A priori voting power : what is it all about?

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What Is It All About?

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ABSTRACT

In this account, we explain the meaning of a priori voting power and outline how it is measured. We distinguish two intuitive notions as to what voting power means, leading to two approaches to measuring it. We discuss some philosophical and pragmatic objections, according to which a priori (as distinct from actual) voting power is worthless or inapplicable.

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1 Introduction

The current state of research into a priori voting power and the controversy around it can only be understood in the context of the somewhat tortuous history of the subject. This Introduction will therefore be partly historical.[1]

We shall outline some of the concepts and results of the theory of voting power. In he main text we avoid as far as we can the use of technical jargon and mathematical formulas: these are mostly relegated to notes, which also contain references to the literature for additional technical matters such as rigorous statements and proofs of theorems. A reader uninterested in these technicalities may skip these notes.

As far as we know, the first properly scientific study of voting power was undertaken by LS Penrose (1946).[2] The subject is further developed in his booklet (1952). Penrose proposed a probabilistic measure of absolute a priori voting power. Although this measure is defined in the context of a special class of decision rules – 'decisions made by [ordinary] majority vote'[3] – it is clearly much more general, and can be applied to a very broad class of rules, which we now proceed to describe and define.

1.1 Binary decision rules

Such a rule applies to an *assembly* of *voters*.[4] When the assembly is called upon to make a decision on a bill, each of its members votes either 'yes' or 'no'. This creates a *division* of the assembly into two camps: the 'yes' voters and the 'no' voters. For an assembly consisting of two voters, *a* and *b*, there are four possible divisions (both vote 'yes', both vote 'no', *a* votes 'yes' and *b* votes 'no', or *a* votes 'no' and *b* votes 'yes'); for an assembly of three voters, the number of possible divisions is eight; an assembly of four gives rise to 16 divisions; and so on.[5]

Any division must have one of two possible outcomes: *positive*, in which case the bill is passed; or *negative*, in which case the bill is blocked. The function of a decision rule for a given assembly is to specify the outcome – positive or negative – of each possible division of the assembly.

Decision rules of the kind just described are *binary* because each voter has just two possible inputs into a decision: 'yes' or 'no' (thus, abstention is not admitted as distinct option); and the outcome must also be clear-cut positive or negative. Almost the whole of the literature on voting power confines itself to binary decision rules, and we shall also do so in what follows, so for brevity we shall henceforth omit the qualification 'binary'.[6]

1.1.1. Weighted rules The most common way of specifying a decision rule is by means of a *weighting system*: each voter is assigned a non-negative number of votes, or *weight*; and a positive number, which is not greater than the sum of all weights, is specified as *quota*. What determines the outcome of a division is the total weight of its 'yes' voters: the outcome is positive just in case the total weight of the 'yes' camp equals or exceeds the quota. For a real-life illustration see Table 1: it shows the main decision rule – known as *qualified majority voting* (QMV) – of the EU's most important decision-making body, the Council of Ministers (CM), in the first five periods of the Union.[7]

Note that in principle a voter can be assigned weight 0 (which is of course a non-negative number); but in practice this would be silly, because it would make such a voter a *dummy*, whose vote – whether 'yes' or 'no' – would not make the slightest difference to the outcome. However, a voter having positive weight can also be a dummy. In fact, this was the case with Luxembourg under the QMV in the period 1958–72, when its weight was 1: as the reader can easily check by consulting the column headed '1958' in Table 1, Luxembourg's vote could never affect the outcome.

1.1.2. Decision rules in general[8] More generally, and put rather abstractly, a decision rule for a given assembly is *any* classification of all its possible divisions into those with positive or negative outcome, subject only to the following three conditions:

(1) The division in which all the voters unanimously vote 'yes' must have positive outcome.

(2) The division in which all the voters unanimously vote 'no' must have negative outcome.

(3) If a division has positive outcome, and a voter is transferred from its 'no' camp to the 'yes' camp, then the resulting division must likewise have positive outcome.

These conditions exclude perverse would-be rules. The meaning of (1) and (2) is obvious. Condition (3) can be paraphrased as stipulating that increased support for a bill cannot hurt it.[9]

It is easy to see that any weighted decision rule satisfies these three conditions. However, not all decision rules are – or even can be – presented as weighted rules.[10]

1.2 Penrose's measure of voting power

By the *a priori voting power* of a voter under a given decision rule we mean, roughly speaking, the amount of potential influence over the possible outcomes of divisions, which the voter possesses by virtue of the rule. This last qualification – 'by virtue of the rule' – is of vital importance: it serves to distinguish a priori from actual (or a posteriori) voting power. We shall return to this in Section 3. Since in this paper we are concerned for the most part with a priori voting power, we shall often refer to it briefly as 'voting power' *simpliciter*, except where the qualification 'a priori' is essential.

