CONCEPTUAL STRUCTURALISM AND THE CONTINUUM

Solomon Feferman

PHILMATH INTERSEM 2010
Université Paris Diderot-Paris 7
June 8, 2010

http://math.stanford.edu/~feferman

Is the Continuum Hypothesis (CH) a Definite Mathematical Problem?

- My view: No; in fact it is essentially indefinite ("inherently vague").
- That is, the concepts of arbitrary set and function as used in its formulation are essentially indefinite.
- This comes from my general view of the nature of mathematics, that it is humanly based and that it deals with more or less clear conceptions of mathematical structures; for want of a better word, I call that view conceptual structuralism.

The Opposite Point of View: Ontological (Platonic) Realism

- Under this view, Kurt Gödel, in his article "What is Cantor's continuum problem?" (1947/1964), asserted that CH is a definite mathematical problem, though one that may require new axioms of set theory in order to settle it.
- Gödel's program(s) for new axioms: intrinsic and extrinsic.
- Currently only high hopes for the extrinsic program.

Mathematical Structuralism

- Modern mathematics dominated by structuralist views (abstract algebra, topology, analysis; Bourbaki, category theory, etc.)
- Explicit inception often credited to Dedekind.
- But mathematicians have implicitly always been structuralists.
- "Mathematics is in its most general sense the science of relationships, in which one abstracts from any content of the relationships."(C. F. Gauss, *Werke* X/1)

Mathematical Structuralism (Cont'd)

- Hilbert, *Grundlagen der Geometrie*.
- In the Hilbert-Frege exchange, Frege is the odd man out.
- "Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if those objects are replaced by others, provided that the relations do not change. ...they are interested by form alone." (Poincaré, *Science and Hypothesis)*

Structuralist Philosophies of Mathematics

- Paul Benacerraf, "What numbers could not be" (1965)
- Geoffrey Hellman, *Mathematics Without Numbers* (1989) (modal structuralism)
- Michael Resnik, *Mathematics as a Science of Patterns* (1997) (holistic realism)
- Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology* (1997) (ante rem structuralism)

Structuralist Philosophies of Mathematics (Cont'd)

- Charles Chihara, *A Structural Account of Mathematics* (2004) (nominalistic structuralism)
- Charles Parsons, *Mathematical Thought and its Objects* (2008)
- Daniel Isaacson, "The reality of mathematics and the case of set theory" (2008) (quasi-conceptual structuralism)
- Their positions on CH

Conceptual Structuralism
Thesis 1

- The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways (counting, ordering, matching, combining, separating, and locating in space and time).

Thesis 2

- Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable.

Thesis 3

- The basic conceptions of mathematics are of certain kinds of relatively simple ideal-world pictures which are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations. They are communicated and understood prior to any axiomatics or systematic logical development.

Thesis 4

- Some significant features of these structures are elicited directly from the world-pictures which describe them, while other features may be less certain. Mathematics needs little to get started and, once started, a little bit goes a long way.

Thesis 5

- Basic conceptions differ in their degree of clarity. One may speak of what is true in a given conception, but that notion of truth may only be partial. Truth in full is applicable only to completely clear conceptions.

Theses 6 and 7

- What is clear in a given conception is time dependent, both for the individual and historically.
- Pure (theoretical) mathematics is a body of thought developed systematically by successive refinement and reflective expansion of basic structural conceptions.

Theses 8 and 9

- The general ideas of order, succession, collection, relation, rule and operation are pre-mathematical; some implicit understanding of them is necessary to the understanding of mathematics.
- The general idea of property is pre-logical; some implicit understanding of that and of the logical particles is also a prerequisite to the understanding of mathematics. The reasoning of mathematics is in principle logical, but in practice relies to a considerable extent on various forms of intuition.

Thesis 10

- The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other.
- Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics.
- The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality.

