Deduction and Definability in Infinite Statistical Systems

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Abstract

Classical accounts of intertheoretic reduction involve two pieces: first, the new terms of the higher-level theory must be definable from the terms of the lower-level theory, and second, the claims of the higher-level theory must be deducible from the lower-level theory along with these definitions. The status of each of these pieces becomes controversial when the alleged reduction involves an infinite limit, as in statistical mechanics. Can one define features of or deduce the behavior of an infinite idealized system from a theory describing only finite systems? In this paper, I change the subject in order to consider the motivations behind the definability and deducibility requirements. The classical accounts of intertheoretic reduction are appealing because when the definability and deducibility requirements are satisfied there is a sense in which the reduced theory is forced upon us by the reducing theory and the reduced theory contains no more information or structure than the reducing theory. I will show that, likewise, there is a precise sense in which in statistical mechanics the properties of infinite limiting systems are forced upon us by the properties of finite systems, and the properties of infinite systems contain no information beyond the properties of finite systems.

Keywords: algebraic quantum theory, C*-algebras, category theory, reduction

1 Introduction

Classical accounts of intertheoretic reduction involve two pieces: first, the new terms of the higher-level theory must be definable from the terms of the lower-level theory, and second, the higher-level theory must be deducible from the lower-level theory along with these definitions. One notable limitation of such accounts of reduction is that they make controversial certain standard practices in physics involving infinite limits—practices that at least intuitively appear "reductive". In particular, when one considers the possible reduction of thermodynamics to statistical mechanics, one must face the standard practice of taking infinite limits in order to construct thermodynamic quantities out of statistical quantities. It will help to have an example in mind to illustrate this practice, which we will refer back to in what follows.

- **Example: The Spin Chain**
  In order to understand phase transitions in ferromagnets, one can start with a simple model of a spin-chain involving N evenly spaced spin-$\frac{1}{2}$ particles in a lattice. But one finds that to account for critical behavior, one must extend this to a model involving a countable infinity of particles evenly spaced in the lattice $\mathbb{Z}^d$.

The presence of this infinite limit is a distinguishing feature that appears to thwart fitting examples from statistical physics into philosophical accounts of intertheoretic reduction, especially in the presence of so-called "singular limits". In these cases, the quantities of the infinite limiting system have properties that, in a sense, go beyond the properties of finite subsystems. So it appears, at least at first glance, that the infinite limit is neither definable from nor deducible from (in some relevant sense) the properties of finite systems.

However, it is my contention that one can understand the limiting procedures in statistical physics as preserving two important motivating features behind the requirements of definability and deducibility. The features I'm interested in are as follows. The requirements of definability and deducibility together demand that (i) the higher-level models are, in a sense, *forced upon us* by the lower-level models we use, and (ii) the higher-level models, in a sense, *contain no more information* than the lower-level models.

Of course, it would be nice if we could ask straightforwardly whether the higher-level models of infinite systems are definable or deducible from the lower-level models of finite systems, but since the examples from statistical physics are not typically formulated in a formal language, the notions of deduction and definability from mathematical logic are not straightforwardly applicable. This should give us reason to at least consider posing the question in a different way: are the higher-level models of infinite systems forced upon us by the lower-level models of finite systems? Do the higher-level models of infinite systems contain more information than the lower-level models of finite systems? I will not provide an argument for this way of thinking; instead, I hope to demonstrate its utility by showing that it helps us understand how these infinite limits share features with classical philosophical accounts of reduction, and by showing that it helps us understand infinite limits in statistical physics quite generally.

I will show that there is a precise sense in which the infinite models in statistical physics are forced upon us by the finite models we start with through these infinite limiting procedures. Specifically, I will show that the relevant limits—*weak limits* in a C*-algebra—are unique. Furthermore, I will show that there is a precise sense in which the infinite models in statistical physics contain no more information than the finite models they derive from. Specifically, I will show senses in which the relevant category of C*-algebras for representing physical quantities of finite systems has at least as much structure as two relevant categories of *weakly complete* C*-algebras (i.e., C*-algebras containing all of their weak limit points) that can be used for representing the physical quantities of infinite limiting systems.

Section 2 gives a brief review of the classical accounts of intertheoretic reduction in terms of definability and deducibility, with reference to contemporary debates in the philosophy of physics about reduction and emergence. Section 3 sets up a framework for thinking about reduction and emergence in terms of limits in an algebra of operators. Sections 4 and 5 show the senses in which the quantities of infinite limiting systems are forced upon us and contain no more information than the quantities of finite systems. Section 6 points out some connections to recent work in general philosophy of science on the structure of scientific theories and some possible implications of the results here for the debate between "Algebraic Imperialism" and "Hilbert Space Conservatism" in quantum theory. This serves to provide future directions for work if one takes at least the motivations behind the argument of this paper seriously. Section 7 concludes with a discussion of the significance of these results for the debates about reduction and emergence in physics.

2 Reduction, Deduction, and Definability

The goal of this section is to explain the controversy surrounding classical accounts of intertheoretic reduction in the face of "singular limits" in statistical physics.

Classical accounts of intertheoretic reduction trace back to Nagel (e.g., 1961). As is well known, on Nagel's model of reduction we are to think of scientific theories as sets of sentences in some formal language. One theory is then said to reduce to another just in case it is possible to *derive*, via deductive arguments, the law sentences of the reduced theory from the reducing theory. In the case that the reduced theory involves terms that do not appear in the reducing theory, one must in addition specify *bridge laws*, which define the terms of the reduced theory using only the vocabulary of the reducing theory.

On Nagel's account, the reduced theory is *forced upon us* in the sense that if the reducing theory and bridge laws are true, then the reduced theory must also be true. And the reduced theory contains *no more information or structure* than the reducing theory in the sense that every sentence that is derivable from the reduced theory is also derivable from the reducing theory along with the bridge laws.

Even before philosophers began debating the significance of "singular limits", some noticed that the presence of limiting operations in reductions in physics bears on our conception of reduction. Nickles (1975) argues that there are really two distinct concepts of reduction: one to capture the notion Nagel describes and another—which Nickles claims does not involve definability and deducibility—to capture limiting relations in physics. On the other hand, Schaffner (1967) argues for a unified framework in which to fit reductions with and without limiting relations. Schaffner's generalized model allows for the replacement of the reduced theory with a "strongly analogous" theory that is derivable from the reducing theory. Allowing for the presence of only a "strong analogy" means that we need not be able to derive the reduced theory exactly from the reducing theory. Instead, for example, we might only be able to derive from the reducing theory that the reduced theory is *approximately true*.[1] It is my hope that analyzing the notion of approximation available in the mathematical details of the

[1] See also Dizadji-Bahmani et al. (2010) for more on Schaffner's approach applied to statistical mechanics.

limiting relations discussed in this paper will help us understand how the kinds of approximations we encounter in reductions in statistical physics relate to the classical accounts just mentioned.[2]

However, when we look at the details of reductions via infinite limits in statistical physics, we encounter further challenges for an understanding of reduction as definability and deducibility. We'll see that the presence of "singular limits" makes it difficult to see how the properties of an infinite system could be forced upon us or determined by the properties of a system's finite components. And furthermore, "singular limits" threaten our conception of the properties of an infinite system as containing no more information or structure than the properties of its finite components. For example, models of infinite statistical systems—the systems with these "singular limits"—involve global quantities that represent properties of an entire infinite system and which seem to go beyond the properties of any finite system.

These global properties can display features of the infinite system that no finite system could possibly have. As discussed in Batterman (2002, 2005, 2009), Callender (2001), and Butterfield (2011a,b), the thermodynamic quantities of a finite system always vary continuously while the thermodynamic quantities of an infinite system can display discontinuities. It is this feature that leads some to label certain statistical systems as involving "singular limits". A "singular limit" is present whenever a limiting system displays different features (in some sense) from the sequence of systems on the way to the limit.

It is sometimes argued that, because discontinuities in thermodynamic quantities are necessary for a system to be able to undergo a phase transition, only infinite systems can display phase transitions while finite systems are incapable of doing so. For example, Batterman (2002) argues that this amounts to the emergence of qualitatively different behavior in phase transitions, and Butterfield (2011a,b) argues that we see the appearance of novel and robust phenomena.[3] Given these allegedly emergent and novel phenomena, one might become worried that one cannot use these infinite limits in a reduction,

[2]See Fletcher (2016) and Rosaler (2015) for a discussion of limiting relations, reduction, and approximation in other areas of physics. See also Norton (2012) for a discussion of the role of approximation and idealization.

[3]There is a further point of contention in the philosophy of physics literature regarding the significance of the renormalization group (see, e.g., Batterman (2010)). I will not touch upon renormalization techniques in this paper; I think further work is required to determine whether my results are applicable to the limits taken during renormalization.

or at least not without rejecting the classical accounts of reduction. My goal in this paper, however, is to show that the presence of these "singular limits" need not thwart a reduction. I do not intend to dispute the claims about emergent or novel phenomena appearing in the limit, but only to ask: in what sense are these phenomena that we see in the infinite limit novel or emergent?

One of the prominent features of the classical accounts of reduction is that a reduction is supposed to show a sense in which a reduced theory cannot contain novel features when compared to the reducing theory. If a reduced theory is definable and deducible from some reducing theory, then the reduced theory is forced upon us or completely determined by the reducing theory, and the reduced theory contains no more information or structure than the reducing theory. I will argue in the remainder of this paper that the infinite limits in statistical mechanics, even when they are "singular", have features in common with this way of understanding the classical accounts.

Let me stress from the start, though, that I have no intention of arguing that the features the infinite limits in statistical mechanics have in common with classical accounts of reduction are sufficient for judging that thermodynamics reduces to statistical mechanics, or even that these particular thermodynamic systems undergoing phase transitions reduce to their statistical mechanical counterparts. I only want to show that the features of "singular limits" that other authors have pointed to need not be understood as thwarting reduction. For this, it suffices to show that there is *a sense* in which the "singular limits" have something in common with classical accounts of reduction, even though there may be *other senses* in which these examples fail to fit into the classical account. In any case, I believe some will find the features "singular limits" have in common with classical reduction worth examining in their own right.

I will argue in what follows that, however novel or emergent the phenomena we see at the infinite limit, they are still, in a precise sense, forced upon us and completely determined by the physics of finite systems. And furthermore, I will argue that there is a precise sense in which our models of infinite systems contain no more information or structure than our models of finite systems. Together, these claims show that even the presence of "singular limits" in infinite quantum statistical systems does not remove some of the important features behind intertheoretic reduction.

