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The Modality of Finite (Graded Modalities VII)

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Abstract. We prove a completeness theorem for K_f, an extension of K by the operator $\Diamond_f$ that means "there exists a finite number of accessible worlds such that ... is true", plus suitable axioms to rule it. This is done by an application of the method of consistency properties for modal systems as in [4] with suitable adaptations. Despite no graded modality is invoked here, we consider this work as pertaining to that area both because $\Diamond_f$ is a definable operator in the graded infinitary system $K^0_{\omega_1}$ (see [4]), and because this idea was the original source for the development of graded modalities.

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1 Introduction

During one of the annual meetings in Siena, in the first eighties (maybe in 1982), R. MAGARI said (almost literally) that a logic can be considered mature when it can treat the finite. The first author then realized that modal logic, which he considered a well mature one, had not accomplished yet that task. A first result in this spirit, on the semantic side, arrived soon: in [2] F. BELLISSIMA and M. MIROLLI characterize any given finite Kripke frame by traditional means (it's worth noting this line of research was carried on: see [1] so far).

Contemporarily, and independently, the first author considered a syntactical way to treat the finite in modal logic, by imagining a modality like $\Diamond_f$ to be read as "there exists a finite number of accessible worlds such that ... is true". It was soon apparent that such a modality was a really intriguing one, being a mix of necessity (being empty is a way to be finite, for a set of accessible worlds!) and of an infinite number of degrees of possibility (having one element, having two elements, ...). Furthermore it was easy to build counterexamples to compactness, so to forbid the use of such a classical tool as canonical models.

Then, as a first attempt, he envisaged what he called 'graded modalities' as a kind of approximation of $\Diamond_f$. At that time (1983) he had good reasons to believe this idea was entirely new (in the preceding ten years nothing similar appeared): later on he was made acquainted by W. VAN DER HOEK [7] that K. FINE [5] and

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D. KAPLAN [8] preceded him, but - rather surprisingly - both authors seemed to neglect a development of a so promising line of research.

So a series of works by the first author and many of his pupils was published and a well structured theory grew: the main references are to be found in [4]. Eventually S. GRASSOTTI and himself studied in [4] an infinitary modal system, inside which $\Diamond_f$ could be defined in the obvious way, as a countable disjunction of graded operators. This was in a sense coherent with the general feeling that 'the finite can be treated only by means which transcend it'.

Nevertheless he thought that modal logic is so powerful and so mature (in the MAGARI's sense) to allow a *finitary* treatment of the finite, in the strong sense of a system with formulas of *finite length* with a *finite set of axioms*. The present work describes how this challenge was accepted by the first author and another student of him (the second author), with success.

We regret R. MAGARI is no more with us: we would have been glad to offer this work to him and we think he would have appreciated at least this offer. All we can do now is to dedicate it to His memory and to testify – once more – the fertility of His mathematical thought.

2 The syntax and the semantics of K_f

The language of K_f has the usual set of symbols: a countable set of atomic sentences, including a specific symbol for "false" ($\bot$), a basic set of connectives, the usual modal operator of possibility $\Diamond$, and the new modal operator $\Diamond_f$. Formulas are defined in the usual way, so as the other connectives and the dual operators of the basic ones. The set of all the formulas of K_f will be denoted by $\mathrm{Fml}(K_f)$.

The axioms of K_f are all instances of the following schemata:

Ax.1 classical tautologies;

Ax.2 $\Box(A \to B) \to (\Box A \to \Box B)$;

Ax.3 $\Box(A \to B) \to (\Diamond_f B \to \Diamond_f A)$;

Ax.4 $\neg\Diamond A \to \Diamond_f A$;

Ax.5 $\Diamond_f A \wedge \Diamond_f B \to \Diamond_f (A \vee B)$.

The basic rules of inference of K_f are the usual ones, i. e. modus ponens and necessitation.

We may note that the first two axioms provide an axiomatization of K, while the other three ones are a list of the very basic properties of the finite (relatively to the set of the worlds which are accessible from a fixed one).

As to semantics, the intended class of models of K is the whole class of all the Kripke models $M = (W, R, V)$. The definition of truth is the standard one, plus the clause

$$(M,w) \models \Diamond_f A \quad \text{iff} \quad |\{w' \in W \,:\, wRw' \text{ and } (M,w') \models A\}| < \omega,$$

so that

$$(M,w) \models \neg\Diamond_f A \quad \text{iff} \quad |\{w' \in W \,:\, wRw' \text{ and } (M,w') \models A\}| \geq \omega,$$

$$(M,w) \models \Box_f A \quad \text{iff} \quad |\{w' \in W \,:\, wRw' \text{ and } (M,w') \not\models A\}| \geq \omega.$$

Obviously a soundness theorem holds for K_f. On the contrary, compactness fails for K_f. For the sake of readibility we recall that compactness here is intended as the following statement: given a set of formulas S, S is satisfiable (in a world of a model) iff every finite subset of S is satisfiable (in a world of a model). In fact, if we take a countable sequence $B, A_0, A_1, \ldots$ of atomic formulas and define

$$C_0 = B \wedge A_0,$$
$$C_1 = B \wedge \neg A_0 \wedge A_1,$$
$$\ldots$$
$$C_n = B \wedge \neg A_0 \wedge \neg A_1 \wedge \cdots \wedge \neg A_{n-1} \wedge A_n,$$
$$\ldots$$

we see at once that the set $S = \{\Diamond C_n : n \in \mathbb{N}\} \cup \{\Diamond_f B\}$ is not satisfiable while any of its finite subsets is. This is essentially known already (see e. g. [3] and [4]).

