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PLATONISM AND MATHEMATICAL INTUITION IN KURT GÖDEL'S THOUGHT

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The best known and most widely discussed aspect of Kurt Gödel's philosophy of mathematics is undoubtedly his robust realism or platonism about mathematical objects and mathematical knowledge. This has scandalized many philosophers but probably has done so less in recent years than earlier. Bertrand Russell's report in his autobiography of one or more encounters with Gödel is well known:

> Gödel turned out to be an unadulterated Platonist, and apparently believed that an eternal "not" was laid up in heaven, where virtuous logicians might hope to meet it hereafter.[1]

On this Gödel commented:

> Concerning my "unadulterated" Platonism, it is no more unadulterated than Russell's own in 1921 when in the *Introduction to Mathematical Philosophy* ... he said, "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features." At that time evidently Russell had met the "not"

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[1] Russell [19, page 356].

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> even in this world, but later on under the influence of Wittgenstein he chose to overlook it.[2]

One of the tasks I shall undertake here is to say something about what Gödel's platonism is and why he held it.

A feature of Gödel's view is the manner in which he connects it with a strong conception of mathematical intuition, strong in the sense that it appears to be a basic epistemological factor in knowledge of highly abstract mathematics, in particular higher set theory. Other defenders of intuition in the foundations of mathematics, such as Brouwer and the traditional intuitionists, have a much more modest conception of what mathematical intuition will accomplish. In this they follow a common paradigm of a philosophical conception of mathematical intuition derived from Kant, for whom mathematical intuition concerns space and time as forms of our sensibility. Gödel's remarks about intuition have also scandalized philosophers, even many who would count themselves platonists. I shall again try to give some explanation of what Gödel's conception of intuition is. It is not quite so intrinsically connected with his platonism as one might think and as some commentators have thought. I hope to convince you that even though it is far from satisfactory as it stands, there are at least genuine problems to which it responds, which no epistemology for a mathematics that includes higher set theory can altogether avoid. I will suggest, however, that Gödel aims at what other philosophers (in the tradition of Kant) would call a theory of reason rather than a theory of intuition. Gödel is, however, evidently influenced by a pre-Kantian tradition that does not see these two enterprises as sharply distinct and that admits "intuitive knowledge" in cases that for us are purely conceptual.[3]

In connection with these explanations I shall try to say something about the development of Gödel's views. Late in his career, Gödel indicated that some form of realism was a conviction he held already in his student days, even before he began to work in mathematical logic. Remarks from the 1930's,

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[2]From a draft reply to a 1971 letter from Kenneth Blackwell, quoted in Wang [25, page 112]. The quotation is from Russell [18, page 169]. Gödel was fond of this particular quotation from Russell. In commenting on it in *1944*, however, he stated erroneously (p. 127 n.) that it had been left out in later editions of *Introduction to Mathematical Philosophy*. See Blackwell [3]. Evidently Russell himself did not pay close attention to Gödel's footnote. The specific issue about "not" is not pursued elsewhere in Gödel's writings, and I shall not pursue it here. Gödel also remarks that Russell's statement gave the impression that he had had many discussions with Russell, while he himself recalled only one.

[3]It is possible that Gödel was influenced by the remarks about intuitive knowledge in Leibniz's "Meditations on knowledge, truth, and ideas" [11]. Knowledge is intuitive if it is *clear*, i.e., it gives the means for recognizing the object it concerns, *distinct*, i.e., one is in a position to enumerate the marks or features that distinguish an instance of one's concept, *adequate*, i.e., one's concept is completely analyzed down to primitives, and finally one has an immediate grasp of all these elements.

however, indicate that at that time his realism fell short of what he expressed later. But it appears in full-blown form in his first philosophical publication, "Russell's mathematical logic" *1944*. The strong conception of mathematical intuition, however, seems in Gödel's published writings to come out of the blue in the 1964 supplement to "What is Cantor's continuum problem?" Even in unpublished writings so far available it is at most hinted at in writings before the mid-1950's. In what follows I will trace this development in more detail.

§**1.** Speaking quite generally, philosophers often talk as if we all know what it is to be a realist, or a realist about a particular domain of discourse: realism holds that the objects the discourse talks about exist, and are as they are, independently of our thought about them and knowledge of them, and similarly truths in the domain hold independently of our knowledge. One meaning of the term "platonism" which is applied to Gödel (even by himself) is simply realism about abstract objects and particularly the objects of mathematics.[4]

The inadequacy of this formulaic characterization of realism is widely attested, and the question what realism is is itself a subject of philosophical examination and debate. One does find Gödel using the standard formulae. For example in his Gibbs lecture of 1951, he characterizes as "Platonism or 'Realism' " the view that "mathematical objects and facts (or at least something in them) exist independently of our mental acts and decisions" (**1951*, p. 311) and that "the objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world" (p. 312 n. 17). In "Russell's mathematical logic"—as I have said the first avowal of his view in its mature form—he does not use this language to characterize Russell's (earlier) "pronouncedly realistic attitude" of which he approves, but he does in his well-known criticism of the vicious circle principle, where he says that the first form of the principle "applies only if the entities involved are constructed by ourselves. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be described ... only by reference to this totality" (136).[5]

Gödel is concerned in the Russell essay to argue for the inadequacy of Russell's attempts to show that classes and concepts can be replaced by "constructions of our own" (152), and the Gibbs lecture contains arguments against the view that mathematical objects are "our own creation", a view

[4]For a general discussion of mathematical platonism, see Maddy [12].

[5]Cf. also: "For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually ... " (*1964* p. 262).

maybe more characteristic of nineteenth-century thought about mathematics than of that of Gödel's own time.

Rather than exploring how Gödel himself understands these characterizations, I will note some points that are more distinctive of Gödel's own realism. Introducing the theme in "Russell's mathematical logic," he quotes the statement from Russell [18] quoted above and then turns to an "analogy between mathematics and natural science" he discerns in Russell:

> He compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that ... this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future (127).

In other places, as is well known, Gödel claims an analogy between the assumption of mathematical objects and that of physical bodies:

> It seems to me that the assumption of such objects [classes and concepts] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions (ibid., 137).

In *1964* the question of the "objective existence of the objects of mathematical intuition" is said (parenthetically) to be "an exact replica of the question of the objective existence of the outer world" (272).

Thus a Gödelian answer to the question what the "independence" consists in is, for example, that mathematical objects are independent of our "constructions" in much the same sense in which the physical world is independent of our sense-experience. Gödel does not address in a general way what the latter sense is, although some evidence of his views can be gleaned from his writings on relativity. The main thesis of his paper *1949a* is that relativity theory supports the Kantian view that time and change are not to be attributed to things as they are in themselves. But this thesis is specific to time and change; it is perhaps for that reason that he is prepared in one place to gloss the view by saying that they are *illusions*, a formulation that Kant expressly repudiates.[6] Gödel is not led by the considerations he advances to reject a realist view of the physical world in general; for example he does not suggest that space-time is in any way ideal or illusory. In fact, he frequently

[6] *1949a*, pp. 557–8; Kant, *Critique of Pure Reason*, B69.

reproaches Kant for being too subjectivist.[7] But he is quite cautious in what little he says about how far we can be realists about knowledge of the physical world. But in his discussion of Kant, he clearly thinks that modern physics allows a more realistic attitude than Kant held; for example he remarks that "it should be assumed that it is possible for scientific knowledge, at least partially and step by step, to go beyond the appearances and approach the things in themselves."[8]

§**2.** I now want to approach the question of the meaning of Gödel's realism by inquiring into its development. One distinctive feature of Gödel's realism is that it extends to what he calls concepts (properties and relations), objects signified in some way by predicates. These would not necessarily be reducible to sets, if for no other reason because among the properties and relations of sets that set theory is concerned with are some that do not have sets as extensions.[9] It may be that this feature arose from convictions with which Gödel started. In an (unsent) response to a questionnaire put to him by Burke D. Grandjean in 1975, Gödel affirmed that "mathematical realism" had been his position since 1925.[10] In a draft letter responding to the same questions, Gödel wrote, "I was a conceptual and mathematical realist since about 1925."[11] The term "mathematical realism" occurs in Grandjean's question; the term "conceptual" is introduced by Gödel.

