Enriched Stratified systems for the Foundations of Category Theory

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Abstract Four requirements are suggested for an axiomatic system S to provide the foundations of category theory: (R1) S should allow us to construct the category of all structures of a given kind (without restriction), such as the category of all groups and the category of all categories; (R2) It should also allow us to construct the category of all functors between any two given categories including the ones constructed under (R1); (R3) In addition, S should allow us to establish the existence of the usual basic mathematical structures and carry out the usual set-theoretical operations; and (R4) S should be shown to be consistent relative to currently accepted systems of set theory. This paper explains how all but parts of (R3) can be met using a system S extending NFU enriched by a stratified pairing operation; to meet more of (R3) a stronger system S^* is introduced, but there are still some real obstacles to meeting this requirement in full. For (R4) it is sketched how both S and S^* are shown to be consistent.

1 Introduction

This is the fourth in a series of intermittent papers on the foundations of category theory stretching back over more than thirty-five years. The first three were "Set-theoretical foundations of category theory" (1969), "Categorical foundations and foundations of category theory" (1977), and much more recently, "Typical ambiguity: Trying to have your cake and eat it too" (2004). The present paper summarizes the results from a long (in two senses) unpublished manuscript,"Some formal systems for the unlimited theory of structures and categories" (1974), referred to below simply as "Unlimited". That MS can be found in full on my home page at `http://math.stanford.edu/~feferman/papers/Unlimited.pdf`; the lengthy

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proof of its main consistency result is omitted here but the methods involved are outlined briefly in the Appendix below.

I have been concerned in these papers with set-theoretical foundations of category theory not because I am a proponent of set theory — on the contrary I am opposed to it on fundamental philosophical grounds[1] — but rather because it is currently widely accepted, its ins and outs are well understood, and it has dealt successfully with the problems surrounding objects that are somehow "too large". It is just such problems in what is sometimes called "naïve" category theory that require foundational attention. Namely, objects like the category of all groups, the category of all topological spaces, etc., seem natural enough mathematically, but what about the category of all categories? And, further, what about the category of all functors between any two categories? Several proposals have been made for dealing with these within the general framework of axiomatic set theory, most notably the familiar ones of Mac Lane (1961, 1971) and Grothendieck (in Gabriel (1962)). This is one reason that alternatives, such as my (1969; 2004) and the present one, are best explored within the same framework for purposes of comparison. I do think that the foundations of category theory ought also to be explored within other frameworks such as those of constructive or semi-constructive mathematics of various stripes, but the directions those might take is not touched on here.

There are some workers in the field who think that category theory does not need foundations and in fact that it is category theory itself that provides the proper foundations for all of mathematics, including itself; see, for example, Lawvere (1966), Bénabou (1985) and, more recently, McLarty (2004). In my 1977 paper cited above, I have argued against that position in a way that I think is no less compelling now than then. Those arguments are not repeated here; in addition to my (1977), the interested reader should also see their extension by Hellman (2003). But there are further objections to be made. It is not clear what exactly is meant by categorical foundations for category theory and how it proposes to handle the problem of the category of all categories and that of arbitrary functor categories. There is also a specific mathematical objection that has been raised by Rao (2006) concerning the construction of localizations in homotopical algebra that make use of transfinite induction and recursion. As he says, "[i]t is not clear how to formulate these in categorical terms....Solving these problems [by such means] looks remote at the moment."

2 What the various proposals do and don't do

In my 1977 paper cited above, on pp. 154-156 I suggested three requirements on a system S for the foundations of category theory.[2] Rephrased from there, S should:

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[1] See, for example, my collection of essays, *In the Light of Logic* (Feferman, 1998).

[2] Bénabou (1985) proposes more specific requirements which need to be considered for a full scale foundation of naïve category theory.

(R1) Allow us to construct the category of all structures of a given kind, e.g. the category **Grp** of all groups, **Top** of all topological spaces, and **Cat** of all categories.
(R2) Allow us to construct the category B^A of all functors from A to B, where A and B are any categories.
(R3) Allow us to establish the existence of the usual basic mathematical structures and carry out the usual set-theoretical operations.

A further requirement was not stated there, but is implicit in the above:

(R4) S should be established to be consistent relative to currently accepted systems of set theory.

Let us see how the existing proposals stack up against these requirements. Mac Lane's proposal was to work in the Bernays-Gödel (BG) system of sets and classes, using the distinction between "small categories" and "large categories" according to whether the categories are sets or proper classes. This meets (R1) in a rather modified form: one can only talk about the large categories $\mathbf{Grp}_{\mathrm{sm}}$, $\mathbf{Top}_{\mathrm{sm}}$, and $\mathbf{Cat}_{\mathrm{sm}}$ of all small groups, small topological spaces and small categories, respectively. (R2) can be met only for A small, since the class of all functions from one proper class into another does not exist in BG. (R3) and (R4) are of course met as is: BG is a conservative extension of Zermelo-Fraenkel set theory ZF, and the same holds when the Axiom of Choice AC is added to both systems.