Finally we want to stress that K_f (identified to the set of its theorems) can be thought of as a subsystem of $K^0_{\omega_1}$ (identified to the set of its theorems), via the natural embedding which takes $\Diamond_f$ to the definable modality $\Diamond_{\mathrm{fin}}$ of $K^0_{\omega_1}$ (see [4]).

2 Consistency properties for K_f

We introduce now the consistency properties for K_f (that will be the same as for K) in the style of [4] (i. e. of [6]).

Definition 1. Let C be a non empty family of sets of sentences of K_f. The family C is a *consistency property* for K_f (from now on, simply called a consistency property) iff it is a consistency property for K (as in [6], but closed under subsets; see [6, Chap. 2, Lemma 5.4]), i. e. iff for any $\Delta \in C$ and any formulas A, B of K_f, the following conditions are satisfied:

(o) if $\Omega \subseteq \Delta$, then $\Omega \in C$;

(i) $\perp \notin \Delta$, and if $A \in \Delta$, then $\neg A \notin \Delta$;

(ii) if $A \wedge B \in \Delta$, then $\Delta \cup \{A\} \in C$ and $\Delta \cup \{B\} \in C$;

(iii) if $A \vee B \in \Delta$, then either $\Delta \cup \{A\} \in C$ or $\Delta \cup \{B\} \in C$;

(iv) if $A \rightarrow B \in \Delta$, then $\Delta \cup \{A\} \in C$ implies $\Delta \cup \{B\} \in C$;

(v) if $A \leftrightarrow B \in \Delta$, then $\Delta \cup \{A\} \in C$ iff $\Delta \cup \{B\} \in C$;

(vi) if $\Diamond A \in \Delta$, then $\Delta^\# \cup \{A\} \in C$, where

$$\Delta^\# = \{A : \Box A \in \Delta\} \cup \{\neg A : \neg\Diamond A \in \Delta\}.$$

The elements of C will be often called *C-consistent sets* and the maximal elements of C w.r.t. $\subseteq$ (if any) will be called *C-maximal sets*, or *C-maximals*, or simply *maximals* (when no ambiguity about C can arise). To stress a set of C as a C-maximal it will be indexed by C. The reader may have noted that in Definition 1 $\Diamond_f$ doesn't appear: as a matter of fact, the notion of a consistency property for K – but for sets of formulas of the language of K_f – suffices to do our job.

Definition 2. Let Γ be a set of sentences of the language of K_f. A consistency property C is called *Γ-compatible* if for any $\Delta \in C$ and any $A \in \Gamma$, $\Delta \cup \{A\} \in C$. When $\Gamma = \mathrm{Th}(K_f)$ (= the set of all the theorems of K_f) we shall write *K_f-compatible*, or simply *compatible*, instead of $\mathrm{Th}(K_f)$-compatible.

As an useful substitute of the axiom of choice we shall use here (as in [6]) the Teichmüller-Tukey Lemma, which has to do with families (of sets) of finite character. The reader who needs a memory's refreshment about those topics can see e.g. [9].

As a matter of fact one can assume – without loss of generality – a consistency property is of finite character, also saving, if necessary, the Γ-compatibility (see [6, Chap. 2, Prop. 5.6]). In such a case the following theorem holds.

Theorem 1 (LINDENBAUM's Lemma for C). *Let C be a consistency property of finite character. For any $\Sigma \in C$ there exists $\Phi \in C$ such that $\Sigma \subseteq \Phi$ and Φ is maximal in C.*

Proof. Let $\Sigma \in C$ and $\mathcal{F} = \{\Delta \subseteq \mathrm{Fml}(\mathrm{K_f}) : \Sigma \cup \Delta \in C\}$. We show that $\mathcal{F}$ is of finite character. Assume $\Delta \in \mathcal{F}$ and $\Theta \subseteq_{\mathrm{f}} \Delta$ (that is, Θ is a finite subset of Δ). Then $\Theta \in \mathcal{F}$, because $\Sigma \cup \Theta \subseteq \Sigma \cup \Delta \in C$ and C is closed under subsets. Vice versa let Δ be such that $\Theta \in \mathcal{F}$ for every $\Theta \subseteq_{\mathrm{f}} \Delta$. We show that $\Delta \in \mathcal{F}$. Now $\Delta \in \mathcal{F}$ iff $\Sigma \cup \Delta \in C$, and C is of finite character. Then $\Sigma \cup \Delta \in C$ if for any $\Lambda \subseteq_{\mathrm{f}} \Sigma \cup \Delta$ one has $\Lambda \in C$. But if $\Lambda \subseteq_{\mathrm{f}} \Sigma \cup \Delta$, then $\Lambda \subseteq_{\mathrm{f}} \Sigma \cup \Theta$ for some $\Theta \subseteq_{\mathrm{f}} \Delta$, so such that $\Theta \in \mathcal{F}$. Then $\Sigma \cup \Theta \in C$ and $\Lambda \in C$, because C is closed under subsets.

