The classical limit of a state on the Weyl algebra

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This paper considers states on the Weyl algebra of the canonical commutation relations over the phase space $\mathbb{R}^{2n}$. We show that a state is regular iff its classical limit is a countably additive Borel probability measure on $\mathbb{R}^{2n}$. It follows that one can "reduce" the state space of the Weyl algebra by altering the collection of quantum mechanical observables so that all states are ones whose classical limit is physical.

I. INTRODUCTION

In quantum theories we often restrict attention to *regular* states, thereby ruling out non-regular states as unphysical or pathological.[58] Although this restriction is often seen as instrumental, this paper will propose a principled justification. We will show that the regular states play a privileged role in explaining the success of classical physics.

We explain the success of classical physics through the *classical limit*. The classical limit of a quantum state, in the sense discussed in this paper, is the result of looking at approximations on larger and larger scales until the effects of quantum mechanics disappear. The central result of this paper shows that we can recover all of the classical states of a classical theory by taking the classical limits of only regular states. The classical limits of non-regular states play no role in classical physics.

We will deal only with theories with finitely many degrees of freedom modeled by a phase space $\mathbb{R}^{2n}$, in which case the physical states of the classical theory are countably additive Borel probability measures on $\mathbb{R}^{2n}$. Quantum states are then positive, normalized linear functionals on the C*-algebra of (bounded) physical magnitudes, or *observables*, satisfying the canonical commutation relations. This algebra is known as the Weyl algebra over $\mathbb{R}^{2n}$. The central purpose of this paper is thus to establish the following claim:

Theorem. *A state on the Weyl algebra over $\mathbb{R}^{2n}$ is regular iff its classical limit is a countably additive Borel probability measure on $\mathbb{R}^{2n}$.*

The intended interpretation is that one need only use the classical limits of regular quantum states to explain the success of classical physics.

Moreover, this discussion has import for the construction of quantum theories. We construct quantum theories, at least in the algebraic approach, by constructing their algebra of observables. But in many cases there is no widespread consensus about precisely which algebra we should use to represent the observables of a given quantum system.[4,16,19] Although the Weyl algebra over $\mathbb{R}^{2n}$ is one commonly used tool, it always allows for non-regular states. If one has the desire to restrict attention to regular states, then one might look for another algebra that allows for only regular states. We demonstrate that such an algebra exists.

It is known that each C*-algebra uniquely determines its state space.[1] And if this state space is "too large" in the sense that it contains unphysical or pathological states, then there is a general procedure (established in previous work) for constructing a new C*-algebra whose state space consists in precisely the collection of physical states:[42]

Theorem 1 (Feintzeig[19])**.** *Let $\mathfrak{A}$ be a C*-algebra and let $V \subseteq \mathfrak{A}^*$. Then there exists a unique C*-algebra $\mathfrak{B}$ and a surjective *-homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{B}^* \cong V$ with the isomorphism given by $\omega \in \mathfrak{B}^* \mapsto (\omega \circ f) \in V$ iff the following conditions hold:*

(i) for all $A, B \in \mathfrak{A}$,

$$\sup_{\omega \in V; \|\omega\|=1} |\omega(AB)| \leq \sup_{\omega \in V; \|\omega\|=1} |\omega(A)| \sup_{\omega \in V; \|\omega\|=1} |\omega(B)|$$

(ii) V is maximal *in the sense that*

$$\textit{for all } \omega \in \mathfrak{A}^*, \textit{ if } \bigcap_{\rho \in V} Ker(\rho) \subseteq Ker(\omega) \textit{ , then } \omega \in V$$

To prove Thm. 1, one shows the C*-algebra $\mathfrak{B}$ is *-isomorphic to $\mathfrak{A}/\mathcal{N}(V)$, where $\mathcal{N}(V)$ is the ideal given by $\mathcal{N}(V) := \bigcap_{\rho \in V} Ker(\rho)$. This allows one to choose a subspace of $\mathfrak{A}^*$ satisfying (i) and (ii) and construct a unique C*-algebra with precisely the desired dual space. Thus, we can use Thm. 1 to find a new algebra—constructed from the Weyl algebra—which allows for only regular states. The quantum states on this new algebra suffice for explaining the success of classical physics through the classical limit.

The structure of the paper is as follows. In §II, I define the Weyl algebra over $\mathbb{R}^{2n}$ and the notion of a regular state. In §III, I clarify the notion of the "classical limit" of a state on the Weyl algebra using a continuous field of C*-algebras. In Section IV, I establish some small lemmas to characterize the countably additive Borel measures on $\mathbb{R}^{2n}$ in terms of algebraic structure, and I apply Thm. 1 to the purely classical system with phase space $\mathbb{R}^{2n}$. Finally, in Section V, I prove the main result and discuss its significance. The results of this paper involve little, if any, mathematical novelty. I hope, however, that the perspective I provide on the construction of new quantum theories is of interest.

II. PRELIMINARIES

The Weyl algebra over $\mathbb{R}^{2n}$ is formed by deforming the product of the C*-algebra $AP(\mathbb{R}^{2n})$ of complex-valued almost periodic functions on $\mathbb{R}^{2n}$. The C*-algebra $AP(\mathbb{R}^{2n})$ is generated by functions $W_0(x) : \mathbb{R}^{2n} \to \mathbb{C}$ for each $x \in \mathbb{R}^{2n}$ defined by

$$W_0(x)(y) := e^{ix \cdot y}$$

for all $y \in \mathbb{R}^{2n}$, where $\cdot$ is the standard inner product on $\mathbb{R}^{2n}$. Polynomials (with respect to pointwise multiplication, addition, and complex conjugation) of functions of the form $W_0(x)$ for $x \in \mathbb{R}^{2n}$ are norm dense in $AP(\mathbb{R}^{2n})$ with respect to the standard supremum norm.[45]

The *Weyl algebra* over $\mathbb{R}^{2n}$, denoted $\mathcal{W}_h(\mathbb{R}^{2n})$, for $h \in (0, 1]$ is generated from the same set of functions by defining a new multiplication operation. The symbol $W_h(x) \in \mathcal{W}_h(\mathbb{R}^{2n})$ is used now to denote the element $W_0(x)$ as it is considered in the new C*-algebra. Define the non-commutative multiplication operation on $\mathcal{W}_h(\mathbb{R}^{2n})$ by

$$W_h(x)W_h(y) := e^{\frac{ih}{2}\sigma(x,y)}W_h(x + y)$$

for all $x, y \in \mathbb{R}^{2n}$, where σ is the standard symplectic form on $\mathbb{R}^{2n}$ given by

$$\sigma((a, b), (a', b')) := a' \cdot b - a \cdot b'$$

for $a, b, a', b' \in \mathbb{R}^n$ and $\cdot$ is now the standard inner product on $\mathbb{R}^n$. The Weyl algebra is the norm completion in the minimal regular norm[6,41,53] of polynomials of elements of the form $W_h(x)$ for $x \in \mathbb{R}^{2n}$ with respect to the non-commutative multiplication operation.