1.2.1. Voter's probability of success Penrose's first key idea was simple: *the more powerful a voter is, the more often will the outcome go the way s/he votes*. In other words, a more powerful voter is a more successful one, one that is more often on the victorious side of a division.[11]

So consider a given decision rule and a given voter a. Let us denote by r_a the proportion of all possible divisions of the assembly in which a is successful:
a votes 'yes' and the outcome is positive, or votes 'no' and the outcome is negative. This can also be expressed in terms of probabilities: assuming that all the voters vote independently and at random (for example, each voter flips a true coin and votes 'yes' if it shows heads and 'no' if it shows tails)[12] then r_a is the probability that a will be on the victorious side.

For example, take the 1958 QMV rule shown in Table 1. Here the assembly has six voters and there are $2^6 = 64$ possible divisions. As the reader can easily verify by listing these 64 divisions, France is successful in 42 of them; so $r_{\text{France}} = 21/32$, and the same of course holds for Germany and Italy. Also, Belgium is successful in 38 of the 64 divisions, so $r_{\text{Belgium}} = 19/32$, and the same holds for the Netherlands. Finally, Luxembourg is successful in 32 divisions, so $r_{\text{Luxembourg}} = 1/2$.

1.2.2. Penrose's definition Clearly, the voting power of voter a ought to be directly related to r_a. But r itself is not a very convenient measure of voting power, because it runs together luck and genuine influence.[13] Indeed, even for a dummy $r = 1/2$, because even a dummy finds itself, by sheer luck, in the successful camp in half of all divisions, as we have just seen in the case of Luxembourg under the 1958 QMV. So Penrose proposed

$$\square_a := 2r_a - 1 \qquad (1)$$

as measure of a's voting power.[14] This takes the element of luck out of it: for a dummy $\square = 0$, whereas for a 'dictator' (a voter whose vote completely determines the outcome, and who is therefore invariably successful $\square = 1$.

Thus under the 1958 QMV (see Table 2) we have $□_{\text{France}} = □_{\text{Germany}} = □_{\text{Italy}}$ $= 5/16$, $□_{\text{Belgium}} = □_{\text{Netherlands}} = 3/16$ and $□_{\text{Luxembourg}} = 0$.

Penrose also observed that □ can be interpreted directly in the following striking way: *it is the probability that the given voter can be decisive*;[15] in other words, $□_a$ is the probability of occurrence of a division in which voter *a* can reverse the outcome by reversing his or her vote.[16]

For example, the 64 possible divisions of the 1958 CM can be arranged in 32 pairs, so that the two divisions in each pair differ from each other in one respect only: the vote of Belgium. In 26 of these pairs, the outcome under QMV is the same for both divisions of the pair; so in these divisions Belgium's vote makes no difference. But in the remaining 6 pairs Belgium's vote makes all the difference: in one division of the pair, in which Belgium votes 'yes', the outcome is positive; while in the other division, in which Belgium votes 'no', the outcome is negative. So Belgium is decisive in 6 out of 32 pairs of divisions (or in 12 divisions out of 64). Therefore the probability of Belgium being decisive is 3/16, which is indeed exactly the value of $□_{\text{Belgium}}$.

1.3 Relative voting power: the Banzhaf index

Penrose's pioneering work lay for many years unnoticed or forgotten by main-Stream writers on social choice. But his ideas on measuring voting power are so natural, so compelling, that they forced themselves on several other scholars who tackled the problem of measuring voting power: without knowing of Penrose's – or of one another's – work, they re-invented some of his ideas.

The first among them (as far as we know) who began to address this problem
Was the American jurist John F Banzhaf (1965). (See also Banzhaf, 1966; 1968).[17] Banzhaf approached the problem in much the same way as Penrose. But since he was interested in voting power not as an absolute magnitude, but only in the *ratio* of one voter's power to another's, the *Banzhaf index of voting power* named after him and denoted by □ (the Greek letter *beta*), gives only the *relative* power of each voter. The value of □ for any voter can be obtained very simply from □ by dividing the value of □ for that voter by the sum of the values of □ of all the voters in the assembly.[18] So, unlike □ the values of □ for all the voters in an assembly always add up to 1. The values of □ for all member states under QMV in the first five periods of the EU are shown in Table 3. Note that this table can be obtained from Table 2 by dividing the figures in each column of the latter table by the total shown at the bottom of the column.