Grothendieck's proposal was to work in ZFC (= ZF + AC) with the addition of a strong axiom of "universes". Roughly speaking, a universe U is a transitive set that contains the set ω of natural numbers, is closed under subsets, satisfies the ZFC axioms, and in addition satisfies the inaccessibility condition that whenever $a \in U$ and $f : a \to U$ then the range of f is in U. These conditions imply that the cardinal number of U is a strongly inaccessible cardinal. The Grothendieck axiom is that there are arbitrarily large universes, i.e. for every set a there is a universe U with $a \in U$. Again, requirements (R1) and (R2) are met only in a restricted form. Namely, for any universe U, we may speak only of the category of all categories that lie in U; it belongs to any larger universe U'. Also if A and B are two categories whose objects and morphisms all lie in U, then B^A lies in U'. (R3) is met by assumption; (R4) is met by the reduction to ZFC + "there exist arbitrarily large strongly inaccessible cardinals."[3]

In my papers (1969; 2004), I worked in ZFC with one or more additional constant symbols for universes U that are transitive, closed under subsets and satisfy the reflection principle, i.e. the scheme for each formula $\varphi(x_1 \ldots x_k)$ of the language of ZFC (without the additional symbols):

$$\forall x_1 \ldots \forall x_k [x_1 \ldots x_k \in U \to (\varphi^{(U)}(x_1 \ldots x_k) \leftrightarrow \varphi(x_1 \ldots x_k))]$$

[3] Though inaccessible cardinals are not met in ordinary mathematical practice, working set-theorists accept their existence without hesitation as constituting a natural extension of the ZFC axioms, and indeed as only the first in a series of progressively stronger extensions. Gödel (1947) was an early proponent of this idea.

This scheme insures that U satisfies the ZFC axioms. But it is *not* assumed that U is closed under the inaccessibility condition. Requirements (R1) and (R2) are met as in the Grothendieck proposal. (R3) is met by assumption. For (R4) it is shown in both the cited papers that the system is conservative over ZFC. Thus one need not assume the existence of inaccessible cardinals, though a few applications (such as the Kan Extension theorem) apparently need U to satisfy the inaccessibility condition. The principal advantage of my proposal over Grothendieck's is a conceptual one: any U satisfying the above reflection condition looks, from the point of view of the set-theoretical language, exactly like the universe V of all sets, and thus serves as a stand-in for it. Thus anything we can contemplate doing over V can already be done over U and in that way be fully expressed in ZFC.[4] This approach is also taken by Rao (2006).

Though each of these solutions is adequate for normal applications of category theory as in Mac Lane (1971), none of them satisfies (R1) and (R2) without modification. The purpose of this paper is to show how those two requirements can be met in full by working in certain systems of set theory extending Quine's idea of stratification, as explained in the next section. These systems are shown to be consistent relative to standard systems of set theory (as proved in detail in the "Unlimited" MS and outlined in the Appendix), so (R4) is also met. Finally, while (R3) is met to a considerable extent, we shall see that there are two ubiquitous set-theoretical constructions that can't be carried out in these systems without ad hoc modification: passage to equivalence classes under an equivalence relation, and formation of the Cartesian product of an indexed family of classes. In addition, certain basic results of category theory such as the Cartesian closedness of the category of all sets and Yoneda's Lemma can't be formulated unrestrictedly. These drawbacks are the price paid under the existing stratified approach in order to satisfy (R1) and (R2) in full. It may be that there can be no solution to (R1)-(R4) without such trade-offs, but nothing I know currently excludes that.

What is to be emphasized from all this work is not that naïve category theory ought further to be pursued within the framework of stratified systems (nor, equally, that it ought *not* to be pursued in that way), but rather that it serves to illustrate how one can meet at least some of the basic requirements without restriction, unlike current standard set-theoretical approaches. Thus emboldened, one should seek ways to meet *all* of the requirements without restriction.

3 The system NFU with stratified pairing

The system NF of "New foundations for mathematical logic" has a single sort of variable and the basic relations $=$ and $\in$; its axioms are Extensionality and Stratified Comprehension. For reasons below, I shall use capital letters $A, B, C, \ldots, X, Y, Z$ for

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[4] Just one universe of this kind is assumed in my 1969 paper; that is all one needs for the applications. In the 2004 paper, I assumed a sequence of such universes $U_n \in U_{n+1}$ for each $n \in \omega$, in order to relate the idea more directly to Russell's idea of typical ambiguity.

its variables; the objects these range over will be called *classes*.[5] A formula φ is said to be *stratified* if it results from a formula of simple type theory by suppressing type distinctions, equivalently if it is possible to assign natural number *type* superscripts to each variable in φ in such a way that (i) each variable is assigned the same type at all its occurrences, (ii) for each subformula of φ of the form $X = Y$, the variables X and Y are assigned the same type, and (iii) for each subformula of φ of the form $X \in Y$, the variable Y is assigned type $n+1$ when X is assigned type n.

Examples: $X \in Y$ and $Y \in X$ are stratified (for X, Y distinct variables) but $X \in X$ is not and $(X \in Y \wedge Y \in X)$ is not.

The Stratified Comprehension Axiom scheme consists of (the universal closures of) all formulas of the form

$$(\mathrm{SCA}) \quad \exists A \forall X [X \in A \leftrightarrow \varphi]$$

where φ is stratified and the variable A does not occur in φ.