3 Preliminaries

What mathematical structure do we need to assume to model an infinite system in quantum statistical mechanics like the infinite spin chain? We'll start by assigning three physical quantities to any individual spin-$\frac{1}{2}$ particle: σ_x, σ_y, and σ_z. These quantities will represent the magnetic moment of the particle in the x, y, and z directions, respectively. For such a particle, these quantities must obey the canonical commutation relations:

$$[\sigma_x, \sigma_y] = i\sigma_z \qquad [\sigma_y, \sigma_z] = i\sigma_x \qquad [\sigma_z, \sigma_x] = i\sigma_y$$

$$(\sigma_x)^2 + (\sigma_y)^2 + (\sigma_z)^2 = 3I$$

These relations are fundamental quantum mechanical assumptions about the quantities of a spin-$\frac{1}{2}$ particle. In the infinite spin system, we will have an infinite collection of such quantities $\sigma_x^k, \sigma_y^k, \sigma_z^k$, which, for fixed k, satisfy the above constraints. Here, we will take $k \in \mathbb{Z}$ so that we have countably many particles arranged in an evenly spaced one-dimensional lattice. Moreover, for distinct particles in the infinite system, we have

$$[\sigma_a^j, \sigma_b^k] = 0 \text{ for } a, b = x, y, z \text{ and } j \neq k$$

The quantities $\sigma_x^k, \sigma_y^k, \sigma_z^k$ represent the magnetic moment of the kth particle in the x, y, and z directions, respectively.

We need to construct further physical quantities from these magnetic moments for the purposes of writing down a theory of the ferromagnet. For example, we use the multiplication and addition operations to construct arbitrary polynomials of magnetic moment quantities to plug into a Hamiltonian and define a dynamics. Moreover, we need to take limits of infinite sequences of quantities; continuous functions of magnetic moments are, in general, limits of infinite sequences of polynomials of magnetic moments, and we likewise need these continuous functions to describe the dynamics of certain systems.

To construct these further functions of magnetic moment quantities, we must assume the collection of physical quantities or observables forms a *C*-algebra*.[4] This means that one can add and multiply

[4]For more on C*-algebras and W*-algebras, see Kadison and Ringrose (1997), Sakai (1971), and Landsman (1998). For more on algebraic quantum theory, see Haag (1992), Bratteli and Robinson (1996), Emch (1972), and Wald (1994). For philosophical introductions, see Halvorson (2006) and Ruetsche (2011b).

observables, and multiply observables by scalars. In addition, a C*-algebra $\mathfrak{A}$ carries an operation of involution that is a generalization of complex conjugation. A C*-algebra $\mathfrak{A}$ also carries a norm that is compatible with the involution, meaning it is required to satisfy the following identity:

$$\|A^*A\| = \|A\|^2$$

for all $A \in \mathfrak{A}$. The presence of this norm is necessary in order to take the limits of sequences alluded to above in the *norm topology*, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges to A in the norm topology[5] iff

$$\|A_i - A\| \to 0$$

where the convergence is now in the standard topology on $\mathbb{R}$. The C*-algebra $\mathfrak{A}$ is required to be complete with respect to this topology in the sense that for every Cauchy net $\{A_i\} \subseteq \mathfrak{A}$, i.e., for every net such that

$$\|A_i - A_j\| \to 0$$

there is an $A \in \mathfrak{A}$ such that $A_i \to A$ in the norm topology. Completeness here ensures that the limit point of any net that "appears to converge" in norm, in the sense that it is a Cauchy net, is contained in the C*-algebra of quantities.

More generally, for any infinite system, one can start with the C*-algebra $\mathfrak{A}_0$ for a single component and construct the algebra of a system composed of an infinite number of those identical (finite) components. In the infinite spin chain, the algebra $\mathfrak{A}_0$ is the C*-algebra generated by the quantities $\sigma_x, \sigma_y, \sigma_z$ for a single particle. The total algebra of the infinite system (again, of countably many components arranged on a one-dimensional lattice $\mathbb{Z}$) is then $\mathfrak{A} = \bigotimes^{\mathbb{Z}} \mathfrak{A}_0$. This C*-algebra $\mathfrak{A}$ for the infinite spin chain gives the collection of spin quantities for all finite components in the infinite system.

This algebra $\mathfrak{A}$ of magnetic moment quantities for the spin chain is naturally understood as the algebra generated by *local* quantities to each finite segment of the spin chain. By local quantities, I mean quantities that represent physical features of finite components of the infinite system. The norm limits of these local quantities are sometimes said to be *quasi-local*.

[5] One could restrict attention here to sequences because the norm topology is second countable, but for the weak topologies considered later, which are not second countable, one must work with arbitrary nets.

This algebra suffices to define a collection of physical states. Since $\mathfrak{A}$ is a vector space, we can also consider the dual space $\mathfrak{A}^*$ of bounded (i.e., norm continuous) linear functionals $\rho : \mathfrak{A} \to \mathbb{C}$. A *state* on a C*-algebra $\mathfrak{A}$ is just a particular kind of element of the dual space $\mathfrak{A}^*$—namely one that is positive and normalized.[6] A state is called *pure* iff it cannot be written as a convex combination of distinct states. Otherwise, a state is called *mixed*. Pure states represent the possible states of an individual system while mixed states are typically taken to represent some sort of probabilistic combination (whether it be via an ensemble interpretation or mere epistemic uncertainty).

One small caveat before preceding. Throughout this paper, I will understand a physical system as (partially) represented by an algebra of quantities. To obtain a full representation of a physical system, one might be interested in further pieces of mathematical apparatus including, for example, a structured space of physical states and a dynamics.[7] For the purposes of this paper, however, I will ignore all pieces of structure except the algebra of quantities for two reasons. First, as described above, an algebra of quantities already determines a structured space of physical states.[8] Second, one can generate the worries about reduction and "singular limits" already by considering only the algebra of quantities without any specification of further information like dynamics. Although additional complications may arise when considering dynamics (and in particular, when considering the role of the renormalization group), for the purposes of this paper I will focus only on the initial worry concerning limits of physical quantities.

Now even once we have the C*-algebra of quasi-local quantities for the spin chain, there are still further quantities we need to define in order to be able to describe the thermodynamic behavior of the ferromagnet. In other words, there are physically significant quantities of the spin chain that are not contained in the C*-algebra we've defined. For example, the total average magnetization in the z-direction, m_z^{tot} is neither a local nor quasi-local quantity. To define m_z^{tot}, we start by

[6]A linear functional $\rho \in \mathfrak{A}^*$ is *positive* if $\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and *normalized* if $\|\rho\| = 1$.

[7]Thanks to an anonymous reviewer for this point.

[8]See Alfsen and Shultz (2001) for more on the mathematical structure on the space of states. See Ruetsche and Earman (2011) and Ruetsche (2011a) for more on the physical interpretation of certain kinds of states. And see Feintzeig (2017a) for more on the relationship between an algebra and its collection of allowed states.

taking the sequence of finite average magnetization quantities

$$m_z^n = \frac{1}{2n+1} \sum_{k=-n}^{k=+n} \sigma_z^k$$

For each $n \in \mathbb{N}$, m_z^n is a quasi-local quantity in the algebra $\mathfrak{A}$. We would like to define m_z^{tot} as the limit of this sequence as n goes to infinity. But here we face a problem: this sequence does not converge to a limit in the norm topology. The total average magnetization is thus a quantity missing from our C*-algebra of quasi-local quantities.

In order to take this infinite limit to obtain the total magnetization, we must use the dual space $\mathfrak{A}^*$ to define an alternative to the norm topology on $\mathfrak{A}$, called the *weak topology*. The weak topology is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{A}$ converges in the weak topology to $A \in \mathfrak{A}$ iff for every $\rho \in \mathfrak{A}^*$,

$$\rho(A_i) \to \rho(A)$$

where the convergence is now in the standard topology on $\mathbb{C}$. The weak topology is the coarsest topology on $\mathfrak{A}$ that makes all of the linear functionals in $\mathfrak{A}^*$ continuous. It turns out the sequence m_z^n of average magnetization quantities does converge in the weak topology. But there's a catch: the sequence converges to a quantity that lies outside the C*-algebra $\mathfrak{A}$. This is the topic of the next section.

4 Weak Limits

In order to take the required weak limits to obtain the total average magnetization, we need to find a new algebra that contains the limits of weakly converging sequences. The reason is that the limits of weakly converging sequences, like the global magnetization quantity, will not in general belong to our original quasi-local C*-algebra $\mathfrak{A}$. The algebra that contains these weak limits is instead a *W*-algebra*.

A W*-algebra is a C*-algebra $\mathfrak{R}$ with a *predual*, i.e., a vector space $\mathfrak{R}_*$ such that $(\mathfrak{R}_*)^* = \mathfrak{R}$.[9] We can understand the elements of the predual as canonically embedded in the dual space $\mathfrak{R}^*$ by the map $\rho \in \mathfrak{R}_* \mapsto \hat{\rho} \in \mathfrak{R}^*$ with $\hat{\rho}$ defined by

$$\hat{\rho}(A) = A(\rho)$$

[9]Here, $\mathfrak{R}$ as a Banach space is to be understood as the Banach space dual to $\mathfrak{R}_*$, but the algebraic operations of multiplication and involution are not determined by $\mathfrak{R}_*$.

for all $A \in \mathfrak{R}$. The elements of the predual $\mathfrak{R}_*$ define a further topology on $\mathfrak{R}$, called the weak* topology, which is characterized by the following condition for convergence. A net $\{A_i\} \subseteq \mathfrak{R}$ converges in the weak* topology to $A \in \mathfrak{R}$ iff for every $\rho \in \mathfrak{R}_*$,

$$A_i(\rho) \to A(\rho)$$

where the convergence is now in $\mathbb{C}$. The weak* topology is the coarsest topology with respect to which every element of the predual $\mathfrak{R}_*$ is continuous when considered as a linear functional on $\mathfrak{R}$. Weak* continuous states on $\mathfrak{R}$ are called *normal states.*

