Predicative foundations of arithmetic

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0 Introduction

Predicative mathematics in the sense originating with Poincaré and Weyl begins by taking the natural number system for granted, proceeding immediately to real analysis and related fields. On the other hand, from a logicist or set-theoretic standpoint, this appears problematic, for, as the story is usually told, impredicative principles seem to play an essential role in the foundations of arithmetic itself.[1] It is the main purpose of this paper to show that this appearance is illusory: as will emerge, a predicatively acceptable axiomatization of the natural number system can be formulated, and both the existence of structures of the relevant type and the categoricity of the relevant axioms can be proved in a predicatively acceptable way.

Indeed, there are three aspects of the set-theoretical foundations of arithmetic which might appear to involve impredicative principles in an essential way, either explicitly or implicitly. These may be listed as follows:

(a) Axiomatization: The axiom of *mathematical induction* ((N-III) below) is *prima facie* in full second-order form.

(b) Categoricity: As usually formulated, the proof of categoricity employs an impredicative construction, namely in the specification of an isomorphism between two models (of arithmetic with the full second-order induction axiom) as the intersection of all one-one maps between their domains preserving the zeros and succession.

(c) Existence: To construct a set satisfying the axioms of (b), assuming one has a set satisfying the usual first-order axioms on successor, one takes the intersection of all sets closed under successor; this is an impredicative definition.

Concerning these points, we make the following remarks:

Ad (a): As is well known, no first-order axiomatization is categorical. The axiom (N-III) is on its face second-order, so the usual conclusion is that we need "full" second-order logic to underly the use of (N-III).

Ad (b) : This kind of proof is standard and is given, for example, in Shapiro [1991], pp. 82–83.

Ad (c) : The question of existence is raised in a logicist or modified logicist (set-theoretical) approach to the foundations of arithmetic, where one is also required to "construct" the natural numbers. This played an important role for Dedekind [1888], whose "existence proof" calls for special comment and reformulation (cf. e.g.

Hellman [1989], Ch. 1). Predicativists such as Poincaré and Weyl, however, did not require this step (cf. Feferman [1988]).

Thus, we may ask, to what extent can *(a) - (c)* be accounted for on predicatively acceptable grounds?

Re (a): Unrestricted second-order variables may be considered to range over predicatively defined classes satisfying weak, predicatively acceptable closure conditions.

Re (b) : We assume that the notion of *finite set* is predicatively understood, governed by some elementary closure conditions.[2] Instead of defining the isomorphism between two "N-structures" (as Dedekind's "simply infinite systems" or the set theorist's "ω-sequences" will be called below) as the *intersection of* (*certain*) *classes*, we obtain it as the *union of* (*certain*) *finite sets*.[3]

Re (c) : Even if we pursue the logicist challenge to "construct" the natural numbers, this can in fact be carried out within a predicatively justified system.

In sum, the moral is that appearances are deceiving!

Our work will be carried out within a three-sorted system, EFSC, standing for "Elementary theory of Finite Sets and Classes", and a certain extension EFSC*. The three sorts are (general) *individuals*, *finite sets*, and *classes*. The notion of *N-structure* can be formulated in the language of EFSC, and it is proved there that any two N-structures are isomorphic (§1). EFSC* is obtained from EFSC by adding an axiom to the effect that any (truly) finite set is Dedekind finite. EFSC* proves the existence of an N-structure (§3). It will be shown that EFSC* contains the first-order system of Peano Arithmetic, PA (under suitable definition of its basic notions) and is a conservative extension of PA (§4). The system PA is a part of what is predicatively acceptable, granted the conception of the totality of natural numbers (cf. [Feferman 1964, 1968]).[4]

1 The system EFSC

The *language* $\mathcal{L}$(EFSC) contains:

Individual variables: $a, b, c, u, v, w, x, y, z$, with or without subscripts;
Finite set variables: A, B, C, with or without subscripts;
Class variables: $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, with or without subscripts;
Operation symbol: $(,)$;
Individual terms, s, t, ... are generated from the individual variables by the operation $(,)$;
Atomic formulas: $s = t$, $s \in A$, $s \in \boldsymbol{X}$;
Formulas $\phi, \psi, \ldots$ are generated from atomic formulas by &, $\vee$, $\neg$, $\rightarrow$ and $\forall$, $\exists$ applied to any one of the three sorts of variables.
WS-formulas: *Predicative* or *Weak Second-Order Formulas* are those in which there are no *bound* class variables.

The underlying logic is classical three-sorted (first-order) predicate calculus with equality (in the first sort).

Definition: $\boldsymbol{X} \subseteq \boldsymbol{Y} \equiv^{df} \forall x[x \in \boldsymbol{X} \rightarrow x \in \boldsymbol{Y}]$; $\boldsymbol{X} = \boldsymbol{Y} \equiv^{df} \quad \boldsymbol{X} \subseteq \boldsymbol{Y} \;\&\; \boldsymbol{Y} \subseteq \boldsymbol{X}$. Similarly defined are $\boldsymbol{X} \subseteq A$, $A \subseteq \boldsymbol{X}$, $\boldsymbol{X} = A$, $A \subseteq B$, $A = B$.