Objectivity in Social Reality

- John Searle, *The Construction of Social Reality* (1995)
- " There are portions of the real world, objective facts in the world, that are only facts by human agreement. In a sense there are things that exist only because we believe them to exist. ...
- ... things like money, property, governments, and marriages. Yet many facts regarding these things are 'objective' facts in the sense that they are not a matter of [our] preferences, evaluations, or moral attitudes." (Searle 1995, p.1)

Objectivity in Social Reality: Examples

- I am a citizen of the United States.
- I have voted in every U.S. presidential election since I became eligible by age to do that.
- I have a PhD in Mathematics from the University of California.
- My wife and I own our home in Stanford, California; we do not own the land on which it sits.

More Examples

- Rafael Nadal won the 2008 men's Wimbledon finals match, and the 2009 Australian Open.
- In the game of chess, it is not possible to force a checkmate with a king and two knights against a lone king.
- There are infinitely many prime numbers.

The Basic Conceptions of Mathematics as Social Constructions

- The objective reality that we ascribe to mathematics is simply the result of intersubjective objectivity about those conceptions and not about a supposed independent reality in any platonistic sense.

- This view does not require total realism about truth values. It may simply be undecided under a given conception whether a given statement has a determinate truth value.

- Example: the presidential line of succession in the U.S. government is undetermined past a certain point.

Conceptions of Sequential Generation

- The most primitive mathematical conception is that of the positive integer sequence represented by the tallies: I, II, III, ...
- Our primitive conception is of a structure $(N^+, 1, Sc, <)$
- Certain facts about this structure are evident (if we formulate them at all): $<$ is a total ordering, 1 is the least element, and $m < n$ implies $Sc(m) < Sc(n)$.

Open-ended Schematic Truths and Definite Properties

- At a further stage of reflection we may recognize the least number principle: if P(n) is any definite property of members of N^+ and there is some n such that P(n) then there is a least such n.
- The schema is open-ended. What is a definite property? This requires the mathematician's judgment.
- The property, "n is the number of grains of sand in a heap" is not a definite property.
- What about the property, "GCH does not hold at n"?

Reflective Elaboration of the Structure of Positive Integers

- Concatenation of tallies immediately leads us to the operation of addition, $m + n$, and that leads us to $m \times n$ as "n added to itself m times".
- The basic properties of the + and × operations such as commutativity, associativity, distributivity, and cancellation are initially recognized only implicitly.
- One goes on to the relations $m|n$, "n is a prime number", $m \equiv n \pmod{p}$, etc.
- Soon have a wealth of expression and interesting problems (primes, perfect numbers, etc., etc.)

Truth in Number Theory

- The conception of the structure $(N^+, 1, Sc, <, +, \times)$ is so clear that there is no question in the minds of mathematicians as to the definite meaning of such statements and the assertion that they are true or false, independently of whether we can establish them one way or the other.
- N^+ is recognized as a definite totality and the logical operation $(\forall n \in N^+)\ P(n)$ is recognized as leading from definite properties to definite statements that are true or false.
- In other words we accept realism in truth values, and the application of classical logic in reasoning about such statements is automatically legitimized.

Further Reflection

- Further reflection on the structure of positive integers led to the structure of natural numbers (N, 0, Sc, <, +, ×), then the integers Z and the rational numbers Q. Though not basic conceptions we are no less clear in our dealings with them than for the basic conception of N^+.
- More advanced reflection leads to the general open-ended scheme of proof by induction on N, $P(0) \wedge \forall n[P(n) \rightarrow P(Sc(n))] \rightarrow \forall n\, P(n)$.
- That is then used to justify definition by recursion on N.

The Unfolding of Arithmetic

- There is a general notion of unfolding of open-ended schematic systems. (Feferman, 1996)
- The unfolding of a basic schematic system for the natural numbers is equivalent in strength to predicative mathematics (Feferman, Strahm 2000).
- But beyond that, the scheme of induction ought to be accepted for any definite property P that one will meet in the process of doing mathematics.

Multiple Sequential Generation

- Finite generation under more than one successor operation Sc_a where a is an element of an index collection A.
- We may conceive of the objects of the resulting structure as “words on the alphabet A”, with $Sc_a(w) = wa$ in the sense of concatenation.
- In the case that A = {0, 1} we also conceive of the words on A as the finite paths in the binary branching tree.