Extensionality (Ext) is stated as usual. Thus NF = Ext + SCA. To this day it is not known whether NF is consistent. For an exposition of the considerable work that has been done exploring NF and some of its variants, see Forster (1995) and Holmes (1998). The variant that occupies our attention here and that *has* been shown to be consistent by Jensen (1969) is called NFU, because it allows for the possible existence of more than one "urelement", i.e., a class which has no members. This is done by weakening Extensionality as follows to apply to non-urelements:

$$(\mathrm{Ext}') \quad \exists X (X \in A) \wedge \forall X (X \in A \leftrightarrow X \in B) \rightarrow A = B$$

Thus NFU = Ext′ + SCA. NFU is very weak as systems go; Jensen proved its consistency relative to Peano Arithmetic, PA. One cannot prove the existence of an infinite class in NFU.[6]

By (SCA) there is at least one empty class; fix any such and denote it by Λ. For each stratified φ with free variables included in $\{X, Y_1, \ldots, Y_n\}$, we define $\{X \mid \varphi(X, Y_1, \ldots, Y_n)\}$ to be the unique A satisfying SCA for φ if $\exists X \varphi(X, Y_1, \ldots, Y_n)$, otherwise Λ.

In particular, we can define the familiar set-theoretical operations as usual in NFU: $\{Y\}$, $\{Y_1, Y_2\}$, $Y_1 \cup Y_2$ and $Y_1 \cap Y_2$; more generally we can define the union of any class of classes $\bigcup Y$ as $\{X \mid \exists Z (Z \in X \wedge X \in Y)\}$. Writing $X \subseteq Y$ for $\forall Z (Z \in X \rightarrow Z \in Y)$, we can also define $\wp(Y) = \{X \mid X \subseteq Y\}$. Constructions that are distinctive to classes are $-Y = \{X \mid X \notin Y\}$ and $V = \{X \mid X = X\}$; we have $-\Lambda = \bigcap \Lambda = V$. Also, self-membership makes its first appearance with $V \in V$.

When dealing with relations R in a typed or stratified set-up, for example those that are binary, it is natural to consider them as classes of ordered pairs (X, Y) such that

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[5] Lower case letters will also be used for classes in some contexts below.

[6] Actually, NFU is quite weak, proof-theoretically, compared to PA (Solovay, unpublished). As shown by Enayat (2004), one can obtain an extension of NFU equivalent in strength to PA by adding "every set is finite" and "every Cantorian set is strongly Cantorian" as axioms (cf. the final section below for the notions of Cantorian and strongly Cantorian sets in the framework of NFU).

(*) the types assigned to X, Y, and (X, Y) are all the same.

The usual way of defining pairs in set theory as $(X,Y) = \{\{X\},\{X,Y\}\}$ is not stratified in this sense. Quine (1945) showed how to define a pairing operation in NF to satisfy (*), but his definition requires full Extensionality and an Axiom of Infinity (Rosser (1952)). Let L_p be the language L augmented by a binary operation $(.,.)$ symbol. By the Pairing Axiom in L_p is meant the following:

(P) $(X_1,X_2) = (Y_1,Y_2) \rightarrow X_1 = Y_1 \wedge X_2 = Y_2$.

The terms $s,t,\dots$ of L_p are generated from the variables by closing under the pairing operation: whenever s,t are terms, so also is (s,t). The system NFU_P consists of $\mathrm{Ext}' + \mathrm{SCA} + \mathrm{P}$, where now the notion of stratification in SCA has to be expanded to accord with (*); this can be achieved by modifying the definition of a formula φ being stratified as follows:

1. Each term t occurring in φ is assigned a natural number as type
2. The type assigned to a term t of φ is the same as the type assigned to each variable occurring in t
3. Each variable of φ has the same type assigned to it at all occurrences
4. For each subformula of φ of the form $s = t$, the types assigned to s and t are the same
5. For each subformula of φ of the form $s \in t$ and type n assigned to s, the type assigned to t is $n+1$.

Examples: $(X,Y) \in Z$ is L_p stratified, but not $[(X,Y) \in Z \wedge X \in Y]$.

Theorem 1 NFU_P *is consistent.*

This theorem may be proved by a straightforward modification of the proof of consistency of NFU + Inf in Theorem 1 of Jensen (1969), where Inf is an Axiom of Infinity. Consistency of a much stronger system than NFU_P is stated in the penultimate section below and an outline of how that is proved is given in the Appendix.[7]

4 First-order structures in NFU_P

For any classes A,B, define $A \times B$ to be the class of all (X,Y) with $X \in A \wedge Y \in B$. Define n-tuples inductively by $(X_1) = X_1$ and $(X_1,\dots,X_n,X_{n+1}) = ((X_1,\dots,X_n),X_{n+1})$. Then for any A,n, define A^n to be the class of all n-tuples $(X_1,\dots,X_n)$ with $X_i \in A$. An n-ary relation R on A is a subclass of A^n. A function F on A into B, in symbols, $F : A \rightarrow B$, is a subclass of $A \times B$ such that for each $X \in A$ there is exactly one Y with $(X,Y) \in F$; we write $F(X) = Y$ in this case. Note that $B^A = \{F \mid F : A \rightarrow B\}$ exists by SCA. An n-ary function from A to B is an $F : A^n \rightarrow B$.

