What kind of logic is "Independence Friendly" logic?

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1. Two kinds of logic. To a first approximation there are two main kinds of pursuit in logic. The first is the traditional one going back two millennia, concerned with characterizing the logically valid inferences. The second is the one that emerged most systematically only in the twentieth century, concerned with the semantics of logical operations. In the view of modern, model-theoretical eyes, the first requires the second, but not vice-versa. According to Tarski's generally accepted account of logical consequence (1936), inference from some statements as hypotheses to a statement as conclusion is logically valid if the truth of the hypotheses ensures the truth of the conclusion, in a way that depends only on the form of the statements involved, not on their content. Interpreted model-theoretically this means that every model of the hypotheses is a model of the conclusion. However, there is an ambiguity in Tarski's explication, as he himself emphasized, since for the specification of form one needs to determine what are the logical notions. Once those are isolated and their semantical roles are settled, one can see how the truth of a statement (in a given model and relative to given assignments) is composed from the truth of its basic parts, in whatever way those are specified. The problem of what are the logical notions is an unsettled and controversial one (cf. Feferman 1999, Gómez-Torrente 2002). In the classical truth-functional perspective, proposals range from those of first-order logic to generalized quantifiers to second and higher-order quantifiers to infinitary languages and beyond. Many of these stronger semantical notions have been treated in the volume *Model Theoretic Logics* (Barwise and Feferman 1987).

In a series of singular, thought-provoking publications in recent years, Jaakko Hintikka has vigorously promoted consideration of an extension of first-order logic called IF logic, along with claims that its adoption promises to have revolutionary consequences. My main purpose here is to examine in what sense it deserves to be called a logic. On the face of it, IF logic fits squarely into the semantic approach, but I shall argue both that the fit is problematic, and that the neglect of the inferential aspect of logic in its use constitutes a serious defect. Along the way, I shall raise concerns of a philosophical nature concerning its underlying semantics.

2. IF logic. The primary reference that I shall use in the following for Hintikka's work on IF logic is *The Principles of Mathematics Revisited* (Hintikka 1996, referred to below as *PMR*). The basic idea is very simple, and is sketched here in order to make this piece reasonably self-contained.

When sentences of the first-order predicate calculus are put in prenex normal form, the usual semantics makes each existential quantified variable y dependent on all the universally quantified variables in whose scope it lies, i. e. which precede it in the prefix. This dependence is made explicit in the language of Skolem functions, whose use in general in the semantics requires the Axiom of Choice. To illustrate, if a sentence S takes the following prenex form,

(1) $\forall x\ \exists y\ \forall z\ \exists u\ R(x, y, z, u)$

where R is the quantifier-free matrix, its Skolem equivalent is of the form

(2) $\exists f\ \exists g\ \forall x\ \forall z\ R(x, f(x), z, g(x, z))$.

In the language of game theoretic semantics, S is true in a given structure $\underline{M} = \langle M, ...\rangle$ just in case V ("Verifier") has a winning strategy (f, g) in the associated evaluation game (cf. van Benthem, this volume). At each move by V in this game over M the choice of next move is based on complete information about the preceding moves by F ("Falsifier") who first chooses an $a \in M$ to which V responds with a choice of some $b \in M$; following that F chooses a $c \in M$, to which V responds finally with a choice of some $d \in M$. For this sequence of choices, V wins if R(a, b, c, d) is true in $\underline{M}$, otherwise F wins.

Motivated in part by games with imperfect information, Hintikka and Sandu (1989) proposed consideration of semantic games where V's choices do not depend on all (or, indeed, any) of F's prior choices. In the case of (1) above, this leads to the following possible independencies from earlier universally quantified variables: (i) y can be independent of x and (ii) u can be independent of the variables in a subset of {x, z}, for example it can be independent of z. These particular independence relations are indicated by:

(3) $\forall x\ \exists y/\forall x\ \forall z\ \exists u/\forall z\ R(x, y, z, u)$

or equivalently,

(4) $\exists y\ \exists g\ \forall x\ \forall z\ R(x, y, z, g(x))$.

Since this depends on *declarations of independence*, Hintikka called the resulting semantics of sentences such as (3), *Independence Friendly Logic,* or *IF logic* for short.[i] More recently, Hintikka (2002) has proposed to call this *Hyperclassical Logic* instead, but I shall follow the earlier designation since it is more suggestive of the basic idea involved, and also because it was used in all the prior publications on the subject. One difference here: in *PMR* Hintikka refers to this consistently as *IF first-order logic*, but since that is tendentious and the main bone of contention in this piece--see below--I will omit the "first-order" part except when quoting directly. Even the use of "logic" may be considered tendentious, in view of my arguments below, but I shall at least follow Hintikka in retaining that part of the name for the subject.