The Axioms of EFSC are (WS-CA), (Sep), (FS-I), (FS-II), and (P-I), (P-II), as follows:

(WS-CA) $$\exists \boldsymbol{X} \forall x[x \in \boldsymbol{X} \leftrightarrow \phi]\,,$$

where ϕ is a WS-formula lacking free '$\boldsymbol{X}$'. ('WS-CA' is for 'Weak Second-order Comprehension Axiom'.)

Notation: We write $\{x|\phi\}$ for an $\boldsymbol{X}$ such that $x \in \boldsymbol{X} \leftrightarrow \phi$. Since $\boldsymbol{X}$ is determined up to definitional $=$, this is an inessential extension of the symbolism. As special cases, we define: $\boldsymbol{X} \cap \boldsymbol{Y} = \{x|x \in \boldsymbol{X} \;\&\; x \in \boldsymbol{Y}\}$, $\boldsymbol{X} \cup \boldsymbol{Y} = \{x|x \in \boldsymbol{X} \;\vee\; x \in \boldsymbol{Y}\}$, $-\boldsymbol{X} = \{x|x \notin \boldsymbol{X}\}$, $\boldsymbol{X} \times \boldsymbol{Y} = \{z|\exists x, y(x \in \boldsymbol{X} \;\&\; y \in \boldsymbol{Y} \;\&\; z = (x, y))\}$, $\boldsymbol{V} = \{x|x = x\}$, $\Lambda = \{x|x \neq x\}$, $A \cap \boldsymbol{X} = \{x|x \in A \;\&\; x \in \boldsymbol{X}\}$, $\{a\} = \{x|x = a\}$, etc.

(Sep) $$\forall A \exists B \forall x[x \in B \leftrightarrow x \in A \;\&\; \phi]\,,$$

ϕ a WS-formula, 'B' not free in ϕ.

(FS-I)(*Empty*) $$\exists A \forall x[x \notin A]\,.$$

(FS-II)(*Adjunction*) $$\forall a \forall A \exists B \forall x[x \in B \leftrightarrow x \in A \;\vee\; x = a]\,.$$

(P-I) $$(x_1, x_2) = (y_1, y_2) \leftrightarrow x_1 = y_1 \;\&\; x_2 = y_2\,.$$

(P-II) $$\exists u \forall x, y[(x, y) \neq u]\,.$$

Remarks on the axioms:
1. Define $Fin(\boldsymbol{X}) \equiv \exists A[\boldsymbol{X} = A]$, "$\boldsymbol{X}$ is (truly) finite". Then under (WS-CA), the Separation Axiom (Sep) is equivalent to $\forall A, \boldsymbol{X}[\boldsymbol{X} \subseteq A \rightarrow Fin(\boldsymbol{X})]$. However, we can consider variant formulations of EFSC without class variables. Then we need (Sep) as above. (See remark 7 and Metatheorem 1, below.)
2. Under (WS-CA), the first Finite Set Axiom (FS-I) is equivalent to $Fin(\Lambda)$.
3. Under (WS-CA), (FS-II) is equivalent to $\forall A, a[Fin(A \cup \{a\}]$ or $\forall \boldsymbol{X}, a[Fin(\boldsymbol{X}) \rightarrow Fin(\boldsymbol{X} \cup \{a\}]$.

4. We use pairs so as to define relations, functions, etc., in terms of classes. Alternatively, we can dispense with pairs by introducing more sorts of variables for binary, ternary, ... relations, and for (truly) finite relations of each "arity".
5. The axiom (P-II) says that there are "urelements" under pairing.
6. The system EFSC has a certain analogy to the NBG (von Neumann-Bernays-Gödel) theory of sets and classes. In fact, there is a simple interpretation of EFSC in NBG: both individual and finite set variables are taken to range over all sets in NBG, and (x, y) is defined as usual, $(x, y) = \{\{x\}, \{x, y\}\}$. We can take u in (P-II) to be the empty set Λ. Class variables are taken to range over classes in NBG. This interpretation does not require the axiom of infinity. Note that NBG – $\{Inf\}$ has a standard model in which the set variables range over the hereditarily finite sets V_ω, i.e., $\cup_{n<\omega} V_n$, where $V_o = \Lambda$ and $V_{n+1} = V_n \cup \mathcal{P}(V_n)$, and the class variables range over the definable subsets of V_ω.
7. Let EFS be EFSC without (WS-CA) in the language without class variables.

Metatheorem 1: EFSC is a conservative extension of EFS.
Proof (*model-theoretic*): By Gödel's completeness theorem, it suffices to show how any model $\mathcal{M}$ of EFS can be expanded to a model $\mathcal{M}'$ of EFSC. We can simply take the range of the class variables to be all the WS-definable subsets of $\mathcal{M}$ (from parameters for individuals and "finite" sets in $\mathcal{M}$.)

A proof-theoretic argument can also be given. This is analogous to the conservation result of NBG over ZF, which was established by proof-theoretic means in [Shoenfield, 1954].

