Gödel's *Dialectica* interpretation and its two-way stretch*

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1 The *Dialectica* paper.

In 1958, Gödel published in the journal *Dialectica* an interpretation of intuitionistic number theory in a quantifier-free theory of functionals of finite type; this subsequently came to be known as Gödel's functional or *Dialectica* interpretation. The article itself was written in German for an issue of that journal in honor of Paul Bernays' 70^{th} birthday. In 1965, Bernays told Gödel of a plan to publish an English translation by Leo F. Boron of his 1958 paper, again in *Dialectica*. However, Gödel was dissatisfied with certain aspects of the original, and set out to revise the translation. A year after doing so to his apparent satisfaction, Gödel changed his mind and decided instead to add a new series of extensive footnotes by way of improvement and amplification. The result was sent to the printer in 1970 after much help and encouragement by Bernays and Dana Scott, but when the proof sheets were returned,

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(Use for reference data only, since the published version was mercilessly mangled by editorial misapplication of their LaTeXprogram despite its having been submitted in accordance with all instructions.)

Gödel was again dissatisfied, especially with two of the added notes. Though he apparently worked on rewriting these until 1972, the paper was never returned in final form for publication. The corrected proof sheets found in his *Nachlass* were reproduced for the first time in Volume II of Gödel's *Collected Works* [1990], where they appear as *1972*. The full story of the vicissitudes of this paper is told by A.S. Troelstra in his introductory note to *1958* and *1972* in that volume.

There is also a long and interesting prior history to the development of Gödel's functional interpretation, much of which has only emerged in recent years through the study of previously unpublished lecture texts going back to 1933 and found in Gödel's *Nachlass*; these texts are to appear in Volume III of his *Collected Works*, which is nearing completion as this is being written. It seemed to me fitting to use the present occasion to trace the development of these ideas to which Gödel devoted repeated attention over such a long stretch of time.

2 Toward the functional interpretation: 1933–1938.

The previously unpublished lecture texts referred to above are three in number; they are:

A. "The present situation in the foundations of mathematics" – an invited lecture delivered in December 1933 to a meeting of the Mathematical Association of America held jointly with the American Mathematical Society in Cambridge, Massachusetts.

B. "Vortrag bei Zilsel" – a lecture in January 1938 for an informal seminar organized by Edgar Zilsel in Vienna.

C. "In what sense is intuitionistic logic constructive?" – a lecture at Yale University in April 1941.

The texts for the Cambridge and Yale lectures were found fully written out in Gödel's *Nachlass* and required very little editorial work to be established for Volume III of the Gödel *Works*. That for the Zilsel seminar was a different matter altogether and required a great deal of arduous effort; the notes in this case were found in the *Gabelsberger* shorthand employed by Gödel. They were initially transcribed by Cheryl Dawson and worked up in collaboration with her to a relatively coherent text by Charles Parsons and Wilfried Sieg, who then translated it into English. Introductory notes to all of these items are also to appear with their forthcoming publication in the Gödel *Works*, Vol. III: the first of these is mine, the second is by Parsons and Sieg, and the third is by Troelstra. I have drawn on all three of these notes in the following, and am indebted to Parsons, Sieg, and Troelstra for what I have learned from their exegetical work. I will give a synopsis of the relevant portions of the Cambridge and Vienna lectures in this section; the Yale lecture will be taken up in the next section.

In the 1933 Cambridge lecture (A), Gödel says that the problem of providing a foundation for mathematics falls into two parts: the first is to represent the methods of proof actually used by mathematicians in a deductive system reduced to a minimum number of axioms and rules of inference, and the second is to give a justification in some sense or other for these axioms. After arguing that the first part has been successfully accomplished via systems of axiomatic set theory (such as that of Zermelo-Fraenkel), Gödel turns to the second part of the foundational project with the surprising statement:

> The result... is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent (A, p. 19).[1]

While Gödel says that it is very likely these axioms *are* consistent, he says a proof of freedom from contradiction must use utterly unobjectionable methods– and thus must exclude such problematic features of set theory as the use of non-constructive reasoning in existence proofs and of impredicative definitions. Thus one must seek a consistency proof by constructive methods; however, here there is a choice, as there are different layers of constructivity. The lowest of these is that of finitism, which is distinguished by three features:

[1]This doesn't seem to square with Gödel's unequivocal assertions in various sources from the 1960's and 1970's that he had held a Platonistic philosophy of mathematics since his student days in Vienna. Cf. [Feferman 1984] pp. 549–552 for information about Gödel's retrospective claims; the discrepancy is discussed further in my forthcoming note to A.

1. The application of the notion of 'all' or 'any' is to be restricted to those infinite totalities for which we can give a finite procedure for generating all their elements [such as the integers]...
2. Negation must not be applied to propositions stating that something holds for all elements, because this would give existence propositions...[these] are to have a meaning in our system only in the sense that we have found an example but, for the sake of brevity, do not state it explicitly...
3. And finally we require that we should introduce only such notions as are decidable for any particular element and only such functions as can be calculated for any particular element. (A, p. 23)

But, contrary to Hilbert's expectation, it appears to Gödel hopeless to demonstrate even the consistency of classical arithmetic [Peano Arithmetic, PA] by finitist methods, in view of his second incompleteness theorem and the fact that all known or prospective such methods can easily be carried out in that system. This leads one to consider the possible use of intuitionistic methods in the wider sense of Brouwer and Heyting. And, although one has a reduction of classical to intuitionistic arithmetic [Heyting Arithmetic, HA] by [Gödel 1933], this does not meet the desired goals, since intuitionistic principles violate the above criteria in two essential respects. Namely, in intuitionism one allows the formation of negation of arbitrary propositions by giving to $\neg p$ the meaning that one has a (constructive) demonstration that any proof of p leads to an absurd conclusion. This violates the criterion 1, since "any" here ranges over the inherently vague totality of arbitrary constructive proofs (which cannot be limited to formal proofs in any one formal system); and it violates criterion 2 by allowing the formation of (and reasoning with) negations of universal propositions; $\neg\forall x\phi(x)$. At the end of his 1933 Cambridge lecture, Gödel concludes that "[the] foundation of classical arithmetic by means of the notion of absurdity is of doubtful value", but he is hopeful that "one may find other and more satisfactory methods of construction beyond the limits of [finitism]" to found at least classical arithmetic and then analysis.

