WHAT'S DEFINITE? WHAT'S NOT?

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Is the Totality of All Sets an Indefinite Totality?

- Definite totalities are set-like. If definite totalities are sets then the totality of all sets is indefinite (Russell).
- Zermelo (1930) seems to view the totality of all sets in this way: "[T]he transfinite number series...in its unrestricted progression features no real conclusion, but only relative stopping points."
- Dummett: The concept of set is "indefinitely extensible" (1963 and on).
- Similar considerations for finitism, predicativity.

Some Recent Philosophical Literature

- Hellman, "Maximality vs. Extendability: Reflections on structuralism and set theory" (2002)
- Shapiro, "All sets great and small: And I do mean All" (2003)
- Linnebo, "Sets, properties and unrestricted quantification" (2005)
- Hellman, "Against 'absolutely everything'!" (forthcoming)

A Formal Distinction Between Definite and Indefinite Concepts

- "What's definite is the domain of classical logic, what's not is that of intuitionistic logic."
- In the case of predicativity, consider systems in which quantification over natural numbers is governed by classical logic, while quantification over sets of natural numbers (and sets more generally) is governed by intuitionistic logic.
- In the 1970s, I used such systems as intermediate tools in my work applying functional interpretation with non-constructive operators.

A Formal Distinction (Continued)

- In the case of set theory, where every set is conceived to be a definite totality, but the universe of sets is an indefinite totality, accept classical logic for bounded quantification while use intuitionistic logic for unbounded quantification.
- Some early case studies on relatively strong semi-intuitionistic subsystems of ZF: Poszgay (1971, 1972), Tharp (1971), Friedman (1973), Wolf (1974);
- and on a relatively weak system: Friedman (1980), "A strong conservative extension of Peano Arithmetic" (the system ALPO).

A General Pattern for Studies

- Start with a system S formulated in fully classical logic, and consider an associated system SI formulated in a mixed, semi-intuitionistic logic.
- Ask whether there is any essential loss in proof-theoretical strength when passing from S to SI.
- In the cases that are studied, it turns out that there is no such loss.

A General Pattern (Continued)

- But there can be an advantage in going to such a semi-intuitionistic system SI:
- Namely, we can beef it up to a semi-constructive system SC without changing the proof-theoretical strength from that of S (the original classical system), by the adjunction of certain principles that go beyond what is admitted in SI.

The Case of Admissible Set Theory

- Start with S = KPω, the classical system of admissible set theory (including the Axiom of Infinity)
- SI has the same axioms as KPω, but is based on intuitionistic logic plus the Law of Excluded Middle for bounded formulas,
- (Δ_0-LEM) $\varphi \vee \neg\varphi$, for all Δ_0 formulas φ.
- SI = IKPω + (Δ_0-LEM)

A Semi-Constructive System of Admissible Set Theory

- Beef up SI to a system SC that includes the Full Axiom of Choice Scheme for sets (AC_{Set}),

$$\forall x \in a \exists y\ \varphi(x,y) \rightarrow \exists r[\mathrm{Fun}(r) \wedge \mathrm{dom}(r)=a \wedge \forall x \in a\ \varphi(x,r(x))]$$

for φ an arbitrary formula,

- Then SC proves the Full Collection Axiom Scheme,

$$\forall x \in a \exists y\ \varphi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b\ \varphi(x,y),$$ for φ

arbitrary, while this holds only for Σ_1 formulas in SI.

Some Other Principles for SC

- Bounded Omniscience Scheme(BOS),

$(\forall x\in a)[\varphi(x)\vee\neg\varphi(x)] \rightarrow$

$(\forall x\in a)\varphi(x)\vee (\exists x\in a)\neg\varphi(x)$, for *all* formulas $\varphi(x)$.

- Markov's Principle (MP),
  $\neg\neg\exists x\ \varphi \rightarrow \exists x\ \varphi$, for all Δ_0 formulas φ.

- Independence of Premises (IP),
  $(\forall x\ \varphi\rightarrow\exists y\ \psi) \rightarrow \exists y(\forall x\ \varphi\rightarrow\psi)$,
  for all Δ_0 φ, ψ.