Another source of motivation for IF logic is the study of branching quantifiers, introduced by Henkin and pursued by Walkoe (1970), Enderton (1970) and Barwise (1979), among others. The paradigm example is given by

(5) $$\left.\begin{matrix}\forall x\exists y\\ \\ \forall z\exists u\end{matrix}\right\} R(x,y,z,u)$$

where y depends only on x and u only on z. Such quantifiers are subsumed by IF logic in terms of the independence notation, e. g. as

(6) $\forall x\ \forall z\ \exists y/\forall z\ \exists u/\forall x\ R(x, y, z, u)$.

Because ordinary first-order logic lacks the capacity to indicate such relations of independence between bound variables, Hintikka calls it *dependence handicapped* or *independence challenged* (Hintikka 2002, p. 408). By comparison, he asserts that

> [u]nder any term [IF logic] is the general unrestricted first-order logic. Some philosophers have been so blindly committed to the "ranging over" idea as the whole truth about quantifiers that where this idea fails, as it fails in IF logic, they have jumped to the conclusion that such a logic must somehow be higher-order. This is nonsense by their own criteria, for the only reasonable way of making the first-order vs. higher-order distinction is in terms of the entities one's quantified

variables range over. And by this criterion IF logic first-order logic is indeed first-order. (Hintikka 2002, p. 409).

This remarkable claim cannot go unchallenged.[ii]

Before getting into these issues, I need to introduce some more precise terminology concerning IF logic, in this respect following *PMR* only in part. As Hintikka explains, the logic in general applies to sentences in negation normal form (i.e., in which negation appears at most applied to atomic formulas) that are built up by means of the operations $\wedge$, $\vee/\forall\underline{x}$, $\forall u$ and $\exists y/\forall\underline{x}$, under the following restrictions: $\underline{x}$ is a sequence of variables, u and y are any variables, y is not in $\underline{x}$, and each $\vee/\forall\underline{x}$ and $\exists y/\forall\underline{x}$ is in the (eventual) scope of all the $\forall x_i$ for each x_i in $\underline{x}$. (Slashes are dropped if the sequence $\underline{x}$ is empty.) Formulas generated along the way are called *IF-formulas*, and those without free variables are called *IF-sentences*. The semantics of the "slashed" disjunctions is illustrated by the case of an IF-sentence of the form

(7) $$\forall x\forall z\, [A \vee/\forall z\, B],$$

where A and B are IF-formulas. Given a distinguished constant 0, to show independence of the disjunction from the variable z, (7) is taken to hold just in case

(8) $$\exists f\forall x\forall z[\, (f(x) = 0 \wedge A) \vee (f(x) \neq 0 \wedge B)\,].$$

This has the same truth conditions as the IF sentence,

(9) $$\forall x\forall z\exists u/\forall z[\, (u = 0 \wedge A) \vee (u \neq 0 \wedge B)\,].$$

In view of these equivalents, for simplicity we ignore slashed disjunctions in the following, and take IF-sentences to be built up from atomic formulas and their negations by $\wedge$, $\vee$, $\forall u$ and $\exists y/\forall\underline{x}$, under the restrictions on variables given above. Every IF-sentence can be brought as usual to a prenex normal form in which there is an initial quantifier prefix consisting of quantifiers of the form $\forall u$ and $\exists y/\forall\underline{x}$, in which the latter occur within the scope of earlier $\forall x_i$ for each x_i in $\underline{x}$; the matrix of such a formula is quantifier-free. These are called here *prenex IF-sentences*; the initial sequence of quantifiers is called the

IF-quantifier prefix of such a sentence. The *Skolem form* of a prenex IF-sentence S is of the form

(10) $\exists \underline{f} \forall \underline{u}\, R(\underline{f}, \underline{u})$,

where $\underline{f}$ is a sequence of function variables f_i of various numbers of arguments (possibly zero); each f_i is associated with a (possibly) slashed existential quantifier $\exists y_i/\forall \underline{x}^{(i)}$ in the original IF-quantifier prefix of S and each occurrence of that y_i in the matrix of S is replaced by $f_i(\underline{w}^{(i)})$, where $\underline{w}^{(i)}$ is the list of variables in $\underline{u}$ other than those in $\underline{x}^{(i)}$.