It is this C*-algebra (sometimes known as the *CCR algebra*, or the *Weyl form of the CCRs*) that is often used to model the physical magnitudes, or *observables*, of a quantum mechanical system constructed from a classical system with phase space $\mathbb{R}^{2n}$. One can then take the positive, normalized linear functionals, or *states*, on the C*-algebra of physical magnitudes to model the physically realizable states of the quantum system.[8,9,25,29]

However, it is known that many of the states on $\mathcal{W}_h(\mathbb{R}^{2n})$ are pathological, in the sense that they violate continuity conditions some believe to be necessary for physics.[3,26,27,58] Among the states one might consider pathological are the *non-regular* states. A bounded linear functional ω on $\mathcal{W}_h(\mathbb{R}^{2n})$ is called *regular* just in case for all $x \in \mathbb{R}^{2n}$, the mapping

$$t \in \mathbb{R} \mapsto \omega(W_h(tx))$$

is continuous. The well known Stone-von Neumann theorem[53,59] tells us that if (and only if) a state is regular, its GNS representation is quasiequivalent to the ordinary Schrödinger representation on the Hilbert space $L^2(\mathbb{R}^n)$, and hence leads to a reconstruction of the orthodox formalism for quantum mechanics. In this case, one sees immediately that the restriction to regular states is desirable for applications in physics.

When one moves, however, to the context of quantum field theory, the story so far breaks down. The Stone-von Neumann theorem is no longer applicable to the phase space of a field-theoretic system, which is infinite-dimensional, and hence, fails to be locally compact. Instead, infinitely many (even regular) unitarily inequivalent irreducible representations of the Weyl algebra appear, and one has no principled way of using any particular irreducible representation to motivate a restriction to some subset of states.

The goal of this paper, however, is to provide a principled method for restricting the state space of the Weyl algebra by looking at its purely algebraic structure, rather than its Hilbert space representations. In particular, by understanding the algebra $\mathcal{W}_h(\mathbb{R}^{2n})$ as part of a *strict and continuous deformation quantization* of $AP(\mathbb{R}^{2n})$, one can use purely algebraic tools to show that the regular states are already privileged. With this quantization, for any state ω on $\mathcal{W}_h(\mathbb{R}^{2n})$, one can construct the continuous field of states that form a "constant" section of linear functionals. We will call the value of this section at $h = 0$ the *classical limit* of the state ω. The privileged states on $\mathcal{W}_h(\mathbb{R}^{2n})$ for our purposes will be the ones whose classical limits can be used to explain the success of classical physics. We will understand the classical limit of a quantum state to be useful for explaining the success of classical physics just in case it is a physical state on $AP(\mathbb{R}^{2n})$.

A natural candidate for the collection of physical states of a classical theory with phase space $\mathbb{R}^{2n}$ is the collection of countably additive Borel probability measures on $\mathbb{R}^{2n}$. But these states form a proper subset of the state space of $AP(\mathbb{R}^{2n})$. Hence we will think of the countably additive Borel probability measures on $\mathbb{R}^{2n}$ as privileged classical states for our purposes. To explain the success of classical physics, we only need to find enough quantum states to recover the privileged classical states in the classical limit. Our main result shows that if one accepts that the countably additive Borel probability measures are the only physical states of the classical theory with phase space $\mathbb{R}^{2n}$, then only the regular states on $\mathcal{W}_h(\mathbb{R}^{2n})$ are needed to explain the success of classical physics through the classical limit.

III. CLASSICAL LIMITS

In this section, we make precise the notion of the *classical limit* of a quantum state. Define the "classical" Weyl algebra as $\mathcal{W}_0(\mathbb{R}^{2n}) := AP(\mathbb{R}^{2n})$. Consider the family of C*-algebras $\{\mathcal{W}_h(\mathbb{R}^{2n})\}_{h\in[0,1]}$ with "quantization" maps, denoted $\mathcal{Q}_h : \mathcal{W}_0(\mathbb{R}^{2n}) \to \mathcal{W}_h(\mathbb{R}^{2n})$, given by the unique linear norm continuous extension of

$$\mathcal{Q}_h(W_0(x)) := W_h(x)$$

for all $x \in \mathbb{R}^{2n}$ and each $h \in [0,1]$. For each $h \in [0,1]$, the map $\mathcal{Q}_h$ has a norm dense range in $\mathcal{W}_h(\mathbb{R}^{2n})$ and satisfies

$$\mathcal{Q}_h(A^*) = \mathcal{Q}_h(A)^*$$

for all $A \in \mathcal{W}_0(\mathbb{R}^{2n})$.

Moreover, it is known that:

(i) For all $A, B \in \mathcal{W}_0(\mathbb{R}^{2n})$,

$$\lim_{h\to 0} \|\frac{i}{h}[\mathcal{Q}_h(A), \mathcal{Q}_h(B)] - \mathcal{Q}_h(\{A, B\})\| = 0$$

where $\{\cdot,\cdot\}$ is the standard Poisson bracket on $\mathbb{R}^{2n}$ corresponding to the symplectic form σ and $[\cdot,\cdot]$ is the commutator (defined by $[X, Y] := XY - YX$).

(ii) For all $A, B \in \mathcal{W}_0(\mathbb{R}^{2n})$,

$$\lim_{h\to 0} \|\mathcal{Q}_h(A)\mathcal{Q}_h(B) - \mathcal{Q}_h(AB)\| = 0$$

(iii) For all $A \in \mathcal{W}_0(\mathbb{R}^{2n})$, the mapping $h \mapsto \|\mathcal{Q}_h(A)\|$ is continuous.