Axioms of KPω

1. Extensionality
2. Unordered pair
3. Union
4. Infinity
5. Δ_0-Separation
6. Δ_0-Collection
7. The $\in$-Induction Axiom Scheme

An Intermediate Reduction

- SI = IKPω + (Δ_0-LEM)
- **Theorem.** KPω $\leq$ SI + (MP)
- **Proof.** By adaptation of the Gödel-Gentzen Negative or "double-negation" interpretation. Use (Δ_0-LEM) + (MP) to take care of the Δ_0-Collection Axiom, where φ is a Δ_0 formula:

$$(\forall x \in a)\neg\neg\exists y\ \varphi(x, y) \to \neg\neg\exists b(\forall x \in a)(\exists y \in b)\varphi(x, y).$$

The Semi-Constructive System

- Take SC = SI +(AC_{Set})+(BOS)+(MP)+(IP)
- SC proves Full Collection and Full Replacement
- To prove SC $\leq$ KPω, will pass through an intermediate functional finite type extension FSC$\uparrow$ via an adaptation of Gödel's *Dialectica* (D-)interpretation
- For simplicity, will only present the type 1 (over sets) part FSC of FSC$\uparrow$ with some type 2 operators.

The Language of FSC

- FSC has both set variables a, b, c, x, y, z,... (variables of type V) and function variables f, g, h,… (variables of type $V \to V$).
- FSC has constants of various types, to begin with the set constants 0 and ω,
- and "logical operation" constants E, M, D, N, C, which serve to reduce every Δ_0 formula to an equation and prove Δ_0-LEM.

The "Logical" Axioms of FSC

1. (Atomic decidability) $x=y \vee x \neq y$

2. (Equality) $E(x, y)=0 \leftrightarrow x=y$

3. (Membership) $M(x, y)=0 \leftrightarrow x \in y$

4. (Disjunction) $D(x, y)=0 \leftrightarrow x=0 \vee y=0$

5. (Negation) $N(x)=0 \leftrightarrow x \neq 0$

6. (Bounded choice)

$$(\exists x \in a) f(x)=0 \leftrightarrow C(a,f) \in a \wedge f(C(a,f))=0.$$

First Consequence

- **Lemma 1**. For each Δ_0 formula $\varphi(\underline{x})$ of set theory with at most $\underline{x} = x_1,\ldots,x_n$ free we have a term t_φ such that the following is provable in FSC: $t_\varphi[\underline{x}] = 0 \leftrightarrow \varphi(\underline{x})$.

“Semi-logical” axioms of FSC

- Markov’s Principle (MP)
  $\neg\neg\exists x\ f(x)=0 \rightarrow \exists x\ f(x)=0.$

- Independence of Premises (IP)
  $[\forall x\ f(x)=0 \rightarrow \exists y \forall z\ g(x,y)=0] \rightarrow$
  $\exists y[\forall x\ f(x)=0 \rightarrow \forall z\ g(x,y)=0]$

- Axiom of Choice for Functions (AC_{Fun})
  $\forall x \exists y\ \varphi(x, y) \rightarrow \exists f\ \forall x\ \varphi(x, f(x))$, for all $\varphi(x, y)$.

Set-theoretical Constants and Axioms of FSC

- Extensionality
- Axioms for 0, ω, P (Unordered pair), U (Union)
- S (Separation), with axiom
  $x \in S(a, f) \leftrightarrow x \in a \wedge fx=0$
- R (Range), with axiom $y \in R(a, f) \leftrightarrow (\exists x \in a)\ f(x)=y$
- ($\in$−Induction)
  $\forall x[(\forall y \in x)\varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x\ \varphi(x)$, for all $\varphi(x)$.

Cartesian Product

- Notation: $\{x, y\} = P(x, y)$, $\{x\} = \{x, x\}$, $x \cup y = \bigcup\{x,y\}$, $x' = x \cup\{x\}$, $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

- **Lemma** 2. There is a closed term Prod such that FSC proves $z \in \underline{\text{Prod}}(a, b) \leftrightarrow \exists x, y[z =\langle x, y\rangle \wedge x\in a \wedge y\in b]$. Proof.

  $\underline{\text{Prod}}(a, b)$ $(= a \times b) = \bigcup(R(a, f))$ where
  $f = \lambda x.\, R(b, \lambda y.\langle x, y\rangle)$ $(= \lambda x.\, \{x\} \times b)$

Function Restriction

Corollary. There is a closed term <u>Res</u> such that FSC proves

$z \in \underline{Res}(a, f) \leftrightarrow (\exists x \in a)(\exists y \in R(a, f)[z = \langle x, y \rangle \wedge fx = y]$.

Proof. <u>Res</u>(a, f) is formed by Δ_0-Separation (S) from the Cartesian product $a \times R(a, f)$. It is the graph of f restricted to a, considered as a set.

<u>Notation</u>: $f|a$ for <u>Res</u>(a, f).

Lemma 3. The system SC is contained in FSC.

