An Infinitary Graded Modal Logic (Graded Modalities VI)[1]

Maurizio Fattorosi-Barnaba and Silvano Grassotti[2]

Dipartimento di Matematica, Università di Roma "La Sapienza",
Piazzale Alberto Moro 2, I – 00185 Roma, Italy[3]

Abstract. We prove a completeness theorem for $K_{\omega_1}^0$, the infinitary extension of the graded version K^0 of the minimal normal logic K, allowing conjunctions and disjunctions of countable sets of formulas. This goal is achieved using both the usual tools of the normal logics with graded modalities and the machinery of the predicate infinitary logics in a version adapted to modal logic.

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0 Introduction

Infinitary extensions are not usual for systems of modal logic with traditional modalities, surely this happens because such a theme would not be sufficiently motivated in that context. On the contrary, systems involving graded modalities seem to offer strong motivation to study their infinitary extensions.

Here we study $K_{\omega_1}^0$, the infinitary extension of K^0 (the basic graded normal system) which allows countable disjunctions and conjunctions, and we prove a completeness theorem for it. Such a system has an obvious interest firstly because it allows to treat as modalities so traditional mathematical concepts as finite and infinite, letting

$$\Diamond_{\mathrm{fin}}A = \bigvee\{\Diamond!_nA : n \in \mathbb{N}\}, \qquad \Diamond_{\mathrm{inf}}A = \neg\Diamond_{\mathrm{fin}}A,$$

and secondly because the infinite modality $\Diamond_{\mathrm{inf}}$ is essentially the same as the (already in [5] introduced) $\Diamond_\omega$, that is the first and basic example of a graded modality with an infinite grade.

The strategy we shall follow consists in trying to repeat the line of proof of preceding works in this area, as [4] and [2], using suitable sets of sentences which are not maximal for the system but have their essential properties with respect to a countable fragment of it.

This kind of treatment is usual in infinitary predicate logic and, for the same reason (the incompactness of the system), we shall use the notion of a consistency property

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as a tool to define models. But this is not enough: having in mind to define and use sets of sentences with particular properties, we need to consider limits (i. e. unions of chains of sets) of a consistency property and to assure they have suitable closure properties. This is made by considering also the notion of a validity property and to assure its compatibility with the consistency property which we are working with. Moreover, the notion of a consistency property we need must be one adapted to modal logic.

For all of these reasons we borrow from [7], [1] and [6] notions and ideas, with suitable adaptations to the context of graded modal logic.

We assume the reader is sufficiently acquainted with the quoted works, but, for the sake of readibility, we shall repeat without proofs or discussions some crucial definitions or results.

1 The syntax of K_{ω_1}

We start by defining the syntactic features of our logic.

D e f i n i t i o n 1.1. An *infinitary language with graded modalities* is the collection of the following objects:

(a) a set of power ω_1 of atomic sentences: $\mathcal{S} = \{P_\lambda : \lambda < \omega_1\} \cup \{\top, \bot\}$,

(b) the connectives $\neg$, $\wedge$, $\vee$ (where $\bigwedge$ and $\bigvee$ are preferably used before sets of formulas),

(c) the graded modal operators $\Diamond_n$ and $\Box_n$ ($n \in \mathbb{N}$),

(d) a finite number of auxiliary symbols: parentheses, commas, etc.

The set $(\mathcal{S})$ of well-formed formulas is defined as usual, where countable conjunctions and disjunctions are also allowed: if Δ is a countable set of formulas, then $\bigwedge\Delta$ and $\bigvee\Delta$ are formulas.

The remaining finitary connectives $\rightarrow$ and $\leftrightarrow$ have the usual definition. We shall use also the exponentiation of the graded modal operators $\Diamond_n^m$ and $\Box_n^m$ ($m, n \in \mathbb{N}$) defined in the obvious way.

D e f i n i t i o n 1.2. Consider numbers $r, n \in \mathbb{N}$, a sequence $\sigma = \langle m_0, m_1, \ldots, m_r \rangle$ of natural numbers of length $r + 1$ and a sequence of formulas $\langle Z_0, Z_1, \ldots, Z_r \rangle$ of equal length. Then we set

$$\Diamond_n^\sigma \langle Z_0, Z_1, \ldots, Z_r \rangle = \Diamond_n^{m_0}(Z_0 \wedge \Diamond_n^{m_1}(Z_1 \wedge \cdots \wedge \Diamond_n^{m_{r-1}}(Z_{r-1} \wedge \Diamond_n^{m_r} Z_r) \ldots)).$$

Let us stress some particular cases: If $r = 0$, the sequence σ has length 1 ($\sigma = \langle m_0 \rangle$) and correspondingly the sequence of formulas is $\langle Z_0 \rangle$, so that $\Diamond_n^\sigma \langle Z_0 \rangle = \Diamond_n^{m_0} Z_0$. Obviously, the formula Z_0 itself is a particular case of the preceding form by letting $m_0 = 0$, i. e., $\Diamond_n^{\langle 0 \rangle} \langle Z_0 \rangle = \Diamond_n^0 Z_0 = Z_0$. This means that any formula can be written in the above form, but we shall be concerned with a special kind of such formulas: let us call *formula of type* C or C-*formula* any formula written in the above form with $Z_r = \bigvee\{A_i : i \in I\}$ ($|I| < \omega_1$) (the letter "C" stands for "choice", because a disjunctive formula forces to choose one of the disjuncts, as we shall see later). So C-formulas are a sort of generalization of the formulas of the type $\bigvee\{A_i : i \in I\}$ ($|I| < \omega_1$), letting e. g. $\sigma = \langle 0, \ldots, 0 \rangle$ and $Z_0 = \ldots = Z_{r-1} = \top$.

Definition 1.3. For any $n \in \mathbb{N}$ let

$$\Diamond!_n A = \begin{cases} \neg \Diamond_0 A & \text{if } n = 0, \\ \Diamond_{n-1} A \wedge \neg \Diamond_n A & \text{if } n > 0. \end{cases}$$

As usual in the infinitary contexts we introduce the definition of the "right-hand negation" $A\neg$ of a formula A:

Definition 1.4.

$\top\neg$	is	$\bot$,	$\bot\neg$	is	$\top$,
$P_\lambda\neg$	is	$\neg P_\lambda$,	$(\neg A)\neg$	is	A,
$\bigvee\{A_i : i \in I\}\neg$	is	$\bigwedge\{\neg A_i : i \in I\}$,	$\bigwedge\{A_i : i \in I\}\neg$	is	$\bigvee\{\neg A_i : i \in I\}$,
$(\Diamond_n A)\neg$	is	$\Box_n(\neg A)$,	$(\Box_n A)\neg$	is	$\Diamond_n(\neg A)$.

Definition 1.5. The *axioms* of the infinitary normal logic with graded modalities $\mathsf{K}^0_{\omega_1}$ are all instances of the following schemata:

(Ax1) any tautology of the classical propositional calculus;

(Ax2) $\Diamond_{n+1} A \to \Diamond_n A$ $(n \in \mathbb{N})$;

(Ax3) $\Box_0(A \to B) \to (\Diamond_n A \to \Diamond_n B)$ $(n \in \mathbb{N})$;

(Ax4) $\Diamond!_0(A \wedge B) \to ((\Diamond!_{n_1} A \wedge \Diamond!_{n_2} B) \to \Diamond!_{n_1+n_2}(A \vee B))$ $(n_1, n_2 \in \mathbb{N})$;

(Ax5) $\bigwedge\{A_i : i \in I\} \to A_i$ $(i \in I, |I| < \omega_1)$;

(Ax6) $\neg\bigvee\{A_i : i \in I\} \leftrightarrow \bigwedge\{\neg A_i : i \in I\}$ $(|I| < \omega_1)$;

(Ax7) $\neg\bigwedge\{A_i : i \in I\} \leftrightarrow \bigvee\{\neg A_i : i \in I\}$ $(|I| < \omega_1)$;

(Ax8) $\Box_n A \leftrightarrow \neg\Diamond_n\neg A$ $(n \in \mathbb{N})$.

The basic *inference rules* of $\mathsf{K}^0_{\omega_1}$ are

Modus Ponens (MP): $\dfrac{A \quad A \to B}{B}$,

Conditioned Conjunction (CC): $\dfrac{B \to A_i \quad (i \in I)}{B \to \bigwedge\{A_i : i \in I\}}$ $(|I| < \omega_1)$,

Normality (N): $\dfrac{\bigwedge\{A_i : i \in I\} \to A}{\bigwedge\{\Box_0 A_i : i \in I\} \to \Box_0 A}$ $(|I| < \omega_1)$.

We write $\vdash A$ to indicate that A is a theorem of $\mathsf{K}^0_{\omega_1}$. For instance, for any formula A one has $\vdash \neg A \leftrightarrow A\neg$.

We list below some theorems and derived rules which will be useful in the sequel. For short, we skip the obvious proofs.

Introduction of $\neg$ in $\leftrightarrow$ (I$\neg\leftrightarrow$): $\dfrac{A \leftrightarrow B}{\neg A \leftrightarrow \neg B}$.

Elimination of $\neg$ from $\leftrightarrow$ (E$\neg\leftrightarrow$): $\dfrac{\neg A \leftrightarrow \neg B}{A \leftrightarrow B}$.

Introduction of "right-hand $\neg$" in $\leftrightarrow$ (I$\leftrightarrow\neg$): $\dfrac{A \leftrightarrow B}{A\neg \leftrightarrow B\neg}$.

Elimination of "right-hand ¬" from ↔ (E ↔ ¬): $\dfrac{A\neg \leftrightarrow B\neg}{A \leftrightarrow B}$.

Introduction of $\bigwedge$ in ↔ ($\bigwedge$ ↔): $\dfrac{A_i \leftrightarrow B_i \quad (i \in I)}{\bigwedge\{A_i : i \in I\} \leftrightarrow \bigwedge\{B_i : i \in I\}}$ $(|I| < \omega_1)$.