Furthermore, for each $h \in [0,1]$, $\mathcal{Q}_h$ is injective and $\mathcal{Q}_h[\mathcal{W}_0(\mathbb{R}^{2n})]$ is closed under the product in $\mathcal{W}_h(\mathbb{R}^{2n})$. This structure $(\mathcal{W}_h, \mathcal{Q}_h)_{h\in[0,1]}$ is thus a *strict deformation quantization*.[7,12,37,39,40,56,46]

In fact, this structure gives rise to a *continuous bundle of C*-algebras* $(\{\mathcal{W}_h\}_{h\in[0,1]}, \mathcal{K})$ over the base space $[0,1]$.[14,37,39] The C*-algebra of continuous sections $\mathcal{K}$ is given by the unique C*-subalgebra of $\prod_{h\in[0,1]} \mathcal{W}_h(\mathbb{R}^{2n})$ containing the elements of the form

$$h \mapsto \mathcal{Q}_h(A)$$

for all $A \in \mathcal{W}_0(\mathbb{R}^{2n})$. Defining the global quantization map $\mathcal{Q} : \mathcal{W}_0(\mathbb{R}^{2n}) \to \mathcal{K}$ by

$$\mathcal{Q}(A) := [h \mapsto \mathcal{Q}_h(A)]$$

for all $A \in \mathcal{W}_0(\mathbb{R}^{2n})$, we have that $(\{\mathcal{W}_h(\mathbb{R}^{2n})\}_{h\in[0,1]}, \mathcal{K}; \mathcal{Q})$ is a *continuous quantization.*[7,37,39,47]

Now we can consider *continuous fields of linear functionals*[39] as families $\{\omega_h\}_{h\in[0,1]}$, where $\omega_h \in \mathcal{W}_h(\mathbb{R}^{2n})^*$ for each $h \in [0,1]$, and for each continuous section $\varphi \in \mathcal{K}$,

$$\lim_{h\to 0} \omega_h(\varphi(h)) = \omega_0(\varphi(0))$$

In particular, for a fixed $H \in [0,1]$ and $\omega \in \mathcal{W}_H(\mathbb{R}^{2n})^*$, since $\mathcal{Q}_H$ is norm continuous we can define $\hat{\omega} \in \mathcal{W}_0(\mathbb{R}^{2n})^*$ by

$$\hat{\omega} := \omega \circ \mathcal{Q}_H$$

And we then construct the "constant" continuous field of functionals $\{\omega_h\}_{h\in[0,1]}$ through $\omega_0 = \hat{\omega}$ and $\omega_H = \omega$ by defining $\omega_h : \mathcal{W}_h(\mathbb{R}^{2n}) \to \mathbb{C}$ as the unique norm continuous extension of

$$\omega_h(\mathcal{Q}_h(A)) := \hat{\omega}(A)$$

for all $A \in \mathcal{W}_0(\mathbb{R}^{2n})$. It is easy to see that $\{\omega_h\}_{h\in[0,1]}$ is indeed a continuous field of functionals, so we will define $\hat{\omega}$ as the *classical limit* of ω. In particular, since for any $h \in [0,1]$, the quantization map $\mathcal{Q}_h$ is positive, we know that when $\omega \in \mathcal{W}_h(\mathbb{R}^{2n})^*$ is a state, its classical limit $\hat{\omega}$ is a state as well.[48] We will use this notion of the classical limit in what follows.

Before proceeding, a caveat is in order: the $\hbar \to 0$ limit described in this section is of course only *one* way, among many others, to make precise the classical limit of quantum theories. Furthermore, when one uses the $\hbar \to 0$ limit as outlined above, the classical limit of a state is understood only *relative* to the continuous quantization considered. Nevertheless, we choose to investigate classical limits defined relative to the above continuous quantization. The current paper demonstrates this notion of the classical limit is fruitful at least for understanding the significance of regular and non-regular states.

IV. CLASSICAL STATES

In this section, we look in more detail at the structure of the state space of a classical theory. This will help us understand the import of Thm. 1 for C*-algebras of physical magnitudes, and it will lead to some lemmas required for the proof of the main result. We focus on a classical theory whose phase space[49] is given by a locally compact Hausdorff topological space $\mathcal{M}$. Although in the next section we restrict attention to finite-dimensional symplectic vector spaces, for the moment we make no such restriction.

Prima facie, it's not clear which C*-algebra of functions on $\mathcal{M}$ we should take to model the bounded physical magnitudes of the classical system. Candidates include:

- $B(\mathcal{M})$, the algebra of bounded Borel measurable complex-valued functions on $\mathcal{M}$.
- $C_b(\mathcal{M})$, the algebra of bounded continuous complex-valued functions on $\mathcal{M}$.
- $C_0(\mathcal{M})$, the algebra of continuous complex-valued functions on $\mathcal{M}$ vanishing at infinity.

Of course, we have the following inclusion relations:

$$C_0(\mathcal{M}) \subseteq C_b(\mathcal{M}) \subseteq B(\mathcal{M})$$

The states on $C_0(\mathcal{M})$ are precisely the countably additive Borel probability measures on $\mathcal{M}$.[22,39] So if we think the physical classical states consist in precisely the countably additive Borel probability measures, then this gives us reason to use $C_0(\mathcal{M})$ to model the physical observables of the classical system. If we choose a larger C*-algebra, then we will in general allow for more states. However, we know that $B(\mathcal{M})$ is the bidual, or universal enveloping W*-algebra, of $C_0(\mathcal{M})$.[21] This means that elements of $B(\mathcal{M})$ can be approximated as weak (pointwise) limits of nets of physical magnitudes in $C_0(\mathcal{M})$. And furthermore, the normal states of $B(\mathcal{M})$ are precisely the states on $C_0(\mathcal{M})$. So we can also use $B(\mathcal{M})$, understood as a W*-algebra, to model possibly idealized (even discontinuous) physical magnitudes, where we understand elements of the normal state space of $B(\mathcal{M})$ to model physical states.

However, in the quantization procedure of §III we focused on a different algebra, the algebra of almost periodic functions.[50] When $\mathcal{M}$ is a locally compact abelian group, we know that $AP(\mathcal{M}) \subseteq C_b(\mathcal{M})$, but $AP(\mathcal{M})$ and $C_0(\mathcal{M})$ are in general not identical. As such, the state space of $AP(\mathcal{M})$ will in general not be equal to that of $C_0(\mathcal{M})$.[28,57] In this section, we will characterize the countably additive Borel measures as a subspace of the bounded linear functionals on $AP(\mathcal{M})$.