The reader must be warned that the Banzhaf index □ can only be used for comparing the voting powers of several voters *under the same decision rule*; it is not a reliable yardstick for comparing the voting powers of different voters, or even of the same voter, under two different decision rules. *For the latter purpose the Penrose measure □ must be used.*

For example, from Table 3 we can reliably infer that under the 1981 QMV Greece had twice as much voting power as Ireland. But the fact that the value of □ for Ireland was greater in 1986 than in 1981, does not mean that Ireland's voting power increased when Spain and Portugal were admitted to the EU. Turning to Table 2, we see that the admission of Spain and Portugal caused *all* ten old members to lose some power. However, Ireland and Denmark happened to lose much less than all the others, so their *relative* positions improved *compared to the others'*. It is of course hardly surprising if an old member of the EU loses some power when new members are admitted. In fact, the only member that actually gained power on such occasions was Luxembourg, to which it happened three times: in 1973, 1981 and 1995; but that was mostly a result of blunders in allocating weights.

Another pitfall against which we must warn the reader is that of treating □ or □ as an *additive* quantity like money, for example. If Frances and Gerry have £0.3125m each and Bella has £0.1875m, then it makes perfect sense to say that the three jointly have £0.8125m, because 2 □ 0.3125 + 0.1875 = 0.8125. But it would be a mistake to infer from Table 2 that under the 1958 QMV the joint voting power of France, Germany and Belgium was 0.8125. This is because □ is computed under the assumption that voters act independently of each other. If these three countries were to conclude a pact binding them to vote always together, as a bloc, then instead of the decision rule shown in the '1958' column of Table 1 we would have a rather different rule: three individual voters with weights 4, 4 and 2 would be replaced by a single bloc voter with weight 10; and a simple calculation shows that under this rule the voting power □ of the bloc would be 0.75 rather than 0.8125.[19]

1.3 Two vital distinctions

In Subsection 1.3 we have run ahead of the historical sequence. In fact, at this point the tale becomes tangled and controversial. Scientific as well as semi-popular discourse on voting power has been bedevilled by lack of clarity about two issues.

First, while Penrose's work lay unnoticed by mainstream social-choice theory, and before his ideas had been re-invented by others, an alternative approach to the measurement of a priori voting power, derived from cooperative game theory, was proposed by Lloyd S Shapley and Martin Shubik (1954). This was based on an intuitive (pre-formal) notion of voting power that differed fundamentally from Penrose's. When Penrose's ideas were reinvented by Banzhaf and others, the two underlying intuitive notions got conflated and confounded with each other; this led to much conceptual confusion and some technical errors. We shall discuss this alternative approach and the vital distinction between the notion underlying it and Penrose's in Section 2.

Second, much ill-founded criticism, opposition and even hostility – mostly by practitioners but also by some political scientists – against the theory of a priori voting power has been caused by lack of clarity about the significance of and justification for the aprioristic assumptions (alluded to at the beginning of Subsection 1.2 and in endnote 12), and hence about the vital distinction between a priori (or theoretically potential) and a posteriori (or actual) voting power. We shall address this issue in Section 3.

2 Alternative notions of voting power

Some approaches to the measurement of voting power – most importantly that proposed by Shapley and Shubik (1954) – are derived from the mathematical theory of *cooperative games with transferable utility*.

Following each play of such a game, every player receives some *payoff.* : an amount – which can be positive, negative, or 0 – of *transferable utility* – briefly, TU – (roughly speaking, money or a money-like substance). The amount that a given player receives depends on the strategies chosen by all the players. Before each play, any subset ('coalition') of the players can negotiate and conclude a binding pact whereby they agree to coordinate their strategies and re-distribute their payoffs. (A coalition that makes such a pact is said to be 'formed'.)

In general, the rules of the game (the strategies available to the players and the payoff they receive for each choice) do not determine with certainty which coalition will be formed and how it will re-distribute its total payoff to its members. Therefore also the amount of payoff that each player will end up with cannot be predicted with certainty. The quantity known as *the Shapley value of the game to a given player* is accepted by many game-theorists as a prior probabilistic estimate of the payoff that the player can expect, on the average.[20]

The acceptance of the Shapley value as a correct estimate of a player's expected payoff is by no means unproblematic or non-controversial. The point is that there is no compelling and realistic general theory of bargaining for cooperative games with more than two players. Shapley (1953) got round this by proceeding axiomatically: he stipulated as axioms a few conditions that a reasonable bargaining theory (if it exists) ought to satisfy, and proved that these axioms determine a unique set of expected payoffs. This justification depends not only on accepting that Shapley's axioms are indeed reasonable, but also that there is no other axiom that is at least as reasonable for a bargaining theory and which is inconsistent with Shapley's axioms.[21]