Sentences of the form (10) are said to be in Σ^1_1-form. Walkoe (1970) and Enderton (1970) showed how to associate with every Σ^1_1 sentence a prenex IF-sentence to whose Skolem form it is equivalent, in the sense that they are true in the same models. When the Skolem form of an IF-sentence S is satisfied, the realization of the function quantifiers encodes the winning strategy for Verifier in the (possibly) imperfect information game for S. Such semantics is seemingly "top-down" or "from the outside in", in contrast to usual model-theoretic Tarskian style semantics which is "bottom up" or "from the inside out", i.e. is *compositional*. On the face of it--as Hintikka repeatedly stresses (and argues as a virtue)--compositional semantics is *not* in general available for IF-sentences built up from IF-formulas. For, without the universal quantification of the variables in $\underline{x}$ preceding a slashed existential quantifier $\exists y/\forall \underline{x}$, no explanation of the semantics for the latter can be given by a recursive definition of satisfaction in the usual way. However, as has been shown by Hodges (1997), (1997a) *there is a perfectly reasonable compositional semantics for IF-formulas;* this is obtained by taking the satisfying objects to be *sets* of sequences of individuals, rather than sequences of individuals in the ordinary way following Tarski. Hodges' work has been extended by Väänänen (2002) to show that the semantics of IF-formulas can be treated in terms of suitable *games of perfect information*.

3. Generalized first-order logical operations. Let us follow up the assertion that "the only reasonable way of making the first-order vs. higher-order distinction is in terms of the entities one's quantified variables range over," quoted from Hintikka (2002) above. Syntactically, a generalized first-order logical operation O applies to predicates of individual variables $P_1,\ldots,P_k$ of $n_1,\ldots,n_k$ arguments respectively. As defined by Lindström (1966), the semantics for such an operation can be specified by a collection K of relational structures $\underline{M} = \langle M, R_1,\ldots,R_k\rangle$ closed under isomorphism, in which the domain M is non-empty and each R_i is an n_i -ary relation between elements of M. This determines, as follows, an operation $O_K\underline{x}^{(1)},...,\underline{x}^{(k)}$ $(P_1(\underline{x}^{(1)}),...,P_k(\underline{x}^{(k)}))$ where $\underline{x}^{(i)}$ is a sequence of n_i distinct variables and $\underline{x}^{(i)}$ is disjoint from $\underline{x}^{(j)}$ when i and j are different. For any structure $\underline{M} = \langle M, R_1,...,R_k\rangle$ providing an interpretation of each P_i by an n_i-ary relation R_i between elements of M, $O_K\underline{x}^{(1)},...,\underline{x}^{(k)}$ $(P_1(\underline{x}^{(1)}),...,P_k(\underline{x}^{(k)}))$ is satisfied in $\underline{M}$ if and only if $\underline{M}$ is in K.

We call such O_K *generalized quantifiers*; the usual quantifiers $\forall$ and $\exists$ can be treated as special cases by taking K to be the class of all $\langle M, R\rangle$ with $R \subseteq M$ such that $R = M$, and $R \neq \emptyset$, respectively. By allowing the n_i to be 0, all the usual propositional operations also fall out as generalized quantifiers in this sense. Further familiar examples are determined by the following classes of structures:

(1) For any infinite cardinal number κ, let K be the class of all $\langle M, R\rangle$ with $R \subseteq M$ and $\mathrm{card}(R) \geq \kappa$. Then $O_Kx\ (P(x))$ expresses that there are at least κ x's such that P(x). This operation O_K is usually denoted $\exists_{\geq\kappa}$.

(2) Let K be the class of all $\langle M, R_1, R_2\rangle$ with $R_1 \subseteq M$ and $R_2 \subseteq M$ such that $\mathrm{card}(R_1) \geq \mathrm{card}(R_2)$. Then $O_Kx,y\ (P(x), Q(y))$ expresses that there are at least as many x's such that P(x) as there are y's such that Q(y).

(3) Let K be the class of all $\langle M, R\rangle$ with $R \subseteq M^2$, such that $\exists f \forall n\ [\ (f(n + 1), f(n)) \in R\ ]$. Then $O_Kx,y\ (P(x, y))$ expresses that the relation determined by P is not well-founded.

It is evident from these examples that though the Lindström generalized quantifiers are *syntactically first-order* insofar as the quantified variables are first-order, they may be *semantically higher-order*. Indeed this is the case for the operations determined by (1)-(3), since the notions of cardinality and well-foundedness are essentially higher order concepts, requiring either implicitly or explicitly quantification over arbitrary functions.

The IF-quantifier prefixes may be used to determine generalized quantifiers in Lindström's sense. For example, with (3) and (4) of the preceding section in mind, the prefix $\forall x\ \exists y/\forall x\ \forall z\ \exists u/\forall z$ may be considered to be the quantifier O_K, where K is the class of structures $\langle M, R\rangle$ with $R \subseteq M^4$ and

(4) $$\exists y\ \exists g\ \forall x\ \forall z\ [\ (x, y, z, g(x)) \in R\ ].$$

As such, the IF quantifier prefixes are semantically no more first-order than the generalized quantifier (3) above, expressing non-well-foundedness. The issue as to whether IF logic deserves to be called a first-order logic is pursued further in the next section.