In fact, we will work with a more general collection of algebras of functions. For the remainder of this section, let $\mathfrak{C} \subseteq C_b(\mathcal{M})$ be an abelian C*-subalgebra of the bounded continuous complex-valued functions on $\mathcal{M}$ satisfying the constraints: (i) $\mathfrak{C}$ separates points of $\mathcal{M}$; and (ii) $\mathfrak{C}$ contains the constants. So $\mathfrak{C}$ may be the almost periodic functions, or it may be another algebra altogether. We characterize the countably additive Borel measures as a subset V of $\mathfrak{C}^*$, and we show that applying Thm. 1 with this choice of V transforms $\mathfrak{C}$ into $B(\mathcal{M})$.

To perform this procedure, we need to notice that $\mathfrak{C}$ is both "too small" and "too large". It is "too small" in the sense that it does not contain discontinuous functions like projections, which are needed for the spectral theory that allows us to interpret physical magnitudes. It is "too large" in the sense that it allows for "states at infinity", which may be considered pathological for physical applications. Thus, we will enlarge the algebra $\mathfrak{C}$ by completing it in its weak topology to form $\mathfrak{C}^{**}$. Then we will reduce the algebra by restricting ourselves to a priveleged collection of states on $\mathfrak{C}^{**}$.

Recall that $\mathfrak{C}$ is *-isomorphic to $C(\mathcal{P}(\mathfrak{C}))$, the continuous functions on the pure state space of $\mathfrak{C}$ with the weak* topology, which is a compact Hausdorff space. Furthermore, there is a continuous injection $k : \mathcal{M} \to \mathcal{P}(\mathfrak{C})$, called a *compactification*,[51] such that for any $f \in \mathfrak{C}$ and its surrogate $\hat{f} \in C(\mathcal{P}(\mathfrak{C}))$, $\hat{f} \circ k = f$. The map k sends each point in $\mathcal{M}$ to the pure state it determines; k is defined for all $p \in \mathcal{M}$ by

$$k(p)(f) := f(p)$$

for all $f \in \mathfrak{C}$. Similarly, all states (even mixed states) can be represented as Borel measures on $\mathcal{P}(\mathfrak{C})$. The Riesz-Markov theorem[54] implies that each $\omega \in \mathfrak{C}^*$ corresponds to a unique positive normalized Borel measure μ_ω such that[52]

$$\omega(f) = \int_{\mathcal{P}(\mathfrak{C})} \hat{f} d\mu_\omega$$

The weak completion of $\mathfrak{C}$ is then $\mathfrak{C}^{**} \cong B(\mathcal{P}(\mathfrak{C}))$, the bounded Borel functions on the pure state space of $\mathfrak{C}$, where the Borel structure is determined from the weak* topology on $\mathcal{P}(\mathfrak{C})$.

First, we need to establish that $\mathfrak{C}^{**}$ is indeed "large enough" to contain all of the discontinuous functions on $\mathcal{M}$ we desire.

Lemma 1. *If $S \subseteq \mathcal{M}$ is a Borel set in $\mathcal{M}$, then $k[S]$ is a Borel set in the weak* topology on $\mathcal{P}(\mathfrak{C})$.*

Proof. Since the Borel sets are generated by the open and closed sets, it suffices to prove the claim for the case where S is an open or closed set. So suppose $S \subseteq \mathcal{M}$ is either an open set or a closed set. Since k is a homeomorphism from $\mathcal{M}$ to $k[\mathcal{M}]$ in the subspace topology, $k[S]$ is either an open set or a closed set in $k[\mathcal{M}]$ in the subspace topology. Hence, there is either an open set or a closed set $S' \subseteq \mathcal{P}(\mathfrak{C})$ such that $S' \cap k[\mathcal{M}] = k[S]$. But Thm. 3.5.8 of Engelking (p. 169)[17] shows that $k[\mathcal{M}]$ is open, so $k[S]$ must be a Borel set. □

Lemma 2. *Suppose $f : \mathcal{P}(\mathfrak{C}) \to \mathbb{C}$ has support contained in $k[\mathcal{M}]$. Then f is Borel measurable on $\mathcal{P}(\mathfrak{C})$ iff $f \circ k$ is Borel measurable on $\mathcal{M}$.*

Proof. ($\Rightarrow$) Suppose f is Borel measurable on $\mathcal{P}(\mathfrak{C})$. Let S be a Borel set in $\mathbb{C}$. Then $f^{-1}[S]$ is Borel measurable in $\mathcal{P}(\mathfrak{C})$. It follows that $(f \circ k)^{-1}[S] = k^{-1}[f^{-1}[S]]$ is Borel measurable in $\mathcal{M}$ since k is continuous.

($\Leftarrow$) Suppose $f \circ k$ is Borel measurable on $\mathcal{M}$. Let S be a Borel set in $\mathbb{C}$. Then $(f \circ k)^{-1}[S]$ is Borel measurable in $\mathcal{M}$. Let $k[\mathcal{M}]^C$ denote $\mathcal{P}(\mathfrak{C}) \setminus k[\mathcal{M}]$. Since k is an injection and supp $f \subseteq k[\mathcal{M}]$, we know that $f^{-1}[S] = k[(f \circ k)^{-1}[S]] \cup k[\mathcal{M}]^C$ if $0 \in S$. On the other hand, if $0 \notin S$, then $f^{-1}[S] = k[(f \circ k)^{-1}[S]]$. The set $k[(f \circ k)^{-1}[S]]$ is Borel measurable by Lemma 1, and $k[\mathcal{M}]^C$ is Borel measurable because $k[\mathcal{M}]$ is measurable by Lemma 1. Hence $f^{-1}[S]$ is Borel measurable. □

Now that we know $\mathfrak{C}^{**}$ contains surrogates for all of the Borel functions on $\mathcal{M}$ and hence is "large enough", we can apply Thm. 1 to obtain the algebra $B(\mathcal{M})$ by restricting attention to an appropriate state space.

We define

$$V_0^C := \{\omega \in \mathfrak{C}^* |\ \mu_\omega(\mathcal{P}(\mathfrak{C}) \setminus k[\mathcal{M}]) = 0\}$$

where μ_ω is the measure on $\mathcal{P}(\mathfrak{C})$ determined by ω through the Riesz-Markov theorem. Furthermore, we define

$$V^C := \{\omega \in \mathfrak{C}^{***} |\ \bigcap_{\rho \in V_0^C} Ker(\rho) \subseteq Ker(\omega)\}$$

It follows that V^C is maximal in the sense of Thm. 1. We will show in Prop. 2 that V_0^C is the collection of linear functionals that can be represented by countably additive Borel measures on $\mathcal{M}$. But first we show that V^C is precisely the collection of functionals that reduce the algebra $\mathfrak{C}^{**}$ to $B(\mathcal{M})$.