Gödel's response to Grandjean would suggest that he was prepared to affirm in 1975 that the realism associated with him was a position he had held since his student days. Moreover, in letters to Hao Wang quoted extensively in Wang [24], Gödel emphasized that realistic convictions, or opposition to what he considered anti-realistic prejudices, played an important role in his early logical achievements, in particular both the completeness and the incompleteness theorems.[12]

[7] E.g., *1964*, p. 272. However, he interprets Kant's conception of time as a form of intuition as meaning that "temporal properties are certain relations of the things to the perceiving subject" (**1946/9-B2*, p. 231), and he finds that there is at least a strong tendency of Kant to think that, interpreted in that way, temporal properties are perfectly objective.

[8] **1946/9-C1*, p. 257; cf. **1946/9-B2*, p. 240. Of course it is quantum mechanics that has been in our own time the main stumbling block for realism about our knowledge in physics. Gödel says little on the subject; what little he does say (e.g., **1946/9-B2*, notes 24 and 25) indicates a definitely realistic inclination without claiming to offer or discern in the literature an interpretation that would justify this.

[9] Thus "property of set" is counted as a primitive notion of set theory (*1947*, p. 520 n., or *1964*, p. 264 n.). This notion corresponds to Zermelo's notion of "definite property" (cf. Gödel *1940*, p. 2).

[10] Wang [25, pp. 17–18].

[11] Ibid., p. 20.

[12] Köhler [9] contains interesting suggestions about the influences on Gödel as a student that might have encouraged realistic views. They are not specific enough as regards mathematics to bear on an answer to the questions of interpretation considered in the text. In

Before I turn to these statements, let me mention the remarks of Gödel from the 1930's, to which Martin Davis and Solomon Feferman have called attention, that do not square with the platonist views expressed in *1944* and later. We have the text of a very interesting general lecture on the foundations of mathematics that Gödel gave to the Mathematical Association of America in December 1933. Much of it is devoted to the axiomatization of set theory and to the point that the principles by which sets, or axioms about them, are generated naturally lead to further extensions of any system they give rise to. When he turns to the justification of the axioms, he finds difficulties: the non-constructive notion of existence, the application of quantifiers to classes and the resulting admission of impredicative definitions, and the axiom of choice. Summing up he remarks,

> The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent (**1933o*, p. 50).

It is clear that Gödel regards impredicativity as the most serious of the problems he cites and notes (following Ramsey) that impredicative specification of properties of integers is acceptable if we assume that "the totality of all properties [of integers] exists somehow independently of our knowledge and our definitions, and that our definitions merely serve to pick out certain of these previously existing properties" (ibid.). That is clearly a major consideration prompting him to say that acceptance of the axioms "presupposes a kind of Platonism."[13]

The other remarks are glosses on his work on constructible sets and the consistency of the continuum hypothesis. In the first announcement of his consistency results Gödel says,

> The proposition A [i.e., $V = L$] added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way (*1938*, p. 557).

Acceptance of $V = L$ as an axiom of set theory would not be incompatible with the *philosophical* realism Gödel expressed later, although it would be

discussing Gödel's relations with the Vienna Circle, Köhler writes as if he already held at the beginning of the 1930's the position of *1944* and later writings. The evidence does not support that.

[13]The cautious and qualified defense of a kind of platonism in Bernays [2] was delivered as a lecture about six months later. We think of one of the influential tendencies in foundations of the time, logicism after Frege and Russell, as a platonist view. That was not the way its proponents saw it in the 1930's.

with the mathematical views he expressed in connection with the continuum problem. But regarding the concept of an arbitrary infinite set as a "vague notion" certainly does not square with Gödel's view in 1947 that the continuum problem *has* a definite answer.[14]

Another document from about this time indicates that, after proving the consistency of the continuum hypothesis and probably expecting to go on to prove its independence, Gödel did not yet have the view of the significance of this development that he later expressed. In a lecture text on undecidable diophantine sentences, probably prepared between 1938 and 1940, Gödel remarks that the undecidability of the sentences he considers is not absolute, since a proof of their undecidability (in a given formal system) is a proof of their truth. But then he ends the draft with the remarkable statement:

> However, I would not leave it unmentioned that apparently there do exist questions of a very similar structure which very likely are really undecidable in the sense which I explained first. The difference in the structure of these problems is only that variables for real numbers appear in this polynomial. Questions connected with Cantor's continuum hypothesis lead to problems of this type. So far I have not been able to prove their undecidability, but there are considerations which make it highly plausible that they really are undecidable (**193?*, p. 175).

It is hard to see what Gödel could have expected to "prove" concerning a statement of the form he describes other than that it is consistent with and independent of the axioms of set theory, say ZF or ZFC, and that this independence would generalize to extensions of ZFC by axioms for inaccessible cardinals in a way that Gödel asserts that his consistency result does. There seems to be a clear conflict with the position of *1947*; it's hard to believe that at the earlier time he thought that exploration of the concept of set would yield new axioms that would decide them. Moreover the statement is a rather bold statement. I don't think it can be explained away as a manifestation of Gödel's well-known caution in avowing his views.

Let me now turn to the most informative documents about Gödel's early realism, the letters to Wang. There he explains the failure of other logicians to obtain the results obtained by him as due to philosophical prejudices, in particular against the use of non-finitary methods in metamathematics, deriving from views associated with the Hilbert school, according to which non-finitary reasoning in mathematics is justified "only to the extent to which it can be 'interpreted' or 'justified' in terms of a finitary metamathematics"

[14]Martin Davis notes that in *1940* Gödel refers to $V = L$ as an axiom, indicating that he still held the view expressed in the above quotation from *1938*. (See his introductory note to **193?* in CW III, at p. 163.) It would confirm, however, only the first of the two distinguishable aspects of the *1938* view.

(Wang [24, p. 8]). This is applied to the completeness theorem, of which the main mathematical idea was expressed by Skolem in 1922. Gödel also asserts that his "objectivistic conception of mathematics and metamathematics in general" was fundamental also to his other logical work; in particular "the highly transfinite conception of 'objective mathematical truth', as *opposed* to that of 'demonstrability' " is the heuristic principle of his construction of an undecidable number-theoretic proposition (ibid., p. 9).

It should be pointed out that only one of the examples Gödel gives essentially involves impredicativity and thus conflicts sharply with the view of **1933o*: his own work on constructible sets. Where the conflict lies is of course in accepting the conception of the constructible sets as an intuitively meaningful conception, but it's on this that Gödel lays stress rather than on the fact that at the end of the process one can arrive at a finitary relative consistency proof. Gödel is said to have had the idea of using the ramified hierarchy to construct a model quite early; whether by the time of the MAA lecture he had seen that it "has to be used in an *entirely nonconstructive way*" (Wang [24, p. 10]) is not clear. It seems to have been only in 1935 that he had a definite result even on the axiom of choice.[15]

It seems we cannot definitely know whether Gödel in December 1933 already thought the "kind of Platonism" he discerned more acceptable than he was prepared to say. But it seems extremely likely that, with whatever conviction he embraced impredicative concepts in first developing the model of constructible sets in the form we know it, his confidence in this point of view would have been increased by his obtaining definite and important results from it. The remarks from 1938 show that there was already a further step to be taken; one possible reason for his taking it may have been reflection on the consequences of $V = L$ for descriptive set theory, which could have

[15]Wang writes ([25, p. 97]:

> From about 1930 he had continued to think about the continuum problem .... The idea of using the ramified hierarchy occurred to him quite early. He then played with building up enough ordinals. Finally the leap of taking the classical ordinals as given made things easier. It must have been 1935, according to his recollection in 1976, when he realized that the constructible sets satisfy all the axioms of set theory (including the axiom of choice). He conjectured that the continuum hypothesis is also satisfied.