Then $\mathcal{F}$ is of finite character and not empty ($\emptyset \in \mathcal{F}$), i.e. $\mathcal{F}$ has a maximal element Δ. Therefore $\Phi = \Sigma \cup \Delta$ is maximal in C: if $\Sigma \cup \Delta \subseteq \Sigma \cup \Delta \cup \Theta \in C$, then $\Delta \cup \Theta \in \mathcal{F}$, so $\Delta = \Delta \cup \Theta$. □

Observation 1. *The maximals are closed w.r.t. Modus Ponens.*

Proof. We have to prove that if $A \in \Delta_C$ and $A \to B \in \Delta_C$, then $B \in \Delta_C$. This follows immediately from the maximality of Δ_C and Definition 1(iv). □

If C is a compatible consistency property and Δ_C is a maximal of C, one has clearly $\mathrm{Th}(\mathrm{K_f}) \subseteq \Delta_C$. This fact has important consequences: we prove, in this situation, that the maximals have the main properties of the maximal consistent sets of the classical logic.

Observation 2. *If C is a compatible consistency property, then $A \vee B \in \Delta_C$ iff either $A \in \Delta_C$ or $B \in \Delta_C$.*

Proof. The left-to-right implication is an immediate consequence of the maximality of Δ_C and of Definition 1(iii). As to the vice versa, let us note that $A \to A \vee B$, $B \to A \vee B \in \mathrm{Th}(\mathrm{K_f})$ and, being $\mathrm{Th}(\mathrm{K_f}) \subseteq \Delta_C$, the statement follows from Observation 1. □

As a consequence we obtain that for any sentence A, since $A \vee \neg A \in \mathrm{Th}(\mathrm{K_f})$, one has either $A \in \Delta_C$ or $\neg A \in \Delta_C$, but not both (Definition 1(i)). Therefore,

Observation 3. *If C is a compatible consistency property and A is a sentence of $\mathrm{K_f}$, then $A \in \Delta_C$ iff $\neg A \notin \Delta_C$.*

Finally, we have

Observation 4. *If C is a compatible consistency property and $A \wedge B$ is a sentence of $\mathrm{K_f}$, then $A \wedge B \in \Delta_C$ iff $A \in \Delta_C$ and $B \in \Delta_C$.*

Proof. The left-to-right implication is an immediate consequence of the maximality of Δ_C and of Definition 1(ii). As to the vice versa, if $A \wedge B \notin \Delta_C$, then $\neg(A \wedge B) \in \Delta_C$ (by Observation 3), so that $\neg A \vee \neg B \in \Delta_C$ (by compatibility) and the Observations 2 and 3 imply that either $A \notin \Delta_C$ or $B \notin \Delta_C$. □

When a compatible consistency property is assumed to be also of finite character, a stronger version of clause (ii) of Definition 1 holds.

Observation 5. *Let C be a compatible consistency property of finite character and $A \wedge B$ a sentence of* $\mathrm{K_f}$. *If $A \wedge B \in \Delta \in C$, then $\Delta \cup \{A, B\} \in C$.*

Proof. Let Γ_C be a maximal of C, extending Δ. Since $A \wedge B \in \Gamma_C$, one has A, $B \in \Gamma_C$ (Observation 4), and $\Delta \cup \{A, B\} \in C$, by closure under subsets. □

4 The restriction of a consistency property

Here we introduce the key tool to obtain our main results. Firstly we remember that a set Ω of sentences of $\mathrm{K_f}$ is called *closed under subformulas* when, if B is a subformula of $A \in \Omega$, then also $B \in \Omega$.

Definition 3. Let C be a consistency property and $\Omega \subseteq \mathrm{Fml}(\mathrm{K_f})$. The *restriction* $C|_\Omega$ of C to Ω is the collection of the sets $\Delta \cap \Omega$, where $\Delta \in C$. Note that $C|_\Omega \subseteq C$.

The following is the key result about restrictions of consistency properties.

Proposition 1. *Let C be a consistency property and Ω a set of sentences closed under subformulas and deletion of negated diamonds (i. e. if $\neg\Diamond A \in \Omega$, then $\neg A \in \Omega$). Then $C|_\Omega$ is a consistency property. Furthermore if C is of finite character, the same holds for $C|_\Omega$.*

Proof. For the first statement, let us verify clauses (o) – (vi) of Definition 1.

(o) is trivial: if $\Delta \subseteq \Sigma \in C|_\Omega \subseteq C$, then $\Delta \in C$ by Defintion 1(o); but $\Delta \subseteq \Omega$, so $\Delta \in C|_\Omega$. For ease of reference we stress the (quite trivial) fact which this argument is based on:

(*) for every $\Sigma \in C$, if $\Sigma \subseteq \Omega$, then $\Sigma \in C|_\Omega$.

For (i), let $\Sigma \in C|_\Omega$. If $A, \neg A \in \Sigma$, then $\{A, \neg A\} \in C$; contradiction with Definition 1(i). Analogously, $\perp \notin \Sigma$.