Remarks (cont'd):
8. As will be seen, the system EFSC allows for a natural development of arithmetic and a general proof of categoricity, since it allows for quantification over general structures. (In particular, individuals may be any objects whatever, and are not restricted to objects in the range of numerical quantifiers.) The alternative approach of adopting number-theoretic axioms directly along with those of EFSC just presented−a system that we have dubbed "Predicative Dedekind Arithmetic" ("PDA")−has the awkwardness of being able to treat only structures with "numbers" as individuals. An advantage of PDA, however, that we forego in the present approach, is that the implication from (true) finiteness to Dedekind finiteness (TF→ DF)−adopted as a further axiom (Card) below−can be derived in PDA from the more intuitive axiom that "truly finite sets are bounded" (in the natural ordering of natural numbers). That implication (TF→ DF), however, is elementary enough for present purposes, and so we shall pursue the present systems, in which the number-theoretic structures of interest may be both characterized and proved to exist.

Definition: $Func(\boldsymbol{X}) \equiv \forall x, y_1, y_2[(x, y_1) \in \boldsymbol{X} \; \& \; (x, y_2) \in \boldsymbol{X} \rightarrow y_1 = y_2]$.
We use f, g, h with or without subscripts to range over functions. Define $Dom(f) = \{x | \exists y[(x, y) \in f]\}$; $Ran(f) = \{y | \exists x[(x, y) \in f]\}$; we write $f(x) = y$ for $(x, y) \in f$; $f : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ means $Dom(f) = \boldsymbol{X} \; \& \; Ran(f) \subseteq \boldsymbol{Y}$. We shall use functional notation freely in the following.

Definition: $(x_1, ..., x_n, x_{n+1}) = ((x_1, ..., x_n), x_{n+1})$;
$\boldsymbol{X}^{n+1} = \boldsymbol{X}^n \times \boldsymbol{X}$.
Thus, functions of n arguments may be reduced to unary functions on Cartesian powers.

Definition: A *pre-N-structure* is a triple $\mathcal{M} =< \boldsymbol{M}, a, g >$, where $a \in \boldsymbol{M}$, $g : \boldsymbol{M} \to \boldsymbol{M}$, and
(N-I) $\forall x \in \boldsymbol{M}[g(x) \neq a]$
(N-II) $\forall x, y \in \boldsymbol{M}[g(x) = g(y) \to x = y]$.

Definition: An N-*structure* is a pre-N-structure $< \boldsymbol{M}, a, g >$ such that
(N-III) (*Induction*) $\forall \boldsymbol{X} \subseteq \boldsymbol{M}[a \in \boldsymbol{M} \ \& \ \forall x(x \in \boldsymbol{X} \to g(x) \in \boldsymbol{X}) \to \boldsymbol{M} \subseteq \boldsymbol{X}]$.

By (WS-CA), the scheme of induction for WS-formulas, relativized to any N-structure, can be derived from (N-III). In particular, for any N-structure $< \boldsymbol{N}, 0,' >$, one can prove

(Pr) (*Predecessors*): $x \neq 0 \to \exists y(x = y')$.

Definition: Let $\mathcal{M} =< \boldsymbol{M}, a, g >$ be a pre-N-structure; we define
$y \leq_{\mathcal{M}} x \equiv \forall A[x \in A \ \& \ \forall z(g(z) \in A \to z \in A) \to y \in A]$.
Where no confusion will arise, the subscript $\mathcal{M}$ may be dropped. It will also contribute to intuitive clarity to work with the notation $< \boldsymbol{N}, 0,' >$ for an arbitrary but fixed pre-N-structure.

Theorem 2: EFSC proves the following for any pre-N-structure $< \boldsymbol{N}, 0,' >$, where $x, y \in \boldsymbol{N}$:
(i) $x \leq x$;
(ii) $w \leq y \leq x \to w \leq x$;
(iii) $y' \leq x \to y \leq x$;
(iv) $y \leq 0 \leftrightarrow y = 0$;
(v) $y \leq x' \leftrightarrow y \leq x \vee y = x'$.
Proof: (i), (ii), (iii), *and* $\leftarrow$ *of* (iv) and (v) are immediate.
(iv) $\to$: Suppose $y \leq 0$. Let $A = \{0\}$, which exists by (FS-I), (FS-II), *i.e.*, $\forall z(z \in A \leftrightarrow z = 0)$. Then $0 \in A$ and $\forall z(z' \in A \to z \in A)$, since $z' \neq 0$ (by (N-I). So, by hypothesis, $y \in A$, whence $y = 0$.
(v) $\to$: Suppose $y \leq x'$ but not $y \leq x$; to show $y = x'$. By hypothesis, $\exists A[x \in A \ \& \ \forall z(z' \in A \to z \in A) \ \& \ y \notin A]$. Let $B = A \cup \{x'\}$, *i.e.*, $z \in B \leftrightarrow z \in A \vee z = x'$. Then $x' \in B \ \& \ \forall z(z' \in B \to z \in B)$. Since $y \leq x'$, it follows that $y \in B$. But $y \notin A$, so $y = x'$.
Let $< \boldsymbol{N}, 0,' >$ be an N-structure and let $\boldsymbol{M}$, a, and g be arbitrary satisfying $a \in \boldsymbol{M}$ and $g : \boldsymbol{M} \to \boldsymbol{M}$. We seek to introduce f satisfying the recursion equations

$$(1) \qquad \begin{cases} f(0) = a \\ f(x') = g(f(x)). \end{cases}$$

Let $Rec(A, g, a, x)$ be the following formula, which expresses that the finite binary relation A is the graph of f restricted to $\{z | z \leq x\}$:

$$Rec(A,g,a,x) \equiv \forall z,w[A(z,w) \to z \le x] \mathbin{\&} \forall w[A(0,w) \leftrightarrow w = a] \mathbin{\&}$$
$$\forall z < x \forall w[A(z',w) \leftrightarrow \exists u(A(z,u) \mathbin{\&} g(u) = w)].$$

(Here and in the following two theorems, it simplifies notation to use x, y, z as individual variables ranging over $\boldsymbol{N}$ and u, v, w as individual variables ranging over M.)