Gödel returned to modified forms of Hilbert's program in his lecture for the Zilsel seminar (B), to meet certain criteria such as 1–3 above, but now with more definite candidates for constructive consistency proofs. The possible routes considered are: 1° The use of higher-type functionals, 2° a "modal-logical" approach, and 3° use of transfinite induction and transfinite recursion. In the first of these, Gödel follows [Hilbert 1926] by giving examples of schemata for primitive recursive functionals of finite type; however, he gives no indication that he is in possession of an interpretation of HA in such a system. Nevertheless, this should be considered a first step toward the *Dialectica* interpretation. Under 2°, Gödel sketched an abstract theory of constructive proofs as a foundation of intuitionistic reasoning. In this he anticipates [Kreisel 1962] (which however met various difficulties, cf. [Goodman 1970]). Finally, under 3°, Gödel considers Gentzen's consistency proof for classical arithmetic ([Gentzen 1936]). Here he explains the essentials of Gentzen's reduction procedures in terms of a functional interpretation using functionals just of type level ≤ 2. The idea is illustrated by means of an example; consider

$$\forall x \exists y \forall z \exists u R(x, y, z, u) \tag{1}$$

where R is decidable. Constructively one would have a counterexample to (1) if for some a, f we have

$$\forall y, u \neg R(a, y, f(y), u). \tag{2}$$

Then an interpretation of (1) is that any proposed counterexample can be blocked by computable functionals $Y(f, a), U(f, a)$, such that

$$\forall f, a R(a, Y(f, a), f(Y(f, a)), U(f, a)). \tag{3}$$

Moreover, Gödel suggests how Gentzen's assignments of ordinals in his proof can be used to define such realizing functionals for provable statements of classical arithmetic by transfinite recursion on ordinals up to ϵ_0. Here Gödel anticipated the "no-counterexample interpretation " (n.c.i.) introduced by Kreisel [1951, 1952] for arithmetic (itself obtained by an analysis of Ackermann's version of Gentzen's consistency proof). It should also be noted that the n.c.i. was later shown (by Kreisel) to be a consequence of Gödel's translation of PA into HA followed by his functional interpretation of HA.

3 The Yale Lecture: Gödel's heuristics for the functional interpretation.

Gödel lectured at Yale University on April 15, 1941 to a joint meeting of the Mathematics and Philosophy Clubs,[2] under the title "In what sense is intuitionistic logic constructive?" (C, in the preceding section). The text of this lecture is more informal and leisurely than the later *Dialectica* publication, and it explains (as the latter does not) the heuristic reasoning that led him to the functional interpretation. Again, he sets down three criteria for constructivity which are close to those of the 1933 lecture in Cambridge but differ in one essential respect:

> 1. All primitive (undefined) functions...must be calculable for any given arguments and all primitive relations must be decidable for any given arguments.
> 2. Existential assertions must have a meaning only as abbreviations for actual constructions...
> 3. Universal propositions can be negated in the sense that a counterexample exists in the sense just described... Therefore, leaving out abbreviations, universal propositions can't be negated at all... (C, pp. 5–6)

The essential difference from the former criteria is that Gödel no longer requires universally quantified variables to range over totalities whose elements are generated by some finite procedure. Gödel calls a system "strictly constructive" or "finitistic" if it satisfies these three conditions, though he is troubled by the latter appellation. In fact, Gödel described later (in 1958) his use of functionals of finite type as a (novel) extension of finitistic methods. In any case, he says that intuitionism does not (on the face of it) meet these three conditions. Nevertheless, he is able to show that "in its application to definite mathematical systems intuitionistic logic can be reduced to finitistic systems" (in the above sense). This is illustrated by his interpretation of Heyting Arithmetic, HA, in a system Σ of functionals of finite type, which has variables

[2] I learned of these auspices from John Dawson. Kreisel [1987] p. 104 erroneously states that this lecture was on the occasion of an honorary doctorate; Gödel was awarded an Honorary D. Litt. by Yale ten years later in June 1951.

ranging over natural numbers, functions of numbers, functions of functions (functionals), etc. Σ provides for two basic means of introducing specific functions and functionals: Explicit Definition, and [Primitive] Recursive Definition, where in $F(0) = G, F(x+1) = H(x, F(x))$, the values of $F(x)$ may lie in any given type. The axioms and rules of Σ allow for ordinary propositional calculus applied to decidable (quantifier-free) formulas, the rule of induction, and rules for equality and substitution. The "meaningful propositions" of Σ have the form $(\exists x_1, ..., x_n)(\forall y_1, ..., y_m)R(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where R is quantifier-free and $x_1, \ldots, x_n, y_1, \ldots, y_m$ may be of arbitrary type. This comes with the understanding that one can assert such a statement in Σ only if specific instances $t_1, ..., t_n$ have been found such that $R(t_1, ..., t_n, y_1, ...y_m)$ is established to hold for arbitrary $y_1, ..., y_m$. Gödel argues that the requirements 2 and 3 above force one to limit the statements in this way, since propositional operations (in particular negation) may not be applied to universal statements, and existential quantification is only regarded as an abbreviation for successful instantiation.