Seen in light of the remarks in **1933o*, the "leap of taking the classical ordinals as given" was a decisive step in the development of Gödel's realism about set theory. Wang's remarks (evidently based on Gödel's much later recollection) suggest, but do not explicitly say, that this leap was taken close enough to 1935 to be probably later than December 1933. On the other hand Feferman conjectures that the rather casual treatment in **1933o* itself of the problem of the axiom of choice may have been due to Gödel's having an approach to proving its consistency. (See his introductory note to **1933o* in CW III.)

It can be documented that Gödel obtained the essentials of the proof of the consistency of CH in June 1937. See Feferman, [note s (CW I 36)].

convinced him that $V = L$ is false. But it should be pointed out that the idea that some mathematical propositions are absolutely undecidable is one that Gödel still entertained in his Gibbs lecture in 1951, and in itself it is not opposed to realism.[16]

There is another more global and intangible consideration that could lead one to doubt that Gödel's views of the 1930's were the same as those he avowed later. This is the evidence of engagement with the problems of proof theory, in the form in which the subject evolved after the incompleteness theorem. Gödel addresses questions concerning this program in the MAA lecture **1933o* and more thoroughly and deeply in the remarkable lecture **1938a* given in early 1938 to a circle organized by Edgar Zilsel. This lecture shows that he had already begun to think about a theory of primitive recursive functionals of finite type as something relative to which the consistency of arithmetic might be proved; it is now well known that he obtained this proof in 1941 after coming to the United States. The lecture at Zilsel's also contains a quite remarkable analysis of Gentzen's 1936 consistency proof, including the no-counter-example interpretation obtained later by Kreisel (see Kreisel [10]). What he says about the philosophical significance of consistency proofs such as Gentzen's is not far from what was being said about the same time by Bernays and Gentzen, in spite of somewhat polemical remarks about the Hilbert school in this text and in others.[17]

§3. I shall not try to trace the development of Gödel's realism further independently of the notion of mathematical intuition. As I said, it is firmly avowed in *1944* and further developed in *1946*, *1947*, and **1951*. It is thus during the period from 1943 or 1944 through 1951 that it becomes Gödel's public position.[18]

[16]Note that in *1946* Gödel explores the idea of absolute provability. In this connection it is reasonable to ask whether Gödel is a realist by one criterion suggested by the work of Dummett, according to which realism admits truths that are "recognition-transcendent", that is obtain whether or not it is even in principle possible for humans to know them. In the sphere of mathematics, an obstacle to this view for Gödel is his confidence in reason; he expresses in places the Hilbertian conviction of the solvability in principle of every mathematical problem. See Wang [24, pp. 324–325] (on which see footnote 49 below), cf. **1961/?*, pp. 378, 380.

However, the discussion in **1951* makes clear that Gödel regards the existence of recognition-transcendent truth as meaningful, since if the mathematical truths that the human mind can know can be generated by a Turing machine, then the proposition that this set is consistent would be a mathematical truth that we could not know. And this is presumably what is decisive for Dummettian realism rather than whether recognition-transcendent truths in fact exist, which Gödel was inclined to believe they did not, at least in mathematics.

[17]I owe this observation to Wilfried Sieg. Cf. our introductory note to **1938a* in CW III, at p. 85.

[18]The conversation that was the basis of Russell's remark quoted on p. 44 above would have taken place near the beginning of this period.

I have discussed elsewhere the position of *1944*.[19] It is not easy to discern a definite line of argument for realism (which would in turn clarify the position itself); the form of a commentary on Russell works against this. A very familiar argument which is already present in **1933o* (as well as in Bernays [2]) is that particular principles of analysis and set theory are justified if one assumes a realistic view of the objects of the theory and not otherwise. Gödel applies this point of view particularly in his well-known analysis of Russell's vicious circle principle, where he argues from the fact that "classical mathematics" does not satisfy the vicious circle principle that this is to be considered "rather as a proof that the vicious circle principle is false than that classical mathematics is false" (135).

When Gödel says that assuming classes and concepts as "real objects" is "quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence" (137, quoted above), his claim is that classical mathematics is committed to such objects and moreover it must be interpreted so that the objects are independent of our constructions. Gödel reinforces this claim by his analysis of the ramified theory of types in the present paper and by discussions elsewhere in his writings such as the criticism of conventionalism in **1951* and **1953/9* (actually briefly adumbrated at the end of *1944*). In a way this is hardly controversial today; an impredicative theory with classical logic is the paradigm of a "platonist" theory. But Gödel's rhetoric has certainly led most readers to think that his reasoning is not just to be reconstructed as an application of a Quinean conception of ontological commitment. Why is this so?

One reason is certainly Gödel's remarks about intuition, of which we are postponing discussion. But that conception plays virtually no role in *1944*. Another reason more internal to that text is that Gödel makes clear that his realism extends to concepts as well as classes (which in this discussion he does not distinguish from sets). Standard set theories either quantify only over sets or, if they have quantifiers for (proper) classes, allow a predicative interpretation of class quantification. Thus at most realism about sets seems to be implied by what is common to Gödel and philosophers who have followed Quine. Gödel makes clear that he sees no objection to an impredicative theory of concepts (139–40), and the paper contains sketchy ideas for such a theory, which apparently Gödel never worked out in a way that satisfied him. But Gödel does not directly argue for a realism about concepts that would license such a theory; in particular he does not argue that classical mathematics requires such realism.

[19] In my introductory note in CW II; on realism see particularly pp. 106–110.

In what sense does *1947* offer a further argument for realism?[20] The major philosophical claim of *1947*, that the independence of the continuum hypothesis should in no way imply that it does not have a determinate truth-value, is rather an inference *from* realism. Gödel makes such an inference in saying that if the axioms of set theory "describe some well-determined reality," then "in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality" (520). But Gödel then proceeds to give arguments for the conclusion that the continuum problem might be decided. The first is the point going back to **1933o* about the open-endedness of the process of extending the axioms. The second is that large cardinal axioms have consequences even in number theory. Here he concedes that such axioms as can be "set up on the basis of principles known today" (i.e., axioms providing for inaccessible and Mahlo cardinals) do not offer much hope of solving the problem.[21] The further statement, that axioms of infinity and other kinds of new axioms are possible, was more conjectural, and of course the stronger axioms of infinity that were investigated later (already taken account of to some degree in the corresponding place in *1964*) were shown not to decide CH. The third consideration is that a new axiom, even if it cannot be seen to have "intrinsic necessity," might be verified inductively by its fruitfulness in consequences, in particular independently verifiable consequences. It might be added that Gödel's plausibility arguments for the falsity of CH constitute an argument for the suggestion that axioms based on new principles exist, since any such axiom would have to be incompatible with $V = L$.

Another point, which hardly attracts notice today because it seems commonplace, is that the concept of set and the axioms of set theory can be defended against paradox by what we would call the iterative conception of set. In 1947, to say that this conception offers a "satisfactory foundation of Cantor's set theory in its whole original extent" (518) was a rather bold statement. Even the point (made in Gödel *1944*, p. 144) that axiomatic set theory describes a transfinite iteration of the set-forming operations of the simple theory of types was not a commonplace. Of course in what sense we do have a "satisfactory foundation" was and is debatable. But I think it would now be a non-controversial claim that, granted certain basic ideas (ordinal and power set) in a classical setting, the iterative conception offers

[20] I pass over *1946*, which might, like *1947*, be described as an application of Gödel's point of view to concrete problems. This is not uncharacteristic of Gödel; also in *1944* he often seems to treat realism as a working hypothesis.