Theorem 3 (EFSC):
(i) $Rec(A,g,a,x) \to \forall z[z \le x \to \exists! w\ A(z,w)]$;
(ii) $Rec(A,g,a,x) \mathbin{\&} Rec(B,g,a,x) \to A = B$;
(iii) $\forall x \exists A\ Rec(A,g,a,x)$;
(iv) $Rec(B,g,a,x') \mathbin{\&} \forall z,w[A(z,w) \leftrightarrow B(z,w) \mathbin{\&} z \le x] \to Rec(A,g,a,x)$;
(v) $Rec(A,g,a,x) \mathbin{\&} Rec(B,g,a,y) \mathbin{\&} x \le y \to A \subseteq B$.
Proof: (i) is by induction (in $\boldsymbol{N}$) on z. For (ii), by induction on z, $\forall w[A(z,w) \leftrightarrow B(z,w)]$. ($iii$) is by induction on x (for any given g, a), invoking (Sep), (FS-I), (FS-II). If $Rec(A,g,a,x)$ and $A(x,u)$, then $Rec(B,g,a,x')$, where $B = A \cup \{(x', g(u))\}$. ($iv$) is immediate. ($v$) is by induction on x.

Now we can define f satisfying (1) by a WS-formula:

(2) $\big(f(x) = u\big) \equiv \exists A[Rec(A,g,a,x) \mathbin{\&} A(x,u)]$.

The preceding theorem then yields

Theorem 4 (EFSC): Let $< \boldsymbol{N}, 0, ' >$ be an N-structure and let $\boldsymbol{M}$, a, g, satisfy $a \in \boldsymbol{M}$ and $g : \boldsymbol{M} \to \boldsymbol{M}$; then
(i) $\forall x \exists! u \exists A[Rec(A,g,a,x) \mathbin{\&} A(x,u)]$;
(ii) for f defined by (2), the equations (1) hold, *i.e.*,
$\exists f\{f : \boldsymbol{N} \to \boldsymbol{M} \mathbin{\&} f(0) = a \mathbin{\&} \forall x \in \boldsymbol{N}[f(x') = g(f(x))]\}$.

Theorem 5 (*Categoricity*) (EFSC): If $< \boldsymbol{N}, 0, ' >$ and $< \boldsymbol{M}, a, g >$ are any two N-structures, then $< \boldsymbol{N}, 0, ' > \cong < \boldsymbol{M}, a, g >$, that is, there exists a bijection f between $\boldsymbol{N}$ and $\boldsymbol{M}$ such that
(i) $f(0) = a$;
(ii) $f(x') = g(f(x))$.
Proof: By Theorem 4 (ii), there exists f from $\boldsymbol{N}$ into $\boldsymbol{M}$ satisfying (i) and (ii), *i.e.* preserving the "zeros" and "succession". It remains to show that f is a bijection between $\boldsymbol{N}$ and $\boldsymbol{M}$. By induction on x (in $\boldsymbol{N}$), it follows from the suppositions on $\boldsymbol{M}$ and g and (i) and (ii) that $f(x) \in \boldsymbol{M}$. Further we prove $\forall y[f(x) = f(y) \to x = y]$, also by induction on x : Setting $x = 0$, $\forall y[f(0) = f(y) \to 0 = y]$ is equivalent to $\forall y(f(y) = a \to y = 0)$. Now if $y \neq 0$, find z such that $y = z'$ (by (Pr) above); then $f(y) = f(z') = g(f(z))$. But $g(f(z)) \neq a$, so we have a contradiction. Now suppose $\forall y(f(x) = f(y) \to x = y)$; we must show $\forall y[f(x') = f(y) \to x' = y]$, that is $\forall y[g(f(x)) = f(y) \to x' = y]$. Here if $y = 0$, we again get a contradiction. So $y = z'$ and $f(y) = f(z') = g(f(z))$. But then $f(x) = f(z)$, whence $x = z$, and

then $x' = z' = y$. Finally, to show $\forall u \in \boldsymbol{M} \exists x(f(x) = u)$, we proceed by induction in $\boldsymbol{M}$, using $\boldsymbol{X} = \{w | w \in \boldsymbol{M} \ \& \ \exists x(f(x) = w)\}$. Clearly $a \in \boldsymbol{X}$. If $u \in \boldsymbol{X}$, then $\exists x(f(x) = u)$, so $g(f(x) = f(x') = g(u)$. Hence $\boldsymbol{M} \subseteq \boldsymbol{X}$, *q.e.d.*

For purposes of comparing EFSC with well-known systems, the following theorem establishing closure under primitive recursion on any N-structure will also be useful:

Theorem 6 (EFSC): *If* $< \boldsymbol{N}, 0,' >$ *is* any $\boldsymbol{N}$-structure and $h : \boldsymbol{N}^n \to \boldsymbol{N}$, $g : \boldsymbol{N}^{n+2} \to \boldsymbol{N}$, then there exists $f : \boldsymbol{N}^{n+1} \to \boldsymbol{N}$ such that

$$f(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$$
$$f(x_1, \ldots, x_n, x') = g(x_1, \ldots, x_n, x, f(x_1, \ldots, x_n, x)).$$

Proof: Exactly analogous to that of Theorem 3 above.