Statements of Σ are indicated in the form $\exists x \forall y R(x, y)$, where x, y are understood to be sequences of variables of arbitrary type. The interpretation of intuitionistic logic and arithmetic in Σ is obtained by associating with each formula $A(z)$ of arithmetic, a formula $\exists x \forall y R(x, y, z)$ of Σ. This is done by induction on A; for simplicity, the parameters 'z' are suppressed in the following. The most complicated association (and, in Gödel's words, "the most important") is with $A \to B$, where $\exists x \forall y R(x, y)$ is the interpretation of A and $\exists u \forall v S(u, v)$ is that of B. Gödel says that constructively

$$\exists x \forall y R(x, y) \to \exists u \forall v S(u, v) \tag{1}$$

can only mean

$$\forall x \exists u [\forall y R(x, y) \to \forall v S(u, v)].^3 \tag{2}$$

This then is converted to the form

$$\exists f \forall x [\forall y R(x, y) \to \forall v S(f(x), v)]. \tag{3}$$

[3] However, the equivalence of (1) and (2) is not accepted in intuitionistic logic, since the choice of u could conceivably depend on a proof of $\forall y R(x, y)$.

The problem then is how to interpret the implication in brackets; the simplest way, Gödel says, is to consider its contrapositive,

$$\neg\forall vS(f(x),v) \rightarrow \ \neg\forall yR(x,y), \tag{4}$$

and here one would show how to convert any counterexample to the hypothesis into one for the conclusion, i.e.

$$\exists g_1\forall v[\neg S(f(x),v) \rightarrow \ \neg R(x,g_1(v))] \tag{5}$$

Since the formulas R, S are decidable, classical propositional calculus then leads us from (3) via (4) and (5) to:

$$\exists f\forall x\exists g_1\forall v[R(x,g_1(v)) \rightarrow S(f(x),v)] \tag{6}$$

and finally to

$$\exists f,g\forall x,v[R(x,g(x,v)) \rightarrow S(f(x),v)] \tag{7}$$

as the interpretation of $A \rightarrow B$. Gödel then goes on to show how to interpret $\neg A, A \vee B, A \wedge B, \exists zA(z)$ and $\forall zA(z)$, given those for A and B. Treating $\neg A$ as $A \rightarrow (0=1)$ yields $\exists g\forall x\neg R(x,g(x))$ as its interpretation; the rest are treated in the obvious way and will be shown explicitly in the next section.

As examples, Gödel shows how the interpretation of $(A \rightarrow A)$ is provable in Σ and how the rule of modus ponens is preserved by the interpretation. Both of these use only the Explicit Definition principle. He says that the proofs for the other axioms and rules of HA are a little longer but quite straightforward. Gödel also remarks that his interpretation can be extended to other intuitionistic systems whose primitive functions (relations) are calculable (decidable). In addition, one obtains constructive consistency proofs of certain classical systems by first translating them into corresponding intuitionistic systems by the method of [Gödel 1933] and then applying the functional interpretation. In particular, the system PA is thus reduced to Σ.

4 The Dialectica paper.

The title of Gödel's *Dialectica* paper, "Über eine noch nicht benützte Erweiterung des finiten Standpunktes",[4] signals his principal foundational concern, developed in the discussion with which the paper begins. Here Hilbert's finitism is characterized as the mathematics of finite combinations of concretely representable and directly visualizable objects such as numbers and symbols. But, finitary mathematics is insufficient to establish the consistency of classical number theory, let alone of classical mathematics more generally. For that, continues Gödel, certain abstract notions are needed; one can retain the constructive component of the finitary standpoint while admitting such notions, as for example in intuitionistic logic. But the notion of computable function(al) of finite type, while also abstract, is more definite than the abstract notion of proof that underlies intuitionistic reasoning. The system T of the *Dialectica* paper [previously referred to as Σ in the 1941 Yale lecture] embodies directly evident principles for the functionals of finite type, and this extension of finitism may be used to prove the consistency of classical arithmetic.

As in the Yale lecture, the basic axioms and rules of T are only indicated; the functional interpretation is spelled out, but no details are given of the proof that HA is interpreted in T. In the supplemental footnotes to his 1972 version of the *Dialectica* paper, Gödel filled in various of those details. But he also endeavored there to strengthen the case for the foundational progress achieved by his interpretation, apparently without arriving at a formulation that he considered sufficiently convincing.

Gödel's functional interpretation was brought to the attention of the logic community in a lecture by Georg Kreisel at the Summer Institute in Symbolic Logic held at Cornell in 1957; of this, more below. From then on, a number of researchers worked out Gödel's interpretation in detail and extended it (as he had expected) to a variety of other systems. To my mind, the best introduction to Gödel's own work and the subsequent literature (up to 1990) is provided by Troelstra's introductory notes to *1958* and *1972* in Volume II of the *Collected Works* (pp.217–241). For further study of the technicalities involved, [Troelstra 1973] sec. III.5 is required reading. [Kreisel 1987] pp.104–

[4] Or, in the English translation in [Gödel 1990], "On a hitherto unutilized extension of the finitary standpoint."

120 provides a discursive assessment of Gödel's interpretation and some of its extensions.

Due to limitations of space, I can only touch here on a few results that illustrate the kind of information that may be drawn from the *Dialectica* interpretation and which illustrate its adaptability to a variety of situations. We begin with setting down the interpretation in full and its first consequences, following the expositions by Troelstra just mentioned. Many of the necessary syntactic preliminaries found in those sources are omitted.