2.1 The Shapley–Shubik index

The Shapley–Shubik index, usually denoted by □, is just the Shapley value applied to decision rules, which can be dressed up as cooperative games of a special kind: so-called *simple* games. The idea is that the voters in an assembly can be regarded as 'players' in a game. A play of the game consists in bringing about a division of the assembly. If the outcome of a division is positive, the camp of 'yes' voters is awarded a fixed prize. (If the outcome is negative, no prize is awarded.) The prize – the spoils of victory – is a fixed amount of TU, which the victors share among themselves according to a prior binding agreement, arrived at through bargaining, concluded in advance of the division.[22] The Shapley–Shubik index of a voter ('player') under a given decision rule ('simple game') is then presumed to be a prior estimate of that voter's expected payoff. For convenience, the value of the fixed prize to be shared out is set as 1 unit; so the sum of the values of this index for all the voters is always 1. The underlying idea here is that the voter's expected payoff is a measure of that voter's voting power.

It should be stressed that the Shapley–Shubik index does not rely for its justification on any model of bargaining and coalition formation. Its justification depends on that of the Shapley value (of which it is a special case), discussed above.[23]

The Shapley–Shubik index is widely used – alongside the Penrose measure or its 'relativized' form, the Banzhaf index – as a measure of voting power. In many cases the Shapley–Shubik and Banzhaf indices have fairly similar values; but in general they behave quite di.erently.[24]

Other indices of a priori voting power have been proposed, but are much less widely used than the ones just mentioned. We shall discuss them briefly in Subsection 2.4.

2.2 Conceptual confusion; Coleman's critique

As mentioned in Subsection 1.4, the Penrose and Shapley–Shubik approaches to measuring voting power were rooted in fundamentally different intuitive (pre-formal) notions of what voting power consists in. To facilitate our discussion, we shall again step out of the historical sequence and introduce here a useful (if somewhat inelegant) terminology proposed by Felsenthal, Machover and Zwicker (1998).

By *I-power* we mean voting power conceived of as a voter's potential influence over the outcome of divisions of the decision-making body: whether proposed bills are adopted or blocked. Penrose's approach was clearly based on this notion, and his measure of voting power is a proposed formalization and quantification of a priori I-power.

By *P-power* we mean voting power conceived of as a voter's expected relative share in a fixed prize available to the winning coalition under a decision rule, seen in the guise of a simple TU cooperative game. The Shapley–Shubik approach was evidently based on this notion, and their index is a proposed quantification of a priori P-power.

In Subsection 2.3 below we shall elaborate further on the conceptual distinction between these two notions of voting power. For the moment we only need to point out that whereas P-power is essentially a purely relative magnitude, I-power is primarily an absolute one.

As Penrose's work was ignored or unnoticed by mainstream – predominantly American – social-choice theorists, Shapley and Shubik's 1954 paper was seen as inaugurating the scientific study of voting power. Far from being a mere matter of misattributed priority – which would have been of no more than historiographic interest – this mistake had a rather profound substantive effect on the evolution of voting-power theory itself.

Because the Shapley–Shubik paper was wholly based on cooperative game theory, it induced among social scientists an almost universal unquestioning belief that the study of voting power was necessarily and entirely a branch of that theory. Because the Shapley–Shubik index conceptualized voting power as P-power, it was generally taken for granted that this is necessarily how voting power must be conceived of. Later, when some of Penrose's ideas were re-invented by Banzhaf and others, the viewpoint of cooperative game theory was imposed on them, leading to much confusion and error. In particular, the Banzhaf index □ was mistakenly regarded as quantifying a voter's P-power – similar in meaning to the Shapley–Shubik index □, simply a rival way of measuring the same thing. This in turn led to the fallacies discussed in Subsection 1.3. Since P-power is a purely relative concept, the true meaning of □, which is a measure of absolute I-power, was not generally understood, and it was not realized that □ is merely a derivative quantity, obtained from □ by normalization.[25]

Of course, before 1998 it was not generally realized that there are in fact two quite distinct underlying notions of voting power; they were simply conflated with each other.

A lone voice challenging the Shapely–Shubik index and the P-power notion underlying it was that of James S Coleman (1971). Coleman argued that while this notion, based as it is on TU cooperative game theory, may be reasonable in some unusual cases,[26] it is not ordinarily so:

> . . . for the usual problem is not one in which there is a division of the spoils among the winners, but rather the problem of controlling the action of the collectivity. The action is ordinarily one that carries its own consequences or distribution of utilities, and these cannot be varied at will, i.e. cannot be split up among those who constitute the winning coalition. Instead, the typical question is the determination of whether or not a given course of action will be taken or not [*sic*], that is, the passage of a bill, a resolution, or a measure committing the collectivity to an action.[27]

Here Coleman foreshadows the conceptual distinction between the two notions of voting power; however, he does not elaborate on it or introduce a terminology for referring to it.