Introduction of $\bigvee$ in ↔ ($\bigvee$ ↔): $\dfrac{A_i \leftrightarrow B_i \quad (i \in I)}{\bigvee\{A_i : i \in I\} \leftrightarrow \bigvee\{B_i : i \in I\}}$ $(|I| < \omega_1)$.

Conjunction (C): $\dfrac{A_i \quad (i \in I)}{\bigwedge\{A_i : i \in I\}}$ $(|I| < \omega_1)$.

Necessitation (Nec): $\dfrac{A}{\Box_0 A}$.

Left introduction of $\bigvee$ in → ($\bigvee$ →): $\dfrac{A_i \to B \quad (i \in I)}{\bigvee\{A_i : i \in I\} \to B}$ $(|I| < \omega_1)$.

Graded possibilization of → (P →): $\dfrac{A \to B}{\Diamond_n A \to \Diamond_n B}$ $(n \in \mathbb{N})$.

Iteration of P → (P^m →): $\dfrac{A \to B}{\Diamond_n^m A \to \Diamond_n^m B}$ $(n, m \in \mathbb{N})$.

Graded possibilization of ↔ (P ↔): $\dfrac{A \leftrightarrow B}{\Diamond_n A \leftrightarrow \Diamond_n B}$ $(n \in \mathbb{N})$.

Exact graded possibilization of ↔ (P ! ↔): $\dfrac{A \leftrightarrow B}{\Diamond!_n A \leftrightarrow \Diamond!_n B}$ $(n \in \mathbb{N})$.

Graded necessitation of → (Nec →): $\dfrac{A \to B}{\Box_n A \to \Box_n B}$ $(n \in \mathbb{N})$.

Graded necessitation of ↔ (Nec ↔): $\dfrac{A \leftrightarrow B}{\Box_n A \leftrightarrow \Box_n B}$ $(n \in \mathbb{N})$.

Possibilization of → with $\bigvee$ on the right (P→ $\bigvee$):

$$\dfrac{B \to \bigvee\{A_i : i \in I\}}{\Diamond_0 B \to \bigvee\{\Diamond_0 A_i : i \in I\}} \quad (|I| < \omega_1).$$

T1. $\vdash A_i \to \bigvee\{A_i : i \in I\}$ $(i \in I,\ |I| < \omega_1)$.

T2. $\vdash B \wedge \bigvee\{A_i : i \in I\} \leftrightarrow \bigvee\{B \wedge A_i : i \in I\}$ $(|I| < \omega_1)$.

T3. $\vdash \Diamond_0^n \bigvee\{A_i : i \in I\} \leftrightarrow \bigvee\{\Diamond_0^n A_i : i \in I\}$ $(n \in \mathbb{N},\ |I| < \omega_1)$.

T4. $\vdash \bigvee\{\Diamond_n^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, B \wedge A_i \rangle : i \in I\}$
      $\leftrightarrow \Diamond_n^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, B \wedge \bigvee\{A_i : i \in I\}\rangle)$ $(|I| < \omega_1)$.

Conditioned elimination of choice (CEC):

$$\dfrac{B \to \neg \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee\{A_i : i \in I\}\rangle \quad (h \in I)}{B \to \neg \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle} \quad (|I| < \omega_1).$$

T5. $\vdash \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, A \wedge B \rangle$
       $\rightarrow \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, A \rangle \wedge \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, B \rangle.$

As semantics we shall use the well-known Kripke models $\mathsf{M} = (W, R, P)$. The definition of truth is standard. In particular we recall that for $w \in W$ and $|I| < \omega_1$,

$(\mathsf{M}, w) \vDash \bigvee \{A_i : i \in I\}$ iff   there exists $i \in I$ such that $(\mathsf{M}, w) \vDash A_i$,
$(\mathsf{M}, w) \vDash \bigwedge \{A_i : i \in I\}$ iff   for every $i \in I$, $(\mathsf{M}, w) \vDash A_i$

and for $w \in W$ and $n \in \mathbb{N}$,

$(\mathsf{M}, w) \vDash \Diamond_n A$ iff $|\{w' \in W : w\,R\,w' \text{ and } (\mathsf{M}, w') \vDash A\}| > n$,
$(\mathsf{M}, w) \vDash \Box_n A$ iff $|\{w' \in W : w\,R\,w' \text{ and } (\mathsf{M}, w') \nvDash A\}| \leq n$,
$(\mathsf{M}, w) \vDash \Diamond!_n A$ iff $|\{w' \in W : w\,R\,w' \text{ and } (\mathsf{M}, w') \vDash A\}| = n$.

The soundness theorem has a tedious but obvious proof that we skip.

2 Fragments and consistency properties

We follows essentially [7] both for the notion of fragment and for its properties. The main differences are related to the circumstance that we have to do with a propositional calculus (with operators) instead of a predicative one (without operators).

It can be useful to recall that all the syntactic features are intended to be coded inside set theory so that one needs a 'good' set to detach from the system an appropriate fragment.

We list below the necessary definitions and properties without comments or proofs that can be found in [7].

Definition 2.1. A set $\mathcal{A}$ is said to be a *good set* if

(i) $\mathcal{A}$ is a non-empty transitive set,

(ii) if $a, b \in \mathcal{A}$, then $\{a, b\}, a \cup b, a \times b \in \mathcal{A}$,

(iii) if $a \in \mathcal{A}$ and α is the least ordinal which is not in the transitive closure of a, then $\alpha \in \mathcal{A}$.

Definition 2.2. If $\mathcal{A}$ is a good set, then $\mathsf{F}_{\mathcal{A}} = (\mathcal{S}) \cap \mathcal{A}$ is called a *fragment* (relative to $\mathcal{A}$) of $\mathsf{K}^0_{\omega_1}$. When either misunderstandings are impossible or the topic does not depend on a particular good set, we shall drop the indication of the good set writing simply F for a fragment of $\mathsf{K}^0_{\omega_1}$.

Proposition 2.3. *If $\mathcal{A}$ is a good set and $\mathsf{F}_{\mathcal{A}}$ the fragment of $\mathsf{K}^0_{\omega_1}$ relative to $\mathcal{A}$, then the following statements hold:*

(i) *$\omega \subseteq \mathcal{A}$;*

(ii) *$\mathsf{F}_{\mathcal{A}}$ is closed with respect to negation, finite conjunctions and disjunctions, and to the operators $\Diamond_n$ and $\Box_n$ ($n \in \mathbb{N}$);*

(iii) *if $A \in \mathsf{F}_{\mathcal{A}}$, then, for any subformula B of A, $B \in \mathsf{F}_{\mathcal{A}}$;*

(iv) *if $\{A_i : i \in I\}$ is a countable set of formulas and $\{A_i : i \in I\} \in \mathcal{A}$, then $\{\neg A_i : i \in I\} \in \mathcal{A}$, and $\bigwedge \{A_i : i \in I\}, \bigvee \{A_i : i \in I\} \in \mathsf{F}_{\mathcal{A}};$*

(v) *if $A \in \mathsf{F}_{\mathcal{A}}$, then $A\neg \in \mathsf{F}_{\mathcal{A}}.$*

Proposition 2.4. *If $\{A_i : i \in I\}$ is a countable set of formulas, then there exists a least countable fragment $F_{\mathcal{A}}$ such that $\{A_i : i \in I\} \subseteq F_{\mathcal{A}}$.*

Definition 2.5. Let F be a fragment. An F-*consistency property* is a family C of sets of sentences of F such that

(i) if $\Omega \in C$ and $\Delta \subseteq \Omega$, then $\Delta \in C$;

(ii) if $A \in \Delta \in C$, then $\neg A \notin \Delta$;

(iii) if $\neg A \in \Delta \in C$, then $\Delta \cup \{A\neg\} \in C$;

(iv) if $\bigwedge\{A_i : i \in I\} \in \Delta \in C$ ($|I| < \omega_1$), then, for any $i \in I$, $\Delta \cup \{A_i\} \in C$;

(v) if $\bigvee\{A_i : i \in I\} \in \Delta \in C$ ($|I| < \omega_1$), then, for some $i \in I$, $\Delta \cup \{A_i\} \in C$;

(vi) if $\Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle \in \Delta \in C$ ($|I| < \omega_1$), then there exists $h \in I$ such that $\Delta \cup \{\Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee\{A_i : i \in I\}\rangle\} \in C$;

(vii) if $\Diamond_0 A \in \Delta \in C$, then $\Delta\# \cup \{A\} \in C$, where

$$\Delta\# = \{A : \Box_0 A \in \Delta\} \cup \{\neg A : \neg\Diamond_0 A \in \Delta\}.$$

The idea of using $\Delta\#$ is borrowed from [6]. The next two definitions, those of a validity property for a fragment and of a set of sentences compatible with a consistency property, are borrowed from [1], where the second one has no specific name and is limited to the case when the set of sentences is a validity property.

Definition 2.6. An F-*validity property* is a set Γ of formulas of F that

(i) contains any instance of the axioms of $K^0_{\omega_1}$ that belongs to F,

(ii) is closed with respect to the inference rules of $K^0_{\omega_1}$, provided that premises and conclusions of the rules belong to F,

(iii) does not contain $A \wedge \neg A$ for any formula A belonging to F.

Definition 2.7. Let Γ be a set of sentences of $K^0_{\omega_1}$. A consistency property C is *compatible with* Γ or Γ-*compatible* if $\Delta \cup \{A\} \in C$ for any $\Delta \in C$ and any $A \in \Gamma$.

The idea of the next definition appears both in [7] and [1], but without any particular relevance. On the contrary we stress explicitly the following notion because it is, in a good sense, the key for the proof of the completeness theorem.