Let L^ω be the language of finite types over that of elementary number theory; in this, each term, and in particular each variable, has a specified type σ. Atomic formulas are supposed to be equations between terms of type 0. Equations between terms of higher type are supposed to be abbreviations for equality at all (variable) arguments (when driven down to type 0). There are constants $K_{\sigma,\tau}$ and $S_{\rho,\sigma,\tau}$ of various types ρ, σ, τ as provided by the finite-typed combinatory calculus, satisfying the equations:

$$(1) \qquad K_{\sigma,\tau}xy = x \quad , \quad S_{\rho,\sigma,\tau}xyz = xz(yz),$$

(for x, y, z variables of appropriate type) which ensure closure under explicit definition. There are also, in the case specifically of Gödel's system T, *recursors* R_σ for each type σ satisfying

$$(2) \qquad R_\sigma xy0 = x \quad , \quad R_\sigma xy(z') = y(R_\sigma xyz)z$$

with x of type σ, y of type $\sigma \to 0 \to \sigma$ and z of type 0. Other applications make use of other kinds of recursions in place of (2) and/or additional constants. QF denotes the class of quantifier-free formulas in L^ω, and for any class $\mathcal{F}$ of formulas, QF-$\mathcal{F}$ denotes the intersection of QF and $\mathcal{F}$.

Sequences of variables (which may be empty) are given by boldface lower-case or upper-case letters, viz $\mathbf{x}$=$(x_1, \ldots, x_n)$ or $\mathbf{X}$= $(X_1, \ldots, X_n)$, where $n \geq 0$. When preceded by $\exists$ or $\forall$, $\mathbf{xy}$ is the concatenation of $\mathbf{x}$ and $\mathbf{y}$; within a formula, $\mathbf{Xy}$ indicates the result of applying each term of $\mathbf{X}$ to the sequence $\mathbf{y}$ (when this is well-typed). With each formula ϕ of L^ω is associated its *Dialectica* (or *D*-) interpretation $\phi^D \equiv \exists\mathbf{x}\forall\mathbf{y}\phi_D(\mathbf{x},\mathbf{y})$ where ϕ_D is QF; the free variables of ϕ will be exhibited only when necessary.[5] This is defined inductively as follows, using also $\psi^D \equiv \exists\mathbf{u}\forall\mathbf{v}\psi_D\ (\mathbf{x},\ \mathbf{y})$:

[5]While Gödel defined ϕ^D only for first-order ϕ, its extension to ϕ in L^ω is immediate.

(3) (i) For ϕ an atomic formula, $\mathbf{x}$, $\mathbf{y}$ are empty and $\phi^D \equiv \phi_D \equiv \phi$,
(ii) $(\phi \wedge \psi)^D \equiv \exists \mathbf{xu} \forall \mathbf{vy}(\phi_D \wedge \psi_D)$,
(iii)$(\phi \vee \psi)^D \equiv \exists z\mathbf{xu} \forall \mathbf{vy}[(z = 0 \wedge \phi_D) \vee (z = 1 \wedge \psi_D)]$ (z of type 0),
(iv) $(\phi \to \psi)^D \equiv \exists \mathbf{UY} \forall \mathbf{x}, \mathbf{v}[\phi_D(\mathbf{x}, \mathbf{Yxv}) \to \psi_D(\mathbf{Ux}, \mathbf{v})]$,
(v) $\forall z\phi(z))^D \equiv \exists \mathbf{X} \forall z, \mathbf{y} \;\; \phi_D(\mathbf{X}z, \mathbf{y}, z)$,
(vi) $(\exists z\phi(z))^D \equiv \exists z\mathbf{x} \forall \mathbf{y} \;\; \phi_D(\mathbf{x}, \mathbf{y}, z)$.

Gödel's motivation for (iv) has been described in the preceding section.

Negation of ϕ is regarded as defined by $\neg\phi \equiv (\phi \to 0 = 1)$. Then one obtains

(4) $$(\neg\phi)^D \equiv \exists \mathbf{Y} \forall \mathbf{x} \neg \phi_D(\mathbf{x}, \mathbf{Yx})$$

and so

(5) $$(\neg\neg\phi)^D \equiv \exists \mathbf{X} \forall \mathbf{Y} \neg\neg \phi_D(\mathbf{XY}, \mathbf{Y}(\mathbf{XY})).$$

This is important to take note of, since Gödel's 1933 double-negation or "negative" translation of classical into intuitionistic systems replaces $\exists z\psi(z)$ by $\neg\neg\exists z\psi(z)$ and $\psi \vee \theta$ by $\neg\neg(\psi \vee \theta)$. In particular, we have:

(6) $$(\neg\neg\exists \mathbf{z} R(\mathbf{z}))^D \equiv \exists \mathbf{z} \neg\neg R(\mathbf{z}) \text{ for } R \in QF.$$

Let HA^ω be the system extending HA in L^ω by the combinatory axioms (1) and recursor axioms (2), with the induction scheme extended to arbitrary formulas of L^ω.[6] The quantifier-free part of QF-HA^ω is just another form of Gödel's system T [or his 1941 Σ];[7] in the following we shall use PR^ω as a more suggestive denotation for this theory of primitive recursive functionals of finite type. Gödel showed that the D-interpretation carries HA into PR^ω, in the sense that for every provable ϕ with $\phi^D \equiv \exists \mathbf{x} \forall \mathbf{y} \phi_D(\mathbf{x}, \mathbf{y})$ we can find

[6] Troelstra [1990] uses 'WE-HA^ω' for our HA^ω, where 'WE' stands for 'Weakly Extensional'; there is a slight risk of confusion since he uses 'HA^ω' for a different system, but that is not needed here.

[7] Except that Gödel can be understood as assuming basic decidable equality relations at each type σ in 1958; this is the so-called *intensional* version of T (cf. [Troelstra 1990], pp. 221–222).

a sequence of terms $\mathbf{t}$ (directly from a proof of ϕ) with $\phi_D(\mathbf{t},\mathbf{y})$ provable in PR^ω. With hardly any additional work, one obtains:

$$\text{(7)} \qquad \mathrm{HA}^\omega \vdash \phi \Rightarrow \mathrm{PR}^\omega \vdash \phi_D(\mathbf{t},\mathbf{y}) \text{ for some } \mathbf{t}.$$

The proof of (7) breaks into two parts. The first, quite general, part is to show that the axioms and rules of intuitionistic logic in L^ω are preserved under the D-interpretation (in the sense of (7)); this uses only the axioms (1) for explicit definition and the underlying logic of PR^ω. The second part is to show that the non-logical axioms of HA^ω are preserved under the D-interpretation; here the main step is to use the recursor axioms (2) to verify the D-interpretation of the induction scheme. The latter involves a nice exercise in the application of the definition of $(\phi \to \psi)^D$.