For (ii), let $A \wedge B \in \Sigma \in C|_\Omega$. Then, being $A \wedge B \in \Sigma \in C$, one has, by Definition 1(ii), $\Sigma \cup \{A\} \in C$ and $\Sigma \cup \{B\} \in C$. On the other hand, $\Sigma \subseteq \Omega$, and $A \wedge B \in \Omega$ implies A, $B \in \Omega$ (because of its closure under subformulas). So $\Sigma \cup \{A\}$, $\Sigma \cup \{B\} \subseteq \Omega$ and (*) gives the desired conclusion.

Clauses (iii) – (v) are shown in a similar way, so we skip the detailed verifications.

For (vi), let $\Sigma \in C|_\Omega$ and $\Diamond A \in \Sigma$. We have to prove that $\Sigma^\# \cup \{A\} \in C|_\Omega$. Note that, being Ω closed under subformulas,

$$\Sigma^\# = \{B : \Box B \in \Sigma\} \cup \{\neg B : \neg\Diamond B \in \Sigma\} \subseteq \Omega,$$

and the same is true for A (which is a subformula of a formula of $\Sigma \subseteq \Omega$). But, from $C|_\Omega \subseteq C$ it follows by Definition 1(vi) that $\Sigma^\# \cup \{A\} \in C$. Then, by (*), $\Sigma^\# \cup \{A\} \in C|_\Omega$.

Now we prove the second statement: restriction saves the finite character. What we have to prove is that if every finite subset of a set of sentences Σ belongs to $C|_\Omega$, then $\Sigma \in C|_\Omega$. Since $C|_\Omega \subseteq C$, every finite subset of Σ, if it belongs to $C|_\Omega$, then it belongs to C. So, being C of finite character, $\Sigma \in C$. To conclude that $\Sigma \in C|_\Omega$ it is enough (by (*)) to show that $\Sigma \subseteq \Omega$. If $A \in \Sigma$, then $\{A\} \subseteq_{\mathrm{f}} \Sigma$ and $\{A\} \in C|_\Omega$, i.e. there exists $\Delta \in C$ such that $\{A\} = \Delta \cap \Omega$, hence $A \in \Omega$. □

Definition 4. Let Σ be a set of sentences. We define $\overline{\Sigma}$ as the set obtained by substituting in all the formulas of Σ every occurrence of the operators $\Diamond$ and $\Diamond_f$ with an occurrence of the respectively equivalent ones $\neg\Box\neg$ and $\neg\Box_f\neg$. Finally define $[\Sigma]$ as the closure under subformulas of $\Sigma \cup \overline{\Sigma}$. Then $[\Sigma]$ will be called the *associated set* of Σ. Obviously $[\Sigma]$ is closed under deletion of negated diamonds.

5 The satisfiability theorem

Let C be a compatible consistency property of finite character, Ω a set of sentences of K_f closed under subformulas and deletion of negated diamonds, $C|_\Omega$ the restriction of C to Ω, and $\Delta_{C|_\Omega}$ a maximal of its. We want to observe that $C|_\Omega$ is not necessarily compatible (e. g. it is not such when Ω is finite). So we have to reexamine the Observations 2 – 4.

Observation 6. *If $A \vee B \in \Omega$, then $A \vee B \in \Delta_{C|_\Omega}$ iff either $A \in \Delta_{C|_\Omega}$ or $B \in \Delta_{C|_\Omega}$.*

Proof. The left-to-right implication still holds, because it does not depend on compatibility. As to vice versa we show that if e. g. $A \in \Delta_{C|_\Omega}$, then $A \vee B \in \Delta_{C|_\Omega}$. Since $C|_\Omega \subseteq C$, we have $\Delta_{C|_\Omega} \in C$ and, by Lindenbaum's Lemma, there exists a maximal Δ_C that extends it. By Observation 2, since $A \in \Delta_C$ we have $A \vee B \in \Delta_C$. Therefore $\Delta_{C|_\Omega} \cup \{A \vee B\} \in C$. By hypothesis, $\Delta_{C|_\Omega} \cup \{A \vee B\} \subseteq \Omega$, and this assures by $(*)$ that $\Delta_{C|_\Omega} \cup \{A \vee B\} \in C|_\Omega$, and the maximality of $\Delta_{C|_\Omega}$ in $C|_\Omega$ yields $A \vee B \in \Delta_{C|_\Omega}$. □

Observation 7. *If $A, \neg A \in \Omega$, then $A \in \Delta_{C|_\Omega}$ iff $\neg A \notin \Delta_{C|_\Omega}$.*

Proof. The left-to-right implication is a consequence of Definition 1(i) and the fact that $C|_\Omega \subseteq C$. As to vice versa, let us assume that $A \notin \Delta_{C|_\Omega} \in C$. Since C is compatible, $\Delta_{C|_\Omega} \cup \{A \vee \neg A\} \in C$, and either $\Delta_{C|_\Omega} \cup \{A\} \in C$ or $\Delta_{C|_\Omega} \cup \{\neg A\} \in C$ (by Definition 1(iv)). But $\Delta_{C|_\Omega} \cup \{A\} \subseteq \Omega$ and $\Delta_{C|_\Omega} \cup \{\neg A\} \subseteq \Omega$ so that by $(*)$ either $\Delta_{C|_\Omega} \cup \{A\} \in C|_\Omega$ or $\Delta_{C|_\Omega} \cup \{\neg A\} \in C|_\Omega$. The maximality of $\Delta_{C|_\Omega}$ in $C|_\Omega$ implies that either $A \in \Delta_{C|_\Omega}$ or $\neg A \in \Delta_{C|_\Omega}$, so $\neg A \in \Delta_{C|_\Omega}$. □