Definition 2.8. A set Δ of sentences of F is said to be a *good limit* (to distinguish it from a simple limit, i.e. a set which has only the property (i) below) or else a *limit* (since we shall use only the good ones, indeed) of C and will be written with C as index, Δ_C, if the following conditions hold:

(i) there exists a countable chain $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$, with $\Delta_n \in C$ for each $n \in \mathbb{N}$, such that $\Delta_C = \bigcup\{\Delta_n : n \in \mathbb{N}\}$;

(ii) if $\neg A \in \Delta_C$, then $A\neg \in \Delta_C$;

(iii) if $\bigwedge\{A_i : i \in I\} \in \Delta_C$ ($|I| < \omega_1$), then, for any $i \in I$, $A_i \in \Delta_C$;

(iv) if $\bigvee\{A_i : i \in I\} \in \Delta_C$ ($|I| < \omega_1$), then, for some $i \in I$, $A_i \in \Delta_C$;

(v) if $\Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle \in \Delta_C$ ($|I| < \omega_1$), then there exists $h \in I$ such that $\Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee\{A_i : i \in I\}\rangle \in \Delta_C$.

The collection of the limits of C will be indicated by Φ_C.

Note that a limit of C has also the property

(o) if $A \in \Delta_C$, then $\neg A \notin \Delta_C$

as one can see immediately by observing that, if not so, it would happen e. g. $A \in \Delta_n$, $\neg A \in \Delta_m$, and if $n < m$, then $\Delta_n \subseteq \Delta_m$ and $A, \neg A \in \Delta_m$, contradicting $\Delta_m \in C$ (see Definition 2.5(ii)).

The next theorem states, under suitable conditions, the existence of limits and, at the same time, provides a Lindenbaum-like property of them with respect to the elements of C.

T h e o r e m 2.9. *Let* F *be a countable fragment of* $K^0_{\omega_1}$, C *an* F-consistency property *and* $\Gamma \subseteq F$. *Then for any* $\Delta \in C$ *there exists* $\Delta_C \in \Phi_C$ *such that* $\Delta \subseteq \Delta_C$. *Furthermore, if* C *is* Γ-compatible, *then one has also* $\Delta \cup \Gamma \subseteq \Delta_C$.

P r o o f. Let us define a chain of sets of sentences $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$, where $\Delta_n \in C$ for any $n \in \mathbb{N}$.

First let $\Delta_0 = \Delta$. Then, given Δ_n, let us define Δ_{n+1}.

Let $A_0, A_1, A_2, \ldots$ be an enumeration of the sentences of the fragment F. Consider the sentence A_n. If $\Delta_n \cup \{A_n\} \in C$, then let $\Delta'_n = \Delta_n \cup \{A_n\}$. If $\Delta_n \cup \{A_n\} \notin C$, then let $\Delta'_n = \Delta_n$. When the first case occurs one has to examine the structure of A_n.

If A_n has the form $\bigvee\{A_{n,i} : i \in I\}$ with a countable I, enumerate I, pick out the least $i \in I$ such that $\Delta'_n \cup \{A_{n,i}\} \in C$ and define $\Delta_{n+1} = \Delta'_n \cup \{A_{n,i}\}$.

If, in general, A_n is a C-formula $\Diamond^\sigma_0 \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle$ with a countable I, then $\Diamond^\sigma_0 \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle \in \Delta'_n \in C$. By Definition 2.5(vi), there is $h \in I$ such that $\Delta'_n \cup \{\Diamond^\sigma_0 \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee\{A_i : i \in I\}\rangle\} \in C$. The enumeration of I allows to choose h_0 as the least $h \in I$ for which this happens; so define $\Delta_{n+1} = \Delta'_n \cup \{\Diamond^\sigma_0 \langle Z_0, Z_1, \ldots, Z_{r-1}, A_{h_0} \wedge \bigvee\{A_i : i \in I\}\rangle\}$.

Otherwise define $\Delta_{n+1} = \Delta'_n$.

Obviously, $\Delta_n \in C$ for any $n \in \mathbb{N}$. Finally let $\Delta_C = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. We verify that Δ_C is a limit.

The condition (i) is obviously satisfied.

Let us verify the condition (ii). Suppose $\neg A \in \Delta_C$. We have to show $A\neg \in \Delta_C$. Let $A\neg$ have the index m in the above enumeration of the sentences of F, and let $\neg A = A_n$. If $n = m$ (this happens only if $A = P_\lambda$), the condition is trivially satisfied. Let us suppose that $m < n$. Being $\Delta_m \subseteq \Delta_n$, one has $\Delta_m \cup \{A\neg\} \subseteq \Delta_n \cup \{A\neg\}$. But $\Delta_n \cup \{\neg A\} \in C$, so $\Delta_n \cup \{\neg A\} \cup \{A\neg\} \in C$. From $\Delta_m \cup \{A\neg\} \subseteq \Delta_n \cup \{\neg A\} \cup \{A\neg\} \in C$ one gets the conclusion. Suppose now $n < m$. Then $\neg A \in \Delta_{n+1}$ and $\Delta_{n+1} \subseteq \Delta_m$ so that $\neg A \in \Delta_m$ and $\Delta_m \cup \{A\neg\} \in C$. This yields $\Delta_m \cup \{A_m\} \in C$, that is $A\neg = A_m \in \Delta_{m+1} \subseteq \Delta_C$.

Let us consider the condition (iii). Let $\bigwedge\{A_i : i \in I\} \in \Delta_C$ and assume that $\bigwedge\{A_i : i \in I\} = A_n$. We have to verify that the A_i's are in Δ_C. If $A_i = A_m$, then certainly $m \neq n$. Suppose first $n < m$. Then $\bigwedge\{A_i : i \in I\} \in \Delta_{n+1} \subseteq \Delta_m$. This yields $\Delta_m \cup \{A_i\} \in C$ (because $A_i = A_m$), so $A_i \in \Delta_C$. If $m < n$, then $\Delta_n \cup \{\bigwedge\{A_i : i \in I\}\} \in C$, so $\Delta_n \cup \{\bigwedge\{A_i : i \in I\}\} \cup \{A_i\} \in C$. Being $\Delta_m \subseteq \Delta_n$, one has $\Delta_m \cup \{A_i\} \subseteq \Delta_n \cup \{\bigwedge\{A_i : i \in I\}\} \cup \{A_i\} \in C$, C is closed with respect to inclusion (Definition 2.5(i)), so as above we reach the desired conclusion.

As to condition (iv), suppose $\bigvee\{A_i : i \in I\} \in \Delta_C$. From the definition of Δ_C it follows that $\Delta_n \cup \{\bigvee\{A_i : i \in I\}\} \in C$ for some $n \in \mathbb{N}$. Looking at the definition of the chain of the Δ_n, there is an $A_i \in \Delta_{n+1} \subseteq \Delta_C$, so the condition is verified.

The condition (v) is obviously satisfied.

Finally let us verify that if C is a Γ-compatible consistency property, then every Δ_C built as above contains Γ. In fact, if $A_n \in \Gamma$, then $\Delta_n \cup \{A_n\} \in C$, so that $A_n \in \Delta_{n+1} \subseteq \Delta_C$. $\qquad\square$

O b s e r v a t i o n 2.10. *The limit sets are closed with respect to Modus Ponens.*

P r o o f. Suppose that $A \in \Delta_C$ and $A \to B \in \Delta_C$. The latter means $\neg A \vee B \in \Delta_C$ so that either $\neg A \in \Delta_C$ or $B \in \Delta_C$; but $A \in \Delta_C$ and property (o) implies $B \in \Delta_C$. $\square$

O b s e r v a t i o n 2.11. *Let C be an F-consistency property compatible with an F-validity property Γ, and let Δ_C be a limit such that $\Gamma \subseteq \Delta_C$. If $\Diamond_0(A \wedge B) \in \Delta_C$, then $\Diamond_0 A \in \Delta_C$.*

P r o o f. It is obvious that the formulas A, B, $A \wedge B$, $A \wedge B \to A$, $\Box_0(A \wedge B \to A)$, $\Diamond_0(A \wedge B)$, $\Diamond_0 A$, $\Diamond_0(A \wedge B) \to \Diamond_0 A$, $\Box_0(A \wedge B \to A) \to (\Diamond_0(A \wedge B) \to \Diamond_0 A)$ belong to the fragment F. $A \wedge B \to A$ and $\Box_0(A \wedge B \to A) \to (\Diamond_0(A \wedge B) \to \Diamond_0 A)$ are axioms written with formulas of the fragment, so they belong to Γ. Furthermore, from $A \wedge B \to A \in \Gamma$ one gets $\Box_0(A \wedge B \to A) \in \Gamma$, because Γ is closed with respect to the rule (Nec) and both the premiss and the conclusion are formulas of the fragment. By the Γ-compatibility of C, $\Box_0(A \wedge B \to A) \to (\Diamond_0(A \wedge B) \to \Diamond_0 A) \in \Delta_C$ and $\Box_0(A \wedge B \to A) \in \Delta_C$, then, remembering that Δ_C is closed with respect to Modus Ponens, $\Diamond_0(A \wedge B) \to \Diamond_0 A \in \Delta_C$. By hypothesis $\Diamond_0(A \wedge B) \in \Delta_C$, and, by Modus Ponens again, $\Diamond_0 A \in \Delta_C$. $\qquad\square$

This observation will be useful later on, but its principal aim now is to display a style of proof that will be used again and again in the sequel.

The following observations show that the limits have the main properties of the maximal consistent sets with respect to the fragment F.