A single-sorted first-order structure is a tuple

[7] Independently, Holmes (1991) showed that NFU_P is interpretable in NFU + Inf, giving a more direct proof of Theorem 1 assuming Jensen's work.

$$A = (O, R_1, \ldots, R_j, F_1, \ldots, F_k, K_1, \ldots K_l)$$

where the domain O of objects of A is non-empty and each R_i is an n_i-ary relation on O for some n_i, each F_i is an m_i-ary function on O to O for some m_i, and each K_i is a singleton, $K_i = \{C_i\}$ for some $C_i \in O$. This is generalized in the obvious way to many-sorted first-order structures. Given a sentence θ in the first-order language of such structures, we write as usual, $A \models \theta$ to express that A satisfies θ, or is a model of θ. By Model(θ) we mean the class of all A such that $A \models \theta$. Associated with each such θ is an L_P stratified formula $\theta^*(X)$ such that Model$(\theta) = \{X \mid \theta^*(X)\}$

Examples:

i) Consider structures $A = (O, R)$ with R a binary relation on O. The class PO is defined to be the class of all such A that are partially ordered. Then PO $=$ Model$(\theta) = \{X \mid \theta^*(X)\}$ where $\theta^*(X)$ is the following formula:

$$\exists Y, Z[X = (Y, Z) \wedge \exists V(V \in Y)$$
$$\wedge \forall U(U \in Z \to \exists V, W(V \in Y \wedge W \in Y \wedge U = (V, W))$$
$$\wedge \forall V(V \in Y \to (V, V) \in Z)$$
$$\wedge \forall V, W((V, W) \in Z \wedge (W, V) \in Z \to V = W)$$
$$\wedge \forall V, W, U((V, W) \in Z \wedge (W, U) \in Z \to (V, U) \in Z)]$$

θ^* is L_P stratified by assigning type 1 to the variables X, Y, Z and type 0 to the variables U, V, W.

ii) We treat similarly the class Equiv of all $A = (O, R)$ such that R is an equivalence relation on O. $A \in$ Equiv iff $A \models \theta$ where θ is a first-order formula, and then Equiv $=$ Model$(\theta) = \{X \mid \theta^*(X)\}$ with L_P stratified θ^*.

iii) Consider structures $A = (O, F, G, \{E\})$ where F is a binary operation on O, G is a unary operation on O and $E \in O$. The class Grp is defined to be the class of all such A that are groups, in which F is the multiplication operation on O, G is the inverse operation on O, and E is the identity element for F. Then Grp $=$ Model$(\theta) = \{X \mid \theta^*(X)\}$ for a first-order θ as usual.

iv) We here treat categories as two-sorted structures $A = (O, M, C, D_0, D_1)$ where O is the collection of its objects, M is the collection of its morphisms, C is the composition operation on morphisms and D_0, D_1 give the domain and codomain, resp., of a morphism, to tell when composition is defined. Thus each D_i is a function from M to O and the ternary relation $C \subseteq M^3$ is a partial function from M^2 to M, with $C(f, g)$ or fg defined for $f, g \in M$ when $D_1(f) = D_0(g)$.[8] The defining condition for A to be a category is given by a first-order formula θ and we can take Cat $=$ Model$(\theta) = \{X \mid \theta^*(X)\}$ to be the class of all categories.

As a warm-up for meeting requirement (R1) in the next section, consider the statement that the class PO is partially ordered under the substructure relation Sub,

[8] As in Mac Lane (1971) we use lower-case letters $f, g, h, \ldots$ for morphisms in an abstract category, but this does not signal a new kind of variable in NFU$_P$. Similarly, in the next section, where we use $a, b, \ldots$ for objects in a category and η for natural transformations.

where $((O,R),(P,S)) \in \mathrm{Sub}$ iff $O \subseteq P$ and $R \subseteq S$ and $S \cap O^2 = R$. The relation Sub is provably a class in $\mathrm{NFU_P}$ since it is defined by an L_P stratified formula. Then the informal statement can be written as:

$$(\mathrm{PO}, \mathrm{Sub}) \in \mathrm{PO}$$

Similarly, we can re-express the statement that the relation Isom of isomorphism is an equivalence relation on the class Equiv of structures as:

$$(\mathrm{Equiv}, \mathrm{Isom}) \in \mathrm{Equiv}$$

5 Meeting requirements (R1) and (R2) in $\mathrm{NFU_P}$

(R1)

The category of all groups has the form

$$\mathbf{Grp} = (\mathrm{Grp}, \mathrm{Hom}, C, D_0, D_1)$$

where Grp is the class of all groups as in the preceding section, Hom is the class of all $F = (F_0, A, B)$ such that $A = (O_A, \ldots)$ and $B = (O_B, \ldots)$ are groups and $F_0 : O_A \to O_B$ is a group homomorphism from A into B, $D_0(F) = A$ and $D_1(F) = B$, and the composition $C(F,G)$ of F and G in Hom is defined as usual when $D_1(F) = D_0(G)$. Since the classes Grp and Hom and the functions C, D_0, and D_1 exist by SCA, and the structure **Grp** satisfies the conditions to be a category, we may state:

$$\mathbf{Grp} \in \mathrm{Cat}$$

Similarly we can define the category **Top** of all topological spaces and verify that

$$\mathbf{Top} \in \mathrm{Cat}$$

The category of all categories has the form

$$\mathbf{Cat} = (\mathrm{Cat}, \mathrm{Funct}, C, D_0, D_1)$$

where Cat is the class of all categories as in the preceding section, Funct is the class of all $F = (F_0, F_1, A, B)$ such that $A = (O_A, M_A, \ldots)$ and $B = (O_B, M_B, \ldots)$ are categories and the pair $F_0 : O_A \to O_B$, $F_1 : M_A \to M_B$ determines a functor from A into B, $D_0(F) = A$ and $D_1(F) = B$, and the composition $C(F,G)$ of F and G in Funct is defined as usual when $D_1(F) = D_0(G)$. Since **Cat** satisfies the conditions to be a category, we have:

$$\mathbf{Cat} \in \mathrm{Cat}$$

In this way, requirement (R1) is satisfied in $\mathrm{NFU_P}$.