It is also easy to verify that the D-interpretation preserves the Axiom of Choice, as given by the scheme:

$$\text{AC} \qquad \forall x \exists y \phi(x,y) \to \exists f \forall x \phi(x, fx).$$

This principle is generally accepted by intuitionists. However, the following scheme is not (ordinarily), since it generalizes "Markov's Principle":

$$\mathrm{M}' \qquad \neg\forall \mathbf{v} R(\mathbf{v}) \to \exists \mathbf{v} \neg R(\mathbf{v}), \text{ for } R \in \mathrm{QF}.$$

This principle figures in a slightly different analysis (than that given by Gödel in his 1941 lecture) of how one arrives at the D-interpretation of implication; cf. [Troelstra 1990] p. 226. Another such schema which is problematic for intuitionists is a form of "independence of premiss" principle:

$$\mathrm{IP}' \qquad [\forall \mathbf{y} R \to \exists \mathbf{u} \forall \mathbf{v} S] \to \exists \mathbf{u}[\forall \mathbf{y} R \to \forall \mathbf{v} S], \quad \text{for } R, S \in \mathrm{QF}.$$

That was used in the step from (1) to (2) in the preceding section, again in the treatment of implication. Let $\widetilde{\mathrm{HA}}^\omega = \mathrm{HA}^\omega + \mathrm{AC} + \mathrm{M}' + \mathrm{IP}'$; then (7) above can be strengthened to the following:

$$\text{(8)} \qquad \begin{array}{ll} (i) & \widetilde{\mathrm{HA}}^\omega \vdash \phi \leftrightarrow \phi^D \text{ for any } \phi, \text{ and} \\ (ii) & \widetilde{\mathrm{HA}}^\omega \vdash \phi \Rightarrow \mathrm{PR}^\omega \vdash \phi_D(\mathbf{t},\mathbf{y}) \text{ for some } \mathbf{t}. \end{array}$$

(See [Troelstra 1990] p. 232). The result (8) serves to simplify various verifications. As an immediate application of (8)(i) one has, by (6):

(9) $$\widetilde{\mathrm{HA}}^{\omega} \vdash \neg\neg\exists \mathbf{z} R(\mathbf{z}) \leftrightarrow \exists \mathbf{z} R\mathbf{z} \text{ for } R \in \mathrm{QF},$$

since R is decidable; of course, (9) also follows from M'.

5 Application of the D-interpretation to classical systems.

For $\phi \in \mathrm{L}^{\omega}$ considered as a formula in a system with classical logic, let ϕ^{-} be its "negative" translation, obtained by prefixing every disjunctive or existential subformula of ϕ by double negation. Let PA^{ω} be HA^{ω} with classical logic, so PA^{ω} extends PA as HA^{ω} extends HA. [Gödel 1933] showed that the negative translation sends PA into HA; this carries over immediately to the corresponding finite type extensions. In connection with the further application of the D-interpretation, [Howard 1968] p. 115 observed the following useful strengthening of this result. Let $\widetilde{\mathrm{PA}}^{\omega}$ be PA^{ω} together with all formulas of the form $\psi \leftrightarrow (\psi^{-})^{D}$ for $\psi \in \mathrm{L}^{\omega}$. Then:

(1) $$\widetilde{\mathrm{PA}}^{\omega} \vdash \phi \text{ implies } \widetilde{\mathrm{HA}}^{\omega} \vdash (\phi^{-}), \text{ hence } \widetilde{\mathrm{HA}}^{\omega} \vdash (\phi^{-})^{D}.$$

Application of the *Dialectica* interpretation to classical systems involves calculating the effect of $\phi \mapsto (\phi^{-})^{D}$ for various ϕ.[8] Let S^{-} be the set of $(\theta)^{-}$ for $\theta \in S$. Since $(\ )^{-}$ preserves $\wedge$ and $\rightarrow$, we have:

(2) $$\widetilde{\mathrm{PA}}^{\omega} + S \vdash \phi \Rightarrow \widetilde{\mathrm{HA}}^{\omega} + S^{-} \vdash \phi^{-}.$$

The easiest application of this observation is that

(3) $$\widetilde{\mathrm{PA}}^{\omega} + (\mathrm{QF}-\mathrm{AC}) \vdash \phi \Rightarrow \widetilde{\mathrm{HA}}^{\omega} \vdash \phi^{-}.$$

For, an instance $\forall \mathbf{x} \exists \mathbf{y} P(x, y) \rightarrow \exists f \forall \mathbf{x} P(\mathbf{x}, f\mathbf{x})$ of QF-AC, with $P \in \mathrm{QF}$, is sent into $\forall \mathbf{x} \neg\neg\exists \mathbf{y} P(\mathbf{x}, \mathbf{y}) \rightarrow \neg\neg\exists f \forall \mathbf{x} P(\mathbf{x}, f\mathbf{x})$. But the hypothesis is equivalent to $\forall \mathbf{x} \exists \mathbf{y} P(\mathbf{x}, \mathbf{y})$ by (9) of the preceding section and the conclusion then follows from AC in $\widetilde{\mathrm{HA}}^{\omega}$. As a corollary, we obtain, for $R \in \mathrm{QF}$:

[8] [Shoenfield 1967] pp. 219 ff provides an alternative interpretation which avoids passing through the negative translation. This associates with each ϕ a formula ϕ^{S} of the form $\forall \mathbf{x} \exists \mathbf{y} \phi_{S}(\mathbf{x}, \mathbf{y})$ with $\phi_{S} \in \mathrm{QF}$, such that if PA$\vdash \phi$ then $\mathrm{PR}^{\omega} \vdash \phi_{S}(\mathbf{x}, \mathbf{tx})$ for some $\mathbf{t}$. However, this provides no real saving in work in general.