Of course, more nets converge in the weak* topology on $\mathfrak{R}$ than in the weak topology because one only requires convergence of expectation values on a subspace of the dual space $\mathfrak{R}^*$. Nevertheless, the weak* topology is a natural generalization of the weak topology in the special case where the predual of $\mathfrak{R}$ is itself the dual space of a C*-algebra. In this case, one has a C*-algebra $\mathfrak{A}$, its dual space $\mathfrak{A}^*$, and a W*-algebra $\mathfrak{A}^{**}$ called the *bidual.* The original C*-algebra is canonically embedded in its bidual by $J_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^{**}$, defined by $J_{\mathfrak{A}}(A) = \hat{A}$, where $\hat{A}$ is defined for all $\rho \in \mathfrak{A}^*$ by[10]

$$\hat{A}(\rho) = \rho(A)$$

In this section, I will show that the limits of all weakly converging nets of observables in a C*-algebra are contained in the bidual by showing that any W*-algebra is weak*-complete. This will establish that global quantities, like the total average magnetization observable, belong to the bidual, or in other words that the bidual provides us with the resources for taking this weak limit. But one might worry that if the total average magnetization observable does not belong to our original algebra, it might not be completely determined by the finite magnetization observables—there might be other ways of taking the relevant weak limit! This would be a problem because if the limit were

[10]It might be somewhat surprising that $\mathfrak{A}^{**}$, which is just the Banach dual space to a Banach space, has additional algebraic structure that makes it into a C*-algebra. But this is easy to see once one notices that the original C*-algebra $\mathfrak{A}$ is weak* dense in $\mathfrak{A}^{**}$ with respect to this canonical embedding $J_{\mathfrak{A}}$ (See Feintzeig, 2017b; Sakai, 1971), so the algebraic structure of $\mathfrak{A}^{**}$ can be naturally inherited from $\mathfrak{A}$. Multiplication and involution on $\mathfrak{A}^{**}$ are defined as the unique weak* continuous extensions of the operations on $\mathfrak{A}$. (We only require multiplication to be separately weak* continuous in each of its arguments because multiplication in the original C*-algebra $\mathfrak{A}$ is not, in general, jointly weakly continuous.)

not unique, this would show a sense in which a limiting property of the infinite system is not completely determined by the properties of its finite components. However, I will show in this section that the bidual provides us with the unique way to take weak limits and construct the total average magnetization. This will establish that the bidual of a C*-algebra is, in a sense, *forced upon us* by the original C*-algebra, and hence the global properties of the infinite system are forced upon us by the properties of the system's finite components.

The results of this section and the next do not involve novel mathematics; they are hopefully intuitive and obvious to anyone well-versed in the mathematical language of operator algebras. What I hope to add here is a clear discussion of how these results bear on the philosophical issues surrounding reduction, deduction, and definability.

A C*-algebra $\mathfrak{A}$ with its weak topology and the bidual $\mathfrak{A}^{**}$ with the weak* topology are both locally convex vector spaces. All this means is that their topologies are generated by a family of semi-norms L on their underlying vector spaces. In both cases, we have the same family of semi-norms $L = \mathfrak{A}^*$, but it defines two distinct topologies on $X = \mathfrak{A}$ and $X' = \mathfrak{A}^{**}$. A locally convex vector space X with topology generated by a family of semi-norms L is said to be *complete* if every Cauchy net converges to an element of the space. Here a net $\{y_\beta\} \subseteq X$ is *Cauchy* just in case for all $\epsilon > 0$ and all $l \in L$, there is a β_0 such that $|l(y_\beta) - l(y_\gamma)| < \epsilon$ for all $\beta, \gamma \succ \beta_0$.

Cauchy nets are ones that "appear to converge". For example, the sequence of finite average magnetization observables m_z^n defined above is Cauchy in both $\mathfrak{A}$ and $\mathfrak{A}^{**}$. But, as remarked above, its limit does not exist in the original algebra $\mathfrak{A}$, which means that $\mathfrak{A}$ is not complete in the weak topology. The following proposition[11] shows, however, that the bidual always contains the limit points of all Cauchy sequences. So the bidual is always complete in this sense in the weak* topology.

Proposition 1. *Given any Banach space X, its dual X^* is complete in the weak* topology as a locally convex vector space.*

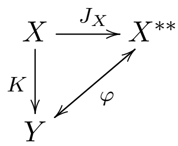
Corollary 1. *Every W*-algebra $\mathfrak{R}$ is complete in its weak* topology. In particular, the bidual $\mathfrak{A}^{**}$ to any C*-algebra $\mathfrak{A}$ is complete in its weak* topology.*

Now we know that if we are given a C*-algebra $\mathfrak{A}$, then there always exists a completion of $\mathfrak{A}$ in the weak topology, i.e., a C*-algebra

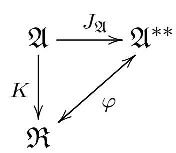
[11] Proofs of all propositions appear in the appendix.

that is complete in the weak* topology and in which $\mathfrak{A}$ can be densely embedded isomorphically and homeomorphically (via the canonical evaluation map $J_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^{**}$). This means that we are guaranteed that there exists a way of taking the limit of the observables m_z^n in order to obtain the total average magnetization observable m_z^{tot}. But could we have completed $\mathfrak{A}$ in the weak* topology in some other way? Could we have taken the limit of m_z^n to get a different global quantity? If so, we would have a case in which the properties of the finite subsystems of the spin chain do not completely determine the global magnetization of the infinite system. The following proposition, however, shows that the answer is no; the bidual $\mathfrak{A}^{**}$ is the unique completion of $\mathfrak{A}$ in the weak* topology (See also Takesaki, 1979).

Proposition 2. *Suppose X is a Banach space and $J_X : X \to X^{**}$ is the canonical evaluation embedding of X in its bidual. Suppose we are given another faithful linear embedding $K : X \to Y$ of X in a complete locally convex vector space Y such that $K(X)$ is dense in Y in the locally convex vector space topology on Y. Suppose, in addition, that K is a homeomorphism from X to $K(X)$ in the weak topology on X and the subspace topology on $K(X)$ generated by the locally convex vector space topology on Y. Then there is a vector space isomorphism $\varphi : Y \to X^{**}$ that is a homeomorphism in the locally convex vector space topology on Y and the weak* topology on X^{**} and such that $J_X = \varphi \circ K$.*



Corollary 2. *Suppose $K : \mathfrak{A} \to \mathfrak{R}$ is a faithful *-homomorphism such that $K(\mathfrak{A})$ is dense in the W*-algebra $\mathfrak{R}$ and K is a homeomorphism from $\mathfrak{A}$ with its weak topology to $K(\mathfrak{A})$ in the subspace topology generated by the weak* topology on $\mathfrak{R}$. Then there is a *-isomorphism $\varphi : \mathfrak{R} \to \mathfrak{A}^{**}$ that is a homeomorphism in the respective weak* topologies and satisfies $J_{\mathfrak{A}} = \varphi \circ K$.*



Hence, the bidual is the unique choice of completion of $\mathfrak{A}$. This means that there is a unique way to take the limit to obtain the total average magnetization observable. In other words, the magnetizations associated with local, finite parts of the spin chain suffice to determine the global magnetization of the infinite system. This shows that global properties of an infinite system, like the total magnetization of the spin chain, although at first absent from our (quasi-)local descriptions of finite components of the infinite system, are forced upon us by the local properties of finite components like the local spin properties.

So our construction has proceeded uniquely from the finite parts of spin systems to the infinite limiting system as follows. The algebra $\mathfrak{A}$ of quasi-local spin properties is forced upon us or completely determined by the local spin properties as the unique norm completion of the algebra of local spin properties. This quasi-local algebra $\mathfrak{A}$ then determines a collection of states that defines the weak topology on $\mathfrak{A}$. Given this weak topology, there is a unique weak limit of any Cauchy sequence in $\mathfrak{A}$, which lives in $\mathfrak{A}^{**}$. The sense in which the total magnetization is forced upon us is that if one wants to define a global property as a limit over finite averages as above, then there will be one and only one way to take that limit. A limiting global property of the infinite system is fully constrained or fully determined by the algebra of quasi-local quantities of finite components of the system.

5 Limits and Structure

At this point, one might object—isn't there a sense in which the bidual $\mathfrak{A}^{**}$ contains more information or more physical content than the C*-algebra $\mathfrak{A}$ that we started with? After all, the bidual contains more physical quantities than our original algebra, including global quantities that are not even continuous functions of the local spin quantities. Isn't there a sense in which the C*-algebra $\mathfrak{A}$ and the bidual $\mathfrak{A}^{**}$ do not provide equivalent theories or descriptions of our physical system? If so, then this would show that the properties of the infinite system, which we need the bidual $\mathfrak{A}^{**}$ to describe, contain more information than the properties of its finite components, which can be described with the algebra $\mathfrak{A}$. In other words, if the answers to the above questions are both "yes", then this would show that an infinite system has structure beyond that of its finite subsystems.

This issue is subtle, but in this section I will answer these questions in the negative. I will show that on three ways of making these questions precise, the bidual of a C*-algebra contains no more structure than the original C*-algebra. To provide this answer, I will use the tools of category theory, which allow us to compare the construction of the bidual to other "natural" constructions in mathematics.[12]

Category theory has recently made its way into the philosophy of science and philosophy of physics literatures as a tool for answering questions about equivalence and the structure of scientific theories. Halvorson (2012, 2016) and Halvorson and Tsementzis (2017) argue that scientific theories can be represented using the tools of category theory rather than mathematical logic or model theory. Barrett (2017a) and Weatherall (2016a,b,c, 2017) argue that the tools of category theory can help us determine when two scientific theories are theoretically equivalent. And Barrett (2015a,b, 2017b) argues that the tools of category theory help us compare the amounts of structure in different scientific theories, or different formulations of a scientific theory. While these works have mostly focused on examples in mathematics, classical mechanics, and spacetime theories, here I will apply the tools of category theory to quantum statistical mechanics. Our approach will be to treat the theory of quantum statistical mechanics as a category of models of quantum statistical mechanics—each of which is an algebra of observables that can be taken to represent (at least features of) a possible physical system.

A *category* is a collection of *objects* with *arrows* between them (sometimes called *morphisms*) and an operation $\circ$ on arrows, called composition. Composition takes a pair of arrows $f : A \to B$ and $g : B \to C$ and sends them to a new arrow $g \circ f : A \to C$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g\circ f}\searrow & \downarrow{\scriptstyle g} \\ & & C \end{array}$$

- **Example: NVec**
  The category **NVec** has normed vector spaces as objects and bounded linear transformations between them as arrows.
- **Example: Ban**
  The category **Ban** has Banach spaces (complete normed vector spaces) as objects and bounded linear transformations as arrows.

[12]For introductions to category theory, see Awodey (2010) and Borceux (1994).

- **Example: C*-Alg**
  The category **C*-Alg** has C*-algebras as objects and *-homomorphisms between them as arrows.
- **Example: W*-Alg**
  The category **W*-Alg** has W*-algebras as objects and weak* continuous *-homomorphisms between them as arrows.