O b s e r v a t i o n 2.12. *Let C be an F-consistency property compatible with an F-validity property Γ, and let Δ_C be a limit such that $\Gamma \subseteq \Delta_C$. If $\bigvee\{A_i : i \in I\} \in F$, then $\bigvee\{A_i : i \in I\} \in \Delta_C$ iff there exists an $i \in I$ such that $A_i \in \Delta_C$.*

P r o o f. The implication from left to right follows immediately by the very definition of a limit. As to the inverse implication, note that $A_i \in F$ $(i \in I)$, $\neg A_i \in F$ $(i \in I)$, $(\bigvee\{A_i : i \in I\})\neg = \bigwedge\{\neg A_i : i \in I\} \in F$, $(\bigwedge\{\neg A_i : i \in I\})\neg = \bigvee\{\neg\neg A_i : i \in I\} \in F$, so that, using various axioms and (both basic and derived) rules, one reconstructs T1 as an element of Γ, i.e. $A_i \to \bigvee\{A_i : i \in I\} \in \Gamma$. Since $\Gamma \subseteq \Delta_C$, the statement follows from Observation 2.10. $\qquad\square$

In particular, for any $A \in F$, $A \vee \neg A \in \Gamma \subseteq \Delta_C$, so that either $A \in \Delta_C$ or $\neg A \in \Delta_C$, but not both (by property (o)). This remark justifies the next observation:

O b s e r v a t i o n 2.13. *Let C be an F-consistency property compatible with an F-validity property Γ, and let Δ_C be a limit such that $\Gamma \subseteq \Delta_C$. If $A \in F$, then $A \notin \Delta_C$ iff $\neg A \in \Delta_C$.*

Finally we have also

O b s e r v a t i o n 2.14. *Let C be an F-consistency property compatible with an F-validity property Γ, and let Δ_C be a limit such that $\Gamma \subseteq \Delta_C$. If $\bigwedge\{A_i : i \in I\} \in F$, then $\bigwedge\{A_i : i \in I\} \in \Delta_C$ iff for any $i \in I$, $A_i \in \Delta_C$.*

3 A model canonically associated with a consistency property compatible with a validity property

In this section we prove that, under certain conditions, one can repeat (with suitable adaptions) the argument developed in [4] and [2] to define a model that is canonically associated with a consistency property compatible with a validity property.

In the sequel we assume as fixed a countable fragment F of $\mathsf{K}^0_{\omega_1}$ and that all consistency and validity properties are relative to F.

L e m m a 3.1. *Let Γ be a validity property, C a Γ-compatible consistency property and Δ_C a limit such that $\Gamma \subseteq \Delta_C$. Then we have:*

(a) *For every $n, m \in \mathbb{N}$ and $A \in F$, if $n \geq m$ and $\Diamond_n A \in \Delta_C$, then $\Diamond_m A \in \Delta_C$.*

(b) *If $A_0, A_1, \ldots, A_k \in F$ $(k \geq 1)$ such that $\bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq k\} \in \Gamma$ and $\Diamond!_{n_i} A_i \in \Delta_C$ for $0 \leq i \leq k$, then $\Diamond!_{n_0 + \cdots + n_k} \bigvee\{A_i : i \leq k\} \in \Delta_C$.*

(c) *If $A \in F$ and $\Diamond!_n A, \Diamond!_m A \in \Delta_C$, then $n = m$.*

(d) *For any $A \in F$ exactly one of the following cases occurs:*

  (i) *for every $n \in \mathbb{N}$, $\Diamond_n A \in \Delta_C$;*

  (ii) *there exists (a unique) $n \in \mathbb{N}$ such that $\Diamond!_n A \in \Delta_C$.*

(e) *If $A \to B \in \Gamma$ and $\Diamond!_m B \in \Delta_C$, then there exists a unique $n \in \mathbb{N}$ such that $\Diamond!_n A \in \Delta_C$, and one has necessarily $n \leq m$.*

P r o o f . The proof of (a) is by induction on $n - m$. If $n - m = 0$, the statement (a) is trivial. Suppose, as inductive hypothesis, that (a) is true for $n - m = k$ and let $n = m + k + 1$. If $\Diamond_{m+k+1} A \in \Delta_C$, being $\Diamond_{m+k+1} A \to \Diamond_{m+k} A \in \Delta_C$, one has $\Diamond_{m+k} A \in \Delta_C$, and the inductive hypothesis yields $\Diamond_m A \in \Delta_C$.

The statement (b) is proved by induction on the number $k + 1$ of formulas. First suppose $k = 1$, i. e. we have the hypotheses $\neg(A_0 \wedge A_1) \in \Gamma$ and $\Diamond!_{n_0} A_0, \Diamond!_{n_1} A_1 \in \Delta_C$. As a validity property, Γ is closed with respect to (Nec) (premiss and conclusion being in the fragment), so that we get $\Box_0 \neg(A_0 \wedge A_1) \in \Gamma$ and $\neg\Diamond_0(A_0 \wedge A_1) \in \Gamma$. But $\neg\Diamond_0(A_0 \wedge A_1) \to ((\Diamond!_{n_0} A_0 \wedge \Diamond!_{n_1} A_1) \to \Diamond!_{n_0 + n_1}(A_0 \vee A_1)) \in \Gamma \subseteq \Delta_C$ and Δ_C is closed with respect to Modus Ponens. Thus the thesis is proved. Now let (b) hold for $k = 1, \ldots, r$. We must prove it when $k = r + 1$. The hypotheses are: $\bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq r + 1\} \in \Gamma \subseteq \Delta_C$ and $\Diamond!_{n_i} A_i \in \Delta_C$ for $i \leq r + 1$. From

$$\bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq r + 1\} \to \bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq r\} \in \Gamma \subseteq \Delta_C$$

and the closure of Δ_C with respect to Modus Ponens one gets

$$\bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq r\} \in \Delta_C$$

so that, by the inductive hypothesis, $\Diamond!_{n_0 + \cdots + n_r} \bigvee\{A_i : i \leq r\} \in \Delta_C$. Now we put $A = \bigvee\{A_i : i \leq r\}$ and $n = n_0 + \cdots + n_r$. By suitable tautologies and applications of Modus Ponens, one proves the following statements:

$$\bigwedge\{\neg(A_i \wedge A_j) : 0 \leq i < j \leq r + 1\} \in \Gamma,$$
$$\bigwedge\{\neg(A_i \wedge A_{r+1}) : i \leq r\} \in \Gamma,$$
$$\neg\bigvee\{A_i : i \leq r\} \vee \neg A_{r+1} = \neg A \vee \neg A_{r+1} \in \Gamma,$$
$$\neg(A \wedge A_{r+1}) \in \Gamma.$$

From the initial hypotheses and from the inductive one, we obtain $\Diamond!_n A \in \Delta_C$ and $\Diamond!_{n_{r+1}} A_{r+1} \in \Delta_C$. So, applying the statement in the case $k = 1$, we conclude $\Diamond!_{n+n_{r+1}}(A \vee A_{r+1}) \in \Delta_C$.

For the proof of (c) we suppose that $n \neq m$, e.g. $n > m$. Surely $n > 0$, so $\Diamond!_n A = \Diamond_{n-1} A \wedge \neg \Diamond_n A \in \Delta_C$. Since $(\Diamond_{n-1} A \wedge \neg \Diamond_n A) \to \Diamond_{n-1} A$ is a tautology formed by formulas of the fragment, $(\Diamond_{n-1} A \wedge \neg \Diamond_n A) \to \Diamond_{n-1} A \in \Gamma \subseteq \Delta_C$, and therefore $\Diamond_{n-1} A \in \Delta_C$. Because $n - 1 \geq m$, from (a) we get $\Diamond_m A \in \Delta_C$. The other hypothesis is $\Diamond!_m A \in \Delta_C$, which implies $\neg \Diamond_m A \in \Delta_C$: contradiction.

To prove (d) note first that the two conditions are incompatible: in fact, if (ii) holds, then $\neg \Diamond_n A \in \Delta_C$, what conflicts with (i). Suppose (i) does not hold. Then there exists an n such that $\Diamond_n A \notin \Delta_C$, i.e. $\neg \Diamond_n A \in \Delta_C$. Let m be the least such number. If $m = 0$, then $\neg \Diamond_0 A = \Diamond!_0 A \in \Delta_C$. If $m > 0$, then $\Diamond_{m-1} A \in \Delta_C$ and $\neg \Diamond_m A \in \Delta_C$. So $\neg \Diamond_m A \wedge \Diamond_{m-1} A = \Diamond!_m A \in \Delta_C$. The uniqueness of m follows from (c).

For the proof of (e) we remark that $A \to B$ and also $\Diamond_n A \to \Diamond_n B$ belong to the fragment, and in that circumstance Γ is closed with respect to the rules of inference. So, for any $n \in \mathbb{N}$, $\Diamond_n A \to \Diamond_n B \in \Gamma$ and, by compatibilty, $\Diamond_n A \to \Diamond_n B \in \Delta_C$ and also $\neg \Diamond_n B \to \neg \Diamond_n A \in \Delta_C$. From $\Diamond!_m B \in \Delta_C$ one gets $\neg \Diamond_m B \in \Delta_C$ and, by Modus Ponens, $\neg \Diamond_m A \in \Delta_C$. From (d) it follows $\Diamond!_n A \in \Delta_C$ for a unique n. So, if $m < n$, then $n > 0$ and $\Diamond_{n-1} A \in \Delta_C$. Moreover $n - 1 \geq m$ and, by (a), $\Diamond_m A \in \Delta_C$ and $\Diamond_m B \in \Delta_C$: contradiction. Therefore, $n \leq m$.   □

Lemma 3.2. *Let Γ be a validity property, C a Γ-compatible consistency property and Δ_C a limit containing Γ. Let $A_0, A_1, \ldots, A_k$ $(k \geq 1)$ be any formulas of Γ. Consider the $r = 2^k$ formulas $A_0 \wedge C_1 \wedge \cdots \wedge C_k$, where C_i is either A_i or $\neg A_i$ $(1 \leq i \leq k)$ and call them $B_0, B_1, \ldots, B_{r-1}$.*

(a) If there exists $n \in \mathbb{N}$ such that $\Diamond!_n A_0 \in \Delta_C$, then there exist also $n_0, n_1, \ldots, n_{r-1}$ such that $n = n_0 + n_1 + \cdots + n_{r-1}$ and $\Diamond!_{n_h} B_h$ for $0 \leq h < r$.

(b) If $\Diamond_n A_0 \in \Delta_C$ for any $n \in \mathbb{N}$, then there exists $h < r$ such that $\Diamond_n B_h \in \Delta_C$ for every $n \in \mathbb{N}$.