He proceeds to define a pair of measures of voting power: first, negative or blocking power, *the power of a member to prevent action*; and second, positive power, *the power of a member to initiate action*. These are unmistakably measures of what we now call I-power: taken jointly, they constitute in effect a useful refinement of Penrose's measure □, containing some important additional information.[28]

However, it was soon pointed out that, when normalized, both of Coleman's measures yield the same *relative* distribution, namely the Banzhaf index.[29] Because at that time the concept of absolute voting power was not generally understood, the true significance of the Coleman measures was not realized, and attention was focused on their normalized derivative forms. Therefore, Coleman's technical contributions of 1971 were not sufficiently appreciated, and were believed mostly to replicate Banzhaf's work, of which the former was apparently unaware. His conceptual critique of the cooperative game theoretic notion of P-power was largely ignored.

2.3 The I-power/P-power distinction

As mentioned above, this distinction was first made explicitly by Felsenthal et al. (1998). It aims to disentangle two quite different notions of what voting power consists in. While both seek to quantify the potential influence that a member of a decision-making body has over the possible outcomes of divisions of that body,[30] they differ fundamentally in what they regard as the outcome of a division. The I-power notion takes the outcome to be the immediate one: passage or defeat of the proposed bill. The P-power view is that passage or defeat of the bill is merely the ostensible and proximate outcome of a division; the real and ultimate outcome is the distribution of a fixed purse – the prize of power – among the victors in case a bill is passed. From this viewpoint, a member's voting power, the extent of his or her control of the ultimate outcome, is to be measured by that member's expected or estimated share in the fixed purse.

This fundamental difference has several aspects and implications, to which we shall now turn.

2.3.1. Motivation of voting behaviour I-power assumes that voting behaviour is motivated by policy seeking. This implies that whether a voter will vote for or against a given bill depends crucially on the specific bill and on the voter's specific preferences regarding the issues addressed by the bill; but does not depend on the decision rule or information about other voters' intentions.[31]

In contrast, P-power assumes that voting behaviour is motivated by office seeking.[32] It is crucial to a voter to be included in the winning coalition that is being formed (see 2.3.3) and gain as much as possible of the winners' prize. Also, since the decision rule affects each voter's bargaining strength, it may well affect also his or her voting behaviour: a voter who has great P-power will have greater incentive as well as greater opportunity to participate in a winning coalition, and hence greater tendency to vote 'yes'.

2.3.2. Mathematical theory The mathematical treatment of P-power is a branch of TU cooperative game theory. Indeed, the very notion of P-power derives from that theory, and could hardly occur to anyone unfamiliar with it.

As far as a priori I-power is concerned, the principal mathematical framework is probability theory; the study of this notion certainly does not belong to cooperative game theory.[33]

When it comes to actual I-power, it is necessary to posit – in addition to a given decision rule, and quite distinct from it – some model of voters' behaviour, incorporating elements such as interactions and influences between voters, voters' preferences, the actual state of the world, and potential changes to the status quo implied by proposed bills.[34] Such models may of course be derived from diverse mathematical theories.

2.3.3. Bargaining; secret voting P-power, rooted in cooperative game theory, presupposes that before a division takes place a winning coalition can be *formed* in the sense explained in the beginning of this section.[35] This involves making a binding pact among the members of this coalition. If voting is secret, there is no way of verifying that these members have voted as agreed; so P-power is inapplicable.

In contrast, the policy-seeking voting behaviour presupposed by I-power does not require any active ‘formation’ of coalitions: voters who have a similar attitude towards a bill simply find themselves on the same side of a division. Thus I-power is applicable whether voting is secret or not.

2.3.4. Nature of payoffs and power P-power assumes that the total payoff ensuing from a division is a private good, and is a TU available for division among the members of a winning coalition that has formed. The absolute amount of the total payoff depends only on the bill in question, and not on the decision rule; but by a suitable choice of units, it is always possible to set the total payoff at 1. The P-power of a voter under a given decision rule is a purely relative magnitude: it is the voter’s expected relative share in the total payoff. Absolute P-power is meaningless.

In contrast, I-power assumes that the outcome of a division is a public good (or bad) affecting all voters (and perhaps others). The payoff that voters (and others) may derive from this outcome may, but need not, be quanta of TU. In any case, there is no connection between these payoffs and the a priori I-power of voters under a given decision rule. I-power is primarily an absolute pure number – a probability. Relative I-power can be defined; but it is a secondary concept, derived from absolute I-power.[36]

2.4 Minor indices

The Penrose measure and the Shapley–Shubik index are by far the most important measures of a priori voting power, in the sense of being the most robust mathematical formalization of the two alternative pre-formal notions. They are also the most widely used (although in the case of the former it is its normalized form, the Banzhaf index, which is still most often used).