Though IF quantifiers can be subsumed under generalized quantifiers in the way just explained, there are some obvious differences between Hintikka's conception of the former and the usual treatment of the latter. Namely, generalized quantifiers can be compounded unrestrictedly with themselves and with other quantifiers and the usual classical propositional operations. For example, taking κ to be the least uncountable cardinal $\aleph_1$, we can express that there are only countably many P's by forming $\neg\exists_{\geq\kappa}x\ (P(x))$, and we can express that there are uncountably many P's or uncountably many Q's by $\exists_{\geq\kappa}x\ (P(x)) \vee \exists_{\geq\kappa}\ y\ (Q(y))$; finally, for example, $\exists x\ \exists_{\geq\kappa}\ y\ P(x, y)$ expresses that for some x there are uncountably many y for which P(x, y). Once the semantics of the quantifier $\exists_{\geq\kappa}$ is specified as in (1) above, the truth conditions of such compounds is determined compositionally.

By contrast, IF operations as given by quantifier prefixes are not compounded with themselves or other operations except in a limited sense, and only to the extent that they can be treated via game-theoretic semantics (allowing imperfect information), for example by taking the conjunction or disjunction of two IF-sentences, brought to a common prenex form. The prime example of an operation which cannot be so treated is that of classical or "contradictory" negation ($\neg$), where $\neg$S is true if and only if S is not true. The operation of contradictory negation does not in general take an IF-sentence to another IF-sentence (up to equivalence); that holds only for first-order sentences S in the ordinary sense (cf. *PMR*, p. 133). Rather a new "dual negation" operation ~S is introduced by Hintikka (*PMR*, Ch. 7), whose semantics is given by the game dual to that for S, i.e. ~S is true if Falsifier has a winning strategy in the game associated with S. We do not in general have S $\vee$ ~S valid since neither Verifier nor Falsifier may have a winning strategy, as is commonly illustrated by the case of the IF-sentence $\forall x \exists y/\forall x\ (x = y)$ when tested in a domain of more than one element. Similarly there are two operations of conditional to be considered in application to IF-formulas S and S′, the one being the classical $S \rightarrow S'$ [denoted $S \supset_T S'$ by Hintikka] whose semantics is the same as that of

$\neg S \vee S'$, while the other, denoted $S \supset S'$, is defined as $\sim S \vee S'$; the latter has game theoretic semantics since the disjunction of two IF formulas can be treated as a single IF formula.

Sentences S^* of the form $\neg S$ for S an IF-sentence can be brought to Π^1_1 form and conversely. When Hintikka considers the expressive power of IF sentences or their negations in this sense, he refers to it as *extended IF logic*. Allowing full compounding of IF-sentences with $\neg$, $\wedge$ and $\vee$ leads to what he calls *truth-functionally extended IF-logic*. As is easily seen, such sentences can be brought to Δ^1_2 form, i.e. are equivalent to sentences in both Σ^1_2 and Π^1_2 form. This is a non-trivial part of full second-order logic; the relation to that is examined more closely in the next section.

Another difference of IF logic from the semantics of generalized quantifiers is that it is not informative to speak of the logic of the latter as a whole. Rather, what is of interest is the logic of one or a few such specific quantifiers considered in combination with the connectives and quantifiers of ordinary first-order logic. This is illustrated by the work descending from Mostowski (1957) in which the center of attention is the logic of $\exists_{\geq \kappa}$ in that sense, for various cardinal numbers κ. For example, as shown by Keisler (1970), the logic of "there exist uncountably many" shares many good properties with usual first-order logic, including a completeness theorem for validity (which happens not to be the case for the logic of "there exist infinitely many"). By contrast, IF logic is simply the logic of *arbitrary* IF quantifier prefixes and their relations to each other.