Proof. For the proof of (a) we first note that $B_h \to A_0 \in \Gamma$ for any $h < r$. From $\Diamond!_n A_0 \in \Delta_C$ and Lemma 3.1(e), there exists $n_h \leq n$ such that $\Diamond!_{n_h} B_h \in \Delta_C$. Now we let $m = n_0 + n_1 + \cdots + n_{r-1}$ and prove that $m = n$. For any h, l such that $0 \leq h < l < r$, $\neg(B_h \wedge B_l)$ is a tautology and a formula of F. Therefore, $\neg(B_h \wedge B_l) \in \Gamma$ and also $\bigwedge\{\neg(B_h \wedge B_l) : 0 \leq h < l < r\} \in \Gamma$. Let $G = \bigvee\{B_h : h < r\}$, which is a formula of the fragment. By Lemma 3.1(b) one has $\Diamond!_m G \in \Delta_C$. Now let $D_h = C_1 \wedge \cdots \wedge C_k$ and $G' = \bigvee\{D_h : 0 \leq h < r\}$. G' is a tautology and is in F, therefore $G' \in \Gamma$. For the same reasons, $A_0 \leftrightarrow A_0 \wedge G' \in \Gamma$ and also $A_0 \wedge G' \leftrightarrow G \in \Gamma$. So we conclude that the formula $G \leftrightarrow A_0$ is a tautology and is in F, therefore $G \leftrightarrow A_0 \in \Gamma$. The formula $\Diamond!_n G \leftrightarrow \Diamond!_n A_0$ can be deduced by tautologies and derived rules and belongs to F. Thus $\Diamond!_n G \leftrightarrow \Diamond!_n A_0 \in \Gamma$ and therefore $\Diamond!_n G \leftrightarrow \Diamond!_n A_0 \in \Delta_C$. But $\Diamond!_n A_0 \in \Delta_C$ so that $\Diamond!_n G \in \Delta_C$. Therefore, $\Diamond!_n G \in \Delta_C$ and $\Diamond!_m G \in \Delta_C$, what implies $n = m$ by Lemma 3.1(c).

Turning to (b), suppose the statement fails. Then, for any $h < r$, there exists n_h such that $\Diamond!_{n_h} B_h \in \Delta_C$ and, arguing as in (a), $\Diamond!_m A \in \Delta_C$. Then it cannot be $\Diamond_n A \in \Delta_C$ for any $n \in \mathbb{N}$ (by Lemma 3.1(d)).   □

Definition 3.3. Let Γ be a validity property and C a Γ-compatible consistency property. The function $m : \Phi_C \times \Phi_C \longrightarrow \omega \cup \{\omega\}$ is defined as follows:

$$m(\Delta_C, \Omega_C) = \begin{cases} \omega & \text{if for any } A \in \Omega_C \text{ and } n \in \mathbb{N}, \; \Diamond_n A \in \Delta_C, \\ \min\{n \in \mathbb{N} : \Diamond!_n A \in \Delta_C \text{ and } A \in \Omega_C\} & \text{otherwise.} \end{cases}$$

Lemma 3.4. *Let Γ be a validity property, C a Γ-compatible consistency property and Δ_C, Ω_C two limits of C containing Γ. The following conditions are equivalent:*

(i) $m(\Delta_C, \Omega_C) \neq 0$.

(ii) *For any $A \in F$, if $A \in \Omega_C$, then $\Diamond_0 A \in \Delta_C$.*

(iii) *For any $A \in F$, if $\Box_0 A \in \Delta_C$, then $A \in \Omega_C$.*

Proof. Suppose (i) and not (ii), i.e. there is $A \in F$ such that $A \in \Omega_C$ and $\Diamond_0 A \notin \Delta_C$. Then $\neg\Diamond_0 A = \Diamond!_0 A \in \Delta_C$, and by the preceding definition this yields $m(\Delta_C, \Omega_C) = 0$: contradiction.

Now let us prove that (ii) implies (iii). First, since $\Box_0 A \in \Delta_C$, A and all the formulas below, built from A, are in F. Because $\Box_0 A \leftrightarrow \neg\Diamond_0\neg A$ is an axiom and is in F, one has $\Box_0 A \leftrightarrow \neg\Diamond_0\neg A \in \Delta_C$. Then $\neg\Diamond_0\neg A \in \Delta_C$, therefore $\Diamond_0\neg A \notin \Delta_C$, and by (ii), $\neg A \notin \Omega_C$ what implies $A \in \Omega_C$.

Finally let us prove that (iii) implies (i). If it were $m(\Delta_C, \Omega_C) = 0$, then there would exist $A \in \Omega_C$ such that $\Diamond!_0 A \in \Delta_C$, that is $\neg\Diamond_0 A \in \Delta_C$. Then $\Box_0\neg A \in \Delta_C$ and, by (iii), $\neg A \in \Omega_C$: contradiction. $\square$

Lemma 3.5. *Let F be a countable fragment, Γ an F-validity property, C a Γ-compatible F-consistency property, and Δ_C a limit of C such that $\Gamma \subseteq \Delta_C$. If $\Diamond_0 A \in \Delta_C$, then there exists $\Omega_C \in \Phi_C$ such that $\Gamma \cup \{A\} \subseteq \Omega_C$ and $m(\Delta_C, \Omega_C) \neq 0$.*

Proof. Let $A = A_0, A_1, \ldots, A_n, \ldots$ be an enumeration of the sentences of F. By this enumeration we shall build a chain $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_n \subseteq \cdots$ of finite elements of C. Those sets will have the further property that if $B_n = \bigwedge \Omega_n$, then $\Diamond_0 B_n \in \Delta_C$.

First we define Ω_0. Let $\Omega' = \{A_0\} = \{A\}$. We examine the structure of A.

If A has the form $\bigvee\{A_i : i \in I\}$, one has $\Diamond_0\bigvee\{A_i : i \in I\} \in \Delta_C$. That is a formula of type C and, by Definition 2.8(v), there is $h \in I$ such that $\Diamond_0(A_h \wedge \bigvee\{A_i : i \in I\})$ belongs to Δ_C. Then define $\Omega_0 = \Omega' \cup \{A_h\} = \{\bigvee\{A_i : i \in I\}, A_h\}$. Let us verify the conditions $\Diamond_0 B_0 = \Diamond_0(A_h \wedge \bigvee\{A_i : i \in I\}) \in \Delta_C$ and $\Omega_0 \in C$. The first one is obvious. As to the second condition, from $\Diamond_0 B_0 \in \Delta_C$ one gets $\Diamond_0 B_0 \in \Delta_n \in C$ for some $n \in \mathbb{N}$, which implies $\Delta_n \# \cup \{A_h \wedge \bigvee\{A_i : i \in I\}\} \in C$. Then there exists a limit Π_C such that $A_h \wedge \bigvee\{A_i : i \in I\} \in \Pi_C$. By Observation 2.14, $\bigvee\{A_i : i \in I\} \in \Pi_C$ and $A_h \in \Pi_C$. So, if $\Pi_0 \subseteq \Pi_1 \subseteq \Pi_2 \subseteq \cdots$ is the chain whose limit is Π_C, for suitable $n, m \in \mathbb{N}$ one has $\bigvee\{A_i : i \in I\} \in \Pi_n$ and $A_h \in \Pi_m$, and if e.g. $n \leq m$, then $\Omega_0 = \{\bigvee\{A_i : i \in I\}, A_h\} \subseteq \Pi_m \in C$ so that $\Omega_0 \in C$ (see Definition 2.5(i)).

It will result useful in the sequel to note that the preceding argument can be easily generalized to obtain the following implication:

(*)   If $\Diamond_0(A_1 \wedge \cdots \wedge A_n) \in \Delta_C$, then $\{A_1, \ldots, A_n\} \in C$.

Now let $A = \Diamond_0^\sigma \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle$ with $\sigma = \langle m_0, m_1, \ldots, m_r\rangle$. Because of $\Diamond_0 A \in \Delta_C$ one has $\Diamond_0^{\sigma'} \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee\{A_i : i \in I\}\rangle \in \Delta_C$, where

$\sigma' = \langle m_0 + 1, m_1, \ldots, m_r \rangle$. From this it follows that there exists $h \in I$ such that $\Diamond_0^{\sigma'} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle \in \Delta_C$. Then we define the set Ω_0 as $\{ \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee \{ A_i : i \in I \} \rangle, \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle) \}$. It is obvious that $\Omega_0 \in C$. Let us verify that $\Diamond_0 B_0 \in \Delta_C$. Obviously, the formula

$$\Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle$$
$$\rightarrow \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee \{ A_i : i \in I \} \rangle$$
$$\wedge \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle$$

belongs to Γ, and as well the formula

$$\Diamond_0 \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle$$
$$\rightarrow \Diamond_0 (\Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee \{ A_i : i \in I \} \rangle$$
$$\wedge \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle).$$

Because $\Gamma \subseteq \Delta_C$ and Δ_C is closed with respect to Modus Ponens, we obtain that $\Diamond_0 B_0 \in \Delta_C$.

In every other case $\Omega_0 = \{ A \}$, and it is obvious that $\Diamond_0 B_0 \in \Delta_C$ and $\Omega_0 \in C$.

Suppose now that we have built $\Omega_n \in C$. Let $B_n = \bigwedge \Omega_n$ and $\Diamond_0 B_n \in \Delta_C$. Let us define $\Omega_{n+1} \in C$ such that $\Omega_n \subseteq \Omega_{n+1}$ and $\Diamond_0 B_{n+1} = \Diamond_0 \bigwedge \Omega_{n+1} \in \Delta_C$. Observe that the last condition implies, by $(*)$, that $\Omega_{n+1} \in C$, so that we shall define $\Omega_{n+1} \supseteq \Omega_n$ such that $\Diamond_0 \bigwedge \Omega_{n+1} \in \Delta_C$.