Every object X in a category has a unique identity arrow 1_X that leaves all other arrows intact under composition.

$$1_A \circlearrowright A \xrightarrow{f = f \circ 1_A} B \circlearrowleft 1_B$$

An arrow is an *isomorphism* if it has an inverse, i.e., an arrow whose composition with it yields the identity.

$$1_A = f^{-1} \circ f \circlearrowright A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B \circlearrowleft 1_B = f \circ f^{-1}$$

These arrows can be understood as “structure-preserving” maps. Bounded linear transformations between vector spaces preserve addition, scalar multiplication, and the norm topology; *-homomorphisms between C*-algebras furthermore preserve multiplication and involution; and weak* continuous *-homomorphisms between W*-algebras even further preserve the predual, or equivalently the weak* topology.

5.1 Forgetfulness

A morphism between categories is called a *functor*. A functor is a pair of maps from the objects and arrows of one category to the objects and arrows of another category, respectively, that preserves arrow composition. By this, I mean that a functor $F : \mathbf{C} \to \mathbf{D}$ between two categories $\mathbf{C}$ and $\mathbf{D}$ must satisfy the following condition: for any two arrows $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$, the corresponding arrows $F(f) : F(A) \to F(B)$ and $F(g) : F(B) \to F(C)$ in $\mathbf{D}$ satisfy

$$F(g \circ f) = F(g) \circ F(f)$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{g \circ f} & \downarrow g \\ & & C \end{array} \qquad\qquad \begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ & \searrow^{F(g \circ f)} & \downarrow F(g) \\ & & F(C) \end{array}$$

A functor allows us to compare one category to another by looking at what a functor "forgets", according to the following definitions (Baez et al., 2004). A functor $F : \mathbf{C} \to \mathbf{D}$ is *full* if for any two objects A and B in $\mathbf{C}$, and any arrow $g : F(A) \to F(B)$, there is an arrow $f : A \to B$ such that $F(f) = g$. A functor $F : \mathbf{C} \to \mathbf{D}$ is *faithful* if for any two objects A and B in $\mathbf{C}$, and any two arrows $f, g : A \to B$, if $F(f) = F(g)$, then $f = g$. A functor $F : \mathbf{C} \to \mathbf{D}$ is *essentially surjective* if for every object D in $\mathbf{D}$, there is an object A in $\mathbf{C}$ such that $F(A)$ is isomorphic to D.

A functor *forgets properties* if it fails to be essentially surjective, *forgets structure* if it fails to be full, and *forgets stuff* if it fails to be faithful. A functor that is full, faithful, and essentially surjective forgets nothing and is called a *categorical equivalence*.

- **Example: Algebra to vector space**
  We can define a functor from **C*-Alg** to **NVec** that takes each C*-algebra and sends it to its underlying normed vector space (regardless of its multiplication and involution operations), and sends every *-homomorphism to its corresponding bounded linear transformation. This functor forgets structure because there are bounded linear transformations (arrows in **NVec**) between C*-algebras that are not *-homomorphisms.

This example illustrates the significance of labeling functors that fail to be full as functors that forget structure. Such a functor forgets the structure preserved by the arrows in its domain category. Whereas the *-homomorphisms are forced to preserve multiplication, bounded linear transformations have no multiplication operation to preserve, so the functor in the above example forgets the multiplication structure of C*-algebras.

- **Example: Banach space completion**
  We can define a functor F_0 from **Ban** to **NVec** that takes each Banach space to itself as a normed vector space and each arrow to itself as a bounded linear transformation. This inclusion functor of **Ban** in **NVec** forgets properties, namely the property of being norm complete, because there are some objects in **NVec** that are not norm complete and hence are not isomorphic to any object in the range of F_0. Similarly, we can define a functor G_0 from **NVec** to **Ban** that takes each normed vector space to its unique Banach space completion and each bounded linear

transformation to its unique norm continuous extension. One can check that this completion functor G_0 forgets nothing.

In the case at hand of C*-Algebras and W*-Algebras, we can define two relevant functors that are analogous to F_0 and G_0 in the previous example. First, we can define a functor F : **W*-Alg** → **C*-Alg** that takes every W*-algebra to itself as a C*-algebra and every weak* continuous *-homomorphism to itself as a *-homomorphism. This functor F forgets properties, namely the property of being complete in the weak* topology, because there are C*-algebras that don't even have a predual and so don't even have a weak* topology to be complete with respect to. Moreover, F forgets structure because there are *-homomorphisms that fail to be weak* continuous.

Comparing W*-algebras to C*-algebras by this functor F provides one sense in which the objection we started this section with is correct. There is a sense in which the bidual $\mathfrak{A}^{**}$, considered as a W*-algebra, has more structure than the C*-algebra $\mathfrak{A}$. Namely, if we compare the category **W*-Alg**, in which $\mathfrak{A}^{**}$ resides, to the category **C*-Alg**, in which $\mathfrak{A}$ resides, by the functor F, then we find that, since F forgets structure, there is a sense in which $\mathfrak{A}^{**}$ contains more structure than $\mathfrak{A}$. This motivates our worry because it shows a sense in which the properties of an infinite system, which we need the algebra $\mathfrak{A}^{**}$ to describe, exhibit more structure than the properties of its finite components, which we can describe using only the algebra $\mathfrak{A}$. So we can now state the objection as follows.

Objection: The physical quantities of an infinite system described by a bidual $\mathfrak{A}^{**}$ contain more information or structure than the physical quantities of the system's finite components described by $\mathfrak{A}$ in the sense that the functor F : **W*-Alg** → **C*-Alg** forgets structure.

But it is worth noticing that such structural comparisons can only be made *relative to a choice of functor between categories*. The functor F is not the only relevant functor, and, in fact, if we choose another relevant functor, then we get the opposite answer.

Above, I claimed that we can define two functors between the categories **W*-Alg** and **C*-Alg** analogous to the functors F_0 and G_0 above. So far, we've seen the first functor F analogous to F_0. Now, let us define another functor G : **C*-Alg** → **W*-Alg** analogous to G_0. We define G as taking each C*-algebra to its bidual, which is its completion in the weak topology. To finish the definition of the functor

G, we need to specify its action on arrows. The following proposition shows that we can define G as mapping each *-homomorphism to its unique weak* continuous extension to the bidual.

Proposition 3. *Suppose $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism between C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then there is a unique weak* continuous extension $\tilde{\alpha} : \mathfrak{A}^{**} \to \mathfrak{B}^{**}$ such that $\tilde{\alpha} \circ J_{\mathfrak{A}} = J_{\mathfrak{B}} \circ \alpha$, where $J_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^{**}$ and $J_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}^{**}$ are the canonical evaluation maps.*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha} & \mathfrak{B} \\ {\scriptstyle J_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathfrak{B}}} \\ \mathfrak{A}^{**} & \xrightarrow[\tilde{\alpha}]{} & \mathfrak{B}^{**} \end{array}$$

Now we see that G is a well-defined functor taking each C*-algebra $\mathfrak{A}$ to its bidual and each *-homomorphism to its weak* continuous extension. The functor G is obviously faithful so it does not forget stuff. It may be surprising, however, to learn that G is not full and so forgets structure. (Prop. 8 below furthermore shows that G is not essentially surjective and so forgets properties.)

Proposition 4. *G is not full, i.e., G forgets structure.*

This is intuitively strange. Since F : **W*-Alg** $\to$ **C*-Alg** forgets structure, we were tempted to say that there is a sense in which W*-algebras have more structure than C*-algebras. But now, since the functor G : **C*-Alg** $\to$ **W*-Alg** also forgets structure, we are also tempted to say that there is a sense in which C*-algebras have more structure than W*-algebras. The structure G forgets is the canonical embedding $J_{\mathfrak{A}}$ that tells us where to find our original C*-algebra $\mathfrak{A}$ in our new W*-algebra $\mathfrak{A}^{**}$. There are arrows $\alpha : \mathfrak{A}^{**} \to \mathfrak{B}^{**}$ in **W*-Alg** between two biduals $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$ for C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that α does not preserve the embeddings $J_{\mathfrak{A}}$ and $J_{\mathfrak{B}}$, i.e., $\alpha \circ J_{\mathfrak{A}}(\mathfrak{A}) \nsubseteq J_{\mathfrak{B}}(\mathfrak{B})$; it is these arrows that do not lie in the range of G. The way in which a C*-algebra is embedded in its W*-algebra completion by J matters for some purposes; it distinguishes quasi-local properties from global properties of a physical system. Thus, G forgets nontrivial structure.

This shows a—perhaps surprising—sense in which the bidual $\mathfrak{A}^{**}$, when considered as an object in **W*-Alg**, contains *less* information or structure than the original algebra $\mathfrak{A}$ with respect to the functor G. Or, in terms of the physical systems we are using these algebras to represent, this shows that the infinite system described by $\mathfrak{A}^{**}$ actually contains less information or structure than the finite components described by $\mathfrak{A}$ when we compare these systems by the functor G. We'll see more on the structure G specifically forgets in the next section, but for now we have the following result.

Result 1: Infinite systems described by a bidual $\mathfrak{A}^{**}$ contain less information or structure than their finite components described by $\mathfrak{A}$ in the sense that the functor G : **C*-Alg** → **W*-Alg** forgets structure.

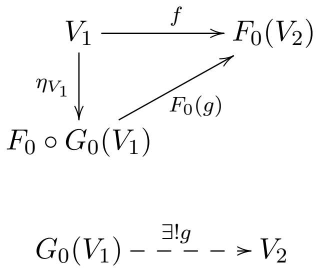
5.2 Adjunction

We've uncovered some subtleties in the relationship between **C*-Alg** and **W*-Alg**. Specifically, we've encountered senses in which C*-algebras have both more and less structure than W*-algebras. At this point, one might be tempted to say that the categories, functors, or notions of forgetfulness we've used don't capture the relevant philosophical notions of structural comparison. But I think there is still something to be gained by looking at the relation of these functors with each other. There is a sense in which the functors F and G, even though they yield unintuitive answers when considered alone, together hint at something intuitive. They hint at a sense in which the biduals in **W*-Alg** are *definable* from the objects in **C*-Alg**.

To understand this notion of definability (which differs from perhaps more familiar notions of definability in mathematical logic), I'd like to return us to a previous example in which certain mathematical structures are intuitively definable from others—namely, the categories **NVec** and **Ban** with the norm completion functor G_0 : **NVec** → **Ban** and the inclusion functor F_0 : **Ban** → **NVec**. In this case, it is natural to think that there is a sense in which Banach spaces are definable from normed vector spaces, and that the completion procedure gives us a way of defining them. After all, in familiar proofs of the existence of Banach space completions, the points of the underlying vector space of the completion are defined as equivalence classes of Cauchy sequences of elements from the original normed vector space. The Banach space completion is literally built out of the ingredients of the original normed vector space. Is the completion of a C*-algebra into a W*-algebra similar? Should we understand W*-algebras as being definable from C*-algebras?