2. Existence of special pre-N-structures in EFSC.

Definition: Let 0 be some fixed urelement under pairing, *i.e.*, suppose

(P-II)$_0$ $\forall x, y[(x, y) \neq 0]$.

Then define $x' =^{df} (x, 0)$.

Lemma: (*i*) $\forall x[x' \neq 0]$;
(*ii*) $\forall x, y[x' = y' \to x = y]$.
Proof: Immediate, by P-I, P-II$_0$.

Corollary: $< \boldsymbol{V}, 0,' >$ is a pre-N-structure.
Proof: Immediate, by the definition of pre-N-structure and the lemma.

Now the question is whether we can construct a pre-N-structure satisfying the induction axiom, (N-III). The obvious way to do this would be to take

$$\boldsymbol{M} = \{x | \forall \boldsymbol{X}[0 \in \boldsymbol{X} \ \& \ \forall y(y \in \boldsymbol{X} \to y' \in \boldsymbol{X}) \to x \in \boldsymbol{X}]\}.$$

However, in EFSC we cannot infer the existence of such $\boldsymbol{M}$ using (WS-CA)−indeed, this is a *prima facie* impredicative definition. We can, however, construct a pre-N-structure satisfying some special conditions which hold in any N-structure; this will serve as a preliminary to the construction of an N-structure in the stronger theory EFSC* of §3.

Definition: $\leq$ *is the* $\leq_{\mathcal{V}}$ *−relation* for the structure $\mathcal{V} =< \boldsymbol{V}, 0,' >$.
For any x, let $Pd_{\leq}(x) =^{df} \{y | y \leq x\}$.

By Theorem 2, $\leq$ *is* reflexive and transitive, $Pd_{\leq}(0) = \{0\}$, and $Pd_{\leq}(x') = Pd_{\leq}(x) \cup \{x'\}$.

Definition: $\boldsymbol{M}$ is *special* just in case:
(*i*) $0 \in \boldsymbol{M}$;
(*ii*) $x \in \boldsymbol{M} \to x' \in \boldsymbol{M}$;
(*iii*) $x \in \boldsymbol{M} \ \& \ y \leq x \to y \in \boldsymbol{M}$;
(*iv*) $x \in \boldsymbol{M} \to Fin(Pd_{\leq}(x))$;
(v) $x \in \boldsymbol{M} \ \& \ x \neq 0 \to \exists y \in \boldsymbol{M}[x = y']$.

If M is special, then the substructure $< \boldsymbol{M}, 0,' >$ of $< \boldsymbol{V}, 0,' >$ is called a *special pre-N-structure.*

Theorem 7 (EFSC): There exist special pre-N-structures.
Proof: Let $\boldsymbol{M} = \{x | Fin(Pd_{\leq}(x))\ \&\ \forall u[u \leq x \rightarrow u = 0\ \vee \exists w(u = w')]\}$.
Then (i) $0 \in \boldsymbol{M}$, because $Pd_{\leq}(0) = \{0\}$ and $\forall u[u \leq 0 \rightarrow u = 0]$.
(ii) Suppose $x \in \boldsymbol{M}$. Then $x' \in \boldsymbol{M}$ because $Pd_{\leq}(x') = Pd_{\leq}(x) \cup \{x'\}$, so $Fin(Pd_{\leq}(x)) \rightarrow Fin(Pd_{\leq}(x'))$, and $\forall u[u \leq x' \rightarrow u = 0 \vee \exists w(u = w')]$ since $u \leq x' \rightarrow u \leq x \vee u = x'$.
(iii) $x \in \boldsymbol{M}\ \&\ y \leq x \rightarrow y \in \boldsymbol{M}$, for first $Pd_{\leq}(y) \subseteq Pd_{\leq}(x)$, so $Fin(Pd_{\leq}(x)) \rightarrow Fin(Pd_{\leq}(y))(cf.$ Remark 1 on the axioms, above). Now if $u \leq y$ then $u \leq x$, so $u = 0\ \vee \exists w(u = w')$.
(iv) $x \in \boldsymbol{M} \rightarrow Fin(Pd_{\leq}(x))$, by construction of M.
(v) $x \in \boldsymbol{M}\ \&\ x \neq 0 \rightarrow \exists y \in \boldsymbol{M}[x = y']$, for $x \leq x$, so $x = 0\ \vee \exists y[x = y']$, by construction of $\boldsymbol{M}$. Suppose $x \neq 0$, and let y be such that $x = y'$. Since $y \leq y' = x$, we have $y \in \boldsymbol{M}$, by (iii).