- **Proof.** Δ_0-LEM follows from Lemma 1 and decidability of =.
- Δ_0-Separation and Δ_0-Collection follow from Lemma 1 together with the use of the S and R operators, resp.
- MP (IP) for Δ_0 formulas follows from Lemma 1 by use of MP (IP) for functions.
- (AC_{Set}) follows from (AC_{Fun}) by using the restriction operation $f|a$.

Lemma 3 Proof (Concluded)

- To obtain the Bounded Omniscience Scheme (BOS), suppose $\forall x \in a[\varphi(x) \vee \neg\varphi(x)]$; then $\forall x \in a \exists y[(y=1 \wedge \varphi(x)) \vee (y=0 \wedge \neg\varphi(x))]$,

  so $\exists f(\forall x \in a)\{[f(x)=0 \vee f(x)=] \wedge [f(x)=1 \leftrightarrow \varphi(x)]\}$.

- $(\exists x \in a)f(x)=0 \leftrightarrow C(a,f) \in a \wedge f(C(a,f))=0$

  by Bounded Choice, and that's decidable,

- so we have $(\forall x \in a)\varphi(x) \vee (\exists x \in a)\neg\varphi(x)$.

The System FSC$\uparrow$

- FSC$\uparrow$ has functional variables of every finite type over V.
- It uses only bounded quantifiers.
- Its "logical" constants and axioms are the same as for SC.
- Its set-theoretical constants (0, ω, P, U, S, R) and axioms are the same as for SC.
- It also has an $\in-$Induction Rule and Recursors in all finite types.

FSC and FSC↑

- <u>NB</u>. FSC↑ does *not* have MP, IP or AC_{Fun} .
- **Theorem 1.** FSC has a D-(*Dialectica* form) interpretation in FSC↑.
- The proof is in my forthcoming paper: "On the strength of some semi-constructive theories," for the Grigori Mints Festschrift in honor of his 70th birthday, June 7, 2009.
- Related results are given there for semi-constructive predicative theories and countable tree ordinals.

Closing the Circle

- **Theorem 2.** We have the following proof-theoretical reductions:

$$\mathrm{KP}\omega \leq \mathrm{SC} \leq \mathrm{FSC} \leq \mathrm{FSC}\!\uparrow\, \leq \mathrm{OST} \leq \mathrm{KP}\omega.$$

[OST is the system from my WoLLIC ’06 paper

“Operational Set Theory and Small Large Cardinals”

to appear in *Information and Computation*.

The last ≤ is proved there, and also by Jäger in

Annals of Pure and Applied Logic 150 (2007).]

Adding the Power Set Axiom

- Let Pow be the axiom $\forall a \exists b \forall x (x \in b \leftrightarrow x \subseteq a)$ in SC.
- In FSC and FSC↑, the axiom Pow, with a new constant symbol $\mathcal{P}$, is written $x \in \mathcal{P}(a) \leftrightarrow x \subseteq a$.
- Pow(ω) is the special case of Pow: $x \in \mathcal{P}(\omega) \leftrightarrow x \subseteq \omega$.

On the Strength of Semi-Constructive Systems with Pow

- **Theorem 3.** We have the following proof-theoretical reductions:

$KP\omega + (Pow) \leq SC+(Pow) \leq FSC+(Pow) \leq$

$FSC\!\uparrow\! +(Pow) \leq OST+(Pow) \leq KP\omega+(Pow)+(V=L)$.

The same holds when we replace (Pow) by the special case, Pow(ω).

(The strength of systems related to SC+(Pow) have been studied by Wolf (1974), Stanford PhD thesis.)

On the Strength with Pow (cont'd)

- The proof of Theorem 3 proceeds along exactly the same lines as for Theorem 2, through the D-interpretation of FSC+(Pow) in FSC↑+(Pow) followed by the interpretation of the latter in OST+(Pow).
- The final reduction, OST+(Pow) $\leq$ KPω+(Pow)+(V=L), is due to Jäger in *APAL* 150 (2007).
- Is KPω+(Pow)+(V=L) $\leq$ KPω+(Pow)?
(The usual argument doesn't work.)

What Properties are Definite?

- From the overall logical point of view taken here, $\varphi(x)$ is formally definite if we have $\forall x[\varphi(x) \vee \neg\varphi(x)]$.
- But looked at more particularly within the kind of framework provided by FSC, using (AC), $\varphi(x)$ is definite just in case $\exists f\ \forall x\ [f(x) = 0 \leftrightarrow \varphi(x)]$.
- By Lemma 1, all Δ_0 formulas are formally definite. But could there be more such formulas?

Definite Properties from the Model-Theoretic Point of View

- A formula $\varphi(x)$ in the language of set theory is model-theoretically definite relative to an axiom system T if φ is invariant under end-extensions in models of T.