To answer this question, let us characterize the situation with **NVec** and **Ban** in more detail. The pair of functors F_0 and G_0 forms an *adjunction*,[13] with left adjoint G_0 and right adjoint F_0. This means that there is a *natural transformation* η called the *unit* from the identity functor $1_{\mathbf{NVec}}$ to $F_0 \circ G_0$ satisfying the following condition. For any V_1 in **NVec**, any V_2 in **Ban**, and any arrow $f : V_1 \to F_0(V_2)$, there is a unique arrow $g : G_0(V_1) \to V_2$ such that

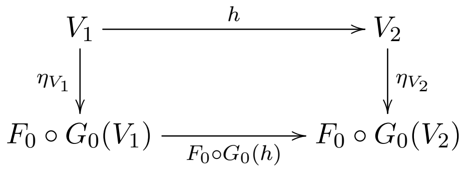
$$f = F_0(g) \circ \eta_{V_1}$$



The arrow g is given by the unique norm continuous extension of f to the Banach space completion $G_0(V_1)$.

That η is a natural transformation (Awodey, 2010, Ch. 7) from $1_{\mathbf{NVec}}$ to $F_0 \circ G_0$ means that it is an assignment of an arrow $\eta_V : 1_{\mathbf{NVec}}(V) \to F_0 \circ G_0(V)$ to each object V in **NVec** such that for any objects V_1 and V_2 in **NVec** and any arrow $h : V_1 \to V_2$,

$$\eta_{V_2} \circ 1_{\mathbf{NVec}}(h) = F_0 \circ G_0(h) \circ \eta_{V_1}$$



It is easy to check that letting $\eta_V = J_V$ for all objects V in **NVec**, where J_V is the dense isometric embedding of any vector space in its norm completion, satisfies all of the above requirements. In this situation we call **Ban** a *reflective subcategory*, meaning it is a full subcategory, whose inclusion functor F_0 in **NVec** has a left adjoint.

Why care about adjunctions? In the presence of an adjunction $G_0 : \mathbf{NVec} \leftrightarrows \mathbf{Ban} : F_0$, for every object V in **NVec** and every

[13] See, e.g., Awodey (2010, p. 214) and Borceux (1994, p. 98) for more on adjunctions.

morphism of the object $G_0(V)$—i.e., structure-preserving map from $G_0(V)$ to some object V' in **Ban**—there is a corresponding morphism of V to $F_0(V')$ in **NVec** (See Prop. 9.4 of Awodey (2010, p. 211)). So there is a sense in which all of the structure in **Ban**, characterized by the structure-preserving morphisms in **Ban**, can likewise be characterized in **NVec** by the corresponding structure-preserving morphisms in **NVec** guaranteed to exist by the adjunction.[14]

The situation for **W*-Alg** and **C*-Alg** looks much the same. The following proposition shows that the functors F and G form an adjunction.

Proposition 5. *F and G form an adjunction, with left adjoint G, right adjoint F and unit J.*

In the case of **Ban** and **NVec**, the presence of an adjunction signals a sense in which Banach spaces are definable from normed vector spaces. The very same feature holds in the case at hand: there is an adjunction between the categories **W*-Alg** and **C*-Alg**, which hints at a sense in which W*-algebras are definable from C*-algebras. In other words, applied to our physical systems, this shows a sense in which an infinite system described by a bidual $\mathfrak{A}^{**}$ in **W*-Alg** contains no more information or structure than its finite components described by $\mathfrak{A}$ in **C*-Alg**.

It is worth mentioning one additional complication concerning adjunctions and definability here, which is the reason I have only claimed that the adjoint functors F and G *hint at* a sense in which infinite systems are definable from finite systems. In the case of **Ban** and **NVec**, one can show a precise sense in which the adjoint functors F_0 and G_0 are accompanied by definitions in the relevant formal language to satisfy the precise conditions of definability in mathematical logic,[15] but I know of no such work for completion in the weak topology. I have every reason to hope that in the case of **C*-Alg** and **W*-Alg** the adjoint functors F and G are similarly accompanied by definitions in the relevant formal language, but I leave this as a conjecture.

Instead of looking for a definition in a formal language in the familiar sense from mathematical logic, we can say that it is at least a

[14]Special thanks to Jim Weatherall and Thomas Barrett for help clarifying the significance of an adjunction. Note that I do not claim that an adjunction immediately signifies definability in the sense of mathematical logic. There is still much further work to be done along these lines—see §6.

[15]Thanks to an anonymous reviewer for pointing this out.

necessary condition for one type of structure to be definable from another that there exist an adjunction between the relevant categories. We might in addition require that the adjunction satisfies certain conditions (namely, being accompanied by definitions in a formal language) to find a sufficient condition for definability. Since I am not interested here in giving a general theory of definability, this goes beyond the scope of the current paper. So, the conclusion of this section should be accordingly restricted: infinite statistical systems satisfy a necessary condition for being definable from their finite components.

I think we can also say slightly more. Even if we do not yet know whether the adjoint functors F and G demonstrate the definability of W*-algebras from C*-algebras in the familiar sense from mathematical logic, the brief argument above concerning the significance of adjoint functors shows that we should think of **W*-Alg** as having no more information or structure than **C*-Alg**. The correspondence between morphisms in **C*-Alg** and **W*-Alg** guaranteed by the adjoint functors F and G shows that the structure of W*-algebras can be fully characterized by corresponding structure in C*-algebras. One can understand this to capture a loose, heuristic notion of definability not associated with a formal language. Now, regardless of the relationship between adjunctions and strict definability in mathematical logic, one still has the following result.

Result 2: Infinite systems described by a bidual $\mathfrak{A}^{**}$ contain no more information or structure than their finite components described by $\mathfrak{A}$ in the sense that the functors

$$G : \textbf{C*-Alg} \leftrightarrows \textbf{W*-Alg} : F$$

form an adjunction with left adjoint G and right adjoint F.

It is worth noting two disanalogies between vector spaces and algebras. First, while **W*-Alg** is a subcategory of **C*-Alg**, it is not a full subcategory because F is not full. On the other hand, **Ban** is a full subcategory of **NVec** because F_0 is full. Second, in **Ban** one loses no structure by throwing away the embedding J; the functor G_0 is full. But in **W*-Alg**, this same move forgets structure in the sense that G is not full. Together, these two differences amount to the following fact. When one considers a Banach space $F_0(V)$ as a vector space and completes it again in norm to $G_0 \circ F_0(V)$, one gets back the same Banach space V. In other words, completing a Banach space in norm adds nothing. On the other hand, when one considers a W*-algebra $\mathfrak{R}$ as a C*-algebra $F(\mathfrak{R})$ and completes it in its weak topology, one gets a larger algebra $G \circ F(\mathfrak{R}) = \mathfrak{R}^{**} \ncong \mathfrak{R}$. The weak topology on $\mathfrak{R}$ is coarser than the weak* topology on $\mathfrak{R}$, so that even though $\mathfrak{R}$, by virtue of being a W*-algebra, is complete in its weak* topology, it is not complete in its weak topology. So, while completions of algebras and normed vector spaces are similar, there is still this difference.

5.3 Equivalence

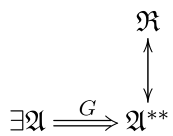
One might get the impression—e.g., because of the subtleties involved in distinguishing the weak topology of an object in **W*-Alg** from its weak* topology—that we are considering the bidual $\mathfrak{A}^{**}$ in the wrong category. Since there are features of W*-algebras that are forgotten by the functors defined in the previous sections, we might look for a different category and different functor that keeps those features of W*-algebras intact. Indeed, I will establish in this section that there is a *different* category of W*-algebras that is equivalent to **C*-Alg**. This is the category of biduals, which one can characterize by the following condition on the predual to a W*-algebra.

Proposition 6. *Let $\mathfrak{R}$ be a W*-algebra and let*

$$I = \{A \in \mathfrak{R} : \rho(A) = 0 \text{ for all } \rho \in \mathfrak{R}_*\}$$

There is a C-algebra $\mathfrak{A}$ such that $\mathfrak{R} \cong \mathfrak{A}^{**}$ iff*

$$\mathfrak{R}_* \cong \{\omega \in \mathfrak{R}^* : \text{ if } A \in I, \text{ then } \omega(A) = 0\} \tag{1}$$



This shows that any W*-algebra satisfying Eq. 1 is the bidual to some C*-algebra. Moreover, the following proposition shows that if Eq. 1 holds for a W*-algebra $\mathfrak{R}$, then there is a sense in which the C*-algebra $\mathfrak{A}$ whose weak completion is $\mathfrak{R}$ is *unique*.

Proposition 7. *Let $\mathfrak{R}$ be a W*-algebra. If $\mathfrak{A}$ and $\mathfrak{B}$ are two C*-algebras such that $\mathfrak{A}^{**} \cong \mathfrak{R} \cong \mathfrak{B}^{**}$, then $\mathfrak{A}$ is *-isomorphic to $\mathfrak{B}$.*

Call a W*-algebra $\mathfrak{R}$ *whole* iff Eq. 1 holds.[16] Notice that while completeness is a topological property that all W*-algebras satisfy in their weak* topology, wholeness is an additional property of the normal state space of a W*-algebra. Let us say (in a sense exactly analogous to ordinary measure theory) that a state ω is absolutely continuous with respect to a collection of states $\mathcal{S}$ just in case whenever all states in $\mathcal{S}$ assign probability zero to some event, ω assigns probability zero to that event, too. A W*-algebra $\mathfrak{R}$ is whole just in case its normal state space contains all of the states absolutely continuous with respect to all the normal states on $\mathfrak{R}$. Thus, one might motivate Eq. 1 along the following lines. Let's say that an event is *guaranteed not to occur* just in case all physically possible states assign it probability zero. If one thinks any state that assigns probability zero to all events that are guaranteed not to occur is a physically possible state, then the physical state space should contain all states absolutely continuous with respect to all physically possible states. Moreover, if the collection of physically possible states is identical with the collection of normal states (See Ruetsche, 2011a), then the normal states should satisfy Eq. 1.

The following proposition shows that wholeness is a substantive constraint. In other words, there are W*-algebras that we set aside if we restrict attention to only whole W*-algebras. (This also shows that the functor G of the previous section forgets properties.)