Thus, though IF logic shares with the logic of generalized quantifiers a model-theoretic perspective, the concerns in most respects are orthogonal to each other. In *PMR*, Ch. 1, in opposition to the traditional *deductive,* inferential function of logic, Hintikka identifies model theory with its *descriptive* function, i.e. with what structures can be described or characterized in terms of given sentences. To be sure, the fundamental relation of model theory is that of *satisfaction*, $\underline{M} \models S$, between structures $\underline{M}$ of a specified kind and sentences S from a specified language L. But that is only the beginning, as any text in model theory reveals (cf., e.g., the classic Chang and Keisler (1990) and the more recent Hodges (1993)). In general, one wants to know--given a set Γ of sentences--what is the class K of models of all S in Γ, and inversely--given a class K of structures--what is the theory of K, i.e. the set Γ of all sentences satisfied in all members of K. The first problem implicitly involves the relation of *logical consequence* in Tarski's model-theoretic sense, since every model of Γ is a model of all consequences of Γ. In other words, the inferential function of logic is implicit in this broad conception of model theory. Of course, without a completeness theorem for a given language L and satisfaction relation for L, there is no assurance that such inference can be conducted on a purely syntactic plane; but the point is that, whether or not one has a completeness theorem, this general model-theoretic problem takes us beyond purely descriptive concerns. Other traditional concerns that are syntactic but not necessarily inferential occupy attention in model theory, e.g. one asks whether the theory of a class K of structures is decidable, or whether its theory can be axiomatized in a sublanguage of L. Beyond this, model theory has been concerned with which properties of structures are preserved under given relations between structures and operations on them. In the opposite (so to speak internal) direction, as pointed out to me by Wilfrid Hodges, mainstream "geometric" model theory is concerned with describing relations within a particular structure, not with defining classes of structures.[iii] None of this is suggested by talk of the descriptive function of logic. In other words, Hintikka's conception of model theory is narrow to the extreme, and is further narrowed by the insistence on dealing with sentences of a very particular form, namely the IF sentences.

The point of departure in this section was whether IF logic deserves to be called a first-order logic according to the criterion offered above by Hintikka, namely "in terms of the entities one's quantified variables range over". I have argued that this does not distinguish IF logic from the logics of generalized quantifiers in Lindström's sense, which by all ordinary measures go beyond first-order logic. The question is whether something more special about IF logic is supposed to make the difference. Of course, one can talk in picturesque terms about playing games with individuals, each play involving only a finite number of choices, as a way of arguing that verification of an IF sentence is a first-order matter. But it is not the particular plays that matter; rather it is whether there is or is not a winning strategy for Verifier in such games, both in any one structure and over all structures in general. And, as we shall see in the next section in pursuit of my argument, *that* lands us squarely in *full second-order logic*.

4. The expressive power of IF and extended IF logic. In Ch. 9 of *PMR*, Hintikka gives a number of examples of mathematical notions which can be expressed by IF-sentences S or by their contradictory negations S*. In ordinary logical terms, this comes down to seeing which notions can be expressed in Σ^1_1 or Π^1_1 form.

The following are standard examples.

(1) The relation of equicardinality is Σ^1_1.

That is, we have a Σ^1_1 sentence S in two unary predicate symbols P and Q, which expresses that P and Q are in one-one correspondence.

(2) The property of being infinite is Σ^1_1.

That is, we have a Σ^1_1 sentence S in one unary predicate symbol P which expresses that P is infinite; using = alone, we can express by such a sentence that the domain of interpretation of the first-order variables is infinite.

(3) The notion of being a non-well-founded binary relation is Σ^1_1. Hence that of being a well-founded relation is Π^1_1; the same applies to the notion of being a well-ordering relation.

(4) There is a Π^1_1 sentence S* which characterizes up to isomorphism the structure of the natural numbers $\langle N, Sc, 0\rangle$, where Sc is the successor relation; the sentence uses one binary symbol P and one constant symbol c.

(5) There is a Π^1_1 sentence S* which characterizes up to isomorphism the two-sorted structure $\langle N, \wp(N), Sc, 0, \in\rangle$ for second-order number theory, where $\wp(N)$ is the set of all subsets of N.

(If preferred, the structure in (5) can be treated as one-sorted by unification of domains). The sentence S* for (5) may be taken to include the statement that every characteristic function $f: N \to \{0, 1\}$ determines a member a of $\wp(N)$ by $\forall x\, [\, x \in a \leftrightarrow f(x) = 0\,]$.

(6) Similarly, there is a Π^1_1 sentence S* characterizing the finite type hierarchy over the natural numbers, obtained by iterating the power set operation $\wp$ up to ω.

(7) Transfinite iterations of the power set operation can also be dealt with in this form, most smoothly within the one sorted-language for the system ZFC of set theory. As pointed out by Väänänen (2001) among others, there is a Π^1_1 sentence S* characterizing (up to isomorphism) the structure for the cumulative hierarchy $\langle V_\kappa, \in\rangle$ up to the first inaccessible cardinal κ,where $V_0 = \varnothing$, each $V_{\alpha+1} = \wp(V_\alpha)$ and V_λ is the union of the V_α

for $\alpha < \lambda$ when λ is a limit ordinal. The same can be done for still larger specific inaccessible cardinals κ.