(R2)

Given any two categories $A = (O_A, M_A, \dots)$ and $B = (O_B, M_B, \dots)$, the category of all functors from A to B has the form

$$B^A = (\mathrm{Funct}(A,B), \mathrm{Nat}, C, D_0, D_1)$$

where $\mathrm{Funct}(A,B)$ is the class consisting of all functors F from A to B, Nat is the class of all natural transformations from one such functor F to another, and C, D_0, D_1 are explained below. As usual we write $f : a \to b$ for $f \in M_A$ and $a, b \in O_A$ when $D_0(f) = a$ and $D_1(f) = b$, and similarly for morphisms in B. The class $\mathrm{Funct}(A,B)$ is the subclass of the class of all pairs (F_0, F_1) for which $F_0 : O_A \to O_B$ is such that for each $a, b \in O_A$ and $f \in M_A$ with $f : a \to b$, we have $F_1(f) \in M_B$ with $F_1(f) : F_0(a) \to F_0(b)$ and the usual conditions on preservation of composition and identity morphisms are satisfied. Natural transformations are taken to be triples (η, F, G) where F, G are two such functors, and $\eta : O_A \to M_B$ in such a way that for each $a \in O_A$, $\eta(a) : F_0(a) \to G_0(a)$ in B and we have the usual commutative square associated with any $f : a \to b$ in A; $D_0(\eta) = F$ and $D_1(\eta) = G$. Composition C of natural transformations is defined in the natural way. Once again we can check that Nat, C, D_0 and D_1 all exist and that B^A is indeed a structure in $\mathrm{NFU_P}$. Moreover it satisfies the conditions to be a category so, finally, we can state

$$B^A \in \mathrm{Cat}$$

as a theorem in $\mathrm{NFU_P}$, just as required by (R2).

6 The requirement (R3); type-shifting problems in $\mathrm{NFU_P}$

One can establish the existence of the class N in $\mathrm{NFU_P}$ and thus the finite type-theoretic hierarchy over N obtained by iterating the power class operation $\wp$ and the construction of function types. More is needed to go to transfinite types as in ZFC; how that is done is dealt with in the next section. Otherwise, for (R3), we have seen in secs. 3 and 4 that many of the standard set-theoretic constructions can be carried out without any obstacle in $\mathrm{NFU_P}$. The fact that Extensionality is weakened to Ext′ does not hinder usual arguments either. Here we concentrate on operations that *can't* be carried out without ad hoc adjustments.

i) *Equivalence classes.* Suppose $(O, E) \in \mathrm{Equiv}$, i.e., E is an equivalence relation on the class O. For each $X \in O$, define $X/E = \{Y \mid (X,Y) \in E\}$ and $O/E = \{X/E \mid X \in O\} = \{Z \mid \exists X (X \in O \wedge \forall Y (Y \in Z \leftrightarrow (X,Y) \in E))\}$. This exists by SCA, assigning type level 1 to O, E, and Z and type level 0 to X and Y. However, the usual function F from O to O/E cannot be shown to exist since it consists of pairs (X, Z) such that $Z = X/E$ is of type level higher than that of X. The ad-hoc modification in this case is to introduce a new kind of function, from the class of singletons associated with O, $\mathrm{Sing}(O) = \{W \mid \exists X(X \in O \wedge W = \{X\}\}$, to O/E. Alternatively, in the presence of a suitably universal choice function (see the penultimate section below for the consistency of that), we can deal in a stratified way with a function to representatives of equivalence classes.

ii) *Cartesian products.* A sequence of classes O_X indexed by X ranging over a class I is given by a function $F : I \to V$ with $F(X) = O_X$ for each $X \in I$. The Cartesian product of this sequence is supposed to be a class $\prod O_X(X \in I)$ whose members are all $G : I \to V$ such that for all $X \in I$, $G(X) \in O_X$. Thus each such G consists of pairs (X,Y) such that $Y \in Z$ where $(X,Z) \in F$; this cannot be arranged in a stratified way in $\mathrm{NFU_P}$. Again, an ad hoc solution is to modify the notion of function, say by taking $F : \mathrm{Sing}(I) \to V$ for the initial sequence of classes.

iii) *Cartesian closedness of* **Class**. In the context of $\mathrm{NFU_P}$, one deals with the category **Class** of all classes in place of the category of all sets in ordinary set-theoretical foundations. The latter is Cartesian closed, one of whose conditions is that we have an adjoint to Cartesian product (cf. Mac Lane, 1971, p. 95). This yields the exponentiation operation with the evaluation morphisms $\mathrm{ev} : b^a \times a \to b$ given by $\mathrm{ev}(f,x) = f(x)$ for each $f : a \to b$. But that can't be done for **Class** in a stratified way in $\mathrm{NFU_P}$. More definitively and more generally, McLarty (1992) showed that Cartesian closedness of **Class** and **Cat** provably fail in NF; his argument works equally well in $\mathrm{NFU_P}$.

iv) *Yoneda lemma.* Given an abstract category $A = (O_A, M_A, \ldots)$, the Hom classes associated with A are the classes

$$\mathrm{Hom}_A(a,b) = \{f \mid f \in M_A \wedge D_0(f) = a \wedge D_1(f) = b\}$$

As in iii), **Class** is the category of all classes, with the usual mappings from one class into another constituting its morphisms. For each $a \in O_A$ we have a functor H^a from A into **Class** given by

$$H^a(b) = \mathrm{Hom}_A(a,b)$$

for each $b \in O_A$, with the obvious choice of $H^a(f) : H^a(b) \to H^a(c)$ whenever $f : b \to c$ in A. What the Yoneda Lemma does is set up a natural isomorphism between $F(a)$ and the natural transformations from H^a into F for each functor F from A into **Class** (cf. Mac Lane, 1971, p. 61). Closer inspection shows that there is a lot of mixing of types here that can't be represented in $\mathrm{NFU_P}$ without ad hoc modifications, to begin with of H^a as a function, since the type level of $H^a(b)$ is one higher than that of its elements $f : a \to b$, which are of the same type level as those of a and b. Like (iii), this is a serious drawback to the use of $\mathrm{NFU_P}$ as it stands as a foundation for category theory.