Proposition 8. *There are W*-algebras that are not whole.*

Since there are W*-algebras that are not whole, and since we already know that there are whole W*-algebras (take the bidual to any C*-algebra), we now know that wholeness is a nontrivial condition. We will use this condition in what follows to characterize a new category of W*-algebras that is equivalent to **C*-Alg**

When we consider only whole W*-algebras, we have the resources to restrict attention to maps that preserve the structure forgotten by G. Call a weak* continuous *-homomorphism $\alpha : \mathfrak{R}_1 \to \mathfrak{R}_2$ between whole W*-algebras $\mathfrak{R}_1$ and $\mathfrak{R}_2$ a **-whomomorphism* if

$$\alpha \circ J_{\mathfrak{A}_1}(\mathfrak{A}_1) \subseteq J_{\mathfrak{A}_2}(\mathfrak{A}_2)$$

where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the unique C*-algebras such that $\mathfrak{R}_1 \cong \mathfrak{A}_1^{**}$ and $\mathfrak{R}_2 \cong \mathfrak{A}_2^{**}$. Notice that one cannot apply the above condition to

[16]Wholeness is just condition (ii) of Thm. 1 of Feintzeig (2017a). Condition (i) of that theorem is always satisfied for the predual of a W*-algebra.

a W*-algebra $\mathfrak{R}$ that is not whole because there will not be a C*-algebra whose bidual is $\mathfrak{R}$. Furthermore, notice that if $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a *-homomorphism between C*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then the unique weak* continuous extension $\tilde{\alpha} : \mathfrak{A}_1^{**} \to \mathfrak{A}_2^{**}$ given by Prop. 3 is a *-whomomorphism.

Consider the category **wW*-Alg** whose objects are whole W*-algebras and arrows are *-whomomorphisms. The arrows in **wW*-Alg** provide enough structure to distinguish the elements of a C*-algebra $\mathfrak{A}$ from the elements of its bidual $\mathfrak{A}^{**}$ because they provide enough structure to define $J_{\mathfrak{A}}$, the map whose image is preserved by all *-whomomorphisms. In the physical cases at hand from quantum statistical mechanics like the spin chain, recall that $\mathfrak{A}$ is the algebra of (quasi)-local quantities, and $\mathfrak{A}^{**}$ is the larger algebra containing also the global quantities. By distinguishing the elements of $\mathfrak{A}$ from the rest of $\mathfrak{A}^{**}$, the category **wW*-Alg** gives us enough mathematical resources to distinguish quasi-local from global quantities. Insofar as one thinks the distinction between quasi-local and global quantities in a W*-algebra is physically significant, one should allow the models of quantum statistical mechanics to be treated as objects in a category with at least as much structure as **wW*-Alg**.

Let $\overline{G}$: **C*-Alg** $\to$ **wW*-Alg** be the functor that takes each C*-algebra to its bidual and each *-homomorphism to its unique weak* continuous *-whomomorphism extension. In other words, the composition of $\overline{G}$ with the inclusion of **wW*-Alg** in **W*-Alg** is just the original completion functor G used in the previous sections. Then the following proposition shows that $\overline{G}$ does not forget structure, and in fact forgets nothing.

Proposition 9. *$\overline{G}$ is a categorical equivalence between **C*-Alg** and **wW*-Alg**.*

Thus, we have our final result showing that the bidual $\mathfrak{A}^{**}$ does not contain more structure or physical content than the C*-algebra $\mathfrak{A}$ we started with. This shows that if one treats the bidual as a W*-algebra with enough structure to distinguish quasi-local from global quantities—that is, if one treats the bidual as an object in **wW*-Alg**—then one assumes no more and no less structure than one assumed in the original C*-algebra.

Result 3: Infinite systems described by a bidual $\mathfrak{A}^{**}$ contain the same amount of information or structure as their finite components described by $\mathfrak{A}$ in the sense that the functor $\overline{G}$: **C*-Alg** $\rightarrow$ **wW*-Alg** is a categorical equivalence.

6 Future Directions

I've argued that there is a sense in which the properties of infinite systems are forced upon us by the properties of finite systems, and that there is a sense in which the properties of infinite systems contain no information beyond the properties of finite systems in quantum statistical mechanics.[17]

The sense in which I showed that properties of infinite systems are forced upon us by properties of finite systems is that limits of Cauchy sequences in the weak topology on a C*-algebra are unique. But it is important to notice that weak limits are not the only relevant infinite limits in this case. A weak limit is the limit of a sequence of observables already contained in the algebra of quantities defined for the infinite system. One can ask further about the limiting process we use to even obtain the algebra of quantities for the infinite system.

It is perhaps surprising to notice that when we consider the limiting process for the entire algebra, there is a sense in which the entire limiting algebra for the infinite system is *not* unique. If one understands the quantities of each finite subsystem of the infinite spin chain to be represented by a C*-algebra, then one can construct a *continuous field of C*-algebras* (See Dixmier (1977) and Landsman (1998, 2013)) to represent the limiting procedure for the entire algebra of quantities. That is, one starts with the quantum theory of a finite number of subsystems and lets the number of subsystems grow larger and larger, analyzing how the entire algebra of quantities changes. It turns out that the algebra for the infinite limiting system is *non-unique* in the sense that there are *two* distinct limit algebras one can use in this continuous field: either the algebra of the infinite quantum system I described in section 3 or the algebra of a classical spin system. Thus, one might worry that the algebras of finite systems do not completely determine the algebra of the infinite system in which the entire discussion of this paper has taken place.

However, there is still hope. One might try to show that if we have more information about the algebras of finite systems, then this forces

[17]The results of this paper also apply to classical statistical systems represented by commutative algebras.

a unique limiting algebra upon us. It is known, for example, that local dynamics converge only in the infinite quantum limit and not the classical limit, and conversely it is known that non-local "mean-field" dynamics converge only in the classical limit and not in the infinite quantum limit. Thus, one might ask whether there is any sense in which the limiting algebra's uniqueness is restored once we choose a dynamics. If so, this would bear on the question of whether the properties of the infinite system are completely determined by the properties (including the Hamiltonian) of its finite subsystems. As far as I know, this is an open question that deserves further attention.

- **Open Question 1**: Under what conditions is there a *unique* limit algebra in a continuous field of C*-algebras on which a given dynamics converges?

An answer to this question would appear to me to have philosophically interesting consequences, whatever it turned out to be. It is my hope that the approach taken in this paper to reduction and limits helps others to see the philosophical significance of the question above.

In addition to my discussion of the uniqueness of limits, I've argued that there are three senses in which the properties of infinite systems contain no information beyond the properties of finite systems. First, I showed that relative to at least one choice of categories and one choice of functor between those categories, the algebras of infinite systems contain less structure than the algebras of finite systems. That is, the functor G from the category **C*-Alg** to the category **W*-Alg** forgets structure. Second, I showed that relative to one choice of categories and a choice of two functors between those categories, the algebras of infinite systems satisfy a necessary condition for being definable from the algebras of finite systems. That is, the weak completion functor G, along with the forgetful functor F, form an adjunction for the categories **C*-Alg** and **W*-Alg**. Third, I showed that relative to another choice of categories and a different functor between them, the algebras of infinite systems contain exactly the same structure that was already present in the algebras of finite systems. That is, the functor $\overline{G}$ is an equivalence between **C*-Alg** and **wW*-Alg**.

Although I have used a rough, informal concept of definability when discussing adjunctions, one might object that this does not match the concepts of definability from mathematical logic employed in the classical accounts of reduction. As mentioned in the previous section, it's not clear at first glance that the concept of an adjunction fully captures what philosophers are looking for when they talk about definability. Some recent work, however, suggests that we might be able to relate the concepts of category theory to concepts of definability in mathematical logic; Barrett (2017b) characterizes the properties of a functor that sends a first order theory to a theory it is explicitly definable from. I think extending these kinds of results to cover the examples of this paper would be philosophically interesting, but it also requires further mathematical work. The examples of completing a topological vector space in the weak topology is naturally understood as formalized in either infinitary or higher order logic. So to determine how the concept of an adjunction relates to more familiar notions of definability, one would need to analyze functorial properties of definability in logics beyond first order logic. Again, this seems to me to lead to interesting open questions.

- **Open Question 2**: Do functors sending theories to theories they are definable from (in some appropriate sense of definability) in logics beyond first order logic form part of an adjunction?

An answer to this question would also be philosophically significant for better understanding both the role of definability and the role of adjunctions in analyses of the relationship between scientific theories.

In addition to the open questions just stated, I think the results of this paper point to interesting further work to be done on the foundations of quantum field theory. In the philosophical literature one finds an interpretive debate between the positions of "Algebraic Imperialism" and "Hilbert Space Conservatism". Roughly, the Algebraic Imperialist believes one can adequately represent the physical quantities of a quantum field system using an abstract algebra alone, while the Hilbert Space Conservative believes one needs a Hilbert space representation of the algebra as well in order to use the weak operator topology of a representation to form a von Neumann algebra.[18] The results of this paper should immediately bear on the Algebraic Imperialist's options of algebras to use for representing physical systems.

Further it is well known that there is a category of von Neumann algebras with appropriately continuous morphisms that is categorically equivalent to the category **W*-Alg**. So there is a sense in which

[18]For more on Algebraic Imperialism and Hilbert Space Conservatism, see Arageorgis (1995), Ruetsche (2002, 2003, 2006, 2011b), Baker (2011), Baker and Halvorson (2013), Lupher (2008, 2016) and Feintzeig (2016, 2017a,c).

these interpretive positions may not be so different after all. However, I suspect there is a different category of von Neumann algebras that might more accurately characterize the position of Hilbert Space Conservatism and its use of a privileged Hilbert space representation. For example, we might define a different category of von Neumann algebras by specifying a different collection of arrows—perhaps unitary equivalences in the natural inclusion representation of a von Neumann algebra on its given Hilbert space. I think it would be interesting to do further work using the tools of category theory to make precise different formulations of the Algebraic Imperialist's and Hilbert Space Conservative's positions in order to compare them.

I hypothesize that there is a formulation of Hilbert Space Conservatism using the tools of category theory that adequately captures the interpretive position and has *more structure* than an adequate formulation of Algebraic Imperialism.

- **Open Question 3**: Can one use the tools of category theory to show that Hilbert Space Conservatism has more structure than Algebraic Imperialism?

An answer to this question would inform future work in the foundations and interpretation of quantum field theory.

