N(\omega, t)$.

In addition to the conditions of Definition 4.1, the valuation function V should satisfy the following condition:

$$V(\omega, \langle\!\langle t\rangle\!\rangle F) = 1 \quad \text{iff} \quad [F] \in N(\omega, t),$$

where $\omega \in W_0$.

Truth in all normal worlds of all FCP CS-subset-neighborhood models is called FCP CS-N-validity.

The function N tells us which propositions t explicitly allows, from ω's perspective.[12] Thus, $\langle\!\langle t\rangle\!\rangle F$ is true at ω just in case the truth set of (or the proposition expressed by) F is explicitly allowed by t (from ω's perspective). From the definition of $[\![\cdot]\!]$, one can obtain the truth condition of the dual of the explicit free choice permission as follows:

$$V(\omega, [\![t]\!]F) = 1 \quad \text{iff} \quad [\neg F] \notin N(\omega, t),$$

where $\omega \in W_0$.

Let us now turn to the completeness of $\mathsf{WoJNoC}^{\mathsf{FCP}}$.

DEFINITION A.2 (Canonical model).
Let CS be an arbitrary constant specification for $\mathsf{WoJNoC}^{\mathsf{FCP}}$. We define the canonical model $\mathcal{M}^C = (W^C, W_0^C, V^C, E^C, N^C)$ by:

- $W^C = \mathcal{P}(\mathcal{L}_J)$.

[12] The function N is similar to the neighborhood function used in [42] and [17]. However, in these papers, the neighborhood semantics are used to present truth condition for justification formulas $t : F$. In [17], the author also used the neighborhood semantics to define the truth condition of the dual of the justification assertions.

- $W_0^C = \{\Gamma \in W^C | \Gamma$ is maximal $\mathsf{WoJNoC}_{\mathsf{CS}}^{\mathsf{FCP}}$-consistent set of formulas$\}$.
- $V^C(\Gamma, F) = 1 \quad$ iff $\quad F \in \Gamma$;
- $E^C(\Gamma, t) = \{\Delta \in W^C | \Delta \supseteq \Gamma/t\}$ where $\Gamma/t := \{F \in \mathcal{L}_J \mid t : F \in \Gamma\}$.
- $N^C(\Gamma, t) = \{|F| \mid \langle\!\langle t \rangle\!\rangle F \in \Gamma\}$.

Here $|F| := \{\Gamma \in W \mid F \in \Gamma\}$ is the *proof set* of F.

Note that in the canonical model $\mathcal{M}^C$ we have

$$|F| = \{\Gamma \in W \mid F \in \Gamma\} = \{\Gamma \in W \mid V^C(\Gamma, F) = 1\} = [F].$$

LEMMA A.3
Let CS be an arbitrary constant specification for $\mathsf{WoJNoC}^{\mathsf{FCP}}$. The canonical model $\mathcal{M}^C$ is an FCP CS-subset-neighborhood model.

PROOF. We first show the condition

$$[A \vee B] \in N^C(\Gamma, t) \Rightarrow [A], [B] \in N^C(\Gamma, t),$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$, and $A, B \in \mathcal{L}_{JF}$.

From $[A \vee B] = |A \vee B| \in N^C(\Gamma, t)$, it follows that $\langle\!\langle t \rangle\!\rangle (A \vee B) \in \Gamma$. By axiom **fcp**, $\langle\!\langle t \rangle\!\rangle A \in \Gamma$ and $\langle\!\langle t \rangle\!\rangle B \in \Gamma$. Thus, $[A], [B] \in N^C(\Gamma, t)$.

We now show the condition

$$[A] \in N^C(\Gamma, t) \Rightarrow E^C(\Gamma, t) \not\subseteq [\neg A],$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$ and $A \in \mathcal{L}_{JF}$.

From $[A] = |A| \in N^C(\Gamma, t)$, it follows that $\langle\!\langle t \rangle\!\rangle A \in \Gamma$. By axiom **con**, $\langle t \rangle A \in \Gamma$. Thus, $\neg t : \neg A \in \Gamma$, and hence $t : \neg A \notin \Gamma$. Therefore, $\neg A \notin \Gamma/t$. Let $\Delta := \Gamma/t$. Then, $\Delta \in E^C(\omega, t)$. From $\neg A \notin \Delta$, it follows that $\Delta \notin [\neg A]$. Hence, $E^C(\omega, t) \not\subseteq [\neg A]$.

We now show the condition

$$[A] \in N^C(\Gamma, t) \Rightarrow [\neg A] \notin N^C(\Gamma, t),$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$ and $A \in \mathcal{L}_{JF}$.

From $[A] = |A| \in N^C(\Gamma, t)$, it follows that $\langle\!\langle t \rangle\!\rangle A \in \Gamma$. By axiom **fcp-wo**, we get $\neg \langle\!\langle t \rangle\!\rangle \neg A \in \Gamma$. Thus, $\langle\!\langle t \rangle\!\rangle \neg A \notin \Gamma$. Therefore, $[\neg A] = |\neg A| \notin N^C(\Gamma, t)$.

We finally show that

$$V^C(\Gamma, \langle\!\langle t \rangle\!\rangle A) = 1 \quad \text{iff} \quad [A] \in N^C(\Gamma, t),$$

where $\Gamma \in W_0^C$, $t \in \mathsf{Tm}$, and $A \in \mathcal{L}_{JF}$.

The proof is as follows:

$$V^C(\Gamma, \langle\!\langle t \rangle\!\rangle A) = 1 \quad \text{iff} \quad \langle\!\langle t \rangle\!\rangle A \in \Gamma \quad \text{iff} \quad |A| = [A] \in N^C(\Gamma, t). \qquad \square$$

Again the truth lemma and the completeness theorem follow easily.

THEOREM A.4 (Soundness and completeness).
Let CS be an arbitrary constant specification for $\mathsf{WoJNoC}^{\mathsf{FCP}}$. For each formula $F \in \mathcal{L}_{JF}$ we have

$$\mathsf{WoJNoC}_{\mathsf{CS}}^{\mathsf{FCP}} \vdash F \quad \text{iff} \quad F \text{ is FCP}\mathsf{CS}\text{-N-valid.}$$

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