There is no obvious modification of the notion of stratification for systems with pairing, that allows pairs (s,t) of mixed type and is consistent. Type-theoretically, the natural thing to try is to assign to (s,t) the type level $\max(n,m)$ when s is assigned n and t is assigned m. The problems i)-iv) all concern situations involving pairs (s,t) where s is assigned type n and t is assigned type $n+1$. However, if SCA were expanded to allow stratification in this sense we would derive a contradiction from

$$\exists A \forall X[X \in A \leftrightarrow \exists Y, Z(X = (Y,Z) \wedge Y \in Z)]$$

and

$$\forall A \exists B \forall Y[Y \in B \leftrightarrow (Y,Y) \notin A]$$

B is just the Russell class. A possible out is to restrict oneself to stratified proofs in a suitable sense, so that the types assigned to a class of pairs don't change in the course of the proof. In the example, the elements of A change from pairs of type $(0,1)$ to pairs of type $(0,0)$. Even if that idea were to lead to a consistent system, it might require keeping track of things in a cumbersome way.

7 The requirement (R3), continued; building in ZFC

In this section we boost NFU$_{\mathrm{P}}$ to incorporate ZFC in a certain way; the resulting system was denoted S* in "Unlimited", and for simplicity of notation, we shall follow that here, too. The language L^* of S* extends the language L_p of NFU$_{\mathrm{P}}$ by the adjunction of a constant symbol V_0 and *set variables* $a,b,c,\ldots,x,y,z$.[9] The terms of L^* are generated from the constant V_0 and both kinds of variables by closure under the pairing operation. The atomic formulas are $s = t$ and $s \in t$, where s,t are arbitrary terms. The notion of a formula φ being L^* stratified is modified as follows from the section that introduced NFU$_{\mathrm{P}}$.

1. Each term t occurring in φ is assigned a natural number as type;
2. the type assigned to a term t in φ is the same as the type assigned to each class variable occurring in t;
3. each class variable of φ has the same type assigned to it at all occurrences;
4. for each subformula of φ of the form $s = t$, the types assigned to s and t are the same; and
5. for each subformula of φ of the form $s \in t$ and type n assigned to s, the type assigned to t is $n+1$.

N.B. No restrictions are made on the types that are assigned to V_0 or set variables; these may be assigned *any* type and may be assigned different types at different occurrences in the same formula.

Examples: The formulas $(V_0 \in X \wedge X \notin V_0)$, $X \in x$, and $(x,X) \in x$ are all L^* stratified, but not the formula $(x,X) \in X$.

Axioms of S*:

1. *Stratified comprehension:* $\exists A \forall X[X \in A \leftrightarrow \varphi]$ for each L^* stratified φ that does not contain the variable A.
2. *Weak extensionality:* $\exists X(X \in A) \wedge \forall X(X \in A \leftrightarrow X \in B) \rightarrow A = B$.
3. *Pairing.* $(X_1,X_2) = (Y_1,Y_2) \rightarrow X_1 = Y_1 \wedge X_2 = Y_2$.

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[9] In the syntax of S* lower case letters are now used only in this way.

4. *Sets and classes.*

   a. $\forall x \exists X(x = X)$
   b. $X \in V_0 \leftrightarrow \exists x(x = X)$
   c. $X \in x \rightarrow X \in V_0$

5. *Empty set:* $\exists! z \forall y(y \notin z)$.
6. *Operations on sets:*

   a. $\{x, y\} \in V_0$
   b. $\bigcup x \in V_0$
   c. $\wp(x) \in V_0$
   d. $(x, y) = \{\{x\}, \{x, y\}\}$

7. *Infinite set:* $\exists a[\exists z(z \in a \wedge \forall y(y \notin z)) \wedge \forall x(x \in a \rightarrow x \cup \{x\} \in a)]$
8. *Replacement:* $\forall x, y_1, y_2[\psi(x, y_1) \wedge \psi(x, y_2) \rightarrow y_1 = y_2] \rightarrow$
   $\forall a \exists b \forall y[y \in b \leftrightarrow \exists x(x \in a \wedge \psi(x, y))]$
9. *Foundation:* $\exists x \psi(x) \rightarrow \exists x[\psi(x) \wedge \forall y(y \in x \rightarrow \neg\psi(y))]$, where $\psi(x, \ldots)$ is any L^* formula that does not contain the variable y.
10. *Universal choice:*

$$\exists C[\forall X, Y_1, Y_2((X, Y_1) \in C \wedge (X, Y_2) \in C \rightarrow Y_1 = Y_2)$$

$$\wedge \forall X(\exists Y(Y \in X) \rightarrow \exists Y(Y \in X \wedge (X, \{Y\}) \in C))]$$

Remarks.