Observation 8. *If $A \wedge B \in \Omega$, then $A \wedge B \in \Delta_{C|_\Omega}$ iff $A \in \Delta_{C|_\Omega}$ and $B \in \Delta_{C|_\Omega}$.*

Proof. Since $A, B \in \Omega$, $\Delta_{C|_\Omega} \cup \{A\} \subseteq \Omega$ and $\Delta_{C|_\Omega} \cup \{B\} \subseteq \Omega$. Then the left-to-right implication is an immediate consequence of Definition 1(ii), $(*)$, and the maximality of $\Delta_{C|_\Omega}$. As to vice versa, let us assume that $A, B \in \Delta_{C|_\Omega} \in C$. Then there exists a maximal Φ_C that extends $\Delta_{C|_\Omega}$. So $A, B \in \Phi_C$ and, by Observation 4, $A \wedge B \in \Phi_C$. Furthermore $\Delta_{C|_\Omega} \cup \{A \wedge B\} \subseteq \Phi_C$ which yields $\Delta_{C|_\Omega} \cup \{A \wedge B\} \in C$. On the other hand, $\Delta_{C|_\Omega} \cup \{A \wedge B\} \subseteq \Omega$, and thus, by $(*)$, $\Delta_{C|_\Omega} \cup \{A \wedge B\} \in C|_\Omega$, and by maximality $A \wedge B \in \Delta_{C|_\Omega}$. □

Observation 9. *If $A, B \in \Omega$, $A \in \Delta_{C|_\Omega}$ and $A \to B \in \mathrm{Th}(K_f)$, then $B \in \Delta_{C|_\Omega}$.*

Proof. We know that $\Delta_{C|_\Omega} \in C$, and by the compatibility of C we obtain that $\Delta' = \Delta_{C|_\Omega} \cup \{A \to B\} \in C$. Furthermore $\Delta' \cup \{A\} = \Delta' \in C$ and Definition 1(iv) implies $\Delta_{C|_\Omega} \cup \{B\} \in C$. Since $\Delta_{C|_\Omega} \cup \{B\} \subseteq \Omega$, we have by $(*)$ $\Delta_{C|_\Omega} \cup \{B\} \in C|_\Omega$ and the maximality of $\Delta_{C|_\Omega}$ assures that $B \in \Delta_{C|_\Omega}$. □

We can now prove the satisfiability theorem. In what follows 'countable' means 'either finite or denumerable'.

Theorem 2. *Let C be a compatible consistency property of finite character and let Σ be a finite set belonging to C and $[\Sigma]$ its associated set. Consider the consistency property $C|_{[\Sigma]}$ and a maximal set $\Gamma \in C|_{[\Sigma]}$. Then there exists a corresponding countable family $\overline{\Gamma}$ of maximal sets in $C|_{[\Sigma]}$ such that*

(a) *if $\Box A \in \Gamma$, then A belongs to every set of $\overline{\Gamma}$;*

(b) *if $\neg\Diamond_{\mathrm{f}} B \in \Gamma$, then there exists a denumerable subfamily of $\overline{\Gamma}$ such that every of its elements contains B;*

(c) *for every formula $\Diamond A \in [\Sigma]$, $\Diamond A \in \Gamma$ iff $|\{\Gamma' \in \overline{\Gamma} : A \in \Gamma'\}| \neq 0$;*

(d) *for every formula $\Diamond_{\mathrm{f}} A \in [\Sigma]$, $\Diamond_{\mathrm{f}} A \in \Gamma$ iff $|\{\Gamma' \in \overline{\Gamma} : A \in \Gamma'\}| < \omega$.*

Proof. Since Σ is finite, so is also $[\Sigma]$; then the same is true for $C|_{[\Sigma]}$ and any of its elements. We divide the proof in three steps.

Step 1. Let us consider the (finite) set of the formulas A such that $\Diamond A \in \Gamma$ and the $C|_{[\Sigma]}$-consistent sets $\Gamma'_A = \Gamma^{\#} \cup \{A\}$. These can be extended to $C|_{[\Sigma]}$-consistent maximal sets Γ_A, which exist by Lindenbaum's lemma, because $C|_{[\Sigma]}$ is of finite character (Proposition 1).