Let us mention briefly three other indices that have been proposed, but are rarely used.

The index proposed by Deegan and Packel (1978) is squarely and explicitly based on the notion of P-power, as can be seen immediately from the very title of the 1982 version of their paper: ‘To the (minimal winning) victors go the (equally divided) spoils… ’. But, unlike the Shapley–Shubik index, it relies for its justification on a specific bargaining model. However, since this bargaining model – like all known bargaining models for cooperative games, even simple ones, with more than two players – is neither realistic nor intuitively convincing for at least some of the games in question, the Deegan–Packel index suffers from rather grotesque pathologies that in our opinion make it quite unacceptable as a credible measure of voting power.[37]

Another index was invented by RJ Johnston (1978). It was obtained by grafting on a perfectly good index of I-power (the Banzhaf index) a 'slight correction' motivated by the ideology of P-power. The result is an index that lacks any coherent justification.

Finally, an index proposed by Holler (1982) and called by him 'the public good index' is, like the Johnston index, a hybrid. But in this case the 'slight correction' goes in the reverse direction. The starting point is the Deegan–Packel index, which as we saw is explicitly based on the notion of P-power; but the modification grafted on it is motivated by regarding the fixed total payoff as a public good, which – as we saw in 2.3.4 – is inconsistent with the notion of P-power. Again, the resulting index lacks coherent justification. Unsurprisingly, both of these hybrid indices suffer from pathologies that in our opinion make them unacceptable for measuring voting power.[38]

While the Shapley-Shubik index is *mathematically* robust, and is the best candidate as an index of P-power – provided this notion is at all coherent – it too suffers from some structural defects that cast doubt on the very coherence of this notion.[39] For these reasons, Felsenthal and Machover (1998, pp. 195–196, 204–207) suggest that in the end it may be impossible to give a consistent mathematical form to this notion, and the Shapley-Shubik index may have to be rejected as an index of any kind of a priori voting power.[40]

3 The concept of a priori voting power

The concept of *a priori* – as distinct from *actual* – voting power is somewhat subtle. Failure to grasp it has given rise to hostile criticism: denying its usefulness or even its legitimacy.

Here we shall first explain what the a priori approach intends to capture, and then deal briefly with the criticism.

3.1 What is meant by a priori voting power?

Looking at the last column in Table 1, we see that the four largest members of the EU have equal weight – 10 votes each – under the 1995 QMV decision rule of the CM.[41] Does this mean that these four member-states have an equal degree of influence over the decisions made by the CM under QMV? Of course not. A member's *actual* influence over these decisions – her actual voting power under QMV – is not reducible to the formalities of the decision rule. It also depends on a complex interaction of real-world factors. Most obvious of these is diplomacy: surely, of two members with the same voting weight, the one with greater diplomatic skill and, frankly, bigger political clout will – other things being equal – be able to exert more influence. But the story does not end here, because other things are in general *not* equal: the ability of member *a* to persuade member *b* to vote as the former wishes depends not only on their relative diplomatic prowess and political muscle but also, very crucially, on their respective preferences regarding the bill in question. Here it is both a matter of how different these preferences are, and how intensely they are held by each of the members. The greater the difference in preferences, the more effort will *a* have to make in order to sway *b*, especially if the latter regards the issue in question as a matter of vital interest; and if *a* does not hold her preference with such great intensity, the effort may not be worth making. All these real-world factors are subject to change: preferences of course vary from issue to issue, but even members' perceived interests – let alone the diplomatic and political resources needed to pursue them – are mutable.

So, when the designers of the 1995 QMV rule assigned equal weights to the four largest member-states, they could not thereby equalize these members' actual voting power; nor was this the designers' intention. What they did – and evidently intended to do – was to equalize *that component of the four members' voting power that derives solely from the decision rule itself* – as distinct from all those real-world interactions outlined in the previous paragraph. This component is what we call 'a priori voting power'.[42] Equality of a priori voting power is thus analogous to equality of rights or equality before the law: it is a formal rather than empirical equality, which does not produce equality of actual attainment, nor is intended to do so.