[21] This had been partly shown by Gödel in extending his consistency proof to such axioms; it was subsequently shown that the independence proof also extended, and the consistency and independence of CH were proved even for stronger large cardinal axioms such as Gödel did not have in mind in 1947.

an intuitive conception of a universe of sets,which, in Gödel's words, "has never led to any antinomy whatsoever" (*1947*, p. 518). I think Gödel wishes to claim more, namely that the axioms follow from the concept of set. That thought is hardly developed in *1947* and anyway belongs with the conception of mathematical intuition.[22] Overall, *1947* was probably meant to offer an indirect argument for realism by applying it to a definite problem and showing that the assumption of realism leads to a fruitful approach to the problem. It is worth noting that he offers arguments for the independence of the continuum hypothesis of which the main ones are plausibility arguments for its *falsity*. An "anti-realist" urging upon us the attempt to prove the independence would presumably dwell more on the obstacles to proving it.

The Gibbs lecture "Some basic theorems of the foundations of mathematics and their implications" (**1951*) seems to complete for Gödel the process of avowing his platonistic position. In some ways, it is the most systematic defense of this position that Gödel gave. At the end it seems to see itself as part of an argument as a result of which "the Platonistic position is the only one tenable" (322–3).[23]

The main difficulty of the Gibbs lecture's defense, however, is not the omission he mentions at the end, of a case against Aristotelian realism and psychologism, but that its central arguments are meant to be independent of one's standpoint in the traditional controversies about foundations; the overall plan of the lecture is to draw implications from the incompleteness theorems. Gödel's main arguments aim to strengthen an important part of his position, which he expresses by saying that mathematics has a "real content".[24] But although this is opposed to the conventionalism that he discerns in the views of the Vienna Circle, and also to many forms of formalism, it is a point that constructivists of the various kinds extant in Gödel's and our own time can concede, as Gödel is well aware. But it is probably a root conviction that Gödel had from very early in his career; it very likely underlies the views that Gödel, in the letters to Wang, says contributed to his early logical work. It would then also constitute part of his reaction to attending sessions of the Vienna Circle before 1930.

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[22] In the revised version *1964*, the discussion of the iterative conception of set is somewhat expanded.

[23] This remark appears to be an expression of a hope that Gödel maintained for many years, that philosophical discussion might achieve "mathematical rigor" and conclusiveness. As he was well aware, his actual philosophical writings, even at their best, did not fulfill this hope, and these remarks are part of an admission that certain parts of the defense of mathematical realism had not been undertaken in the lecture.

[24] This conviction will come up in the discussion of intuition in sections §4 and §5; see also my introductory note to *1944* and Parsons [14].

§4. I now turn to the conception of mathematical intuition, beginning with some remarks about its development. I have outlined above the presentation of Gödel's realism in his early philosophical publications *1944* and *1947* and the lecture **1951*. For a reader who knows *1964*, it is a striking fact about these writings that the word "intuition" occurs in them very little, and no real attempt is made to connect his general views with a conception of mathematical intuition.

In *1944* the word "intuition" occurs in only three places, none of which gives any evidence that intuition is at the time a fundamental notion for Gödel himself. The first (128) is in quotation marks and refers to Hilbert's ideas. The second is in one of the most often quoted remarks in the paper, in which Russell is credited with "bringing to light the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory" (131). Here "intuition" means something like a belief arising from a strong natural inclination, even apparent obviousness. In the following sentence these intuitions are described as "common-sense assumptions of logic." It's not at all clear to what extent "intuition" in this sense is a guide to the truth; it is clearly not an infallible one. In the third place (150), Gödel again speaks of "our logical intuitions," evidently referring to the earlier remarks, and it seems clear that he is using the term in the same sense.

One other remark in *1944* deserves comment. In his discussion of the question whether the axioms of *Principia* are analytic in the sense that they are true "owing to the meaning of the concepts" in them, he sees the difficulty that "we don't perceive the concepts of 'concept' and 'class' with sufficient distinctness, as is shown by the paradoxes" (151). Since "perception" of concepts is spoken of in unpublished writings of Gödel, this seems to be an allusion to mathematical intuition in a stronger sense. But the remark itself is negative; it's not clear what Gödel would say that is positive about perception of concepts.

The word "intuition" does not occur at all in *1946* and only once in *1947*. Concerning constructivist views, he remarks

> This negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition (518).

Since Gödel does not elaborate on his use of "intuition" at all, one can't on the basis of this text be at all sure what he has in mind. But it appears that intuition as here understood, instead of being a basis for possible knowledge of the strongest mathematical axioms, is restricted in its application, so that the demand that mathematical objects be "completely penetrable by our intuition" is a constraint that would strongly limit what objects can be admitted.[25]

The Gibbs lecture is again virtually silent about intuition. I have not found in it a single occurrence of the word "intuition" on its own.[26] But talk of perception where the object is abstract occurs again, this time more positively, but still without elaboration or explanation. Gödel defends the view that mathematical propositions are true by virtue of the meaning of the terms occurring in them.[27] But the terms denote concepts of which he says:

> The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe (320).

At the end, he says of the "Platonistic view":

> Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind (323).

There is nothing in these early writings to rule out the interpretation that the talk of "perception" of concepts is meant metaphorically. The last quoted statement could come down to the claim that the "non-sensual reality" that mathematics describes is *known* or *understood* very incompletely by the human mind. Thus although there are what might be indications as early as *1944* of a strong conception of mathematical intuition, in public documents before 1964 they are less than clear and decisive, and Gödel does not begin to offer a defense of it. Nonetheless the allusions to perception of concepts in *1944* and **1951* are very suggestive in the light of his later writings, and it is reasonable to conjecture that although he was not yet ready to defend his conception of intuition he already had some such conception in mind.

But of course there is one published writing before 1964 in which a concept of intuition figures more centrally, and that is the philosophical introduction to the *Dialectica* paper *1958*. The German word used is the Kantian term *Anschauung*. I shall not discuss this paper in any detail but only state rather dogmatically that what is at issue are conceptions of intuition and intuitive evidence derived from the Hilbert school. Gödel is concerned with the

[25]The meaning of "intuition" here could agree with that of *Anschauung* in *1958*; see below. The phrase is replaced in *1964* by "completely given in mathematical intuition" (262); it is hard to be sure whether Gödel saw this as more than a stylistic change.

[26]There are references to intuitionism (e.g., in n. 15, p. 310), and he does speak (p. 319) of the "intuitive meanings" of disjunction and negation.

[27]This is, of course, a sense in which mathematics could be said to be analytic; for further discussion see Parsons [14].

question of the limits of intuitive evidence, where these limits will clearly be rather narrow. It is contrasted with evidence essentially involving "abstract concepts." Thus the conception of intuition involved is not the strong one, a mark of which is that it yields knowledge of propositions involving abstract concepts in an essential way. There is no doubt that that was Gödel's view of the central concepts of set theory and the axioms involving them. The fact that in *1972 Anschauung* is translated as "concrete intuition" indicates both that in *1958* he was employing a more limited conception of intuition than that of *1964* and that it may be a special case of the latter.

There is, however, a source earlier than *1964* for Gödel's thought about mathematical intuition, the drafts of the paper "Is mathematics syntax of language?" (**1953/9*), which Gödel worked on in response to an invitation from Paul Arthur Schilpp to contribute to *The Philosophy of Rudolf Carnap* but never submitted. Six versions survive in Gödel's *Nachlaß*.

The main purpose of the paper is to argue against the conception of mathematics as syntax that is found in logical positivist writings, especially Carnap's *Logical Syntax of Language*.[28] Gödel had already given a version of his argument in **1951*,[29] in a way that does not use the notion of mathematical intuition, and even sketched the ideas in the discussion of analyticity at the end of *1944*. The basic argument, related to arguments directed at Carnap by Quine, is that in order to establish that interesting mathematical statements are true by virtue of syntactical rules or conventions it is necessary to use the mathematics itself in its straightforward meaning.[30] In arguing, contrary to the view he is criticizing, that mathematics has a "real content," Gödel is, as I have said, affirming one aspect of his realism. It is, however, only one: The same argument would be open to an intuitionist, and Gödel himself argues that certain fallback positions of his opponent still leave him obliged to concede "real content" at least to finitist mathematics.