Step 2. Now we consider the (finite) set of the formulas B such that $\Box_{\mathrm{f}}\neg B \in \Gamma$. We choose such a B and show that the set $\Delta = \Gamma^{\#} \cup \{B\} \cup \{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\}$ belongs to $C|_{[\Sigma]}$. Note that if $\Diamond_{\mathrm{f}} A \in [\Sigma]$, then $\neg\Box_{\mathrm{f}}\neg A \in [\Sigma]$, so that $\neg A \in [\Sigma]$ and $\Delta \subseteq [\Sigma]$. We have to show that $\Delta \in C$ (see (*)). Since $\Gamma \in C$, there exists a maximal Γ_C of C that contains it. We prove that $\Diamond(B \wedge \bigwedge\{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\}) \in \Gamma_C$. Let $D = B \wedge \bigwedge\{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\}$ and suppose $\Diamond D \notin \Gamma_C$. This implies $\neg\Diamond D \in \Gamma_C$. Since C is compatible, $\Box\neg D \in \Gamma_C$ and $\Box(B \to \bigvee\{A : \Diamond_{\mathrm{f}} A \in \Gamma\}) \in \Gamma_C$, i.e. $\Diamond_{\mathrm{f}} \bigvee\{A : \Diamond_{\mathrm{f}} A \in \Gamma\} \to \Diamond_{\mathrm{f}} B \in \Gamma_C$, $\neg\Diamond_{\mathrm{f}} B \to \neg\Diamond_{\mathrm{f}} \bigvee\{A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in \Gamma_C$, and $\Box_{\mathrm{f}}\neg B \to \neg\Diamond_{\mathrm{f}} \bigvee\{A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in \Gamma_C$. By modus ponens (Observation 1) we obtain $\neg\Diamond_{\mathrm{f}} \bigvee\{A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in \Gamma_C$ and $\bigvee\{\neg\Diamond_{\mathrm{f}} A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in \Gamma_C$, what is contradictory. Thus $\Diamond(B \wedge \bigwedge\{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\}) \in \Gamma_C$, so that $\Gamma_C^{\#} \cup \{B \wedge \bigwedge\{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\}\} \in C$ and, by Observation 5, $\Gamma_C^{\#} \cup \{B\} \cup \{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in C$. Since $\Gamma^{\#} \subseteq \Gamma_C^{\#}$ and C is closed under subsets, we have $\Delta = \Gamma^{\#} \cup \{B\} \cup \{\neg A : \Diamond_{\mathrm{f}} A \in \Gamma\} \in C$. Now we extend Δ to a set Γ_B which is a maximal in $C|_{[\Sigma]}$, and we take a denumerable family of copies of Γ_B.

Step 3. Let us define $\overline{\Gamma}$ as the family of all the $C|_{[\Sigma]}$-maximals, obtained as in Steps 1 and 2 from the formulas of the set $\{A : \Diamond A \in \Gamma\} \cup \{B : \Box_{\mathrm{f}}\neg B \in \Gamma\}$. Clearly $\overline{\Gamma}$ is a countable set. We verify that $\overline{\Gamma}$ satisfies the clauses (a) – (d) of the statement.

(a) is true by definition: if $\Box A \in \Gamma$, then A belongs to $\Gamma^{\#}$, so also to any of the sets of formulas from which $\overline{\Gamma}$ is made up.

(b) is true by Step 2: if $\neg\Diamond_{\mathrm{f}} B \in \Gamma$, then, since $\neg\Diamond_{\mathrm{f}} B \to \Box_{\mathrm{f}}\neg B \in \mathrm{Th}(\mathrm{K_f})$ and $\Gamma \in C$, one has (by compatibility) $\Gamma \cup \{\neg\Diamond_{\mathrm{f}} B \to \Box_{\mathrm{f}}\neg B\} \in C$; by the finite character of C, this set is included in a C-maximal Γ_C and this is closed w.r.t. modus ponens, so that $\Gamma \cup \{\Box_{\mathrm{f}}\neg B\} \subseteq \Gamma_C$ and, by closure under subsets, $\Gamma \cup \{\Box_{\mathrm{f}}\neg B\} \in C$. Now, if $\neg\Diamond_{\mathrm{f}} B \in [\Sigma]$, then $\neg\neg\Box_{\mathrm{f}}\neg B \in [\Sigma]$, so that $\Box_{\mathrm{f}}\neg B \in [\Sigma]$. We can conclude that $\Gamma \cup \{\Box_{\mathrm{f}}\neg B\} \in C|_{[\Sigma]}$. Since Γ is maximal in $C|_{[\Sigma]}$, we have $\Box_{\mathrm{f}}\neg B \in \Gamma$. So B belongs to Γ_B and to any of its copies in $\overline{\Gamma}$ (being a denumerable family).

(c) Let $\Diamond A \in [\Sigma]$. If $\Diamond A \in \Gamma$, by Step 1 it is clear that $|\{\Gamma' \in \overline{\Gamma} : A \in \Gamma'\}| \neq 0$. Now suppose that $\Diamond A \notin \Gamma$. Note that $\Diamond A \in [\Sigma]$ implies $\neg\Box\neg A \in [\Sigma]$ and $\Box\neg A \in [\Sigma]$.