In order to endow several members with equal a priori voting power under a weighted decision rule, it is sufficient to assign them equal weights; this is clear without any mathematical analysis. But of course this leaves open the question as to *how much* a priori voting power these members have been granted. This question is especially important if it is intended that some members shall have greater a priori voting power than others. Clearly, the intention of the QMV rule was to make member-states' a priori voting power dependent on their population size: larger member-states are assigned greater weights, though not in strict proportion to their population.[43] So it is fairly obvious that the 1995 QMV rule gives Germany more voting power than Belgium; but *how much* more?

In order to assess how a priori voting power is distributed, one must be able to quantify the amount of voting power that a given decision rule grants to a given member. As the example of Luxembourg discussed in 1.1.1 shows, the weights themselves do not provide a satisfactory measure even of relative, let alone absolute, voting power.[44]

Since our aim is to measure a priori voting power – the power that a member derives exclusively from the decision rule itself – we must go 'behind a veil of ignorance' regarding all other information. Thus we must disregard all the real-world factors mentioned at the beginning of this subsection: diplomacy, political pressures, members' specific interests and preferences, as well as the very nature of the issues to be decided.

In the case of a priori P-power, this means that no payoffs are admitted except those arising from the decision rule itself, regarded as a simple cooperative game; and the only preference a voter is assumed to have is that of maximizing his or her share of the fixed total payoff. (Thus, policy preferences specific to the proposed bill must be ignored.) Similarly, the only bargaining power a voter is assumed to possess is that arising from the structure of the simple game.

In the case of a priori I-power, the informational vacuum implies that we must assign equal prior probability to all possible divisions of the assembly.
This is prescribed by the classical probabilistic *Principle of Insufficient Reason (PIR)*[45].

This was the reasoning that guided Penrose and all those who, independently of one another, re-invented his measure of a priori voting power.[46]

The concept of a priori voting power has come up against two kinds of criticism, to which we shall now turn.

3.2 Objection to PIR

A fundamental – not to say fundamentalist – philosophical objection directed specifically against a priori I-power has recently been raised by Max Albert (2003).[47] He claims that PIR 'has been devastatingly criticized since the nineteenth century', and must therefore be rejected. He does not specify the 'devastating criticism' in question, but refers the reader to Howson and Urbach (1993, ch. 4).

Since, as we have just seen, the Penrose measure of a priori voting power and its variants are indeed based on PIR, its rejection would demolish the foundation of this measure. But Albert's *wholesale* rejection of PIR is far too hasty and quite unjustified. True, PIR is incoherent and may lead to contradiction when applied to *infinite* probability spaces.[48] But it is quite safe and unobjectionable when applied to a *finite* probability space consisting of finitely many clearly distinguished indivisible 'atomic' events. In this special case, PIR is not rejected by the best critical authorities on the subject, including Keynes (1921), and the one cited by Albert himself, Howson and Urbach (1993).[49]

It is this non-controversial special case of PIR that is used to establish the Penrose measure and its variants: here the well-defined clearly distinguished 'atomic' events are the 2^n possible divisions of an assembly of n voters.

3.3 Pragmatic objections to a priori voting power

A more common objection, directed at a priori voting power of any kind – both I-power and P-power – is that it is useless because it makes quite unrealistic assumptions about voters' behaviour and the decision-making process and ignores information about their actual behaviour. This criticism has been made by political scientists – notably Garrett and Tsebelis (1999a, 1999b, 2001)[50] – as well as by a practitioner, Moberg (2002).

This criticism has been rebutted by several authors.[51] All agree that a priori voting power is not a valid measure of the actual current voting power of known voters on known issues. In order to measure the latter, it would be necessary to take into account all those real-world factors mentioned in Subsection 3.1. There are several theoretical models that are designed to do so, at least in part.[52] However, if we do that, we arguably move away from considering power altogether: we move from 'who can get what' to 'who does get what'.

The main purpose of measuring a priori voting power is not descriptive but prescriptive; not empirical but normative. It is indispensable in the proper constitutional design and assessment of decision rules. Here it is important to quantify the voting power each member is granted *by the rule itself*. In order to do so it is necessary to disregard all that is known about voters' specific interactions and preferences, and posit a *purely hypothetical* random voting behaviour.

In addition to this principal use of the theory of a priori voting power, it also has two additional functions, which are of some importance even if one is only interested in empirical, actual voting power.