Consider the tautology $B_n \rightarrow (B_n \wedge A_{n+1}) \vee (B_n \wedge \neg A_{n+1}) \in \Delta_C$. From this one gets $\Diamond_0 B_n \rightarrow \Diamond_0 (B_n \wedge A_{n+1}) \vee \Diamond_0 (B_n \vee \neg A_{n+1}) \in \Delta_C$ and, by Modus Ponens, $\Diamond_0 (B_n \wedge A_{n+1}) \vee \Diamond_0 (B_n \wedge \neg A_{n+1}) \in \Delta_C$. By the properties of the limits, then either $\Diamond_0 (B_n \wedge A_{n+1}) \in \Delta_C$ or $\Diamond_0 (B_n \wedge \neg A_{n+1}) \in \Delta_C$. In the second case we put $\Omega_{n+1} = \Omega_n \cup \{ \neg A_{n+1} \}$; then one has $\Diamond_0 B_{n+1} = \Diamond_0 (B_n \wedge \neg A_{n+1}) \in \Delta_C$ so that $\Omega_{n+1} \in C$. If $\Diamond_0 (B_n \wedge A_{n+1}) \in \Delta_C$, we put $\Omega'_n = \Omega_n \cup \{ A_{n+1} \}$ and proceed as in case $n = 0$ examining the structure of A_{n+1}.

If $A_{n+1} = \bigvee \{ A_i : i \in I \}$, then $\Diamond_0 (B_n \wedge A_{n+1}) \in \Delta_C$ is a formula of type C: $\Diamond_0 (B_n \wedge A_{n+1}) = \Diamond_0^{\sigma} \langle B_n, A_{n+1} \rangle$, where $\sigma = \langle 1, 0 \rangle$. The properties of the limits allow to assert the existence of an $h \in I$ such that $\Diamond_0^{\sigma} \langle B_n, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle = \Diamond_0 (B_n \wedge A_h \wedge \bigvee \{ A_i : i \in I \}) \in \Delta_C$. In this case we join to Ω'_n the sentence A_{h_0} with the least such index h: $\Omega_{n+1} = \Omega_n \cup \{ \bigvee \{ A_i : i \in I \} \} \cup \{ A_{h_0} \}$. This definition implies $\Diamond_0 B_{n+1} \in \Delta_C$ and $\Omega_{n+1} \in C$.

If $A_{n+1} = \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, \bigvee \{ A_i : i \in I \} \rangle$, then $\Diamond_0 (B_n \wedge A_{n+1}) \in \Delta_C$ is also a formula of type C and there exists an $h \in I$ such that

$$\Diamond_0 (B_n \wedge \Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_h \wedge \bigvee \{ A_i : i \in I \} \rangle) \in \Delta_C.$$

Then join to Ω'_n the formula $\Diamond_0^{\sigma} \langle Z_0, Z_1, \ldots, Z_{r-1}, A_{h_0} \wedge \bigvee \{ A_i : i \in I \} \rangle$ with the least such $h = h_0$. One easily verify that $\Diamond_0 B_{n+1} \in \Delta_C$ and therefore $\Omega_{n+1} \in C$.

In every other case let $\Omega_{n+1} = \Omega_n \cup \{ A_{n+1} \}$, and it is obvious that $\Diamond_0 B_{n+1} \in \Delta_C$ and $\Omega_{n+1} \in C$.

To conclude, we let $\Omega_C = \bigcup \{ \Omega_n : n \in \mathbb{N} \}$. What needs to checking is that Ω_C is really a limit and that $m(\Delta_C, \Omega_C) \neq 0$.

The condition (i) of Definition 2.8 is satisfied by definition.

Let us examine the condition (ii). We have to verify that, for any formula $Z \in F$, if $\neg Z \in \Omega_C$, then $Z\neg \in \Omega_C$. The formula $\neg Z$ occurs in the above enumeration of F with an index n and $Z\neg$ occurs with an index m. If $n = m$ (i.e. $A = P_\lambda$), the statement is trivial. If $m = 0$, the statement is trivially true again, because $A = A_0 \in \Omega_C$. Thus we can suppose $n \neq m$ and $m > 0$. By hypothesis one has $\Diamond_0(B_{n-1} \wedge \neg Z) \in \Delta_C$ (if $n = 0$, the formula inside brackets is intended equal to $\neg Z = A_0 = A$). We have to prove that $\Diamond_0(B_{m-1} \wedge (Z\neg)) \in \Delta_C$. Suppose not, then $\Diamond_0(B_{m-1} \wedge \neg(Z\neg)) \in \Delta_C$. As a consequence of $(*)$, $\Omega_m = \{A, \ldots, \neg(Z\neg)\} \in C$. If $m < n$, then $\Omega_m \subseteq \Omega_n$ and $\Omega_n = \{A, \ldots, \neg(Z\neg), \ldots, \neg Z\}$. Since $\neg Z \in \Omega_n$, $\Omega_n \cup \{Z\neg\} \in C$: contradiction. If $n < m$, suppose again $\Diamond_0(B_{m-1} \wedge \neg(Z\neg)) \in \Delta_C$. Then $\Omega_n \subseteq \Omega_m = \{A, \ldots, \neg(Z\neg)\}$. Since $\neg Z \in \Omega_n$, then $\neg Z \in \Omega_m \in C$, so $\Omega_m \cup \{Z\neg\} \in C$: contradiction.

To see that the condition (iii) is satisfied, assume that $\bigwedge\{A_i : i \in I\} \in \Omega_C$. We have to prove that $A_i \in \Omega_C$ for any $i \in I$. If in the fixed enumeration of F, $\bigwedge\{A_i : i \in I\} = A_n$ and $A_i = A_m$, then, of course, $n \neq m$. Therefore $\Omega_n = \{A, \ldots, \bigwedge\{A_i : i \in I\}\}$. Suppose $A_i \notin \Omega_m$, then $\neg A_i \in \Omega_m$. First assume $n < m$. Then from $\Omega_n \subseteq \Omega_m$ it follows $\bigwedge\{A_i : i \in I\} \in \Omega_m \in C$. So one has $\Omega_m \cup \{A_i\} \in C$, but this cannot be accepted, because $\neg A_i \in \Omega_m$. If $m < n$, one has $\neg A_i \in \Omega_n$ and $\bigwedge\{A_i : i \in I\} \in \Omega_n$, but this forces $\Omega_n \cup \{A_i\} \in C$: contradiction.

Conditions (iv) and (v) are immediate consequences of the definition of Ω_C.

So we have verified that Ω_C is a limit. Let us prove now that $\Gamma \subseteq \Omega_C$. If $A_0 = A \in \Gamma$, then, by definition, $A \in \Omega_C$. If $A_{n+1} \in \Gamma$ ($n \in \mathbb{N}$) we can assert $A_{n+1} \in \Omega_{n+1}$ if we can exclude that $\neg A_{n+1} \in \Omega_{n+1}$, what amounts to exclude that $\Diamond_0(B_n \wedge \neg A_{n+1}) \in \Delta_C$ (see above). In fact, if $\Diamond_0(B_n \wedge \neg A_{n+1}) \in \Delta_C$, then, by $(*)$, $\Omega_n \cup \{\neg A_{n+1}\} \in C$, and by the Γ-compatibility of C, $\Omega_n \cup \{\neg A_{n+1}\} \cup \{A_{n+1}\} \in C$: contradiction.

It remains to check the value of $m(\Delta_C, \Omega_C)$. If it were $m(\Delta_C, \Omega_C) = 0$, then there would exist $A_{n+1} \in \Omega_C$ such that $\Diamond!_0 A_{n+1} \in \Delta_C$, that is $\neg\Diamond_0 A_{n+1} \in \Delta_C$ (the index is positive because $\Diamond_0 A_0 \in \Delta_C$). From the definition of the chain whose limit is Ω_C it follows $A_{n+1} \in \Omega_{n+1}$ what yields $\Diamond_0(B_n \wedge A_{n+1}) \in \Delta_C$. By Observation 2.11, $\Diamond_0 A_{n+1} \in \Delta_C$: contradiction. Therefore, $m(\Delta_C, \Omega_C) \neq 0$ and the proof is complete. $\square$

Lemma 3.6. *Let Γ be a validity property, C a Γ-compatible consistency property and let $\Delta_{C,1}, \Delta_{C,2}, \ldots, \Delta_{C,h}$ ($h \geq 2$) be distinct limits of C containing Γ. Then there exist formulas $A_1, A_2, \ldots, A_h$ in F such that $A_k \in \Delta_{C,k}$ ($1 \leq k \leq h$) and $\bigwedge\{\neg(A_k \wedge A_m) : 1 \leq k < m \leq h\} \in \Gamma$.*

Proof. If $h = 2$, we have two distinct limits $\Delta_{C,1}$ and $\Delta_{C,2}$. Then there exists $A \in F$ such that $A \in \Delta_{C,1}$ and $A \notin \Delta_{C,2}$. From the latter one gets $\neg A \in \Delta_{C,2}$. By letting $A_1 = A$ and $A_2 = \neg A$ and the fact that $\neg(A \wedge \neg A)$ is a tautology and is in F, one obtains $\neg(A_1 \wedge A_2) \in \Gamma$.