The presentation of his argument against Carnap in **1953/9* does not similarly eschew reference to mathematical intuition, although in the briefer, stripped down presentation of the argument in version V, it does not figure prominently. Before we go into it we should rehearse some elementary distinctions about intuition. In the philosophical tradition, intuition is spoken of both in relation to objects and in relation to propositions, one might say as a propositional attitude. I have used the terms intuition *of* and intuition *that* to mark this distinction. The philosophy of Kant, and the Kantian paradigm generally, gives the basic place to intuition of, but

[28] Carnap [4] and [5].

[29] A large part of it (pp. 315–319), however, is in a section marked "wegzulassen"; it is possible that this was not included in the lecture as delivered. Cf. editorial note c, p. 315.

[30] For discussion see Parsons [14]. However, I barely touch there on the question whether the position Gödel criticizes is what Carnap actually holds. This is questioned by Warren Goldfarb in his introductory note to **1953/9* in CW III. Cf. Goldfarb and Ricketts [8].

certainly allows for intuitive knowledge or evidence that would be a species of intuition that. But talk of intuition in relation to propositions has a further ambiguity, since in propositional attitude uses "intuition" is not always used for a mode of *knowledge*. When a philosopher talks of his or others' intuitions, that usually means what the person concerned takes to be true at the outset of an inquiry, or as a matter of common sense; intuitions in this sense are not knowledge, since they need not be true and can be very fallible guides to the truth. To take another example, the intuitions of a native speaker about when a sentence is grammatical are again not necessarily correct, although in this case they are, in contemporary grammatical theory, taken as very important guides to truth. In contrast, what Descartes called *intuitio* was not genuine unless it was knowledge. Use of "intuition" with this connotation is likely to cause misunderstanding in the circumstances of today; it may even lead a reader to think one has in mind something like intuitions in the senses just mentioned with the extra property of being infallible. It is probably best to use the term "intuitive knowledge" when one wants to make clear one is speaking of knowledge.[31]

A difficulty in reading Gödel's writing on mathematical intuition is that he uses the term in both object-relational and propositional attitude senses, and in the latter it is not always clear what epistemic force the term is intended to have. Since, where a strong conception is involved, it is mainly concepts that are the objects of intuition, and Gödel does (as we have already seen) speak of perception of concepts, it might be well in discussing Gödel to use the word "perception" where intuition of is in question, and reserve the term "intuition" for intuition that. I will follow that policy in what follows.

In **1953/9* Gödel seems to take the propositional sense as primary. I think it is clear that he has first of all in mind what might be called rational evidence, or, more specifically, autonomous mathematical evidence. Thus in stating the view he is criticizing he writes, "Mathematical intuition, for all scientifically relevant purposes ... can be replaced by conventions about the use of symbols and their application" (version V, 356). Apart from the conventionalism his argument is directed against, the only alternative to admitting mathematical intuition that Gödel considers is some form of empiricism. Thus the deliverances of mathematical intuition are just those mathematical propositions and inferences that we take to be evident on reflection and do not derive from others, or justify on a posteriori grounds, or explain away by a conventionalist strategy.[32]

[31] In the philosophy of mathematics, however, this has the disadvantage that "intuitive knowledge" has a more special sense, for example in Gödel *1958* and *1972*.

[32] One might ask, particularly in the light of later writing in the philosophy of mathematics, about the option of not taking the language of mathematics at face value. The only such option considered in Gödel's writings is if-thenism. Apart from other difficulties, in his view the translations have enough mathematical content to raise again the same questions.

It is clear Gödel has primarily in mind mathematical axioms and rules of inference that would be taken as primitive. He does not, however, distinguish mathematics from logic. An example given in a couple of places is modus ponens.[33] In application to logic, what we have presented up to now of Gödel's position does not differ from a quite widely accepted one, in declining to reduce the evidence of logic either to convention or to other forms of evidence. Such a view is even implied by Quine when he regards the obviousness of certain logical principles as a constraint on acceptable translation, although of course Quine would not agree that this implies an important distinction between logical and empirical principles.

With regard to the epistemic force of Gödel's notion of mathematical intuition, the remarks in the supplement to *1964* have given rise to some confusion. I think this can be largely cleared up by taking account of **1953/9*. I think it is clear that for Gödel mathematical intuition is not *ipso facto* knowledge. In a way the existence of mathematical intuition should be non-controversial:

> The existence, as a psychological fact, of an intuition covering the axioms of classical mathematics can hardly be doubted, not even by adherents of the Brouwerian school, except that the latter will explain this psychological fact by the circumstance that we are all subject to the same kind of errors if we are not sufficiently careful in our thinking (version III, 338 n. 12).[34]

In this context, "intuition" has something like the contemporary philosopher's sense, with perhaps more stability and intersubjectivity: Most of us who have studied mathematics find the axioms of classical mathematics intuitively convincing or at least highly plausible. According to Gödel, Brouwer (or for that matter a conventionalist) should grant this much.[35] Elsewhere, where it is clear that he regards mathematical intuition as a *source* of knowledge, it is still clear that possession of intuition isn't already possession of knowledge, for example when he talks of mathematical intuition producing conviction:

> However, mathematical intuition in addition produces the conviction that, if these sentences express observable facts and were obtained by applying mathematics to verified physical laws (or if they express ascertainable mathematical facts), then these facts will be brought out by observation (or computation) (version III, 340).

[33] Version III, note 34 (p. 347); version V, p. 359.

[34] A parallel passage in version IV is clearer but more controversial in that it introduces the idea of intuition of concepts.

[35] I think Brouwer did grant a good part of what Gödel has in mind here. But to sort this out would be a long story and belong to the discussion of Brouwer rather than Gödel.

> If the possibility of a disproval of mathematical axioms is frequently disregarded, this is due solely to the convincing power of mathematical intuition (version V, 361).

What he calls the "belief in the correctness of mathematical intuition" (version III, 341) is not a trivial consequence of acknowledging its existence.

Gödel does (as we shall see) regard mathematical intuition as significantly like perception, but that someone has the intuition that *p* does not imply *p* in the way that if he sees that *p* that implies *p*. (If someone claims to see that *p* and *p* turns out to be false, then he only seemed to see that *p*.) It is rather more like making a perceptual judgment, which may have a strong presumption of truth but which can in principle be false.

A conclusion I draw from this is that what is at issue between Gödel and his opponents about mathematical intuition is not any basic assertion of its existence, but some questions about its character and especially its ineliminability as an epistemic factor. Gödel attributes to Carnap the view that appeals to mathematical intuition need play no more than an heuristic role in the justification of mathematical claims. Something like this seems also to be true of Quine (although Gödel never comments on Quine's philosophical views).

Even if one grants to mathematical intuition in the sense explained so far a high degree of credence, the question will still arise why it should be called *intuition*. Other philosophers have held that there are non-empirical, non-conventional truths without calling the evidence that pertains to them intuition or using for them a term that could easily be understood as meaning something close to that. A very good example is Frege, who quite on the contrary insists that arithmetical knowledge, because it is a part of logic, does not depend on intuition. For him the term (that is, the German term *Anschauung*) has a roughly Kantian meaning. Gödel himself often speaks of reason in talking of the evident character of mathematical axioms and inferences. This is in agreement with the usage of Frege and others in the rationalist tradition. Yet in speaking of the source of knowledge in these cases as mathematical intuition, without the spatio-temporal connotation of the Kantian tradition, Gödel is not just differing with Frege about terminology.[36]

To analyze the differences between Gödel and Frege would require more exploration of Frege than I can undertake here. But we can see a major difference in the analogy Gödel stresses in places in **1953/9* between what he calls mathematical intuition and sense-perception. His claims about this analogy are strong but not very much developed. As I remarked earlier, he

[36]Although Kantian intuition plays a role in Gödel's writings, it is not altogether clear whether he accepted some version of the notion or simply explored it as part of a philosophy (that of the Hilbert school) he wished to explore because of its connection with proof theory and constructivity.

does not distinguish mathematics from logic. Thus even elementary logic seems to be an application of mathematical intuition:

> The similarity between mathematical intuition and a physical sense is very striking. It is arbitrary to consider "this is red" an immediate datum, but not so to consider the proposition expressing modus ponens or complete induction (or perhaps some simpler propositions from which the latter follows). For the difference, as far as it is relevant here, consists solely in the fact that in the first case a relationship between a concept and a particular object is perceived, while in the second case it is a relationship between concepts (version V, 359).