Proposition 1. *The C^*-algebra $\mathfrak{B} = \mathfrak{C}^{**}/\mathcal{N}(V^C)$ of Thm. 1 is *-isomorphic to $B(\mathcal{M})$.*

Proof. Define $j : (\mathfrak{C}^{**}/\mathcal{N}(V^C)) \to B(\mathcal{M})$ by

$$j(f + \mathcal{N}(V^C)) = f \circ k$$

for any $f \in \mathfrak{C}^{**}$. This is well-defined because $\mathcal{N}(V^C)$ consists only of functions with support on $\mathcal{P}(\mathfrak{C}) \setminus k[\mathcal{M}]$. So for any $f, g \in \mathfrak{C}^{**}$, $f + \mathcal{N}(V^C) = g + \mathcal{N}(V^C)$ iff $f \circ k = g \circ k$. Furthermore, this already establishes that j is one-to-one.

j is onto: for every $\hat{f} \in B(\mathcal{M})$, $\hat{f} = f \circ k$ for

$$f(x) = \begin{cases} \hat{f} \circ k^{-1}(x) & \text{if } x \in k[\mathcal{M}] \\ 0 & \text{if } x \notin k[\mathcal{M}] \end{cases}$$

and it follows from Lemmas 1 and 2 that $f \in \mathfrak{C}^{**}$, i.e. f is Borel measurable on $\mathcal{P}(\mathfrak{C})$.

Since j obviously preserves pointwise algebraic operations, it follows that j is a *-isomorphism. $\square$

Thus, we can use the countably additive Borel measures on $\mathcal{M}$ to reduce the state space of an algebra of continuous bounded functions on $\mathcal{M}$ and in doing so recover the algebra $B(\mathcal{M})$. In particular, when we use the algebra $\mathfrak{C} = AP(\mathbb{R}^{2n})$, we can use the countably additive Borel measures on $\mathbb{R}^{2n}$ to reduce the state space of $AP(\mathbb{R}^{2n})^{**}$ and recover the algebra $B(\mathbb{R}^{2n})$.

The vector space V_0^C has an intuitive physical significance. We now show that V_0^C can indeed be characterized as the collection of countably additive Borel measures on $\mathcal{M}$, which we will use in what follows.

Proposition 2. *Let $\omega \in \mathfrak{C}^*$. Then $\omega \in V_0^C$ iff ω is continuous in the topology of pointwise convergence of functions on $\mathcal{M}$.*

Proof. This follows as a special case of Prop. 10.1.14 of Kadison and Ringrose (p. 722)[32], where we let the representation $\pi : \mathfrak{C} \to \mathcal{B}(\mathcal{H}_U)$ be defined by $\pi(f) := \pi_U(f) \cdot \chi_{k[\mathcal{M}]}$, where $(\pi_U, \mathcal{H}_U)$ is the universal representation of $\mathfrak{C}$ and $\chi_{k[\mathcal{M}]}$ is (pointwise multiplication by) the characteristic function of (all copies of) $k[\mathcal{M}]$, considered as a projection operator in the universal Hilbert space $\mathcal{H}_U = \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} L^2(\mathcal{P}(\mathfrak{C}), d\mu_\omega)$. It is easy to see that the condition $\omega = \chi_{k[\mathcal{M}]}\omega$ from Prop. 10.1.14 of Kadison and Ringrose (p. 722)[32] is equivalent to the condition $\omega \in V_0^C$, and that ultraweak continuity in $\pi(\mathfrak{C})$ is continuity in the topology of pointwise convergence of functions on $\mathcal{M}$. $\square$

This establishes that the elements of V_0^C are precisely the countably additive Borel measures on $\mathcal{M}$. In particular, when $\mathfrak{C} = AP(\mathbb{R}^{2n})$, this shows that V_0^C is the collection of countably additive Borel measures on $\mathbb{R}^{2n}$. We will use this fact to prove our main result.

V. MAIN RESULT

Now we restrict attention to the finite-dimensional phase space $\mathcal{M} = \mathbb{R}^{2n}$. We know from previous results that if we consider the Weyl algebra over $\mathbb{R}^{2n}$ and reduce its state space to the regular states by applying Thm. 1, then we are left with the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a separable Hilbert space $\mathcal{H}$. More precisely, we define

$$V_0^Q := \{\omega \in \mathcal{W}_h(\mathbb{R}^{2n})^* |\ \omega \text{ is regular}\}$$

Just as in §IV, we first need to enlarge the algebra $\mathcal{W}_h(\mathbb{R}^{2n})$ to $\mathcal{W}_h(\mathbb{R}^{2n})^{**}$ so that it is "large enough". Then we consider the maximal set of functionals generated by V_0^Q; we define

$$V^Q := \{\omega \in \mathcal{W}_h(\mathbb{R}^{2n})^{***} |\ \bigcap_{\rho \in V_0^Q} Ker(\rho) \subseteq Ker(\omega)\}$$

Proposition 3 (Feintzeig[19]). *The C*-algebra $\mathfrak{B} = \mathcal{W}_h(\mathbb{R}^{2n})^{**}/\mathcal{N}(V^Q)$ (for $h \neq 0$) of Thm. 1 is *-isomorphic to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space.*

This should be unsurprising because the well known Stone-von Neumann theorem already shows that the regular states form the folium of the Schrödinger representation, which is irreducible. But notice that there is a striking resemblance between the situation in Props. 1 and 3. In both cases, we alter the algebra of observables by choosing an appropriate state space—V^C or V^Q—and applying Thm. 1. We will clarify the relationship between these propositions by characterizing the relationship between V_0^Q, the collection of regular functionals on the Weyl algebra, and V_0^C, the collection of countably additive Borel measures on $\mathbb{R}^{2n}$. To do so, we now prove the main result.

Theorem 2. *Let $\omega \in \mathcal{W}_h(\mathbb{R}^{2n})^*$ (for $h \neq 0$). Then ω is regular iff $\omega \circ \mathcal{Q}_h$ is a countably additive Borel measure on $\mathbb{R}^{2n}$ (i.e., $\omega \in V_0^Q$ iff $\omega \circ \mathcal{Q}_h \in V_0^C$).*

Proof. Some preliminaries. Notice that since the Stone-von Neumann theorem implies the regular functionals are isomorphic to the predual of $\mathcal{B}(L^2(\mathbb{R}^n))$, we know from Thm. 7.4.7 of Kadison and Ringrose (p. 485)[32] that each regular functional can be decomposed into a linear combination of regular states. Hence, it suffices to prove the claim for the case where ω is a state, which we will assume in what follows.