(4) $\widetilde{\mathrm{PA}}^{\omega} + (\mathrm{QF}-\mathrm{AC}) \vdash \forall\mathbf{x}\exists\mathbf{y}R(\mathbf{x},\mathbf{y}) \Rightarrow \mathrm{PR}^{\omega} \vdash R(\mathbf{x},\mathbf{tx})$ for some $\mathbf{t}$.

This shows among other things that every provably recursive function(al) of $\widetilde{\mathrm{PA}}^{\omega}$+ (QF-AC) is definable by a term of PR$^{\omega}$. By formalizing in PA the model of PR$^{\omega}$ in HEO, the hereditarily extensional effective operations, this shows that:

(5) *The provably recursive functions of* PA *are the same as those of* PA$^{\omega}$+QF-AC, *and also the same as the functions of type* 1 *in* PR$^{\omega}$.

It follows from (5) and [Kreisel 1952] that the provably recursive functions of type 1 in PR$^{\omega}$ are exactly the functions definable by effective transfinite recursion on ordinals $< \epsilon_0$ (in its natural ordering). For more direct assignments of ordinals $< \epsilon_0$ to the terms of PR$^{\omega}$ giving the same result, see the references in [Troelstra 1990], p. 238.

6 Functional interpretation of classical analysis.

The historical material described above shows that for Gödel the primary value of the functional interpretation was its use in reducing the consistency of HA to that of PR$^{\omega}$, a system embodying notions which he thought to be closer to those of finitism than those underlying intuitionistic reasoning. He noted in his 1941 Yale lecture that this reduction leads via the negative translation to a proof of consistency of PA on the basis of PR$^{\omega}$, and moreover, that if PA proves $\exists y R(y)$ with R in QF then R(t) holds for some t in PR$^{\omega}$ (an evident precursor of (4) in the preceding section). Toward the end of the same lecture, Gödel speculated that:

> It is perhaps not altogether hopeless to try to generalize these consistency proofs to analysis by means of functions of still higher (i.e. transfinite) type. Future development will show if that is possible at all and in which sense the system necessary to accomplish this proof will be constructed.(C, p.30)

Kreisel [1987] p. 104 reports that Gödel told him, very soon after their first meeting in 1955, about his functional interpretation work in the 1940's, "... later incorporated in the so called *Dialectica* interpretation (with a total shift of emphasis)." Evidently Gödel misremembered: there is really no significant difference in emphasis, though the 1941 lecture mentions a few applications that are not contained in the 1958 *Dialectica* article.[9] Of Gödel's supposed change in emphasis, Kreisel (loc. cit.) goes on to say that "He [Gödel] wanted to fill the superficially principal gap left by his negative translation... He dropped the project after he learned of recursive realizability that Kleene found soon afterward." It is true that the association with each ϕ, provable in HA, of terms $\mathbf{t}$ in PR^ω such that $\phi_D(\mathbf{t}, \mathbf{y})$ holds, is akin to Kleene's original realizability interpretation [1945], though, as Kreisel points out, neither one yields the existential definability property for HA.[10]

Be that as it may, given Kreisel's general interests in constructivity in the 1950's and his prior development of the no-counterexample functional interpretation for classical arithmetic, he was in an excellent position (perhaps uniquely so) to appreciate Gödel's accomplishment in this respect and to exploit it further. Before long he was spreading the word about Gödel's functional interpretation and, by 1957, arrived at an extension to a system for full second-order analysis. Kreisel first presented this work to a large audience of logicians at the Summer Institute for Symbolic Logic held at Cornell in June 1957, under the title "Gödel's interpretation of Heyting's Arithmetic", though most of his lecture was about his own extension of the interpretation to analysis. Later that same summer he lectured on this material at the Colloquium on Constructivity in Mathematics held in Amsterdam; his article for its proceedings appeared in [Kreisel 1959]. The formulation of analysis used there is a second-order system with full Comprehension Axiom scheme CA. In view of the preceding discussion, it is natural to extend this to the language L^ω and to infer CA from the (full) Axiom of Choice scheme AC. Now by (8)(i) of section 4 above we have:

[9] It *is* a question of Gödel's memory, since he wrote Bernays in 1968, "In those days... I set no particular store by the philosophical aspect; rather, it was chiefly the mathematical result that was important to me, while now it is the other way around."([Troelstra 1990], p. 217)

[10] That property was established later by [Kleene 1952] §82 using Γ-realizability (and, for that purpose, simplified still later to "slash" realizability).

(1) $$\mathrm{PA}^\omega + \mathrm{AC} \vdash \phi \leftrightarrow (\phi^-)^D \text{ for each } \phi,$$

since ϕ is equivalent to ϕ^- with classical logic and M′ and IP′ are classically valid. Another way of stating (1) using the notation of the preceding section, is that $\widetilde{\mathrm{PA}}^\omega$ is contained in PA^ω+ AC.

One of the main results of [Kreisel 1959] is that if ϕ is provable in analysis and $(\phi^-)^D \equiv \exists \mathbf{x} \forall \mathbf{y} R(\mathbf{x}, \mathbf{y})$ with $R \in \mathrm{QF}$ then $(\exists \mathbf{x} \in \mathcal{RC})(\forall \mathbf{y} \in \mathcal{C}) R(\mathbf{x}, \mathbf{y})$ holds, where $\mathcal{C}$ consists of the continuous functionals of finite type and $\mathcal{RC}$ of the recursively continuous functionals.[11] The classes $\mathcal{C}$ and $\mathcal{RC}$ had been developed independently by Kreisel and Kleene, with their respective treatments both published in the proceedings of the 1957 Amsterdam conference; however, only Kreisel pursued the connection with Gödel's functional interpretation.[12] A direct strengthening of Kreisel's result to the system in (1) is:

(2) If PA^ω + AC $\vdash \phi$ and $(\phi^-)^D \equiv \exists \mathbf{x} \forall \mathbf{y} \mathrm{R}(\mathbf{x}, \mathbf{y})$ then there exists $\mathbf{f} \in \mathcal{RC}$ such that for all $\mathbf{y}$ in $\mathcal{C}$, $\mathrm{R}(\mathbf{f}, \mathbf{y})$ holds.