Since C is a compatible consistency property, either $\Gamma \cup \{\Diamond A\} \in C$ or $\Gamma \cup \{\neg\Diamond A\} \in C$, i.e. either $\Gamma \cup \{\Diamond A\} \in C$ or $\Gamma \cup \{\Box\neg A\} \in C$. Now both $\Diamond A$ and $\Box\neg A$ belong to $[\Sigma]$. Since Γ is a maximal of $C|_{[\Sigma]}$, one has either $\Diamond A \in \Gamma$ or $\Box\neg A \in \Gamma$. Then $\Box\neg A \in \Gamma$ and, for any $\Gamma' \in \overline{\Gamma}$, $\neg A \in \Gamma'$ (by (a)). So $|\{\Gamma' \in \overline{\Gamma} : A \in \Gamma'\}| \neq 0$.

(d) First of all let us suppose that $\Diamond_f A \in \Gamma$. If it were $|\{\Gamma' \in \overline{\Gamma} : A \in \Gamma'\}| = \omega$, then A would belong to one of the Γ''s of Step 2, but by the definition of those sets, $\neg A \in \Gamma'$. Contradiction. As to the vice versa, let $|\{\Gamma \in \overline{\Gamma} : A \in \Gamma'\}| < \omega$. Suppose that $\Diamond_f A \notin \Gamma$. Arguing as in (c), $\Box_f\neg A \in \Gamma$, so that $|\{\Gamma \in \overline{\Gamma} : A \in \Gamma'\}| = \omega$ (Step 2). Contradiction. Therefore, $\Diamond_f A \in \Gamma$. □

Now we are able to prove the Model Existence Theorem for finite C-consistent sets.

Theorem 3. *Let C be a compatible consistency property of finite character, and let Σ be a finite set of formulas belonging to C. Then there exists a model M and a world Γ in M such that $\Gamma \models \Sigma$.*

Proof. Let Σ be as in the statement, and let $[\Sigma]$ be its associated set. We apply Theorem 2. Obviously $\Sigma \in C|_{[\Sigma]}$. Let Γ_0 be a maximal of $C|_{[\Sigma]}$ containing Σ. Keeping the notation from Theorem 2, let us define

$$\Phi_0 = \{\Gamma_0\}, \ \ldots, \ \Phi_{n+1} = \dot{\bigcup}\{\overline{\Gamma} : \Gamma \in \Phi_n\}, \ \ldots$$

and

$$W = \dot{\bigcup}\{\Phi_n : n < \omega\}.$$

The set W is countable and it may be indexed by a countable set I: $W = \{\Gamma_i : i \in I\}$, where $\Gamma_i \neq \Gamma_k$ if $i \neq k$ $(i, k \in I)$. Now we define a binary relation in W by $\Gamma_i R \Gamma_k$ iff $\Gamma_k \in \overline{\Gamma_i}$. The valuation function is so defined that for any $\Gamma \in W$ and any atomic sentence $P_n \in [\Sigma]$, $V(\Gamma, P_n) = 1$ iff $P_n \in \Gamma$. So we have a model $M = (W, R, V)$, and we want to show that for any $\Gamma \in W$ and $A \in [\Sigma]$,

(+) $\quad V(\Gamma, A) = 1 \quad \text{iff} \quad A \in \Gamma.$

The proof of (+) is by induction on the complexity of $A \in [\Sigma]$.

(i) If $A = P_n \in [\Sigma]$, (+) is true by definition.

(ii) Let $A = \neg B \in [\Sigma]$. By Observation 7, $\neg B \in \Gamma$ iff $B \notin \Gamma$ iff (by induction), $V(\Gamma, B) = 0$ iff $V(\Gamma, \neg B) = 1$.

(iii) Let $A = B \wedge C \in [\Sigma]$. By Observation 8, $B \wedge C \in \Gamma$ iff $B, C \in \Gamma$ iff (by induction) $V(\Gamma, B) = V(\Gamma, C) = 1$ iff $V(\Gamma, B \wedge C) = 1$. Analogously one concludes for the other Boolean connectives.

(iv) Let $A = \Diamond B \in [\Sigma]$. By Theorem 2,

$$\begin{aligned} \Diamond B \in \Gamma \ &\text{iff} \ |\{\Gamma' \in W : \Gamma' \in \overline{\Gamma} \text{ and } B \in \Gamma'\}| \neq 0 \\ &\text{iff} \ |\{\Gamma' \in W : \Gamma R \Gamma' \text{ and } V(\Gamma', B) = 1\}| \neq 0 \ \text{(by induction)} \\ &\text{iff} \ V(\Gamma, \Diamond B) = 1. \end{aligned}$$

(v) Let $A = \Diamond_f B \in [\Sigma]$. By Theorem 2 again,

$$\begin{aligned} \Diamond_f B \in \Gamma \ &\text{iff} \ |\{\Gamma' \in W : \Gamma' \in \overline{\Gamma} \text{ and } B \in \Gamma'\}| < \omega \\ &\text{iff} \ |\{\Gamma' \in W : \Gamma R \Gamma' \text{ and } V(\Gamma', B) = 1\}| < \omega \ \text{(by induction)} \\ &\text{iff} \ V(\Gamma, \Diamond_f B) = 1. \end{aligned}$$

So Σ is true in $\Gamma_0 \in W$ and the proof is complete. □

6 The completeness theorem for $\mathrm{K_f}$

We are almost ready to prove the completeness of $\mathrm{K_f}$. First note the following proposition.