($\Rightarrow$) Suppose ω is a regular state. Consider the GNS representation $(\pi_{\omega\circ\mathcal{Q}_h}, \mathcal{H}_{\omega\circ\mathcal{Q}_h})$ for the state $\omega \circ \mathcal{Q}_h$ on $\mathcal{W}_0(\mathbb{R}^{2n}) \cong C(\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n})))$. Thm. 5.2.6 of Kadison and Ringrose (p.

315)[32] implies there is a projection-valued measure E on $\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$ with the following property. For any $\varphi \in \mathcal{H}_{\omega\circ\mathcal{Q}_h}$, define the Borel measure μ_φ on $\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$ by

$$\mu_\varphi(S) := \langle \varphi, E(S)\varphi \rangle$$

for any Borel set $S \subseteq \mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$. Then we have

$$\langle \varphi, \pi_{\omega\circ\mathcal{Q}_h}(W_0(x))\varphi \rangle = \int_{\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))} \hat{W}_0(x) d\mu_\varphi$$

for any $x \in \mathbb{R}^{2n}$ (where $\hat{W}_0(x)$ is here the surrogate of the function $W_0(x)$ on the space $\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$ of pure states). In particular, for the choice $\varphi = \Omega_{\omega\circ\mathcal{Q}}$, the cyclic vector representing the state $\omega \circ \mathcal{Q}$ from the GNS construction, it follows that μ_φ is the Riesz-Markov measure $\mu_{\omega\circ\mathcal{Q}_h}$ on $\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$ corresponding to $\omega \circ \mathcal{Q}_h$.

Now, since ω is regular, the mapping

$$x \mapsto \pi_{\omega\circ\mathcal{Q}_h}(W_0(x))$$

for all $x \in \mathbb{R}^{2n}$ is a weak operator continuous unitary representation of the topological group $\mathbb{R}^{2n}$, the SNAG theorem (Bratteli and Robinson[8], p. 243) implies that we have a projection-valued measure on $\mathbb{R}^{2n}$:

$$S \mapsto E(k[S])$$

for any Borel set $S \subseteq \mathbb{R}^{2n}$ (Recall $\mathbb{R}^{2n}$ is self-dual as a topological group). Here, the map $k : \mathbb{R}^{2n} \to \mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))$ is the compactification associated to $\mathcal{W}_0(\mathbb{R}^{2n})$. The justification is as follows: for any $\varphi \in \mathcal{H}_{\omega\circ\mathcal{Q}_h}$, define the Borel measure $\hat{\mu}_\varphi$ on $\mathbb{R}^{2n}$ by

$$\hat{\mu}_\varphi(S) := \mu_\varphi(k[S])$$

for any Borel set $S \subseteq \mathbb{R}^{2n}$. We know $\hat{\mu}_\varphi$ satisfies

$$\langle \varphi, \pi_{\omega\circ\mathcal{Q}_h}(W_0(x))\varphi \rangle = \int_{\mathbb{R}^{2n}} W_0(x) d\hat{\mu}_\varphi$$

for any $x \in \mathbb{R}^{2n}$ (where $W_0(x)$ is now considered as a function on $\mathbb{R}^{2n}$).

Hence, since $E \circ k$ is a projection valued measure, it must be the case that

$$\mu_\varphi(k[\mathbb{R}^{2n}]) = \hat{\mu}_\varphi(\mathbb{R}^{2n}) = 1$$

and thus

$$\mu_\varphi(\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n})) \setminus k[\mathbb{R}^{2n}]) = 0$$

This implies that for the choice $\varphi = \Omega_{\omega\circ\mathcal{Q}_h}$,

$$\mu_{\omega\circ\mathcal{Q}_h}(\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n})) \setminus k[\mathbb{R}^{2n}]) = 0$$

Prop. 2 now implies $\omega \circ \mathcal{Q}_h$ is a countably additive Borel measure on $\mathbb{R}^{2n}$.

($\Leftarrow$) Suppose $\omega \circ \mathcal{Q}_h$ is a countably additive Borel probability measure on $\mathbb{R}^{2n}$. By Prop. 2, we know that the Riesz-Markov measure $\mu_{\omega\circ\mathcal{Q}_h}$ corresponding to $\omega \circ \mathcal{Q}_h$ satisfies

$$\mu_{\omega\circ\mathcal{Q}_h}(\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))) \setminus k[\mathbb{R}^{2n}]) = 0$$

Now we know that for any $x \in \mathbb{R}^{2n}$,

$$\begin{aligned} \omega \circ \mathcal{Q}_h(W_0(tx)) &= \int_{\mathcal{P}(\mathcal{W}_0(\mathbb{R}^{2n}))} W_0(tx) d\mu_{\omega\circ\mathcal{Q}_h} \\ &= \int_{k[\mathbb{R}^{2n}]} W_0(tx) d\mu_{\omega\circ\mathcal{Q}_h} \end{aligned}$$

for all $t \in \mathbb{R}$. Since the functions $W_0(tx)$ are uniformly bounded by 1 on the domain $k[\mathbb{R}^{2n}]$, the dominated convergence theorem (Reed and Simon[54], p. 17, Thm. I.11) implies that

$$t \mapsto \omega \circ \mathcal{Q}_h(W_0(tx))$$

is a continuous function, which shows that ω is regular. □

Recall that the state ω determines a constant continuous field of states $\{\omega_h\}_{h\in[0,1]}$ on the deformation quantization discussed in §III with classical limit $\hat{\omega} = \omega \circ \mathcal{Q}_h$. Thus, Thm. 2 establishes the central claim of this paper: a state ω on $\mathcal{W}_h(\mathbb{R}^{2n})$ is regular iff its classical limit $\omega \circ \mathcal{Q}_h$ is a countably additive Borel probability measure on $\mathbb{R}^{2n}$.

VI. DISCUSSION

Thm. 2 shows that we only need the regular quantum states to explain the success of classical physics. As such, this result bears on the question of which quantum states we ought to take to be physically significant in quantum theories. For example, some have been interested in constructing quantum theories using non-regular states and their corresponding representations.[13,27] But these non-regular states are not needed for the explanation of the success of classical physics. I do not claim that this rules out approaches to physics involving non-regular states, but it does demonstrate some of their counterintuitive features.