It was then natural to ask whether the class $\mathcal{RC}$ in (2) could be replaced by a class of functionals determined by schemata extending Gödel's PR^ω. This was pursued intensively by Clifford Spector during the year 1960–61 which he spent at the Institute for Advanced Study in Princeton, where he had frequent contact with Kreisel and also benefited from conversations with Gödel and Bernays. Spector succeeded in achieving this goal by the adjunction of a new scheme to PR^ω, called Bar Recursion:

BR $$F(n, c) = \begin{cases} G(n, \langle c0, \ldots, c(n-1)\rangle) \text{ if } Y(\langle c0, \ldots, c(n-1)\rangle) < n \text{ else} \\ H(\lambda a. F(n+1, \langle c0, \ldots, c(n-1), a\rangle), n, \langle c0, \ldots, c(n-1)\rangle). \end{cases}$$

Here the values $c(k)$ may be at any type σ (with a of the same type) . The scheme (BR) is justified for continuous Y, since $Y(c)$ will be determined by a finite amount of information $\langle c0, \ldots, c(n-1)\rangle$ about c and then for

[11] In fact, according to [Kreisel 1959], ϕ is equivalent to $(\exists \mathbf{x} \in RC)(\forall \mathbf{y} \in \mathcal{C}) R(\mathbf{x}, \mathbf{y})$.

[12] There is a wealth of results and observations in [Kreisel 1959]. Among others, it is noted that the n.c.i. for PA follows from Gödel's functional interpretation. Kreisel also considered the status of the formulas $(\phi^-)^D$ in other classes of functionals of finite type including the hereditarily effective operations and Kleene's recursive functionals.

sufficiently large n one can make $Y(c) = Y(\langle c0, \ldots, c(n-1)\rangle) < n$. Given c, F "searches" for such an n in (BR). The terminology "Bar Recursion" was suggested by that of Brouwer's Bar Theorem, which uses an intuitionistically accepted form of the principle of transfinite induction on well-founded trees of natural numbers. This may be generalized to trees of objects of any type σ as a principle called Bar Induction (BI), and Bar Recursion is then a related principle of definition by recursion on such well-founded trees.

BR^ω is used here to denote the formal quantifier-free system extending PR^ω by adjunction of the defining equations (BR) at each type. The main result of Spector's work may then be formulated as follows:

(3) If $\mathrm{PA}^\omega + \mathrm{AC} \vdash \phi$ and $(\phi^-)^D \equiv \exists\mathbf{x}\forall\mathbf{y}R(\mathbf{x},\mathbf{y})$ with $R \in \mathrm{QF}$ then we can find terms $\mathbf{t}$ of BR^ω such that $\mathrm{BR}^\omega \vdash R(\mathbf{t},\mathbf{y})$.

This appeared in [Spector 1962], brought to publication by Kreisel after Spector's tragic sudden death due to leukemia. Another proof of (3) was later given by William Howard [1968], which perhaps explains better the role of BR in Spector's interpretation of analysis. Howard showed that the scheme AC is implied classically by a subscheme of BI whose negative translation follows from $\widetilde{\mathrm{HA}}^\omega$ + BI; he then used BR to realize the D-interpretation of BI. In other publications (cf. [Troelstra 1990] for references), Howard made further substantial contributions to the analysis and use of the *Dialectica* interpretation for various systems.

While Spector's interpretation did not follow the lines of Gödel's speculation that one might be able to extend his functional interpretation to analysis by the use of transfinite types, it did bear transfinite features by its use of recursion on certain well-founded trees. The main foundational issue following Spector's result became the question as to whether it provided a constructive consistency proof of analysis. That it constituted a real advance in perspicuity over Kreisel's 1957 model in the continuous functionals was clear to all. But what was not evident was the intuitionistic acceptability of Bar Recursion; in this respect, [Spector 1962] p. 2 reported varying degrees of opinion among Bernays, Gödel, Heyting and Kreisel. Spector himself thought that its acceptability to the intuitionists was questionable, and that further work would be required to give it a "suitable foundation"; in this, Kreisel concurred.[13] It should be noted that in this respect, the constructivity of intuitionism itself was taken for granted; this was not an attempt to reduce that to "more constructive" principles as had been Gödel's aim with his previous work.

The question of the constructivity of Bar Recursion was the main subject of a Seminar on the Foundations of Analysis, held at Stanford in the summer of 1963 under the leadership of Kreisel. Contributions to the seminar were prepared as a volume of reports, circulated to interested parties but never published *per se*, whose authors were G. Kreisel, W. A. Howard, W. W. Tait (with several parts each), J. Harrison and R. J. Parikh. While much information was obtained in this seminar about classical and intuitionistic systems of analysis, theories of generalized inductive definitions, the functional interpretations and classes of functionals of finite type, the conclusion about the main question was disappointing. In the words of Kreisel's introduction to Vol. II of these Reports (p. i), in answer to the question: "*Can bar recursion of finite type be constructively justified*?"... "the answer is negative by a wide margin, since not even bar recursion of type 2 can be proved consistent."[14] That assessment seems unchanged to date.