7 Conclusion

In this paper, I took up the question of whether the allegedly novel and emergent behavior in the "singular limit" of certain infinite quantum statistical systems thwarts our attempts to understand reduction through the classical philosophical accounts. Classical accounts of reduction require the reduced theory to be definable and deducible from the reducing theory, so one might look to see whether the quantities of infinite quantum statistical systems are definable or deducible from the quantities of finite quantum statistical systems. But instead of tackling the issues of deduction and definability directly, I changed the subject, focusing instead on some motivating features behind the requirements of deduction and definability.

Namely, I took up the questions: are the quantities of infinite quantum statistical systems forced upon us by the quantities of finite quantum statistical systems? And do the quantities of infinite quantum statistical systems contain more information or structure than the quantities of finite quantum statistical systems? After all, the definability and deducibility requirements together entail a sense in which a reduced theory is forced upon us by the reducing theory and contains no more information or structure than the reducing theory. I answered these questions in the affirmative. I showed that uniqueness theorems for limiting procedures entail a sense in which the quantities of infinite quantum statistical systems are forced upon us by the quantities of finite quantum statistical systems. And I showed that functorial comparisons of categories of models for representing the quantities of infinite systems and finite systems show a sense in which the quantities of infinite quantum statistical systems contain no more information or structure than the properties of finite quantum statistical systems.

It is worth repeating a caveat made in the beginning of the paper about these results. I do not claim that the purely mathematical results presented in this paper are sufficient to show that thermodynamics reduces to statistical mechanics. I am happy to concede that such a reduction would presumably involve the physical semantics, dynamics, and perhaps even further interpretive aspects of the theories at issue (in addition to the mathematical aspects discussed here).[19] My focus in this paper has been only on the idea that a purely mathematical feature—the presence of "singular limits"—might thwart reduction. I have argued that the mathematical results I present in this paper show that these "singular limits" do not present the immediate and insurmountable challenge to reduction that others have thought.

Thus, I conclude that—whatever else we learn from the interesting features of so-called "singular limits"—our practices surrounding infinite quantum statistical systems still share much in common with classical accounts of intertheoretic reduction. Although this does not show that thermodynamics reduces to statistical mechanics, it does show that "singular limits" need not be understood as thwarting a classical reduction.

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[19] Thanks to an anonymous reviewer for this point.

Appendix: Proofs of Results

This appendix contains proofs of all results in §4 and §5.

Proposition 1. *Given any Banach space X, its dual X^* is complete in the weak* topology as a locally convex vector space.*

Proof. Suppose $\{y_\beta\}$ is a Cauchy net in X^*. Define $y : X \to \mathbb{C}$ by

$$y(x) = \lim y_\beta(x)$$

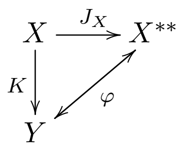
for all $x \in X$. We know that this limit exists because for any $x \in X$, $y_\beta(x)$ is a Cauchy net in $\mathbb{C}$, which means it must converge because $\mathbb{C}$ is complete.

Now we must show that $y \in X^*$, i.e., that y is linear and bounded. The functional y is linear because for any $x, x' \in X$ and $\alpha \in \mathbb{C}$,

$$y(x+\alpha x') = \lim y_\beta(x+\alpha x') = \lim y_\beta(x) + \alpha \lim y_\beta(x') = y(x) + \alpha y(x')$$

Notice that because $|y_\beta(x)|$ is bounded for each $x \in X$, it follows from the principle of uniform boundedness that $\|y_\beta\|$ is bounded (Reed and Simon, 1980). Hence, y is bounded with norm $\|y\| \leq \sup_\beta \|y_\beta\|$. Finally, we must show that y_β converges to y in the weak* topology on X^*. But this holds by construction because for any $x \in X$, $(y - y_\beta)(x)$ converges to zero in $\mathbb{C}$. □

Proposition 2. *Suppose X is a Banach space and $J_X : X \to X^{**}$ is the canonical evaluation embedding of X in its bidual. Suppose we are given another faithful linear embedding $K : X \to Y$ of X in a complete locally convex vector space Y such that $K(X)$ is dense in Y in the locally convex vector space topology on Y. Suppose, in addition, that K is a homeomorphism from X to $K(X)$ in the weak topology on X and the subspace topology on $K(X)$ generated by the locally convex vector space topology on Y. Then there is a vector space isomorphism $\varphi : Y \to X^{**}$ that is a homeomorphism in the locally convex vector space topology on Y and the weak* topology on X^{**} and such that $J_X = \varphi \circ K$.*



Proof. Suppose $K : X \to Y$ is such an embedding. By Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15), the maps $\varphi_0 = J_X \circ K^{-1}$ and $\psi_0 = K \circ J_X^{-1}$ extend uniquely to continuous linear maps $\varphi : Y \to X^{**}$ and $\psi : X^{**} \to Y$ in the weak* topology on X^{**} and the locally convex vector space topology on Y. Since $\varphi_0 \circ \psi_0$ and $\psi_0 \circ \varphi_0$ are the identity operators on $J_X(X)$ and $K(X)$, respectively, it follows that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity operators on X^{**} and Y, respectively. Thus, φ is an isomorphism and a homeomorphism in the weak* topology on X^{**} and the locally convex vector space topology on Y. By construction, we have $\varphi \circ K = \varphi_0 \circ K = J_X \circ K^{-1} \circ K = J_X$. □

Proposition 3. *Suppose $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism between C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then there is a unique weak* continuous extension $\tilde{\alpha} : \mathfrak{A}^{**} \to \mathfrak{B}^{**}$ such that $\tilde{\alpha} \circ J_{\mathfrak{A}} = J_{\mathfrak{B}} \circ \alpha$, where $J_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^{**}$ and $J_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}^{**}$ are the canonical evaluation maps.*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha} & \mathfrak{B} \\ {\scriptstyle J_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathfrak{B}}} \\ \mathfrak{A}^{**} & \xrightarrow[\tilde{\alpha}]{} & \mathfrak{B}^{**} \end{array}$$

Proof. First, we show that α is continuous in the weak Banach space topologies on $\mathfrak{A}$ and $\mathfrak{B}$. Suppose we have a net $\{A_\beta\} \subseteq \mathfrak{A}$ such that $A_\beta \to A$ weakly. Then for all $\rho \in \mathfrak{A}^*$, $\rho(A_\beta) \to \rho(A)$. Consider the net $\{\alpha(A_\beta)\} \subseteq \mathfrak{B}$. For any $\sigma \in \mathfrak{B}^*$, $\sigma \circ \alpha \in \mathfrak{A}^*$, so it follows that $\sigma \circ \alpha(A_\beta) \to \sigma \circ \alpha(A)$. Hence, $\alpha(A_\beta) \to \alpha(A)$ weakly, and it follows that α is weakly continuous.

It follows from Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15) that the map $J_{\mathfrak{B}} \circ \alpha \circ J_{\mathfrak{A}}^{-1} : J_{\mathfrak{A}}(\mathfrak{A}) \to \mathfrak{B}^{**}$ extends uniquely to a weak* continuous map $\tilde{\alpha} : \mathfrak{A}^{**} \to \mathfrak{B}^{**}$. By construction,

$$\tilde{\alpha} \circ J_{\mathfrak{A}} = J_{\mathfrak{B}} \circ \alpha \circ J_{\mathfrak{A}}^{-1} \circ J_{\mathfrak{A}} = J_{\mathfrak{B}} \circ \alpha$$

□

Proposition 4. *G is not full, i.e., G forgets structure.*

Proof. By Prop. 5 below, F and G form an adjunction, and we know G is faithful. So it follows from Prop. 3.4.1 of Borceux (1994, p. 114) that G is full only if the counit of the adjunction $\tilde{1} : G \circ F \to 1_{\mathbf{W^*\text{-}Alg}}$ is a natural isomorphism.

The counit of the adjunction has as its component on any object $\mathfrak{R}$ in **W*-Alg** the unique continuous extension $\tilde{1}_{\mathfrak{R}} : \mathfrak{R}^{**} \to \mathfrak{R}$ of the identity arrow $1_{\mathfrak{R}} : \mathfrak{R} \to \mathfrak{R}$ in the weak topology on the domain and the weak* topology on the codomain, which exists by Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15) since $\mathfrak{R}$ is weak* complete. Notice that $\tilde{1}_{\mathfrak{R}}$ is *not* the identity map from $\mathfrak{R}^{**}$ to itself, because we require continuity in the weak* topology on the codomain rather than the weak topology, so that the codomain is already complete and even after continuously extending $1_{\mathfrak{R}}$ in the relevant topologies, we get a map into the codomain $\mathfrak{R}$. This map $\tilde{1}_{\mathfrak{R}}$ satisfies the universal property for counits, i.e., given any $\mathfrak{A}$ in **C*-Alg**, any $\mathfrak{R}$ in **W*-Alg** and arrow $\tilde{\alpha} : G(\mathfrak{A}) \to \mathfrak{R}$, there is a unique arrow $\alpha : \mathfrak{A} \to F(\mathfrak{R})$ such that $\tilde{\alpha} = \tilde{1}_{\mathfrak{R}} \circ G(\alpha)$. Here, α is just the restriction of $\tilde{\alpha}$ from $G(\mathfrak{A}) = \mathfrak{A}^{**}$ to $\mathfrak{A}$, i.e., $\alpha = \tilde{\alpha}_{|\mathfrak{A}}$.

$$\begin{array}{ccc} G(\mathfrak{A}) & \xrightarrow{\tilde{\alpha}} & \mathfrak{R} \\ {\scriptstyle G(\alpha)}\downarrow & \nearrow {\scriptstyle \tilde{1}_{\mathfrak{R}}} & \\ G \circ F(\mathfrak{R}) & & \end{array}$$

$$\mathfrak{A} \overset{\exists ! \alpha}{\dashrightarrow} F(\mathfrak{R})$$

Indeed, $\tilde{1}$ is a natural transformation because for any two objects $\mathfrak{R}_1$ and $\mathfrak{R}_2$ in **W*-Alg** and any arrow $\alpha : \mathfrak{R}_1 \to \mathfrak{R}_2$, we know that $\alpha \circ \tilde{1}_{\mathfrak{R}_1} = \tilde{1}_{\mathfrak{R}_2} \circ G \circ F(\alpha)$ because there is a unique weakly continuous extension of α from $\mathfrak{R}_1$ to $G \circ F(\mathfrak{R}_1) = \mathfrak{R}_1^{**}$ by Cor. 1.2.3 of Kadison and Ringrose (1997, p. 15).