Further examples mentioned in *PMR*, Ch. 9, concern characterizations of the real numbers in extended IF logic and expressibility of various topological notions. All such examples lead Hintikka to conclude (*PMR*, p. 196) that "...virtually all of classical mathematics can in principle be done in extended IF first-order logic," i.e. can be expressed in Π^1_1 form, and that many mathematical concepts can already be expressed in Σ^1_1 form. However, there is a substantial difference between the two, since Π^1_1 sentences do not admit a direct game-theoretic interpretation. So, what is gained by these expressibility results? Hintikka points out (loc. cit.) a kind of *reduction to* IF logic, which may be formulated more generally as follows. Given a Π^1_1 sentence S* like that indicated above in (5)-(7) for second-order or finite-order number theory, or for the cumulative hierarchy of sets up to the first inaccessible cardinal, and given a mathematical conjecture C expressible in Σ^1_1 form in the language of S*, the implication $S^* \rightarrow C$ is equivalent to $\neg S^* \vee C$, and thence to a Σ^1_1 sentence, or--if one prefers--an IF sentence. Hintikka concludes that

> ...a great many mathematical *problems* can be taken to relate to the logical status of a sentence of an *un*extended IF first-order language. (*PMR,* p. 197, italics in the original).

However, in the use here of the words "logical status" there is a shift from *satisfaction* of Σ^1_1 sentences in some structure or another--Hintikka's main concern when speaking of the descriptive function of logic--to *validity* of such sentences, and this makes a world of difference when it comes to sentences of the form $\neg S^* \vee C$. For, the relevant logical status in these cases is that of validity, not satisfaction in one model or another. An IF-sentence or, equivalently, Σ^1_1 sentence S is valid if it is true in *every* possible interpretation of its non-logical symbols. Here is what Hintikka has to say about the shift in concerns (taking S* from (6) as the example):

> The upshot of this line of thought is thus a kind of reduction of the entire finite theory of types, with standard interpretation, to IF-first order logic. Since most of mathematics can in principle be expressed in a standardly interpreted finite theory of types, this reduction throws some interesting light on mathematics in general. For what can we say of the output sentences [i.e., $\neg S^* \vee C$] of this reduction? They are IF *first-order* sentences. All their bound variables range over individuals. This should warm the heart of every philosophical nominalist. More importantly, their interpretation is completely free of the logical problems that beset the notion of *all subsets* of a given infinite set. An IF first-order sentence is valid if and only if a certain relational structure can't help being instantiated in every model. The problem of whether a given IF first-order sentence is valid or not is therefore a combinatorial problem in a sufficiently wide sense of the term. (*PMR*, p. 198, italics in the original.)

I take it that what Hintikka means by "a certain relational structure [that] can't help being instantiated in every model" $\underline{M}$, for a given IF-sentence S or its Σ^1_1 equivalent $\exists\underline{f}\forall\underline{x}$ $R(\underline{f},\underline{x})$, is a realization of the existentially quantified function variables $\underline{f}$ in $\underline{M}$, if S is true in $\underline{M}$ at all. As to this, Väänänen (2001) p. 519 has proved that *the general question of validity of IF sentences is recursively isomorphic to that for validity in full second-order logic*.[iv] Moreover, he shows there (op. cit., p. 517) that the *set of valid sentences of full second-order logic is a complete Π_2 set (in the sense of the Lèvy set-theoretical hierarchy), hence is not Σ_2 definable*. The two results together imply that *validity of IF sentences is not Σ_2 definable*; that strengthens an old result of Montague (1965), p. 263, according to which *the set of valid Σ^1_1 sentences is not definable in finite type theory over the natural numbers*. Thus, the validity problem for IF sentences is by no means a "combinatorial", nominalistically heart-warming matter. On the contrary, if the question of validity of such sentences is taken to have definite meaning, there is a concomitant commitment to full second-order logic. This seems to be contradicted by Hintikka's statement (*PMR*, pp. 191-192) that "IF first-order logic is equivalent only to a small fragment of second-order logic, namely, the Σ^1_1 fragment." By Väänänen's theorem, that is true only if one considers *satisfiability* of IF sentences, *not* of *validity*.[v]

What we have here is a clear case of trying to have your philosophical cake and eat it too. On the one hand, second-order logic in its supposed standard sense is suspect:

> ...many hard-nosed logicians will not be happy with the proposal of using a second-order language as a medium for their mathematical theorizing, and for a good reason. In order for such a language to serve its purpose, its second-order variables must be taken in their standard sense. They must be taken to range over *all* extensionally possible entities of the appropriate type (sets, functions, etc.) ... But if so, we face all the problems connected with the ideas of arbitrary set and arbitrary function ... I can indicate this kind of commitment to arbitrary higher-order entities by saying that it involves the idea of "all sets". Another way of expressing myself might be to speak of the standard interpretation in Henkin's sense. But whatever the name that this idea passes under, its smell is equally foul to many logicians. And there is a great deal to be said for their perceptions. The idea of the totality of all (sub)sets is indeed a hard one to master. (*PMR,* p. 193, italics in the original.)