Conceptions of the Continuum

- There is no unique concept of the continuum but rather several related ones. (Feferman 2009)
- To clear the way as to whether CH is a genuine mathematical problem one should avoid the tendency to conflate these concepts, especially those that we use in describing physical reality.
- (i)The Euclidean continuum, (ii) The Hilbertian continuum, (iii) The Dedekind real line, (iv) The Cauchy-Cantor real line, (v) The set 2^N, (vi) the set of all subsets of N, S(N).

Conceptions of the Continuum (Cont'd)

- Not included are physical conceptions of the continuum, since our only way of expressing them is through one of the conceptions via geometry or the real numbers.
- Which continuum is CH about? Their identity as to cardinality assumes impredicative set theory.
- Set theory erases the conceptual distinction between sets and sequences.
- CH as a proposition about subsets of S(N) and possible functions (one-one sets of ordered pairs)

The Continuum in Physical Science

- The argument from indispensability for substantial portions of set theory (Quine, Putnam)
- The contrary evidence from case studies for the thesis that all of current scientifically applicable mathematics can be carried out predicatively ("Why a little bit goes a long way...", 1993)
- In fact it can all be done in a system W ("for Weyl") conservative over Peano Arithmetic (Feferman and Jäger 1993/1996)
- What would change this?

Conceptions of Sets

- Sets are supposed to be definite totalities, determined solely by which objects are in the membership relation (∈) to them, and independently of how they may be defined, if at all.
- A is a definite totality iff the logical operation of quantifying over A, $(\forall x \in A)\ P(x)$, has a determinate truth value for each definite property $P(x)$ of elements of A.
- $A \subseteq B$ means $(\forall x \in A)\ (x \in B)$
- Extensionality: $A \subseteq B \wedge B \subseteq A \rightarrow A = B$.

The Structure of "all" Sets

- (V, $\in$), where V is the universe of "all" sets.
- V itself is not a definite totality, so unbounded quantification over V is not justified on this conception. Indeed, it is essentially indefinite.
- If the operation S(.) is conceived to lead from sets to sets, that justifies the power set axiom Pow.
- At most, this conception justifies KP_ω+Pow+AC, with classical logic only for bounded statements (Feferman, t.a.)

The Status of CH

- But--I believe--the assumption of S(N), S(S(N)) as definite totalities is philosophically justified only on platonistic grounds.
- From the point of view of conceptual structuralism, the conception of the totality of arbitrary subsets of any given set is essentially indefinite (or inherently vague).
- For, any effort to make it definite violates the idea of what it is supposed to be.

Gödel's Program and CH

- Gödel's argument (1947/1964) for the meaningfulness of CH and the need for new axioms to settle it.
- The intrinsic program is definitively inadequate (Koellner 2000, 2006).
- The extrinsic program is being vigorously pursued with brilliant metamathematical work (Woodin et al.--cf. Pettitot 2009), but what are the criteria for success?
- Could one prove that CH is essentially indefinite? So far only circumstantial evidence (e.g., the Lévy-Solovay theorem) on the "problem" of CH.

Selected References

- S. Feferman (1998), *In the Light of Logic*, OUP.
- __________ (2009), Conceptions of the continuum, *Intellectica* 51, 169-189.
- __________ (2009), What's definite, what's not?, http://math.stanford.edu/~feferman/papers/whatsdef.pdf (Slides for H. Friedman 60th birthday conference.)

Selected References (cont'd)

- S. Feferman and T. Strahm (2000), The unfolding of non-finitist arithmetic, *APAL* 104, 75-96.
- ______________________ (t.a.), The unfolding of finitist arithmetic, *Review of Symbolic Logic*.
- P. Koellner (2006), On the question of absolute undecidability, *Philosophia Mathematica* 14(2), 153–188.
- J. Petitot (2009), A transcendental view of the continuum: Woodin's conditional platonism, *Intellectica* 51, 93-133.

Historical Note

- An earlier version of this talk was presented at the VIIIth International Ontology Conference in San Sebastián, Oct. 1, 2008.
- A form of the ten theses of conceptual structuralism were first presented in a talk to the Philosophy Department of Columbia University under the title "Mathematics as objective subjectivity." The text was circulated then but never published. Their first publication was in "Conceptions of the continuum" (*Intellectica* 2009).

The End