Furthermore, this result shows that one can use the classical limit of quantum theories, or more specifically quantum states, to guide the construction of quantum theories. Prop. 3 shows that we can transform the Weyl algebra to an algebra whose state space consists precisely of the states whose classical limits are manifestly physically significant. This may be desirable if one believes that the only quantum states that are physical are the ones whose classical limit is physical. The result of this process is precisely the orthodox formulation of non-relativistic quantum mechanics for systems whose phase space is $\mathbb{R}^{2n}$.

The main result of this paper thus suggests a possible methodology for choosing an algebra of observables to use in the construction of new quantum theories. As one will notice from a survey of the literature, there are many different approaches to constructing algebras of quantum observables. One needs to make a choice about what *type* of algebra to use, e.g., *-algebras or C*-algebras (See, e.g., Rejzner[55]). And even once one has made this choice, one has to choose which algebra of a given type is appropriate for modeling a given physical system (See, e.g., Ashtekar and Isham[4]). The methodology the result of this paper suggests is that we look for an algebra that has an appropriate state space—in the sense that the classical limits of the allowed quantum states are physical states in the classical theory. I do not claim that this methodology is guaranteed to work in the quantization of all classical theories, but merely that it works in the simplest case. Since this approach is motivated by the desired explanations of the success of previous theories, I suggest that it might be fruitful to apply the same methodology in the quantization of other classical theories.

I hope that the result of this procedure provides some further understanding of quantization and the classical limit, and that this procedure can be extended to illuminate the quantization of other classical theories. In this vein, I'd like to outline a number of further questions for investigation.

1. Do the algebras $\mathcal{W}_h(\mathbb{R}^{2n})^{**}/\mathcal{N}(V^Q) \cong \mathcal{B}(\mathcal{H})$ with the natural composition of quotient maps with quantization maps provide a strict or continuous quantization[43] of the algebra $\mathcal{W}_0(\mathbb{R}^{2n})^{**}/\mathcal{N}(V^C) \cong B(\mathbb{R}^{2n})$? If so, is this quantization equivalent to the weakly continuous extension of the Berezin quantization of $C_0(\mathbb{R}^{2n}) \subseteq B(\mathbb{R}^{2n})$ to the compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$?

2. Can one extend the results of §IV for field systems whose phase space is not locally compact? In particular, should we understand the countably additive Borel probability

measures on a non-locally compact phase space as modeling the physically significant states? Should we understand the algebra of bounded Borel functions on a non-locally compact phase space as modeling the physically significant observables?[44]

3. Can one apply the methodology of §V to recover the quantization of a system whose phase space is not simply connected? There are known procedures for arriving at the quantization of a theory whose phase space has the form G/H for some locally compact abelian group G and closed subgroup H.[35,36,38] Is it possible to arrive at the same quantum theory by quantizing a system with phase space G and restricting attention to only states with the appropriate classical limit?

I hope that answers to these (or similar) questions might aid both our understanding of the classical limit and the development of tools for constructing strict and continuous deformation quantizations for further systems of physical interest.

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REFERENCES

[1]Alfsen, Erik and Frederic Shultz (2001). *State Spaces of Operator Algebras.* Boston, MA: Birkhauser.

[2]Anzai, H. and S. Kakutani (1943). "Bohr Compacifications of a Locally Compact Abelian Group I." *Proceedings of the Imperial Academy*, *19*, 476–480.

[3]Arageorgis, Aristidis. *Fields, particles, and curvature: Foundations and philosophical aspects of quantum field theory in curved spacetime.* PhD thesis, University of Pittsburgh, (1995).

[4]Ashtekar, A. and C. Isham (1992). "Inequivalent observable algebras. Another ambiguity in field quantisation." *Physics Letters B*, *274*, 393–398.

[5]Berger, C. and L. Coburn (1986). "Toeplitz Operators and Quantum Mechanics." *Journal of Functional Analysis*, *68*, 273–399.

[6]Binz, E., R. Honegger, and A. Rieckers (2004a). "Construction and uniqueness of the C*-Weyl algebra over a general pre-symplectic space." *Journal of Mathematical Physics*, *45*(7), 2885–2907.

[7]Binz, E., R. Honegger, and A. Rieckers (2004b). "Field-theoretic Weyl Quantization as a Strict and Continuous Deformation Quantization." *Annales de l'Institut Henri Poincaré*, *5*, 327–346.

[8]Bratteli, O. and D. Robinson (1987). *Operator Algebras and Quantum Statistical Mechanics*. Volume 1. New York: Springer.

[9]Bratteli, Ola and Derek Robinson (1996). *Operator Algebras and Quantum Statistical Mechanics*. Volume 2. New York: Springer.

[10]Brunetti, R., M. Dütsch, and K. Fredenhagen (2009). "Perturbative Algebraic Quantum Field Theory and the Renormalization Groups." *Advances in Theoretical and Mathematical Physics*, *13*(5), 1541–1599.

[11]Brunetti, R. and K. Fredenhagen (2009). "Quantum Field Theory on Curved Backgrounds." *Lecture Notes in Physics*, *786*, 129–155.

[12]Coburn, L. (1992). "Deformation Estimates for the Berezin-Toeplitz Quantization." *Communications in Mathematical Physics*, *149*, 415–424.

[13]Corichi, A., T. Vukasinac, and J. Zapata (2007). "Polymer Quantum Mechanics and its Continuum Limit." *Physical Review D*, *76*.

[14]Dixmier, Jacques (1977). *C*-Algebras*. New York: North Holland.

[15]Dütsch, M. and K. Fredenhagen (2001). "Perturbative Algebraic Quantum Field Theory, and Deformation Quantization." *Proceedings of the Conference on Mathematical Physics in Mathematics and Physics, Siena*. .

[16]Emch, G. (1997). "Beyond Irreducibility and Back." *Reports on Mathematical Physics*, *40*(2), 187–193.

[17]Engelking, R. (1989). *General Topology*. Berlin: Heldermann Verlag.

[18]Feintzeig, B. (2016). "Unitary Inequivalence in Classical Systems." *Synthese*, *193*(9), 2685–2705.

[19]Feintzeig, B. (2017a). "On the Choice of Algebra for Quantization." *Philosophy of Science*, forthcoming.

[20]Feintzeig, Benjamin (2017b). "On Theory Construction in Physics: Continuity from Classical to Quantum." *Erkenntnis*, forthcoming.

[21]Feintzeig, Benjamin (2017c). "Toward an Understanding of Parochial Observables." *British Journal for the Philosophy of Science*, forthcoming.