Suppose now inductively that $A_1, A_2, \ldots, A_{h-1}$ are formulas in F such that $A_k \in \Delta_{C,k}$ ($1 \leq k \leq h-1$) and $\bigwedge\{\neg(A_k \wedge A_m) : 1 \leq k < m \leq h-1\} \in \Gamma$. The sets $\Delta_{C,k}$ ($1 \leq k < h$) are distinct from $\Delta_{C,h}$. Thus there exist formulas $B_k \notin \Delta_{C,k}$ ($1 \leq k < h$) such that $B_k \in \Delta_{C,h}$. So $\neg B_k \in \Delta_{C,k}$ ($1 \leq k < h$) and $B_1 \wedge B_2 \wedge \cdots \wedge B_{h-1} \in \Delta_{C,h}$. Obviously all the above formulas are in F and the tautologies we used are in Γ. Let $G_k = A_k \wedge \neg B_k$ ($1 \leq k < h$); it is evident that

$G_k \in \mathsf{F}$ and that $G_k \in \Delta_{\mathsf{C},k}$ $(1 \le k < h)$. Moreover let $G_h = B_1 \wedge B_2 \wedge \cdots \wedge B_{h-1}$. The conclusion for the formulas $G_1, \ldots, G_h$ is immediate. □

Definition 3.7. Let Γ be a validity property, C a Γ-compatible consistency property and Δ_{C} a limit containing Γ. The *satisfying family of* Δ_{C} is the set

$$\mathrm{SF}(\Delta_{\mathsf{C}}) = \{\{\Omega_{\mathsf{C}}\} \times m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C}}) : \Gamma \subseteq \Omega_{\mathsf{C}} \in \Phi_{\mathsf{C}}\}.$$

Observation 3.8. *The elements of* $\mathrm{SF}(\Delta_{\mathsf{C}})$ *are pairs as* (Ω_{C}, n) *but they will be often indicated simply by their first component, e. g.* (Ω_{C}, n) *will be written as* Ω_{C}.

Theorem 3.9. *Let* Γ *be a validity property,* C *a* Γ-*compatible consistency property and* Δ_{C} *a limit containing* Γ. *For any formula* $A \in \mathsf{F}$ *and for any* $n \in \mathbb{N}$, $\Diamond_n A \in \Delta_{\mathsf{C}}$ *iff* $|\{\Omega_{\mathsf{C}} \in \mathrm{SF}(\Delta_{\mathsf{C}}) : A \in \Omega_{\mathsf{C}}\}| > n$.

Proof. We prove first that $|\{\Omega_{\mathsf{C}} \in \mathrm{SF}(\Delta_{\mathsf{C}}) : A \in \Omega_{\mathsf{C}}\}| > n$ implies $\Diamond_n A \in \Delta_{\mathsf{C}}$. Two cases can occur:

(i) there exists $\Omega_{\mathsf{C}} \in \mathrm{SF}(\Delta_{\mathsf{C}})$ such that $A \in \Omega_{\mathsf{C}}$ and $m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C}}) > n$;

(ii) for any $\Omega_{\mathsf{C}} \in \mathrm{SF}(\Delta_{\mathsf{C}})$, if $A \in \Delta_{\mathsf{C}}$, then $m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C}}) \le n$.

In case (i), if $\Diamond_n A \notin \Delta_{\mathsf{C}}$, then $\neg\Diamond_n A \in \Delta_{\mathsf{C}}$, so $m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C}}) \le n$: contradiction.

In case (ii) there exist distinct $\Omega_{\mathsf{C},1}, \ldots, \Omega_{\mathsf{C},h} \in \mathrm{SF}(\Delta_{\mathsf{C}})$ $(h \ge 2)$ such that $A \in \Omega_{\mathsf{C},k}$, $0 < n_k = m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C},k}) \le n$ $(k = 1, \ldots, h)$, and $s = n_1 + \cdots + n_h > n$. By Lemma 3.6 there exist formulas $A_1, \ldots, A_h$ in F such that $A_k \in \Omega_{\mathsf{C},k}$ $(1 \le k \le h)$ and $\bigwedge\{\neg(A_k \wedge A_m) : 1 \le k < m \le h\} \in \Gamma$. Now let $B_1, \ldots, B_h$ be formulas of the fragment F such that $B_k \in \Omega_{\mathsf{C},k}$ and $\Diamond!_{n_k} B_k \in \Delta_{\mathsf{C}}$ $(1 \le k \le h)$. Let $D_k = A \wedge B_k \wedge A_k \in \mathsf{F}$ $(1 \le k \le h)$. First note that

(1) $D_k = A \wedge B_k \wedge A_k \in \Omega_{\mathsf{C},k}$ $(1 \le k \le h)$.

Now let us consider the formulas

$$(D_k \wedge D_m) \to (A_k \wedge A_m), \quad \neg(A_k \wedge A_m) \to \neg(D_k \wedge D_m) \ (1 \le k < m \le h).$$

These tautologies are in Γ. Since $\bigwedge\{\neg(A_k \wedge A_m) : 1 \le k < m \le h\} \in \Gamma$, then also $\neg(A_k \wedge A_m) \in \Gamma$, therefore $\neg(D_k \wedge D_m) \in \Gamma$, and then

(2) $\bigwedge\{\neg(D_k \wedge D_m) : 1 \le k < m \le h\} \in \Gamma$.

The tautologies $D_k \to B_k$ $(1 \le k \le h)$ are also in Γ and we recall that $\Diamond!_{n_k} B_k \in \Delta_{\mathsf{C}}$. Thus, by Lemma 3.1(e), there exist $m_k \le n_k$ $(1 \le k \le h)$ such that $\Diamond!_{m_k} D_k \in \Delta_{\mathsf{C}}$. But n_k is the least number p such that D_k can be in the scope of the operator $\Diamond!_p$ (by the definition of $m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C},k})$), so one has necessarily $m_k = n_k$ and

(3) $\Diamond!_{n_k} D_k \in \Delta_{\mathsf{C}}$ $(1 \le k \le h)$.

Now let $F = \bigvee\{D_k : 1 \le k \le h\}$. Obviously $F \in \mathsf{F}$. Furthermore, by (2) and (3) and by Lemma 3.1(b), one has $\Diamond!_s F \in \Delta_{\mathsf{C}}$, so $\Diamond_{s-1} F \in \Delta_{\mathsf{C}}$. $F \to A$ is a tautology in Γ, like $\Box_0(F \to A)$ and $\Diamond_{s-1} F \to \Diamond_{s-1} A$. Since $\Delta_{\mathsf{C}} \supseteq \Gamma$, then $\Diamond_{s-1} F \to \Diamond_{s-1} A \in \Delta_{\mathsf{C}}$ and, by the closure of Δ_{C} with respect to Modus Ponens, $\Diamond_{s-1} A \in \Delta_{\mathsf{C}}$. From $s-1 \ge n$ one gets $\Diamond_n A \in \Delta_{\mathsf{C}}$.

As to the converse, suppose $\Diamond_n A \in \Delta_{\mathsf{C}}$. Then $\Diamond_0 A \in \Delta_{\mathsf{C}}$ and there exists at least one Ω_{C} such that $A \in \Omega_{\mathsf{C}}$ and $m(\Delta_{\mathsf{C}}, \Omega_{\mathsf{C}}) \ne 0$. Without loss of generality we

may suppose that the family of these limits is finite and each of them is such that $m(\Delta_C, \Omega_C) < \omega$ (if not, then obviously $|\{\Omega_C \in \mathrm{SF}(\Delta_C) : A \in \Omega_C\}| > n$). So let $\Omega_{C,1}, \Omega_{C,2}, \ldots, \Omega_{C,h}$ be distinct limits such that $0 < n_k = m(\Delta_C, \Omega_{C,k}) < \omega$ and $A \in \Omega_{C,k}$ $(1 \leq k \leq h)$ and we first suppose that $h \geq 2$. We have to prove that $s = |\{\Omega_C \in \mathrm{SF}(\Delta_C) : A \in \Omega_C\}| = n_1 + \cdots + n_h > n$. By arguing as in point (ii) of the first part of the proof one can find formulas as A_k, B_k, D_k $(1 \leq k \leq h)$ and F with the properties (1), (2), (3) above. Moreover let $G = \neg\bigvee\{B_k \wedge A_k : 1 \leq k \leq h\}$. Clearly $G \in \mathsf{F}$. Now consider the following tautologies (in Γ):

$$A \leftrightarrow (A \wedge G) \vee (A \wedge \neg G),$$
$$(A \wedge \neg G) \leftrightarrow A \wedge \bigvee\{B_k \wedge A_k : 1 \leq k \leq h\},$$
$$(A \wedge \neg G) \leftrightarrow \bigvee\{A \wedge B_k \wedge A_k : 1 \leq k \leq h\},$$
$$(A \wedge \neg G) \leftrightarrow \bigvee\{D_k : 1 \leq k \leq h\},$$

which justify the statement

(4) $\qquad A \leftrightarrow (A \wedge G) \vee F \in \Gamma.$

Furthermore $\neg(D_k \wedge (A \wedge G))$ $(1 \leq k \leq h)$ is also a tautology belonging to Γ. Evidently, D_k and $A \wedge G$ are incompatible. So one can assert

(5) $\qquad \neg(D_k \wedge (A \wedge G)) \in \Gamma \qquad (1 \leq k \leq h).$

One has also

(6) $\qquad \Diamond!_0(A \wedge G) \in \Delta_C.$

In fact, if not, then $\Diamond_0(A \wedge G) \in \Delta_C$ and there would exist $\Omega_C \supseteq \Gamma$ such that $A \wedge G \in \Omega_C$ and $m(\Delta_C, \Omega_C) \neq 0$, and since $A \wedge G \in \Omega_C$, then $A \in \Omega_C$. So Ω_C is one of the previously considered limits. Suppose e. g. $\Omega_C = \Omega_{C,1}$. Now, being $D_1 \in \Omega_{C,1}$ and $\neg(D_1 \wedge (A \wedge G)) \in \Gamma \subseteq \Omega_{C,1}$, one has a conflict with the hypothesis $A \wedge G \in \Omega_{C,1}$. To conclude, let $F' = F \vee (A \wedge G)$ and apply Lemma 3.1(b). As a result one has $\Diamond!_s F' \in \Delta_C$ and, since $A \leftrightarrow F' \in \Gamma$, $\Diamond!_s A \in \Delta_C$. Together with the hypothesis $\Diamond_n A \in \Delta_C$ this implies $s > n$, that is the statement.