3. Existence of *N*-Structures.

The theory EFSC* will be obtained from EFSC by adding the axiom that every (truly) finite set is Dedekind finite.

Definition: *DedFin*$(\boldsymbol{X}) \equiv \forall f[f : \boldsymbol{X} \rightarrow \boldsymbol{X}\ \&\ f$ is one-one $\rightarrow$ *Ran*$(f) = \boldsymbol{X}]$, where "one-one" is defined in the usual way.

(Card) (*Cardinality Axiom*): $\forall A[$*DedFin*$(A)]$.

It follows from (Card) that $\forall \boldsymbol{X}[$*Fin*$(\boldsymbol{X}) \rightarrow$*DedFin*$(\boldsymbol{X})]$ (this is what we called "$TF \rightarrow DF''$ above), and conversely, so this is equivalent to (Card).

We define EFSC* to be EFSC + (Card).
Lemma (EFSC*): Let $< \boldsymbol{N}, 0,' >$ be a special pre-N-structure; then

(i) $\forall x \in \boldsymbol{N} \exists! \boldsymbol{Y} \subset \boldsymbol{N} \forall z \in \boldsymbol{N}[z \in \boldsymbol{Y} \leftrightarrow z \leq x]$;

(ii) (DFIS) $\forall x \in \boldsymbol{N} \forall \boldsymbol{Y} \subseteq \boldsymbol{N}[\forall z \in \boldsymbol{N}(z \in \boldsymbol{Y} \leftrightarrow z \leq x) \rightarrow DedFin(\boldsymbol{Y})]$

("DFIS" is for "*Dedekind finite initial segments*").
Proof: Immediate, by (WS-CA), the definition of "special" and (Card).

Theorem 8 (EFSC*) : Suppose $< \boldsymbol{N}, 0,' >$ is a special pre-N-structure; then it is an N-structure.
Proof: What must be shown is that $< \boldsymbol{N}, 0,' >$ satisfies Mathematical Induction, (N-III). Suppose Induction fails, *i.e.*, for some $\boldsymbol{X} \subseteq \boldsymbol{N}$ we have $0 \in \boldsymbol{X}$, $\forall y(y \in \boldsymbol{X} \rightarrow y' \in \boldsymbol{X})$ but for some z, $z \notin \boldsymbol{X}$. Introduce the predecessor function $p_{\leq z}(y)$ defined for $y \leq z$ and $y \neq 0$, *i.e.* $(p_{\leq z}(y) = u) \equiv (y \leq z\ \&\ y \neq 0\ \&\ u' = y)$. Next introduce the set,

$\boldsymbol{Y} = \{u | u \leq z\ \&\ u \notin \boldsymbol{X}\}$.

Clearly $\boldsymbol{Y}$ is Dedekind-infinite: by (Pr) (Predecessors), $p_{\leq z}$ is a 1-1 function on $\boldsymbol{Y}(0 \notin \boldsymbol{Y}$ by hypothesis that $0 \in \boldsymbol{X})$, with values in $\boldsymbol{Y}$ (by Theorem 2 (iii), and contraposing the second hypothesis of Induction), but $z \notin$ Range$(p_{\leq z})$ by stipulation of $p_{\leq z}$. It follows that $Pd_{\leq}(z) \equiv \{u | u \leq z\}$ is also Dedekind-infinite (by a general argument: define the witnessing function on $Pd_{\leq}(z)$ to be $p_{\leq z} \cup$ the identity on $Pd_{\leq}(z) - \boldsymbol{Y}$). This contradicts the Lemma (*ii*) (DFIS). *q.e.d.*[5]

The following metatheorem is now a direct consequence of Theorem 8.

Corollary: The system PA of Peano Arithmetic is interpretable in EFSC*.
Proof: Fix an N-structure, $< \boldsymbol{N}, 0, ' >$, e.g. as defined in §2 above. By Theorem 6 (§1), all primitive recursive functions are definable on $\boldsymbol{N}$, satisfying their defining equations. In particular, the recursion equations for addition and multiplication hold in $< \boldsymbol{N}, 0, ' >$. Further, any formula $\psi(x, y_1, \ldots, y_n)$ in $\mathcal{L}(PA)$ with parameters $y_1, \ldots, y_n \in \boldsymbol{N}$ defines a class, $\boldsymbol{X} = \{x | x \in \boldsymbol{N} \;\&\; \psi^{(\boldsymbol{N})}(x, y_1, \ldots, y_n)\}$, where in $\psi^{(\boldsymbol{N})}$ all (first-order) quantifiers are relativized to $\boldsymbol{N}$. Hence, by (N-III),

$$< \boldsymbol{N}, 0, ' +, \cdot > \models \forall y_1, \ldots, y_n[\psi(0, y_1, \ldots, y_n) \;\&\; \forall z(\psi(z, y_1, \ldots, y_n) \rightarrow \psi(z', y_1, ..., y_n)) \rightarrow \forall x(\psi(x, y_1, ..., y_n)].$$

This completes the proof.