$$\begin{array}{ccc} G \circ F(\mathfrak{R}_1) & \xrightarrow{G \circ F(\alpha)} & G \circ F(\mathfrak{R}_2) \\ {\scriptstyle \tilde{1}_{\mathfrak{R}_1}}\downarrow & & \downarrow{\scriptstyle \tilde{1}_{\mathfrak{R}_2}} \\ \mathfrak{R}_1 & \xrightarrow[\alpha]{} & \mathfrak{R}_2 \end{array}$$

In general, $\tilde{1}_{\mathfrak{R}}$ will not be an isomorphism. For, if $\mathfrak{R} \ncong \mathfrak{R}^{**}$ (as is the case for infinite dimensional W*-algebras), since $1_{\mathfrak{R}} : \mathfrak{R} \to \mathfrak{R}$ is surjective and $J_{\mathfrak{R}}(\mathfrak{R}) \subsetneq \mathfrak{R}^{**}$, it follows that $\tilde{1}_{\mathfrak{R}} : \mathfrak{R}^{**} \to \mathfrak{R}$ cannot be one-to-one and so cannot be an isomorphism. Thus, the counit $\tilde{1}$ is not a natural isomorphism, and so G is not full. □

Proposition 5. *F and G form an adjunction, with left adjoint G, right adjoint F and unit J.*

Proof. First, we know that J is a natural transformation from $F \circ G$

to $1_{\mathbf{C^*\text{-}Alg}}$ because $F \circ G(\mathfrak{A})$ is just $\mathfrak{A}^{**}$ considered as an object in **C*-Alg**. So for any two objects $\mathfrak{A}$ and $\mathfrak{B}$ in **C*-Alg** and any arrow $\alpha : \mathfrak{A} \to \mathfrak{B}$, it follows by Prop. 3 that

$$J_{\mathfrak{B}} \circ 1_{\mathbf{C^*\text{-}Alg}}(\alpha) = F \circ G(\alpha) \circ J_{\mathfrak{A}}$$

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha} & \mathfrak{B} \\ {\scriptstyle J_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle J_{\mathfrak{B}}} \\ F \circ G(\mathfrak{A}) & \xrightarrow[F \circ G(\alpha)]{} & F \circ G(\mathfrak{B}) \end{array}$$

J serves as the unit because for any objects $\mathfrak{A}$ in **C*-Alg** and $\mathfrak{R}$ in **W*-Alg** and arrow $\alpha : \mathfrak{A} \to F(\mathfrak{R})$, there is a unique arrow $\tilde{\alpha} : \mathfrak{A}^{**} \to \mathfrak{R}$ such that $F(\tilde{\alpha}) \circ J_{\mathfrak{A}} = \alpha$. The arrow $\tilde{\alpha}$ is given by the unique weakly continuous extension of α to the weak completion $G(\mathfrak{A}) = \mathfrak{A}^{**}$.

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\alpha} & F(\mathfrak{R}) \\ {\scriptstyle J_{\mathfrak{A}}}\downarrow & \nearrow{\scriptstyle F(\tilde{\alpha})} & \\ F \circ G(\mathfrak{A}) & & \end{array}$$

$$G(\mathfrak{A}) \overset{\exists! \tilde{\alpha}}{\dashrightarrow} \mathfrak{R}$$

□

Proposition 6. *Let* $\mathfrak{R}$ *be a W*-algebra and let*

$$I = \{A \in \mathfrak{R} : \rho(A) = 0 \textit{ for all } \rho \in \mathfrak{R}_*\}$$

There is a C-algebra* $\mathfrak{A}$ *such that* $\mathfrak{R} \cong \mathfrak{A}^{**}$ *iff*

$$\mathfrak{R}_* \cong \{\omega \in \mathfrak{R}^* : \textit{ if } A \in I, \textit{ then } \omega(A) = 0\} \tag{1}$$

$$\begin{array}{ccc} & & \mathfrak{R} \\ & & \updownarrow \\ \exists \mathfrak{A} & \overset{G}{\Longrightarrow} & \mathfrak{A}^{**} \end{array}$$

Proof. ($\Leftarrow$) Suppose that $\mathfrak{R}$ satisfies Eq. 1. By Prop. 2.11.8 of Dixmier (1977, p. 63) and Cor. 1.8.3 of Dixmier (1977, p. 21), it follows that $\mathfrak{A} := \mathfrak{R}/I$ is a C*-algebra such that $\mathfrak{A}^* \cong \mathfrak{R}_*$. Hence, we know that $\mathfrak{A}^{**}$ is isomorphic to $\mathfrak{R}$ as a Banach space. It suffices to show that the canonical embedding $J_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}^{**}$ induces a map (which we will also call $J_{\mathfrak{A}}$ since no ambiguity will result) from $\mathfrak{R}/I$ to $\mathfrak{R}$ that preserves multiplication in order to show that $\mathfrak{A}^{**}$ and $\mathfrak{R}$ are *-isomorphic. We can choose the isomorphisms so that $J_{\mathfrak{A}}$ induces the map

$$(A + I) \in \mathfrak{R}/I \mapsto A \in \mathfrak{R}$$

We lose no generality in choosing a representative $A \in A + I$ in this way (although we require the axiom of choice). It is easy to check that this map is a faithful *-homomorphism whose range is weak* dense in $\mathfrak{R}$. It follows that $J_{\mathfrak{A}}$ extends to a *-isomorphism from $\mathfrak{A}^{**}$ to $\mathfrak{R}$.
($\Rightarrow$) Suppose that $\mathfrak{R} \cong \mathfrak{A}^{**}$ for some C*-algebra $\mathfrak{A}$. We know from Cor. 1.13.3 of Sakai (1971, p. 30) that $\mathfrak{R}_* \cong \mathfrak{A}^*$. Let

$$I_{\mathfrak{A}} = \{A \in \mathfrak{A}^{**} : \rho(A) = 0 \text{ for all } \rho \in \mathfrak{A}^*\}$$

Then $\mathfrak{R}_* \cong \mathfrak{A}^* \cong (\mathfrak{A}^{**}/I_{\mathfrak{A}})^*$, and by Prop. 2.11.8 of Dixmier (1977, p. 63), we know that

$$(\mathfrak{A}^{**}/I_{\mathfrak{A}})^* \cong \{\omega \in \mathfrak{A}^{***} : \text{ if } A \in I_{\mathfrak{A}}, \text{ then } \omega(A) = 0.\}$$

which implies that $\mathfrak{R}_*$ satisfies Eq. 1. □

Proposition 7. *Let $\mathfrak{R}$ be a W*-algebra. If $\mathfrak{A}$ and $\mathfrak{B}$ are two C*-algebras such that $\mathfrak{A}^{**} \cong \mathfrak{R} \cong \mathfrak{B}^{**}$, then $\mathfrak{A}$ is *-isomorphic to $\mathfrak{B}$.*

Proof. We know from Cor. 1.13.3 of Sakai (1971, p. 30) that $\mathfrak{A}^* \cong \mathfrak{B}^*$. As before, let

$$I_{\mathfrak{A}} = \{A \in \mathfrak{A}^{**} : \rho(A) = 0 \text{ for all } \rho \in \mathfrak{A}^*\}$$

$$I_{\mathfrak{B}} = \{B \in \mathfrak{B}^{**} : \rho(B) = 0 \text{ for all } \rho \in \mathfrak{B}^*\}$$

Again, we know from Prop. 2.11.8 of Dixmier (1977, p. 63) that $(\mathfrak{A}^{**}/I)^* \cong \mathfrak{A}^*$ and $(\mathfrak{B}^{**}/I)^* \cong \mathfrak{B}^*$. Hence, the canonical surjective *-homomorphisms $\mathfrak{A}^{**} \to \mathfrak{A}^{**}/I$ and $\mathfrak{B}^{**} \to \mathfrak{B}^{**}/I$ can serve to define surjective *-homomorphisms from $\mathfrak{R}$ to the C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, who have isomorphic dual spaces. It follows from Thm. 2 of Feintzeig (2017a) that $\mathfrak{A} \cong \mathfrak{B}$. □

Proposition 8. *There are W*-algebras that are not whole.*

Proof. Let $\mathfrak{R} := L^\infty(\mathbb{R})$ be the algebra of equivalence classes of bounded Borel measurable complex-valued functions on the real line that differ only on sets of Lebesgue measure zero, where the algebraic structure is defined by pointwise operations. We know $\mathfrak{R}$ has no pure normal states (See, e.g. Halvorson, 2001) (and any W*-algebra without pure normal states will suffice for the rest of the proof).[20] Suppose, for contradiction, that $\mathfrak{R}$ is whole. Then, by Prop. 6, we know $\mathfrak{R}_* = \mathfrak{A}^*$ is the dual space to some C*-algebra $\mathfrak{A}$. By the Banach-Alaoglu theorem (Cor. 1. 6.6 of Kadison and Ringrose, 1997, p. 46) and the Krein-Milman theorem (Thm. 1.4.3 of Kadison and Ringrose, 1997, p. 32), the state space of $\mathfrak{A}$, which is the normal state space of $\mathfrak{R}$, must contain pure states, which yields a contradiction. □

Proposition 9. *$\overline{G}$ is a categorical equivalence between **C*-Alg** and **wW*-Alg**.*

Proof. $\overline{G}$ is faithful because G is faithful. Prop. 6 shows that $\overline{G}$ is essentially surjective. To show that $\overline{G}$ is full, consider any arrow $\alpha : \mathfrak{R}_1 \to \mathfrak{R}_2$ in **wW*-Alg**, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the unique C*-algebras such that $\mathfrak{A}_1^{**} \cong \mathfrak{R}_1$ and $\mathfrak{A}_2^{**} \cong \mathfrak{R}_2$. To simplify notation, let $J_1 := J_{\mathfrak{A}_1}$ and $J_2 := J_{\mathfrak{A}_2}$. Since α is a *-whomomorphism, we know that the restriction $\alpha_{|J_1(\mathfrak{A}_1)}$ of α to $J_1(\mathfrak{A}_1)$ is a *-homomorphism from the C*-algebra $J_1(\mathfrak{A}_1)$ to the C*-algebra $J_2(\mathfrak{A}_2)$. It follows that $\alpha = \tilde{\alpha}_{|J_1(\mathfrak{A}_1)} = \overline{G}(\alpha_{|J_1(\mathfrak{A}_1)})$, where $\tilde{\alpha}_{|J_1(\mathfrak{A}_1)} : \mathfrak{R}_1 \to \mathfrak{R}_2$ is the unique weak* continuous extension of $\alpha_{|J_1(\mathfrak{A}_1)}$ by Prop. 3. □

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[20]See Ruetsche and Earman (2011) for more on algebras with no pure normal states.

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