7 The D-interpretation as a general proof-theoretical tool.

Gödel's concentration on the consistency problem both in the development of his functional interpretation of arithmetic and in his special interest in Spector's extension of it to analysis, was a direct continuation of the central concern of Hilbert's program, though with the recognized necessity of giving up adherence to strictly finitistic methods. But proof theory in general has had to broaden its concerns and methods. The main line of technique which has been developed from the original work of Herbrand and Gentzen

[13] Spector had originally entitled his paper "Probably recursive functionals in analysis: a consistency proof by an extension of intuitionistic principles." At Gödel's suggestion (cf. [Spector 1962] ftn. 1), "to avoid misunderstandings", this was changed to "...: a consistency proof by an extension of principles formulated in current intuitionistic mathematics."

[14] On the basis of the species of representing functions of continuous functionals, which Kreisel then considered to be the only possible candidate which could be used to provide a justification (op. cit., Vol. I, p. 0.2).

uses normalization of derivations in some sense or other; modern extensions of these methods in many cases make heavy use of infinitary derivations. Except for some of the more traditional proof-theorists, establishing consistency of formal systems has receded as the main goal, to be replaced by a more general *reductive* program. This still takes foundational aims to be primary, but gives up hopes for any supposedly absolute foundation in favor of reductions of various systems to others recognizably more basic in terms of concepts and/or principles (cf. [Feferman 1988, 1993]). At the same time, proof-theoretical tools have been applied to obtain results of a more mathematical character that one may describe as *extractive*, namely to draw explicit or computational information from proofs of statements of existential or universal-existential form. Under the latter one would count the use of proof theory in characterizing the *provably recursive functions* (or *functionals*, where that is appropriate) of a formal system. In practice, both reductive and extractive proof theory work hand in hand, and are related to assessments of proof-theoretic strength in one sense or another, most frequently as measured by the provably recursive well-orderings of a system.

Though the D-interpretation has not been applied as extensively as the Herbrand-Gentzen methods of syntactic transformation, it has proved to be a rather powerful and versatile tool with distinctive advantages, especially for extractive purposes. Applied to intuitionistic systems it takes care of the underlying logic once and for all, verifies the Axiom of Choice AC in all types, and interprets various forms of induction by suitably related forms of recursion. This then leads for such systems to a perspicuous mathematical characterization of the provably recursive functions and functionals. For application to classical systems, one must first apply the negative translation (again taken care of once and for all). Since the D-interpretation verifies Markov's Principle even at higher types (principle M$'$ in sec. 4 above) at least the provably recursive functions and functionals are preserved, as well as QF-AC in all types and induction schemata. The main disadvantage, though, comes with the analysis of other statements whose negative translation may lead to a complicated D-interpretation; special tricks may have to be employed to handle these.

My own experience with the D-interpretation came toward the end of the 1960's, when I used it (among other things) to show that certain reductive results obtained by use of model-theoretic methods by Friedman for subsystems of classical analysis based on Σ^1_n-AC ($n = 1, 2, \ldots$) could be re-obtained and strengthened (e.g. to higher types) by use of the D-interpretation. The trick here was to reduce Σ^1_n-AC to QF-AC by the introduction of Skolem functions which are eventually eliminated (cf. [Feferman 1977], §8, and [Feferman 1987], pp. 466ff). I subsequently came to the conclusion that Herbrand-Gentzen methods worked just as well and were simpler; evidence for this was given in [Feferman and Sieg 1981] and [Feferman and Jäger 1983]. In more recent years, however, I have returned to a more positive view of the merits of the D-interpretation, and want to encourage people to learn how it has worked so far and to get a sense of its potentialities.

Thus, in conclusion, I want to mention some results which show how the D- interpretation may be stretched down to fragments of arithmetic and analysis, i.e. in the direction opposite to the stretch upwards to full analysis described in the preceding section. An early instance of such, and first to be mentioned here, is Parsons' result [1970] that the subsystems Σ°_1-IA and Π°_2-IR of PA are both conservative over PRA (Primitive Recursive Arithmetic) for Π°_2 statements, and hence have exactly the primitive recursive functions as their provably recursive functions. This was established by a D-interpretation into a subclass of the primitive recursive functionals with restricted recursors. Following later model-theoretic work of Friedman which showed that Σ°_1-IA + Δ°_1-CA + WKL is conservative over PRA, [Sieg 1985] showed how to obtain the same result by Herbrand-Gentzen methods, and also strengthened it by replacing Δ°_1-CA by Σ°_1-AC. (WKL is the so called Weak König's Lemma, i.e. for subtrees of $2^{<\omega}$.) A key step in Sieg's treatment of WKL was the use of a majorization argument due to [Howard 1973], originally applied to Gödel's PR^ω functionals for quite different purposes. In unpublished notes [Feferman 1990] I returned to the D-interpretation and showed how this majorization argument could also be applied to systems based on Kalmar-elementary arithmetic as well as Buss' feasible arithmetic S^1_2 ([Buss 1986]), when QF-AC is allowed in all types and WKL is adjoined. I learned then that Cook and Urquhart had already applied the D-interpretation to S^1_2 in 1988 in order to recapture Buss' characterization of its provably recursive functions (in a suitably modified sense) as the polynomial time computable functions; their work is to appear in [Cook and Urquhart 1994]. However, my result for the adjunction of WKL strengthened both that and the result of [Ferreira 1988] for S^1_2 +Δ°_1-CA+WKL. In the meantime, Sieg has independently pursued Herbrand-Gentzen methods to obtain all the same results (cf. [Sieg 1991]). Thus, to a considerable extent, for these purposes it is a matter of taste which approach to prefer. It is my plan to write up the details of my 1990 notes so that interested readers may have a reasonable basis of comparison, also in the hope that this will stimulate finding new applications of the D-interpretation. In any case, I strongly recommend reading the earlier sources referred to in sections 4–6 above, which already demonstrate the considerable elasticity of Gödel's functional interpretation.

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