4. Proof-Theoretic Strength.

Metatheorem 9: EFSC* is of the same proof-theoretic strength as PA, and it is a conservative extension of PA under the interpretation in the preceding Corollary (to Theorem 8).
Proof: By the Corollary, EFSC* is at least as strong as PA. In the other direction, EFSC* is interpretable in the second-order extension ACA_0 of PA ("ACA" stands for "arithmetic comprehension axiom and the subscript '0' indicates that induction is taken as the second- order axiom $\forall \boldsymbol{X}[0 \in \boldsymbol{X} \;\&\; \forall x(x \in \boldsymbol{X} \rightarrow x' \in \boldsymbol{X}) \rightarrow \forall x(x \in \boldsymbol{X})]$").[6] Individuals are interpreted as natural numbers, pairing is a primitive recursive function with $(x, y) \neq 0$, finite sets are interpreted as the codes by numbers of finite sets (with the primitive recursive $\in$-relation–the empty set is coded by 0, the finite non-empty set $\{x_1, ..., x_k\}$ with $x_1 < ... < x_k$ is represented by $2^{x_k} + ... + 2^{x_1}$), and classes are interpreted as sets of natural numbers in the second-order language $\mathcal{L}(\mathrm{ACA}_0)$. Under this translation, the axioms of EFSC* are derivable as theorems in ACA_0. For example, for (Card), one proves in ACA_0 that every finite set is equinumerous with an initial segment. One then proves by induction on k that every initial segment $(0, k)$ is Dedekind finite. Now it is well known that ACA_0 is a conservative extension of PA (either by a direct model-theoretic argument or by a proof-theoretic result similar to [Shoenfield 1954]). Thus, following through the above interpretation of EFSC* in ACA_0, we infer that EFSC* is conservative over PA.

Now, by [Feferman 1964 or 1968], PA is a small part of what is directly predicatively acceptable. Thus, EFSC* is what Feferman has called a *predicatively reducible* system, that is, one which can be reduced by finitary proof-theoretic methods to a directly predicatively justified system.

Remark: Reflecting on the last two theorems, we see that, although EFSC* proves the existence of a (standard) model of PA, this cannot–in virtue of the second Gödel incompleteness theorem–be converted into a consistency proof within EFSC*. Although the relativization to an N-structure of each PA axiom (and indeed PA theorem) can be proved, one by one, in EFSC*, there is no way to introduce *satisfaction* and hence no way to prove a general *soundness* theorem (to the effect that any PA theorem *holds* in any N-structure). Indeed, the introduction of *satisfaction* would require impredicative construction, beyond the power of EFSC*. (Again, the analogy with NBG set theory and ZF is a good one: NBG "proves the existence of a (standard) ZF model, $< V, \in >$ "–again in the sense of proving the relativization to V of each ZF theorem–but, without impredicative class construction, it cannot introduce satisfaction for unbounded set theoretic formulas, and, of course, cannot prove the consistency of ZF.)

We close this section with two questions and a brief discussion:

Question 1: What is the exact proof-theoretic strength of EFSC?

N.B. The proof of the categoricity theorem (§1) uses only a small part of EFSC. This part should be interpretable in the system designated by Friedman RCA_0, *i.e.* the weak fragment of ACA_0 which uses only (relative) Δ^0_1-comprehension. By Friedman's work (cf. [Simpson 1987] or [Sieg 1985]) RCA_0 is a conservative extension of PRA (primitive recursive arithmetic).

Question 2: What is a nice axiomatization of a subsystem of EFSC (*i*) in which categoricity is provable, and (*ii*) which is equivalent in strength to PRA?

Discussion: Even without claiming a strongest possible result, we have seen how the machinery of (a fragment of) weak second-order logic can be exploited to derive both the categoricity of number-theoretic axioms and the existence of arithmetical structures. In both cases, the central idea is that initial segments be finite (as in (*iii*) of Theorem 3 and condition (*iv*) in the notion of a special pre-N-structure), allowing for a predicative construction of isomorphisms (Theorems 4 and 5) and of special pre-N-structures (see the construction of $\boldsymbol{M}$ in the proof of Theorem 7), which then are proved to satisfy full (predicative) induction (Theorem 8). Clearly this procedure bears on the position of Poincaré, according to which induction plays a primitive role in our conception of the natural numbers. As the above demonstrates, however, there is a viable alternative which begins with "Dedekind-finite initial segments" and which actually allows for a "predicative logicist" construction of the natural number system.

The argument that mathematical induction necessarily involves impredicativity (given e.g. by Parsons [1983], p. 137) turns on the evident circularity of attempting to guarantee induction by introducing the natural numbers as "those objects obtainable from 0 by iterating the successor operation *an arbitrary finite number of times."* There is, however, no circularity in beginning with the notion of "finite set" as governed by the axioms of EFSC* (including the link with Dedekind–finitude), which are articulable as above, prior to the construction of the natural number system. (It was indeed one of Dedekind's principal insights [1888] that the concept of finitude *can* be introduced prior to the natural numbers.) Moreover, from the general predicativist standpoint there is independent justification for proceeding in this manner: predicativism restricts itself to domains in which every object is explicitly describable by symbolic expressions of a predicativist language; so the finite subsets of any such domain $\boldsymbol{D}$ are also explicitly describable, by means of disjunctions of the form $x = c_1 \vee x = c_2 \vee \ldots \vee x = c_n$ (where the c_i represent designators (in the predicativist language) of objects in $\boldsymbol{D}$). Again, this is conceptually independent of the natural number sequence.