In this passage and in many others, we find a formulation that is very characteristic of Gödel: In certain cases of rational evidence (of which we can easily grant modus ponens to be one), it is claimed that "perception" of concepts is involved. Indeed, in this passage such perception is even said to be involved in a situation where one recognizes by sense-perception the truth of 'this is red' (with some demonstrative reference or other for 'this'). An inference seems to be made from 'a perceives that ... F ...', where 'F' is a predicate or general term occurring in '... F ...', to 'a perceives (the concept) F'. Gödel does not formulate "the proposition expressing modus ponens," but presumably it would involve the concepts of proposition, being of the form 'if p then q', and implication, so that the claim is that in this case a relation of these concepts is involved. (I am assuming that perceiving a relation between concepts involves perceiving the concepts; I think that can be justified from the texts.[37])

If we ask what the analogy with perception is beyond that of providing an irreducible form of evidence, an appropriate answer is likely to be of the form that certain *objects* are before the mind in a way analogous to that in which physical objects are present in perception. Gödel's answer is "concepts", perhaps concepts of a particular kind. But that in the case of either 'this is red' or elementary logical truths and inferences concepts are present in this way seems to be an assumption, at best part of an explanation of how these things might be evident that is not carried further.[38] This is certainly a point on which Gödel can be criticized.

Gödel actually goes further and sees a close analogy between reason and an "additional sense." After discussing the idea of an additional sense that would show us a "second reality" separated from space-time reality but still describable by laws, Gödel says:

[37] Cf. the formulation of the same point in version IV, ms. p. 19.

[38] Gödel does in one place (**1961/?*, pp. 382, 384) make brief remarks about language learning and suggests that when a child first understands a logical inference this is a step that brings him to a higher state of consciousness. Cf. Parsons [14, section III].

> I even think this comes pretty close to the true state of affairs, except that this additional sense (i.e., reason) is not counted as a sense, because its objects are quite different from those of all other senses. For while through sense perception we know particular objects and their properties and relations, with mathematical reason we perceive the most general (namely the 'formal') concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of particularities without any reference to the formal concepts (version III, 354).[39]

In the corresponding passage in version IV, instead of the last sentence above Gödel has:

> For while with the latter [the senses] we perceive particular things, with reason we perceive concepts (above all the primitive concepts) and their relations (ms. p. 17b).

The difference suggests an important uncertainty or change of mind on Gödel's part, as to the exact sphere of reason (which would include mathematical intuition). In IV any perception of concepts seems to be an application of reason, but in III it seems that the concepts involved in 'this is red' belong rather to sense. The passage from V suggests the position of IV but may have been intended to be noncommittal.

It is disappointing that Gödel's logical example is a general principle that involves quantification over propositions or sentences and characteristically logical concepts like implication. He does not, here or so far as I know elsewhere, answer directly the question whether a particular logical truth such as 'it is raining or it is not raining' or a particular inference (say, by modus ponens) is an application of mathematical intuition. This could depend on the question just mentioned, whether any perception of concepts is an application of reason.

Such elementary logical examples would differ from the example that Gödel was most interested in, the axioms of set theory, in that the claim that they are rather directly and immediately evident has a great deal of plausibility. One couldn't argue for them from more theoretical considerations without using inferences or assumptions of much the same kind.

Gödel had another argument for the analogy between reason and perception, based on what in the Gibbs lecture he called the inexhaustibility of mathematics, which he argued for in two ways: from the incompleteness theorem, which implies that a sound formal system for a part of mathematics

[39]What does Gödel mean by this last assertion? It seems to say, as Warren Goldfarb remarks in his introductory note to the paper, that "the empirical world is fixed independently of mathematics" (CW III 333). Gödel does not suggest such a view in his discussions of physics (**1946/9* and *1949a*), and it is difficult to reconcile with the talk in *1964* of the "abstract elements contained in our empirical ideas" (272).

can always be properly extended, and from the iterative conception of set, where, on his understanding, the conception would always give rise to more sets than a given precise delineation of principles would provide and thus to new evident axioms. Thus there are an unlimited number of independent "perceptions":

> The "inexhaustibility" of mathematics makes the similarity between reason and the senses . . . still closer, because it shows that there exists a practically unlimited number of perceptions also of this "sense" (**1953/9*, version III, 353 n. 43).[40]

The concept of set was doubtless for Gödel the most favorable example of "perception" of concepts. (It is also a case where Gödel argued something I have not stressed here, although it is discussed in Parsons [14], that the propositions known in this way are in a way analytic.) Thus in *1964*, he emphasizes the fact that intuition gives rise to an "open series of extensions" of the axioms (p. 272), and of course the incompleteness theorem implies that any such series generated by a recursive rule would be incomplete and would, indeed, suggest a further reflection that would lead to a still stronger extension. Gödel interpreted these considerations by saying that the "mind, in its use, is not static, but constantly developing" (*1972a*, p. 306). This remark is directed against a mechanist view of mind such as Gödel attributed to Turing. He explicitly offers the generation of new axioms of infinity in set theory as an example.[41] It is interesting that the inexhaustibility of mathematics is used by Gödel both in drawing his analogy between perception and insight into mathematical axioms and in his critical discussion of mechanism.[42] The complex, iterated reflection involved in the uncovering of stronger mathematical axioms and the concepts entering into them strikes me intuitively as very different from perception, and I don't think Gödel has offered more than a rather undeveloped formal analogy. But it is a real problem for what I at the outset called a theory of reason to give a better account.

[40] Gödel follows this remark by remarks about axioms of infinity in set theory. A parallel remark in IV (p. 19) is followed by an appeal to the second incompleteness theorem.

[41] One should, however, compare the version of what in CW II is Remark 3 of *1972a* with the version of the same remark published in Wang [24, p. 325].

[42] On this subject see **1951* (and Boolos's introductory note in CW III), Wang [24, pp. 324–326], and Wang [26]. It would be beyond the scope of this paper to pursue this subject further. But it should be pointed out that what is needed for Gödel's case against mechanism is the inexhaustibility of our potential for acquiring mathematical *knowledge*. He himself makes clear in **1951* that that does not follow simply from the mathematical considerations such as the incompleteness theorems. On the other hand it is not intrinsically connected with platonism as opposed to, say, intuitionism.

§5. Let us now turn to the remarks about mathematical intuition in *1964*. Gödel presents a sketch of his epistemology of mathematics in four paragraphs (pp. 271–272). Some things that are obscure in the first and third paragraphs should be clearer in the light of our discussion so far.

Gödel begins with a remark that is among the most quoted in all his philosophical writing:

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves on us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future (271).

A first problem concerning this passage is how Gödel gets from the axioms "forcing themselves on us as being true", which we might accept as a form of intuition that, to the conclusion that we have something like a perception of the *objects* of set theory, an instance of intuition of. We can see that by "the objects of set theory" Gödel means not just sets but the primitive *concepts* of set theory, "set" itself, membership, what he calls "property of set" (264 n. 18). And it is clear from the above discussion that he understands rational evidence in general as involving perception of the concepts that are the constituents of the proposition in question. This, I think, is the unstated premiss of an inference that at first sight appears to be a *non sequitur*. Although Gödel never so far as I know denies that there is "something like a perception" of *sets*, it isn't on that idea that his conception of our knowledge of axioms of set theory rests.[43] The "new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis", in particular,

[43] Nonetheless there is still a problem, as was pointed out to me by Earl Conee. Perception of "the objects of set theory" does on the face of it include perception of sets, and it is not clear how perception of the concepts explicitly occurring in the axioms of set theory should lead to such perceptions.