[22]Fell, G. and R. Doran (1988). *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles.* Boston: Academic Press.

[23]Fredenhagen, K. and K. Rejzner (2015). "Perturbative Construction of Models in Algebraic Quantum Field Theory." *Advances in Algebraic Quantum Field Theory.* . Springer, 31–74.

[24]Gamelin, T. (1969). *Uniform Algebras.* Englewood Cliffs, NJ: Prentice Hall.

[25]Haag, Rudolf (1992). *Local Quantum Physics.* Berlin: Springer.

[26]Halvorson, Hans (2001). "On the nature of continuous physical quantities in classical and quantum mechanics." *Journal of Philosophical Logic*, *37*, 27–50.

[27]Halvorson, Hans (2004). "Complementarity of Representations in Quantum Mechanics." *Studies in the History and Philosophy of Modern Physics*, *35*, 45–56.

[28]Hewitt, E. (1953). "Linear Functionals on Almost Periodic Functions." *American Mathematical Society*, *74*(2), 303–322.

[29]Honegger, R. (1997). "On the Continuous Extension of States on the CCR Algebra." *Letters in Mathematical Physics*, *42*, 11–25.

[30]Honegger, R. and A. Rieckers (2005). "Some Continuous Field Quantizations, Equivalent to the C*-Weyl Quantization." *Publications of the Research Institute for Mathematical Sciences, Kyoto University*, *41*(113-138).

[31]Honegger, R., A. Rieckers, and L. Schlafer (2008). "Field-Theoretic Weyl Deformation Quantization of Enlarged Poisson Algebras." *Symmetry, Integrability and Geometry: Methods and Applications*, *4*, 047–084.

[32]Kadison, Richard and John Ringrose (1997). *Fundamentals of the Theory of Operator Algebras.* Providence, RI: American Mathematical Society.

[33]Kontsevich, M. (2003). "Deformation Quantization of Poisson Manifolds." *Letters in Mathematical Physics*, *66*(3), 157–216.

[34]Landsman, N. P. (1991). "Algebraic Theory of Superselection and the Measurement Problem in Quantum Mechanics." *International Journal of Modern Physics A*, *6*(30), 5349–5371.

[35]Landsman, N. P. (1993). "Strict deformation quantization of a particle in external gravitational and Yang-Mills fields." *Journal of Geometry and Physics*, *12*, 93–132.

[36]Landsman, N. P. (1995). "The quantization of constrained systems: from symplectic reduction to Rieffel induction." *Proceedings of the XIV Workshop on Geometric Methods in Physics*. Ed. J.P. Antoine et. al. Białowieza: Polish Scientific Publishers PWN.

[37]Landsman, N. P. (2017). *Foundations of Quantum Theory: From Classical Concepts to Operator Algebras*. Springer.

[38]Landsman, Nicolaas P. (1990). "C*-Algebraic Quantization and the Origin of Topological Quantum Effects." *Letters in Mathematical Physics*, *20*, 11–18.

[39]Landsman, Nicolaas P. (1998). *Mathematical Topics Between Classical and Quantum Mechanics*. New York: Springer.

[40]Landsman, Nicolaas P. (2006). "Between Classical and Quantum." *Handbook of the Philosophy of Physics*. Ed. J. Butterfield and J. Earman. New York: North Holland, 417–553.

[41]Manuceau, J., M. Sirugue, D. Testard, and A. Verbeure (1974). "The Smallest C*-algebra for the Canonical Commutation Relations." *Communications in Mathematical Physics*, *32*, 231–243.

[42]For related work, see also Landsman[36].

[43]Here we would of course require a slight technical alteration of the conditions given in Landsman[39] to allow for the weakly continuous extension of the quantization from a Poisson subalgebra rather than merely the norm continuous extension. See Feintzeig[20].

[44]Here, one would hope to make connections with much of the recent work on algebraic quantum field theory.[10,11,15,23,55].

[45]For more on the algebra of almost periodic functions, see Binz et al.[6], Hewitt[28].

[46]These strict quantizations are meant to represent non-perturbative or analytic formulations of the classical limit of quantum theories, as opposed to the *formal deformation quantizations* investigated by, e.g., Waldmann[60], Kontsevich[33].

[47]For the case of a possibly infinite dimensional pre-symplectic phase space, see Binz et al.[7], Honegger and Rieckers[30]. For extensions to larger algebras see Honegger et al.[31].

[48]For more on the relationship between the classical and quantum state spaces, see Berger and Coburn[5], Thm. 13.

[49]A phase space $\mathcal{M}$ also typically has the structure of a symplectic (or at least Poisson) manifold, but that will not be needed for the current section.

[50]For continuous quantizations that start from the classical algebra $C_0(\mathcal{M})$, see, e.g., the discussion of *Berezin quantization* in Landsman[39].

[51]See Gamelin (p. 16)[24] and Kadison and Ringrose (§4.4)[32]. For the case of particular interest for this paper where $\mathfrak{C} = AP(\mathbb{R}^{2n})$, the compactification $\mathcal{P}(AP(\mathbb{R}^{2n}))$ is known as the *Bohr compactification* of $\mathbb{R}^{2n}$; see Anzai and Kakutani[2], Hewitt[28], Rudin[57].

[52]See also Landsman[34] and Feintzeig[18] for similar discussions in the context of classical physics.

[53]Petz, Dénes (1990). *An Invitation to the Algebra of Canonical Commutation Relations.* Leuven: Leuven University Press.

[54]Reed, Michael and Barry Simon (1980). *Functional Analysis.* New York: Academic Press.

[55]Rejzner, K. (2016). *Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians.* New York: Springer.

[56]Rieffel, M. (1994). "Quantization and C*-Algebras." *Contemporary Mathematics*, *167*, 67–97.

[57]Rudin, W. (1962). *Fourier Analysis on Groups.* Wiley & Sons, Inc.

[58]Ruetsche, L. (2011). "Why be normal?." *Studies in the History and Philosophy of Modern Physics*, *42*, 107–115.

[59]Summers, Stephen (1999). "On the Stone-von Neumann Uniqueness Theorem and its Ramifications." *John von-Neumann and the Foundations of Quantum Physics.* Ed. M. Redei and M. Stoeltzner. Dordrecht: Kluwer, 135–152.

[60]Waldmann, S. (2015). "Recent developments in deformation quantization." *Proceedings of the Regensburg Conference on Quantum Mathematical Physics.* `https://arxiv.org/abs/1502.00097v1`.