These suspect notions can be avoided simply by restricting to first-order logic. But ordinary first-order logic is totally inadequate expressively to the task of grounding mathematics because the principal notions of concern such as those listed (1)-(7) above, cannot be characterized in first-order terms. First-order axiomatic set theory provides no solution to this problem since we cannot prevent non-standard interpretations; Ch. 8 of *PMR* is a sustained polemic against axiomatic set theory, aka "Fraenkelstein's monster", primarily on these grounds.

> Since nonstandard interpretations do not help us to deal with the problems of set existence, what can? Here IF logic seems to offer its services to us. As long as we can stay on the level of first-order logic, independence-friendly or not, then problems of set existence do not arise. We do not have to open the Gordian knot of set existence since it was not tied in the first place. (*PMR,* p. 194)

In other words, by *declaring* IF logic to be a first-order logic one can have one's philosophical cake and eat it too. But endlessly declaring it to be so does not make it so. As I have argued, the assimilation of IF logic to first-order logic just doesn't hold water. On the contrary, once validity comes into the picture, and it does come in essentially if one is to account for mathematics in the way that Hintikka proposes above, then one is in the same boat as second-order logic. You cannot buy the one without buying the other.

5. What does it mean to do mathematics? The necessity of inference. We have seen that Hintikka's claim, quoted above, that "virtually all of classical mathematics can be done in extended IF first-order logic" comes down to the validity of Σ^1_1 sentences of the form $S^* \rightarrow C$, where S^* is a Π^1_1 characterization of some substantial part of set theory, be it second-order or higher-order number theory or even the cumulative hierarchy up to the first inaccessible. But that is where the real work for doing mathematics comes into the picture; inference is its *sine qua non*. For, higher mathematics makes essential use of long and involved chains of reasoning from what is already accepted (eventually, special cases of S*) to what is to be established (C). And such reasoning for human mathematicians does not and cannot work directly at the semantic level of the notions involved; it can only proceed syntactically, in a way that is justified by the semantics. In other words, what is needed for the *doing of mathematics* is logic in its traditional deductive sense. And for that it is not an issue whether the model-theoretic notion of validity has a complete axiomatization. This is not to say that deductive logic in the ordinary sense suffices; what separates mathematics from logic is the employment of general notions of set and function that are irreducible to logic, and the assumption of axioms concerning those notions that are accepted on the grounds of what they are supposed to be about. That such assumptions may be seriously problematic philosophically does not mean that they can be ignored; on the contrary (cf. Feferman, et al., 2000). But that is another matter; what is at issue here is whether there is any sense to talking about doing mathematics *without* considering deductive logic. This would hardly need emphasizing except that in *PMR* Hintikka is utterly dismissive of the deductive role of logic in mathematics for reasons that I fail to comprehend.

Given my view of the matter, I have to ask what might be of value in expanding everyday logical reasoning as represented in the ordinary first-order logic to a system of reasoning--necessarily incomplete--making use of the formalism of IF sentences. In *PMR*, Ch. 4, Hintikka gives a few examples in everyday mathematical parlance where the ideas of independence are used loosely, and which can be represented by IF-sentences. But all such examples are equally well taken care of in ordinary first-order terms with the observance of just a little care. And since no substantial fragment of IF logic as an inferential system is on offer, the matter is entirely speculative. If the promoters of IF logic were to grant that some inferential system of reasoning with IF formulas would be of value, they should encourage its pursuit. But I personally think such an effort would be regressive, since it was realized long ago in mathematical practice how to say in precise terms that one quantity is or is not dependent on another, without invoking a new syntax hinging on that idea. Sometimes, all that is required for that is to take care about the order of quantifiers with respect to first-order variables, while other times function quantification is needed; but it is rare in the latter cases that one has to appeal to substantial function existence axioms to reason with such dependence conditions.