Warren Goldfarb has pointed out that in some places in **1953/9* Gödel seems to take "concepts" also to include mathematical objects. (See pp. 332–3 of his introductory note and the texts cited there; of these the most persuasive to me is version III, note 45.) But even if that were Gödel's general usage, it would not solve this particular problem.

It can be said that some sets can be identified individually by concepts, and one, ω, is all but explicitly mentioned in the axiom of infinity. Since Gödel would probably have regarded deduction as leading to further or clearer perceptions of concepts, he could very well have thought that individually identifiable sets are "perceived" by way of perception of the concepts that identify them uniquely. This view would have the consequence that natural numbers are also "perceived". So far as I know Gödel nowhere affirms or denies this.

would be simply insights into the truth of new axioms that would decide the continuum hypothesis.

In the third paragraph Gödel presents ideas that will be familiar to the reader of **1953/9*. He is considering a fallback position where intuition concerning the concepts of set theory is not the guide to knowledge that he himself takes it to be. It is still "sufficiently clear to produce the axioms of set theory and an open series of extensions of them," and this "suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis" (272). That reflection on the concepts of set theory gives rise to intuitions of this kind can hardly be doubted if one studies the work of set theorists, although how clear the intuitions are can be questioned, already concerning the axioms of replacement and power set. The meaning that is thus given to CH is that the progress of set theory could give rise to axioms that are supported by the intuitions of set theorists and decide CH. But of course the particular line of inquiry on which Gödel rested his original hopes, large cardinals, proved fruitful in other respects but has not resolved this particular problem. Whether something like what Gödel hoped for is at all likely to happen through the discovery of axioms of another kind deciding CH is so far as I know open, and I would defer to the judgment of experts in any case. But clearly the question can't be suppressed: Couldn't our intuitions concerning sets be conflicting, so that different axioms were discovered that have their own kind of intuitive support but have opposite implications regarding CH?

The spectre of the concept of arbitrary infinite set being a "vague notion" that needs to be "determined in a definite way" by new axioms isn't easily banished, and then one can't rule out the possibility that it might be determined in incompatible ways.

The opening sentence of the paragraph suggests that mathematical intuition might be developed altogether without any commitment as to the extent to which it is a guide to truth:

> However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here (272).

Commenting on this remark, W. W. Tait compares mathematical intuition to the perceptions of a brain in a vat [23, p. 365]. That would, I think, not square with the view that intuition plays a role in elementary mathematics and logic without which one could not even answer the question whether "intuition" understood more noncommittally is able to decide CH. That Gödel is here making a concession for the sake of argument, and thinking primarily of the intuitions leading to strong axioms of set theory, is suggested by the fact that later in the same paragraph he talks of what "justifies the acceptance of this criterion of truth in set theory."[44] He also, in effect, repeats the point about there being a potentially unlimited number of independent intuitions needed to decide questions not only in set theory but also in number theory.

I now move back to the second paragraph, possibly the most difficult and obscure passage in Gödel's finished philosophical writing. Only a small part of it is much illuminated by the earlier writings that I have studied. The passage presents new ideas, possibly derived from the study of Husserl that Gödel began in 1959. But it is with Kant and perhaps Leibniz that he seems to make a more direct connection.

Since Gödel is making a comparison between mathematical concepts and those referring to physical objects, it may be helpful to recall the most basic elements of Kant's conception of the latter. Knowledge of objects has constituents of two kinds, intuitions and concepts. The former are contributed by the faculty of sensibility (at least at first approximation). But they can't be identified with sensations or sense-data: intuitions are of

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[44]Tait is concerned to argue that mathematical intuition is not "what confers objective validity on our theorems" according to Gödel. I am not sure that Gödel has said enough to make at all clear how he would understand the latter problem. Tait may be denying that according to Gödel mathematical intuition is necessary to us as a ground for asserting mathematical propositions; if so I disagree, and I think the argument of **1951* and **1953/9* would make little sense if the denial is right. The philosophical defense of the objective validity of mathematics is another matter. I agree with Tait that the previous paragraph contains something of what Gödel has to say about that.

Tait seems to reject the interpretation of Gödel as an "archetypical Platonist." In part he is rejecting, certainly rightly, the imputation to Gödel of the postulation of a faculty by which we "interact" with mathematical objects. It still seems to me that there is an important sense in which Gödel is a Platonist and Tait is not. Tait's view that "questions about the legitimacy of principles of construction or proof are not ... questions of fact" and the reasons he gives for this [23, p. 361] are alien to Gödel (see below). Tait himself sees some disagreement (ibid., p. 365, end of footnote 3).

Related remarks concerning Tait's discussion of Gödel are made in Yourgrau [27, pp. 394–5].

Note added in proof. In correspondence, Tait states that the principal concern of note 3 of [23], just discussed, was to criticize the interpretation, expressed by Paul Benacerraf, of Gödel as postulating a faculty with which we "interact" with mathematical objects. I have expressed agreement with Tait's rejection of this interpretation.

Tait raises the question what Gödel means by "the sense which I explained first" in the passage from **193?* quoted on p. 50 above. This is not completely clear, but it is very probable that Gödel refers to the discussion at the beginning of the lecture of Hilbert's conviction of the solvability of every well posed mathematical problem and intends that a proposition undecidable in this sense would contradict Hilbert's conviction.

Some additional confirmation of my reading of this passage and that from *1938* quoted on p. 49 is offered by the remark, in a 1939 Göttingen lecture, that "it is very plausible that with A [i.e., $V = L$] one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidean and non-Euclidean geometry" (**1939b*, p. 155).

objects in space and time, and space and time are a priori contributions of the human mind. Intuition gives knowledge its particular reference, but knowledge is in the end propositional, and something must be predicated of objects. Concepts also have both empirical and a priori dimensions. Any objective knowledge at least subjects its objects to the categories, a priori contributions of the understanding (the faculty of thought). The categories are “concepts of objects in general”; referring our knowledge to objects means applying this abstract and a priori system of concepts.

To return to Gödel, after saying that intuition doesn’t have to be “conceived of as ... giving an *immediate* knowledge of the objects concerned” he says that “as in the case of physical experience we *form* our ideas of these objects on the basis of something else which *is* immediately given.” It is clear that Gödel intends to say that in the case of physical experience something other than sensations is “immediately given.” Here I think he doesn’t mean what most analytic philosophers of today (and also, it should be noted, Edmund Husserl) would say, that in some sense real *objects* are immediately given (to the extent that it is appropriate at all to talk of the “given”). The picture resembles Kant’s, for whom knowledge of objects has as “components” a priori intuition and concepts. It is, to be sure, un-Kantian to think of pure concepts as given, immediately or otherwise. But Gödel’s picture seems clearly to be that our conceptions of physical objects have to be constructed from elements, call them primitives, that are given, and that some of them (whether or not they are much like Kant’s categories) must be abstract and conceptual.

Gödel says, “Evidently the ‘given’ underlying mathematics is related to the abstract elements contained in our empirical ideas.” But the only elaboration of this statement is the remark (footnote 40) that the concept of set, like Kant’s categories, has as its function “ ‘synthesis’, i.e., the generating of unities out of manifolds.”

Anyway, the general idea is that at the foundation of our conceptions of the physical world *and* of mathematics are certain “abstract elements” which appear to be primitive concepts. So far Gödel is in very rough agreement with Kant. What he mysteriously calls “another kind of relationship between ourselves and reality” (than the causal, manifested in the action of bodies on our sense organs) either consists of, or would account for, the fact that these elements represent reality objectively. They are not “purely subjective, as Kant asserted.” Gödel does not offer an interpretation of Kant’s transcendental idealism, but it is pretty clear he means to reject it. But in talking of primitive concepts that are not subjective in Kant’s sense, whatever that is, Gödel may be following the inspiration of Leibniz.