(i) The axioms of S* as presented here are a slight variant of those in “Unlimited”; they are interderivable.
(ii) Axioms 1-3 make S* an extension of NFUP.
(iii) The “ontological” Axiom 4 tells us that the sets are exactly the classes that belong to V_0.
(iv) Since there is a unique empty set by Axiom 5, and Extensionality holds for non-empty classes by 2, we have full Extensionality for sets. As in ZF we use 0 to denote the empty set.
(v) Axiom 6 tells us that V_0 is closed under the operations of unordered pair, union and power as defined for classes above; it also tells us that the ordered pair operation coincides with its usual definition in set theory when restricted to sets.
(vi) From Axiom 7 and Separation (see next), we can define ω as the least set containing 0 and closed under the successor operation $x' = x \cup \{x\}$. Again by Separation (or Foundation) we can apply induction on ω to any formula $\psi(x, \ldots)$ of L^*, not just stratified formulas.
(vii) The Separation scheme for sets in L^* consists of all instances

$$\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \theta(x)]$$

for any formula $\theta(x, \ldots)$ of L^* that does not contain the variables a, b. This is an immediate consequence of Replacement (8), using $\psi(x, y)$ as the formula $\theta(x) \wedge y = x$.

(viii) Foundation (9) is equivalent to transfinite induction on the $\in$-relation restricted to sets for arbitrary formulas $\psi(x,\ldots)$ of L^*:

$$\forall x[\forall y(y \in x \to \psi(x)) \to \psi(x)] \to \forall x \psi(x)$$

(ix) Universal Choice (10) is given in a form appropriate to the use of functions as treated in NFUP in sec. 4. It implies the following form of Universal Choice (UC) for sets:

$$\exists F[F : V_0 \to V_0 \wedge \forall x(x \neq 0 \to F(x) \in x)]$$

for, given C as in axiom X, take $F = \{(x,y) \mid (x,\{y\}) \in C\}$. Then UC implies AC, the usual Axiom of Choice for any set of non-empty sets.

Theorem 2 *i)* S^* *is consistent.*
ii) S^* *is an extension of both* NFUP *and* ZFC.
iii) The system of Morse-Kelley MK *with* UC *is interpretable in* S^*.

The proof of (i) has been given in "Unlimited"; an outline of the ingredients is presented in the Appendix below. (ii) is immediate from the preceding remarks. As to (iii), the system MK is what we obtain from BG by allowing *any* formula $\theta(x)$ in the language of BG to define a class of sets, not just predicative formulas as in BG. We can interpret it in S^* by simply taking the class variables to range over those classes X in S^* with $X \subseteq V_0$.

MK is stronger than BG, since we can establish a notion of truth for the language of ZF, and by its means prove that every theorem of ZF is true; hence MK proves the consistency of ZF, while BG (which is a conservative extension of ZF) does not. Similarly, MK + UC is stronger than ZFC, and a fortiori of BG + AC.

S^* makes up for the defects of NFUP to a certain extent. Obviously we can deal with equivalence classes on sets and Cartesian products on sets as usual as in ZFC. More generally for equivalence relations between classes, the Universal Choice axiom provides the possibility of working with representatives rather than equivalence classes in a stratified way.[10] One can also make the kinds of distinctions used by Mac Lane to secure the applications of category theory by means of the notions of "small categories", "locally small categories", etc. So all of Mac Lane (1971) can be directly represented in S^*. But one should also revisit results like Yoneda's Lemma, the Kan Extension Theorem, the Adjoint Functor Theorem, etc., that so far have been formulated using such distinctions to see whether the use of NFUP over ZFC as provided by S^* gives any additional flexibility or generality. That remains to be done. On the other hand, this type of work returns one to the kinds of distinctions that the aim for a direct foundation of naïve category theory is supposed to avoid. It may be that the use of stratified systems for that purpose cannot be advanced much beyond what has been illustrated here. But at least it shows that the program to satisfy such requirements as (R1)-(R4) is a reasonable one to pursue by some means or another.

[10] This has been suggested to me by Randall Holmes.

8 Cantorian classes and extension of NFU in ZFC

By way of comparison with the preceding, much work has been done beginning in the 1990s on the study of extensions of NFU in which ZFC can be interpreted directly. This centers around the *Cantorian* and *strongly Cantorian* classes, as defined in Holmes (1998): a class A is Cantorian if it is in one-one correspondence with the class $\{Y \mid \exists X(X \in A \wedge Y = \{X\})\}$ of its singletons, and it is strongly Cantorian (or s.c.) if the one-one correspondence is given by the standard map sending X to $\{X\}$ on A. Stratification prevents one from showing that every Cantorian class is strongly Cantorian, let alone that every class has this property; Russell's paradox precludes V, among many other classes, from being strongly Cantorian.[11] The kinds of type-shifting problems met above with the development of category theory in $\mathrm{NFU_P}$ are avoided by restricting to s.c. classes. Thus it is possible that an enhanced development of naïve category theory in a stratified framework could be provided by restriction to s.c. classes and the associated categories defined in terms of them; clearly this would require that there exist "enough" s.c. classes. On the other hand, such a restriction would mean giving up requirement (R1), since the collection of s.c. classes does not form a class (let alone a s.c. class). It is known that the s.c. classes are closed under exponentiation, but this does not help with (R2) for large categories, if (R1) can't be satisfied.