It remains to consider the case $h = 1$. In such a situation one can repeat the above arguments observing that the formulas A_k do not exist, that one has only one formula $B \in \Delta_C$ for which it happens that $\Diamond!_s B \in \Delta_C$, where $s = m(\Delta_C, \Omega_C) = |\{\Omega_C \in \mathrm{SF}(\Delta_C) : A \in \Omega_C\}|$, and which forms only one formula $D = A \wedge B = F$, and that finally $G = \neg B$. The details are left to the reader. □

D e f i n i t i o n 3.10. Let F be a countable fragment of $\mathsf{K}^0_{\omega_1}$, Γ an F-validity property, and C an F-consistency property compatible with Γ. The *model canonically associated with* F, Γ, C is the following Kripke model $\mathsf{M} = (W, R, P)$:

(i) The *set W of worlds* is the set $W = \bigcup\{\{\Delta_C\} \times \omega : \Gamma \subseteq \Delta_C \in \Phi_C\}$; when misunderstandings cannot occur, we simply write the world $\langle \Delta_C, i \rangle$ as Δ_C.

(ii) The *accessibility relation R* is defined as follows:

(a) each set $\{\Delta_C\} \times \omega$ inherits the order of the second factor, so that it has an order of type ω;

(b) let $\langle \Delta_C, i \rangle$ and $\langle \Omega_C, j \rangle$ be two elements of W; we let $\langle \Delta_C, i \rangle \, R \, \langle \Omega_C, j \rangle$ for any $i < \omega$ and $j < m(\Delta_C, \Omega_C)$, when $m(\Delta_C, \Omega_C) \neq 0$; if $m(\Delta_C, \Omega_C) = 0$, it is intended that $\langle \Omega_C, j \rangle$ is not accessible from $\langle \Delta_C, i \rangle$.

(iii) The *function P* which determines the truth value of the atomic sentences is defined by $P : \lambda < \omega_1 \mapsto \{\Delta_C : P_\lambda \in \Delta_C\} \subseteq \mathcal{P}(W)$.

$(M, \Delta_C) \vDash P_\lambda$ means that $P_\lambda \in \Delta_C$, i.e. $\Delta_C \in P(\lambda)$. The usual inductive definition allows the extension of the definition of $\vDash$ to more complex formulas. To be explicit, if $|I| < \omega_1$ and $n \in \mathbb{N}$, we let

$(M, \Delta_C) \vDash \neg A$	iff	$(M, \Delta_C) \nvDash A$;
$(M, \Delta_C) \vDash \bigvee\{A_i : i \in I\}$	iff	there exists $i \in I$ such that $(M, \Delta_C) \vDash A_i$;
$(M, \Delta_C) \vDash \bigwedge\{A_i : i \in I\}$	iff	for every $i \in I$, $(M, \Delta_C) \vDash A_i$;
$(M, \Delta_C) \vDash \Diamond_n A$	iff	$|\{\Omega_C \in W : \Delta_C R \Omega_C \text{ and } (M, \Omega_C) \vDash A\}| > n$.

Observation 3.11. *By the definition of the accessibility relation,* $\Delta_C R \Omega_C$ *iff* $\Omega_C \in \mathrm{SF}(\Delta_C)$.

Proof. If $\Omega_C \in \mathrm{SF}(\Delta_C)$, then we have $m(\Delta_C, \Omega_C) \neq 0$, and this yields $\Delta_C R \Omega_C$. If $\Delta_C R \Omega_C$, then there is j such that $j \in m(\Delta_C, \Omega_C)$, and this implies $m(\Delta_C, \Omega_C) \neq 0$, therefore $\Omega_C \in \mathrm{SF}(\Delta_C)$. □

Theorem 3.12. *Let* F *be a countable fragment of* $K^0_{\omega_1}$, Γ *an* F-*validity property,* C *an* F-*consistency property compatible with* Γ, *and* M *the model canonically associated with* F, Γ *and* C. *Let* Δ_C *a world of this model. For any* $A \in$ F, $(M, \Delta_C) \vDash A$ *iff* $A \in \Delta_C$.

Proof. By induction on the complexity of A. If A is an atomic sentence belonging to F, then the statement holds by definition. If $A = \neg B$, $A = \bigvee\{A_i : i \in I\}$ or $A = \bigwedge\{A_i : i \in I\}$, the statement is an easy consequence by the Observations 2.13, 2.12 and 2.14. When $A = \Diamond_n B$ one has

$(M, \Delta_C) \vDash A$ iff $|\{w \in W : \Delta_C R w \text{ and } (M, w) \vDash B\}| > n$ (by definition of $\vDash$)
iff $|\{\Omega_C \in \mathrm{SF}(\Delta_C) : (M, \Omega_C) \vDash B\}| > n$ (by Observation 3.11)
iff $|\{\Omega_C \in \mathrm{SF}(\Delta_C) : B \in \Omega_C\}| > n$ (by inductive hypothesis)
iff $\Diamond_n B \in \Delta_C$ (by Theorem 3.9).   □

Now we are able to prove the model existence theorem for C-consistent sets:

Theorem 3.13. *Let* F *be a countable fragment of* $K^0_{\omega_1}$, Γ *an* F-*validity property and* C *an* F-*consistency property compatible with* Γ. *Then every* $\Delta \in$ C *is satisfiable.*

Proof. Every C-consistent set Δ is, by Theorem 2.9, contained together with Γ in a limit, which is in turn a world of the model canonically associated with the triple F, Γ, C. The statement follows at once from the preceding theorem.   □

4 The completeness theorem for $K^0_{\omega_1}$

First of all we display as, having a validity property Γ for a countable fragment F, one can define a Γ-compatible consistency property for the same fragment.

Theorem 4.1. *Let* Γ *be a validity property for a countable fragment* F *of* $K^0_{\omega_1}$. *Then* $C = \{\Delta : \Delta \text{ is a finite set of sentences of } F \text{ and } \neg\bigwedge\Delta \notin \Gamma\}$ *is a* Γ-*compatible consistency property.*

Proof. Preliminarily we observe that all the formulas we shall treat below are in the fragment.

If $\Delta \in$ C and $\Omega \subseteq \Delta$, then $\bigwedge\Delta \rightarrow \bigwedge\Omega$ is a tautology and $\Omega \in$ C, i.e. C satisfies 2.5(i).

Turning to condition 2.5(ii), we must prove that, for any formula $A \in F$ and any $\Delta \in C$, we don't have $A \in \Delta$ and $\neg A \in \Delta$. Suppose, on the contrary, that $A \in \Delta$ and $\neg A \in \Delta$. Then $\{A, \neg A\} \in C$. That yields $\neg(A \wedge \neg A) \notin \Gamma$. This is a contradiction, because the formula $\neg(A \wedge \neg A)$ is a tautology of the fragment.

For condition 2.5(iii) we suppose that $\neg A \in \Delta$ and prove that $\Delta \cup \{A\neg\} \in C$. Assume $\Delta \cup \{A\neg\} \notin C$. Then $\neg \bigwedge \{\Delta \cup \{A\neg\}\} \in \Gamma$, i.e. $\neg(\bigwedge \Delta \wedge A\neg) \in \Gamma$, from which one gets $\neg \bigwedge \Delta \vee \neg(A\neg) \in \Gamma$, $A\neg \to \neg \bigwedge \Delta \in \Gamma$, $\neg A \to \neg \bigwedge \Delta \in \Gamma$. Moreover, since $\neg A \in \Delta$, we have $\bigwedge \Delta \to \neg A \in \Gamma$ and $A \to \neg \bigwedge \Delta \in \Gamma$. So $\neg \bigwedge \Delta \in \Gamma$. This is a contradiction, because $\Delta \in C$, i.e. $\neg \bigwedge \Delta \notin \Gamma$.

The proofs of the conditions 2.5(iv), 2.5(v) and 2.5(vii) are left to the reader.

Let us verify condition 2.5(vi). Assume $\Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, \bigvee \{A_i : i \in I\} \rangle \in \Delta \in C$ and there is no $h \in I$ such that $\Delta \cup \{\Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, A_h \wedge \bigvee \{A_i : i \in I\} \rangle\} \in C$. Then, for any $h \in I$, $\neg \bigwedge (\Delta \cup \{\Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, A_h \wedge \bigvee \{A_i : i \in I\} \rangle\}) \in \Gamma$. So, for any $h \in I$, $\bigwedge \Delta \to \neg \Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, A_h \wedge \bigvee \{A_i : i \in I\} \rangle \in \Gamma$. Using rule CEC we obtain $\bigwedge \Delta \to \neg \Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, \bigvee \{A_i : i \in I\} \rangle \in \Gamma$. On the other hand, because of $\Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, \bigvee \{A_i : i \in I\} \rangle \in \Delta$, $\bigwedge \Delta \to \Diamond_0^\sigma \langle Z_0, \ldots, Z_{r-1}, \bigvee \{A_i : i \in I\} \rangle \in \Gamma$. It follows that $\neg \bigwedge \Delta \in \Gamma$: contradiction.

The compability of Γ and C is immediately verified: If $A \in \Gamma$ and $\Delta \in C$ but $\Delta \cup \{A\} \notin C$, then $\neg \bigwedge (\Delta \cup \{A\}) \in \Gamma$ and $A \to \neg \bigwedge \Delta \in \Gamma$, so $\neg \bigwedge \Delta \in \Gamma$, what implies $\Delta \notin C$: contradiction.   □

Now we can prove the completeness theorem.

T h e o r e m 4.2. *For any formula A of $K_{\omega_1}^0$, $\vdash A$ iff $\models A$.*

P r o o f. We prove the hard direction that $\models A$ implies $\vdash A$. Let F be a countable fragment of $K_{\omega_1}^0$ containing A (that surely exists by Proposition 2.4). If $\models A$, then A belongs to all validity properties for that fragment. Indeed, if not, there is a validity property Γ for F such that $A \notin \Gamma$; let C be the Γ-compatible consistency property defined in Theorem 4.1; from $A \notin \Gamma$ one gets $\neg(\neg A) \notin \Gamma$, so $\{\neg A\} \in C$; therefore, by Theorem 3.13, $\neg A$ is satisfiable, what conflicts with our hypothesis $\models A$. Since the set of the theorems of the fragment is a validity property for F, we can conclude that $\vdash A$.   □

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