Taking this idea one step further, it may be of interest (in the spirit of endnote (i)) to set up deductive fragments of second-order logic to formalize dependence relations in practice. In particular, it is often the case that one argues for an implication $\exists\underline{f}\ \forall\underline{x}\ R(\underline{f}, \underline{x}) \rightarrow \exists\underline{g}\ \forall\underline{y}\ S(\underline{g}, \underline{y})$ between Σ^1_1 sentences on the basis of the fact that there is an elementary way H of choosing a witness for $\underline{g}$ from any $\underline{f}$ satisfying $\forall\underline{x}\ R(\underline{f}, \underline{x})$, i.e. the functional $H(\underline{f}) = \underline{g}$ is first-order definable and one has $\forall\underline{x}\ R(\underline{f}, \underline{x}) \rightarrow \forall\underline{y}\ S(H(\underline{f}), \underline{y})$. This way of reasoning immediately suggests a rule of inference:

(1) from $\forall\underline{x}\ R(\underline{f}, \underline{x}) \rightarrow \forall\underline{y}\ S(H(\underline{f}), \underline{y})$, with H first-order,
infer $\exists\underline{f}\ \forall\underline{x}\ R(\underline{f}, \underline{x}) \rightarrow \exists\underline{g}\ \forall\underline{y}\ S(\underline{g}, \underline{y})$.

More generally, it is natural to consider the fragment dealing with implications between *essentially* Σ^1_1 formulas, i.e. those whose prenex form has only existential second-order quantifiers. In particular, this permits formulation of the Axiom of Choice for Σ^1_1 formulas in the form

(2) $\forall z\ \exists\underline{f}\ \forall\underline{x}\ R(\underline{f}, \underline{x}, z) \rightarrow \exists\underline{f}'\ \forall z\ \forall\underline{x}\ R(\underline{f}'(z), \underline{x}, z)$.

A rule like (1) and axiom of the form (2) were featured in one formal system for predicativity (Feferman 1979), under special restrictions on the formulas involved (op. cit., p. 78) but the preceding shows that such principles are clearly meaningful in a much more general context.

6. What's left? Hintikka has offered more reasons than those considered above from logic and mathematics for promoting IF logic. One main claim surrounds the autonomy of the truth definition for IF sentences, in the sense that one has a Σ^1_1 formula T(x) such that for any Σ^1_1 sentence S,

$$T(\#S) \Leftrightarrow S,$$

where #S is the Gödel number of S. The weak link here is the relation $\Leftrightarrow$ of equivalence, where $A \Leftrightarrow B$ is true if A and B are true in the same models; this cannot be treated as a connective of IF logic. For a full critique of the claims on behalf of the self-definability of truth within IF logic the reader is referred to Rouilhan and Bozon (this volume).[vi]

Outside of logic and mathematics, Hintikka appeals among other things to language games, philosophically and in everyday life, and to the IF representation of certain phenomena in natural language (cf. also Hintikka and Sandu 1996). For these, the points disputed in this paper are irrelevant, and a defense of the formalism of IF logic and its associated semantics on such other grounds may well be sustained. In particular, games of imperfect information have a clear interest in their own right, and their investigation (as well as the investigation of the related games of perfect information in Väänänen 2002) merits further study. [vii]

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[i] Interestingly, at least to me, the shift from IF forms such as (3) to modified Skolem forms such as (4), corresponds to a shift from attention to questions of *independence* to those of *dependence*.

[ii] The claim that IF logic is a first-order logic has been challenged by a number of thinkers, including Cook and Shapiro (1998), Hodes (1998), Väänänen (2001), and Rouilhan (2002). Väänänen's results in this respect have been of particular use to me, as will be seen in Section 4 below.

[iii] Personal communication. Hodges added that the usual compositional semantics of first-order formulas is essential for the work on geometrical model theory.

[iv] The argument indicated by Väänänen makes use of a Π^1_1 sentence S* in a binary predicate symbol E and two unary predicate symbols P and Q whose models are exactly those isomorphic to $\langle M, \in, P, Q\rangle$ with Q = $\wp$(P). If A is an IF sentence true in the same models as $\neg$S*, then we can associate with any second-order sentence B a first-order sentence B′ in E, P and Q such that B is valid in full second-order logic if and only if A $\vee$ B′ is a valid IF sentence.

[v] In effect, Hintikka is hoist here by his own petard via the "reduction of the entire finite theory of types, with standard intepretation" to IF logic (as quoted above from *PMR*, p. 198).

[vi] One consequence of the main result of Rouilhan and Bozon, Theorem 2, sec. 4, is that for a great variety of IF languages L with standard interpretation, the relation A $\Leftrightarrow$ B for sentences A, B of L is not definable by any formula of finite order having the same signature as L, nor, a fortiori, by any formula of L. Their full result undermines Hintikka's claims in *PMR* for the autonomy of the model theory of IF languages.

[vii] I wish to thank Wilfrid Hodges, Philippe de Rouilhan and Jouko Väänänen for their comments on a draft of this piece.