Proposition 2. *Let* $C = \{\Delta \subseteq \mathrm{Fml}(\mathrm{K_f}) : \Delta$ *is a consistent set (in* $\mathrm{K_f}$*)*$\}$. *Then* C *is a compatible consistency property of finite character.*

Proof. Let us examine the features that define a consistency property.

(o) and (i) are obviously true.

(ii) Let $A \wedge B \in \Delta \in C$ and suppose that $\Delta \cup \{A\} \notin C$. Then there exists a $\Delta' \subseteq_{\mathrm{f}} \Delta$ such that $A \rightarrow \neg \bigwedge \Delta' \in \mathrm{Th}(\mathrm{K_f})$, so that $\Delta \vdash \neg \bigwedge \Delta'$. Contradiction. Therefore, $\Delta \cup \{A\} \in C$. By replacing A with B, the same argument shows that $\Delta \cup \{B\} \in C$.

(iii) Suppose $A \vee B \in \Delta \in C$, but $\Delta \cup \{A\} \notin C$ and $\Delta \cup \{B\} \notin C$. Then there exist two finite subsets Δ' and Δ'' of Δ such that $A \rightarrow \neg \bigwedge \Delta'$ and $B \rightarrow \neg \bigwedge \Delta''$ are provable in $\mathrm{K_f}$. Then the same holds for $A \vee B \rightarrow \neg(\bigwedge \Delta' \wedge \bigwedge \Delta'')$. So one has $\Delta \vdash \neg(\bigwedge \Delta' \wedge \bigwedge \Delta'')$, where $\Delta' \cup \Delta'' \subseteq_{\mathrm{f}} \Delta$. Contradiction.

(iv) and (v) may be analogously verified: we skip the details.

(vi) If $\Diamond A \in \Delta \in C$ and $\Delta^{\#} \cup \{A\} \notin C$, then there are $B_1, \ldots, B_n, C_1, \ldots, C_m$ such that $\{\Box B_1, \ldots, \Box B_n, \neg\Diamond C_1, \ldots, \neg\Diamond C_m\} \subseteq \Delta$ and

$$\neg \bigwedge(\{B_1, \ldots, B_n, \neg C_1, \ldots, \neg C_m\} \cup \{A\}) \in \mathrm{Th}(\mathrm{K_f}).$$

So

$$\begin{aligned}
&\bigwedge\{B_1, \ldots, B_n, \neg C_1, \ldots, \neg C_m\} \rightarrow \neg A \in \mathrm{Th}(\mathrm{K_f}),\\
&\Box \bigwedge\{B_1, \ldots, B_n, \neg C_1, \ldots, \neg C_m\} \rightarrow \Box\neg A \in \mathrm{Th}(\mathrm{K_f}),\\
&\bigwedge\{\Box B_1, \ldots, \Box B_n, \Box\neg C_1, \ldots, \Box\neg C_m\} \rightarrow \neg\Diamond A \in \mathrm{Th}(\mathrm{K_f}),\\
&\Diamond A \rightarrow \neg \bigwedge\{\Box B_1, \ldots, \Box B_n, \neg\Diamond C_1, \ldots, \neg\Diamond C_m\} \in \mathrm{Th}(\mathrm{K_f}).
\end{aligned}$$

So $\Delta \vdash \neg \bigwedge\{\Box B_1, \ldots, \Box B_n, \neg\Diamond C_1, \ldots, \neg\Diamond C_m\}$. Contradiction.

The finite character of C is obvious.

As to compatibility, let $A \in \mathrm{Th}(\mathrm{K_f})$, $\Delta \in C$, but $\Delta \cup \{A\} \notin C$. Then there exists $\Delta' \subseteq_{\mathrm{f}} \Delta$ such that $\neg \bigwedge(\Delta' \cup \{A\}) \in \mathrm{Th}(\mathrm{K_f})$, so $A \rightarrow \neg \bigwedge \Delta' \in \mathrm{Th}(\mathrm{K_f})$, $\neg \bigwedge \Delta' \in \mathrm{Th}(\mathrm{K_f})$ and $\Delta \notin C$. Contradiction. □

Now we can prove the completeness of $\mathrm{K_f}$.

Theorem 4. *For any sentence* A *of* $\mathrm{K_f}$, *if* $\models A$, *then* $\vdash A$.

Proof. If $\nvdash A$, then $\{\neg A\}$ is C-consistent, C being the consistency property of Proposition 2. Since $\{\neg A\}$ is a finite set of C, by Theorem 3, A is satisfied in a world of a (countable) model. Therefore, $\not\models A$. □

7 Final remarks

Remark 1. As noted before, $\mathrm{K_f}$ can be considered (up to the obvious embedding: see before) as a subsystem of $\mathrm{K}^0_{\omega_1}$ and shares its class of models, i.e. the class of all Kripke models. In this respect, the completeness theorem for $\mathrm{K_f}$ shows that $\mathrm{K}^0_{\omega_1}$ is a conservative extension of $\mathrm{K_f}$.

R e m a r k 2. K_f does not have the finite model property, i.e. it is not complete w.r.t. the class of its finite models. In fact, if it were so, then every formula in the language of K_f, valid in every finite Kripke model, would have to be valid in every Kripke model, which is obviously false (witness is e.g. the formula $\Diamond_f \top$).

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