- **Theorem** (Feferman 1968) This holds just in case there are an essentially Σ_1 formula $\psi(x)$ and essentially Π_1 formula $\theta(x)$ such that

(†) $T \vdash \forall x(\psi(x) \leftrightarrow \theta(x))$ and $T \vdash \forall x(\varphi(x) \leftrightarrow \psi(x))$.

The View from FSC

- **Theorem 4.** If $\varphi(x)$ satisfies (†) for T = FSC then FSC $\vdash \forall x[\varphi(x) \vee \neg\varphi(x)]$.
- **Proof.** By Lemma 1 and AC, there are terms $s(x,y)$ and $t(x,y)$ such that FSC proves
(††) $\forall x[(\exists y)s(x, y)=0 \leftrightarrow (\forall z)t(x, z)=0]$ and

  $\forall x[\varphi(x) \leftrightarrow (\exists y)s(x, y)=0]$.

The Proof of Theorem 4 (cont'd)

Apply IP to $\forall x[(\forall z)t(x, z)=0 \rightarrow (\exists y)s(x, y)=0]$

to obtain $\forall x \exists y[(\forall z)t(x, z)=0 \rightarrow s(x, y)=0]$.

So by AC, there exists a function f such that

$$\forall x[(\forall z)t(x, z)=0 \rightarrow s(x, f(x))=0].$$

We also have $\forall x[(\exists y)s(x, y)=0 \rightarrow (\forall z)t(x, z)=0]$, so

$\forall x[(\exists y)s(x, y)=0 \rightarrow s(x, f(x))= 0]$. Hence $\varphi(x)$ is

equivalent to $s(x, f(x))=0$, and so is formally definite.

Definite Predicates and $\in$−Induction

- Slogan: Definite predicates are those that have a characteristic function.
- Since Separation is supposed to be restricted to definite predicates, shouldn't we do the same with $\in$−Induction?
- That is, shouldn't we restrict it to
  $(I_{Fun})\quad \forall x[(\forall y \in x) f(y)=0 \rightarrow f(x)=0] \rightarrow \forall x[f(x)=0]$?
- FSC_0 is FSC restricted to (I_{Fun}); SC_0 is SC with the $\in$−Induction scheme restricted to Δ_0 formulas.

Friedman’s ALPO

- Friedman, “Analysis based on the Limited Principle of Omniscience,” *The Kleene Symposium* (1980).
- ALPO is a semi-constructive system whose overall logic is intuitionistic.
- It is a fragment of KPU in which the urelements are taken to be the natural numbers equipped with 0 and successor.

The Axioms of ALPO

A. Ontological axioms, B. Urelement extensionality,

C. Successor axioms, D. Infinity, E. Sequential induction,

F. Sequential recursion, G. Pairing, H. Union,

I. Exponentiation, J. Countable choice,

K. Δ_0-Separation, L. Strong Collection, and

M. Limited principle of omniscience.

Friedman's Conservation Theorem

Theorem (Friedman, 1980). ALPO is a conservative extension of PA.

Remark: Friedman's proof makes use of a series of reductions, the last part of which appeals to a model-theoretic argument from his earlier "Set theoretic foundations for constructive analysis", *Annals of Mathematics* 105 (1977).

Conjectures

1. ALPO is proof-theoretically reducible to PA.
2. SC_0 and FSC_0 are proof-theoretically reducible to PA. [Note that ALPO w/o Exp axiom $\subseteq$ SC_0 .]
3. These systems are proof-theoretically reducible to PA when Exp is added but AC is restricted to countable AC . [Full AC plus Exp allows derivation of arithmetical DC.]

Semi-Constructive Mathematics

- How much mathematics can be carried out in SC, SC + Pow(ω), SC + Pow, etc.?
- SC + Pow(ω) looks like an appropriate setting for representing the work of the French school of semi-intuitionists.

What Statements are Definite?

- φ is formally definite in one of our systems if $\varphi \vee \neg\varphi$ is provable there.
- Is the Continuum Hypothesis (CH) definite?
- CH is expressible in SC + Pow(ω) but probably not formally definite there. (How prove?)
  It *is* formally definite in SC + Pow(Pow(ω)).
- Formal definiteness is a very crude criterion of definiteness. Need more refined notions of definiteness/indefiniteness to throw light on whether CH is a definite statement.

The Problem of Large Cardinal Axioms in an Indefinitely Extendible Universe

- What justification, if any, could be given for reflection principles (first order, higher order) in semi-constructive set theories (± Pow)?
- What about stronger "small" large cardinal axioms?
- Is there any place for "large" large cardinal axioms in these theories?

The End