We should not forget that the concept of intuition is not the basis for Gödel’s entire story about mathematical knowledge, since he holds that mathematical axioms can have an a posteriori justification through their consequences. He does not do very much to bring this and the more direct evidence of set-theoretic axioms together; it's as if there were two independent kinds of reasons for which one might accept them. A more holistic view seems to do more justice to the facts and seems even to underlie Gödel's actual argument about the continuum problem. Then one would contrast the more ground-level intuitions (logical inference and elementary arithmetic) with the more theoretical ones. There is a process of mutual adjustment of these. In mathematical practice there are also many "middle-level" intuitions, persuasive propositions about how things should turn out that no one would claim to be evident in themselves or would seriously propose as axioms for a fundamental theory such as set theory.

One can see where Gödel's conception is perhaps eccentric, or at least controversial, by comparing it with another account of rational justification, suggested by John Rawls's views concerning moral and political theories. On this account what would be most properly called "intuitions" are what he calls our "considered judgments" at lower levels, in the moral and political case concrete moral judgments, in particular concerning the justice of social arrangements.[45] Then one constructs theories. Theories may have intrinsic plausibility in their own right and may be defended on theoretical grounds against rivals. But an ineliminable part of their justification is that they yield our considered judgments, or, more likely, a corrected version of them. (If a theory tells us that these judgments are wrong, it explains why they are wrong.) But the process of mutual adjustment of theory and concrete judgment is a dialectical one, which might go through a number of back-and-forth steps. Ideally at least, this process ends in Rawlsian "reflective equilibrium." This view is in two ways more nuanced than Gödel's. First, it allows for a distinction between the kind of intrinsic plausibility possessed by ground-level judgments and that of high-level theoretical principles, and the intrinsic plausibility of the latter is not thought of as analogous to perception. Gödel, in talking about set theory, describes both as instances of intuition and closely analogous to perception. Second, with respect to more theoretical principles, it makes clear that integral to their justification is both their intrinsic plausibility and their ability to yield consequences that square with low-level intuition. It may be that that was Gödel's underlying

[45] Rawls compares them to the intuitions of a native speaker concerning his language; he sees an analogy between their role and that of speakers' intuitions in a theory of grammar. See Rawls [15, p. 47]. In later writings Rawls says that the considered judgments that are relevant are at all levels of generality ([16, p. 8]; [17, p. 8, 28]). Then the comparison with speakers' intuitions is less apt. The view sketched in the text (which still seems to me a reasonable interpretation of the position of [15]) does not take account of this aspect of Rawls's later view. In particular, I do not go into the question of the status on this view of the distinction between principles and considered judgments ([15, p. 20]).

view, but it hardly receives emphasis when he talks about these matters.

If you have begun to think that ideas derived from Rawls offer the kind of theory of reason that the foundations of set theory require, a minimal further look at his writings, not to speak of the extensive controversial literature on them, should disabuse you. At least in later writings than *A Theory of Justice*, Rawls makes clear that the procedure of reflective equilibrium should not be expected to yield a unique theory.[46] In his later writings, Rawls seems to hold that different "comprehensive doctrines" about morality might be developed so as to achieve reflective equilibrium.[47] The nearest analogue in the philosophy of mathematics would be general philosophical and methodological views about mathematics such as constructivism or some kind or other of platonism.

§6. In conclusion, let me return to Gödel's platonism. I suggested at the beginning that the connection between it and Gödel's conception of mathematical intuition would prove not to be as intrinsic as might appear at first sight. One reason is clear: finitary mathematics, intuitionistic mathematics, and classical mathematics without the characteristic concept formations of set theory (say, what is predicative relative to the natural numbers) each have definite and coherent concepts, and Gödel does not deny intuition concerning these concepts. Indeed, part of his case for the indispensability of mathematical intuition is that attempts to reconstruct mathematics without it require taking some mathematics, at least finitary arithmetic, at face value and therefore appeal to intuition at that level. The position that finitism is the limit of what mathematical intuition underwrites may be blind concerning the obvious, and it closes the door to certain extremely natural forms of reflection (such as whatever convinces us that first-order arithmetic is consistent), but it is not logically incoherent.

A second kind of independence of the two views is that Gödel's epistemology of set theory involves not just recognizing the fact of intuition concerning the concepts and axioms, but giving credence to it. Of course he maintains that that is not just an arbitrary judgment. But he clearly admits that the concepts of higher set theory are not so clear that the claim of intuitions concerning the axioms to yield knowledge is as obvious and unquestionable as Descartes intended *intuitio* to be. Even though there is a difference here between Gödel and someone who rejects set theory or who

[46]See Rawls [16, p. 9]. In [15, p. 50], he raises the question whether reflective equilibrium is unique and declines to offer a definite answer. It appears that whether a unique equilibrium is attainable depends on the particular context of application of the method.

[47]For an example see Rawls [17, pp 95–6]. Such a possibility is already mentioned in [15, p. 50]. But Rawls also remarks that "the struggle for reflective equilibrium continues indefinitely, in this case as in all others" ([17, p. 97]), which counters the impression he sometimes gives that fully satisfying the demands of reason is a humanly attainable end.

thinks of the axioms either hypothetically or formalistically, the difference can be overestimated.

There is a third point concerning Gödel's platonism that should be made. Even if we grant Gödel everything he could wish for concerning the clarity of our intuitions concerning the objects of set theory, it is far from clear that he has a case for the transcendental realism concerning these objects that he seems to adhere to, as when he says that the concepts in a mathematical proposition "form an objective reality of their own, which we cannot create or change, but only perceive and describe" (**1951*, p. 320) and that "the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false" (*1964*, pp. 263–4). The widespread impression that Gödel is not just affirming CH $\vee \neg$CH, i.e., allowing the application of the law of the excluded middle here, seems to me correct. The view he is expressing is that even if our grasp of the concept of set is not sufficiently clear to decide CH, the concepts themselves form an independent order that, as it were, guides us in developing set theory.[48]

Such a view clearly goes beyond saying that mathematical intuition is intuition concerning *truth*. In Gödel's conception, it is also the unfolding of certain concepts, and tied to a certain kind of development of the concepts. (Intuition concerning inaccessible cardinals requires a prior understanding of lower-level set theory, say ZF.) It is far from clear that it necessarily contains within itself the means of resolving certain disputes. Mathematical intuition itself doesn't tell us that there *must* be a truth of the matter on questions that intuition and other means of arriving at knowledge do not decide. It also does not tell us that given a question such as the continuum problem, it must be possible to develop our intuitions in such a way that we will arrive at principles sufficient for a solution, although Gödel's conviction appears to have been affirmative in both cases.[49]

Gödel would probably argue that unless they reflect an independent reality, we have no explanation of the convergence and the strength of the intuitions we have. It would require a lengthy exploration of foundational issues in set theory to decide whether this reply has any merit. I will only remark that

[48]Gödel's position as expressed here is analogous to what Rawls calls rational intuitionism in moral theory ([17, pp. 95–6]), not surprisingly since Rawls has given Leibniz as an example of a rational intuitionist. (One cannot take for granted that Leibniz was a conceptual realist; see Mates [13, chapter 10]. Although I cannot justify this here, I believe that Leibniz still offered a model for Gödel's position.)

Still, in actual argument Gödel sometimes steps back from this position or treats it as a working hypothesis. Although I think it was a conviction of his, I doubt that it is a piece of his philosophy that he claimed to have defended at all adequately.

[49]On the second point, see Wang [24, pp. 324–325]. (Wang states ([26, p. 119]), that the passage cited (from p. 324, last line, through the end of the paragraph) was written by Gödel.) But Gödel nowhere claims that this belief itself is a deliverance of mathematical intuition.

it is prima facie an empirical question whether our intuitions in set theory do or do not have a high degree of convergence and strength, a question to be answered in part by investigating the actual development of set theory. To my not very expert eye, the claim I am here attributing to Gödel has not been at all decisively refuted, but the state of the subject leaves a lot of serious questions, in particular those surrounding the continuum problem which already occupied Gödel.

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