By NFUA is meant the system NFU (with stratified pairing) together with the axioms of infinity and choice and the axiom "every Cantorian class is strongly Cantorian". In Holmes (1998), Ch. 20, it is shown how to interpret ZFC in an extension of NFU stronger than NFUA via a certain class of isomorphism types of pointed well-founded extensional relations; this interpretation works in NFUA as well by recent results of Enayat (2004). In fact, much stronger extensions of ZFC come along with that interpretation: in unpublished work, Solovay established the equiconsistency of NFUA with ZFC + "there exist n-Mahlo cardinals" (for each $n \in \omega$); a published proof of that is to be found in Enayat (2004), Theorem 5.5. The strength of the full system in Holmes (1998) has been shown to be that of Morse-Kelley set theory MK plus measurability (in a suitable sense) of the proper class ordinal (the class of all ordinals considered as a virtual ordinal) (Holmes, 2001). An interesting intermediate system designated NFUB has been proved by Solovay (1997) to be of the same strength as MK + "the proper class ordinal is weakly compact."

Compared to these extensions of NFU, the system S^* of section 7 interprets ZFC only by the addition of a constant symbol for a class V_0 and axioms concerning its members. All the members and subclasses of V_0 are automatically strongly Cantorian. It is an open question whether there is a direct interpretation of ZFC in an extension of NFU without such an additional symbol V_0, in which the sets are taken to range over some collection **C** of classes and the membership relation is the restriction of the $\in$ relation to **C**. It is also open what the exact consistency strength is

[11] An axiom stating that all sets are Cantorian was first studied by Henson (1973). A related "axiom of counting" was introduced by Rosser (1953) in order to develop a smooth theory of finite cardinals in NF. It states that the set of finite cardinals is strongly Cantorian; that set is Cantorian in NF and in NFU + Infinity. (I am indebted to Ali Enayat for this background information.)

of S^*; in my original proof, I assumed the existence of two inaccessible cardinals. Since seeing a draft of this paper, Ali Enayat has been looking into that question, and has informed me that — in consistency strength — S^* lies strictly between ZFC + "there exists an inaccessible cardinal" and ZFC + "there exist at least two inaccessible cardinals." He has also pointed out to me that my original proof (outlined below) also establishes the consistency of a strengthening S^{**} of it by an axiom scheme asserting that the extension on V_0 of any property given by an arbitrary formula $\varphi(x,\ldots)$ of the language of S^* is a class: $\exists X \forall x \in V_0[x \in X \leftrightarrow \varphi(x)]$. Furthermore, in the presence of this additional axiom, the Replacement scheme follows from the statement that no partial map from an initial segment of V_0 to V_0 can have a cofinal image, and the scheme of Foundation follows from the usual formulation of Foundation in ZFC (that every non-empty set contains an $\in$-minimal element). It may be more tractable to determine the exact consistency strength of S^{**} than that of S^* in terms of more or less standard extensions of ZFC.

Appendix

The methods used to prove Theorem 2(i), the consistency of S^*, in "Unlimited", are by an extension of those applied by Jensen (1969). They consist of three parts:

1. Specker (1962) reduced the consistency of NF to the existence of models $M_T = (\langle U_i \rangle, \langle \in_i \rangle)_{i \in Z}$ of type theory with types i ranging over the set of all integers, $Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$, where $\in_i \subseteq U_i \times U_{i+1}$, for which M_T satisfies the axioms of typed comprehension and extensionality, and in addition has a type-shifting automorphism $\sigma : U_i \to U_{i+1}$ for all $i \in Z$. The model of NF constructed from M_T is defined to be $M^* = (U_0, \in^*)$ where for $a,b \in U_0$, $a \in^* b \leftrightarrow a \in_0 \sigma(b)$. Jensen observed that if M_T satisfies extensionality only for non-empty classes, then M^* is a model of NFU.
2. Ehrenfeucht and Mostowski (1956) applied the infinite Ramsey theorem to obtain models of first-order theories with indiscernibles $\{c_i\}_{i \in I}$ in given orderings $(I,<)$. When these models are generated by Skolem functions from the indiscernibles we get elementary substructures having automorphisms induced by those of $(I,<)$. Jensen applied the Ehrenfeucht-Mostowski theorem to obtain models M of Zermelo set theory plus the Skolem function axioms having indiscernibles c_i in order type $(Z,<)$ and shifting automorphism induced by $\sigma(c_i) = c_{i+1}$. A Z-typed model as required for the Specker construction of M^* is formed by taking $U_i = \{x \mid x \in_M c_i\}$. Jensen showed that one can also arrange to have M a model of the axioms of Infinity and Choice, which leads to M^* having the same properties. Thus NFU is consistent with Infinity and Choice. In order to satisfy $\mathrm{NFU_P}$ it is only necessary to ensure of the model M that if $x,y \in c_i$ then $\{x\}$ and $\{x,y\} \in c_i$, hence $(x,y) = \{\{x\},\{x,y\}\} \in c_i$.
3. In part II of his paper, Jensen showed how, given any ordinal α, one can construct M^* satisfying these conditions which is an end-extension of α; this uses the Erdös-Rado (1956) generalization of the Ramsey theorem to certain infinite

partitions. These methods were extended in "Unlimited" to construct M^* which are end-extensions of any given transitive set A. The main theorem needed for this and proved in the Appendix of "Unlimited" is in terms of models of $L_{\infty,\omega}$ with indiscernibles satisfying certain prescribed properties. The formulation of that theorem is too technical to present here. The particular transitive set used in the application to Theorem 2(i) above is the cumulative hierarchy up to a strongly inaccessible cardinal κ. The proof also assumes the existence of a strongly inaccessible cardinal δ greater than κ.

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