It should be stressed in this connection that intelligibility of the notion "finite set" does not depend on a prior grasp of the *structure* of finite sets (of a give domain, ordered by inclusion), which is indeed a fairly complicated infinitistic object. It *is*, however, the infinitistic *structure* we call "the natural number sequence" that we seek to ground. Moreover, on a structuralist view, one can say (with Dedekind) that the individual numbers have no identity apart from position in this structure; but this contrasts with the self-standing character of "finite set" we have just noted.

It is true, however, that the weak second-order language is employed in these constructions in an essential way: if one tried to bypass it by substituting the explicitly defined notion "$\boldsymbol{X}$ is Dedekind-finite" for "$\boldsymbol{X}$ is finite"(i.e. $\exists A \forall x(x \in \boldsymbol{X} \leftrightarrow x \in A)$) throughout, one would find oneself appealing to impredicative comprehension, due to the quantifier over general functions in the definition of "Dedekind-finite".

Thus, we can see a far-reaching tradeoff between predicativism–or *predicative logicism*–and *classical logicism*. Classical logicism provides a complete analysis of the concepts "finite", "infinite", and "cardinal number", but at the price of *impredicative comprehension* with all of its attendant "metaphysical" commitments. Predicativism avoids the latter but must presuppose the concept of "finite" in some form or other. However, as the above demonstrates, it can do this in a natural way *without thereby taking the natural number system as given.* On the contrary, it can exploit its assumptions to recover the essential core of Dedekind's analysis.

Notes

[1] For a careful discussion in support of this conclusion, see Parsons [1983]. We shall comment on this below, in the final Discussion section.

[2] It is well known that a categorical theory for the natural number system can be expressed in (monadic) weak second-order logic in its semantical sense: following [Monk 1976], pp. 488-489, letting the variable 'A' range over finite sets of individuals, we can extend Robinson's system Q of arithmetic with the axiom,

$\forall x \exists A[x \in A \,\&\, \forall y(y' \in A \rightarrow y \in A]$

("Every individual belongs to a finite set closed under predecessor"). However, since finite set variables are the only higher-order variables available in this system, neither (the full statement of) mathematical induction nor categoricity can even be expressed, so it is clearly inadequate as a predicativist framework for arithmetic. (Monk's point in considering this system is to demonstrate that even such a minimal enrichment of first-order logic is non-compact and not recursively axiomatizable.)

The systems presented here extend an axiomatizable fragment of weak second-order logic so that mathematical induction, categoricity, and notions of "infinite" and "Dedekind infinite" can be expressed. It then turns out that important metatheorems are provable from within these predicatively justifiable systems.

[3]Interestingly enough, this was also Dedekind's procedure, cf. his "126. Theorem of the definition by induction" [Dedekind 1888, pp. 85-86] used to prove categoricity,

his major "132 Theorem", although indeed Dedekind did not formalize the comprehension principles he employed. Of course, Dedekind worked entirely with the notion of "Dedekind-finite", which involves quantification over general functions, and so a formal version of his proof of Theorem 126 would ultimately appeal to impredicative comprehension.

[4]There is a radical form of predicativism which does not accept the natural numbers as a "completed totality", *i.e.* over which unbounded quantification has a definite truth-functional value. This is the sense of [Nelson 1986]; Nelson's system is *much* weaker than PRA, Primitive Recursive Arithmetic, whereas PRA itself is already acceptable to finitists.

[5]This theorem together with Theorem 5 (Categoricity) realizes in effect a suggestion attributed to Michael Dummett for characterizing the natural numbers in a predicatively acceptable way, as those individuals x belonging to every class containing 0 and closed under successor applied to individuals distinct from x (i.e., $\forall \boldsymbol{X}[0 \in \boldsymbol{X} \; \& \; \forall y(y \in \boldsymbol{X} \; \& \; y \neq x \rightarrow y' \in \boldsymbol{X}) \rightarrow x \in \boldsymbol{X}]$, which specifies the initial segment inclusively up to x), a construction which works as well when the class variable is replaced throughout with a finite-set variable. For a discussion, see [Isaacson 1987], pp. 155-6.

Clearly we disagree with Isaacson's diagnosis of the situation, that "the weak second-order definition does not fare significantly better on the score of avoiding impredicativity than the one based on full second-order logic." (*p.* 156) The reason given is that "an exact representation of the natural number sequence must occur as elements of the domain [of the second-order quantifiers]." But, as maintained here, this assumption–of the existence of finite initial segments (of isomorphisms between ω-sequences (cf. Theorem 3, (*iii*)), as well as of the individual ω-sequences)–is clearly predicatively justified. Moreover, there is a vast difference between presupposing infinitely many finite sets and presupposing an infinite one, especially the very one you are trying to "introduce"! The predicativist can afford to talk like a platonist about (hereditarily) finite sets, but not about the infinite.

[6] For information on ACA_0 and related subsystems of analysis (of a hierarchy explored by Harvey Friedman and others), see *e.g.* [Simpson 1987].

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