First, recall that a priori voting power – the power derived solely from the decision rule – is a component of actual voting power. Therefore the former serves as a benchmark, with which any measurement of the latter may be compared in order to assess the effect of real-world interactions and preferences. Moreover, a necessary condition that a theory of actual voting power must satisfy is that it contains a valid theory of a priori voting power as a special case, to which the more comprehensive theory is reduced in the absence of any information on real-world interactions and preferences.[53]

Second, in making long-term estimates of actual voting power we find ourselves in a situation of genuine ignorance of future real-world factors (as opposed to the self-imposed 'veil of ignorance'). This is particularly true of bodies such as the CM of the EU, in which not only the preferences of the members are subject to change, but also the issues to be decided are extremely varied and hardly predictable in the long term. For the short term we may be able to predict with some confidence which members are likely to find themselves on the same side of a division; but no such prediction can be reliable for the long term. In such long-term estimates of actual voting power, the best we can do is to assume that all divisions are equally likely – which means that a priori voting power is used as an estimator of long-term actual voting power. We are not claiming of course that such prediction is accurate: like all predictions and estimates based on statistical reasoning, it involves some error. But it is the best estimate we can make in the absence of information.

Another pragmatic objection – raised by Garrett and Tsebelis as well as by Moberg – against applying measures of a priori voting power to decision making by the CM of the EU is that these measures do not take into account the realities of the complex institutional framework and procedural aspects of the decision-making process. In reality, the European Commission acts as agenda setter, proposing bills to the CM. Before the latter makes its final decision, the proposed bill goes back and forth between the Commission, the CM and the European Parliament. Moberg (2002, p. 277f.), in particular, stresses the often prolonged process of amendment, and the 'strong consensus culture in the EU, even in matters where the treaty stipulates qualified majority. … Sooner or later after negotiations, consensus or at least a qualified majority will be found.' In fact, it is often the case that by the time the CM makes its decision, the voting is almost a formality.

In response, Felsenthal and Machover (2001b) have pointed out that the theory of a priori voting power admits the concept of *composite* decision rule, which allows the construction of extremely complex decision rules, involving several decision-making bodies, from simpler ones, thus providing models for highly intricate interactions among these simpler components.

Moreover, although the CM is only one component in the composite decision-making procedure, it is important to investigate the distribution of a priori voting power within this component, induced by its QMV decision rule.

Finally, it must be stressed that the voting power of a member of the CM must not be understood too literally, as referring only to the last formal stage at which the decision is finalized. The entire process, sometimes very lengthy, in which a proposed bill is amended and re-amended before the CM approves it, consists of a series of informal straw polls of the CM, performed as it were in the shadow of the formal QMV rule.

4 Conclusion

The development of the theory of voting power was bedevilled by two phenomena. First, the same ideas were reinvented several times over, because researchers were often unaware of relevant work published earlier. Second, the meaning and implications of some of the basic concepts was widely misunderstood. As a result, during much of its 57-years history as an academic discipline, the evolution of the theory was tortuous, with several fits and starts.

In the US academic, as well as the political and legal, interest in the measurement of a priori voting power was quite intense during the 1960s and 1970s but ebbed when the US Courts ruled out the possibility of using weighted voting in various types of local elections in lieu of reapportionment of (unequal) districts every decade. However, there are real-life situations where the use of weighted voting cannot be avoided, for example in international organizations such as the International Monetary Fund (IMF), the European Union, or business corporations which are composed of very unequal partners – be these states or individual shareholders. In such situations the need to establish a decision rule such that each partner will have an 'appropriate' a priori voting power is a fundamental constitutional problem whose solution must rely on a sound theory of measuring a priori voting power.[54]

In Europe interest in the measurement of voting power was rekindled during the past two decades; this was no doubt at least partly due to the subject's direct relevance to the weighted voting rule used in the European Union's most important decision-making body, the Council of Ministers. In particular, the controversy surrounding the re-weighting at the time of the Union's last enlargement (1995) impelled academics (if not politicians) to study the problem in a scientific way. This scientific activity has received fresh impetus from the prospective further enlargement of the EU: the subject of voting power has become once more a political 'hot potato' during the period leading to the 2001 Nice Treaty, in which a new system of weighted voting was adopted for an (enlarged) EU.

In conclusion, we would first like to stress once more the vital distinction between the two different notions of voting power. While the notion of P-power is a construct of cooperative game theory, the older alternative notion of I-power is independent of game theory. Failure to recognize this distinction has led to many errors in appraising the various measures of voting power, and has also confused the arguments regarding the need for an a priori measure in the European context.

As the *n*-person bargaining problem has not been solved conclusively, there is no completely satisfactory and universally agreed index of a priori P-power. On the other hand, Penrose’s measure _ (and its resulting relative Banzhaf index) are satisfactory and intuitively justified from the viewpoint of a priori I-power.

Second, from the foregoing it is quite clear that there is an undeniable practical need for a priori measures of voting power, mainly for constitutional design and analysis.

Finally, the problem of measuring *actual* voting power – which has so far not been solved in a satisfactory *practical* way – should in our view be tackled by a unified method: a method that takes the models used in the aprioristic theory and supplements them by additional structure that allows the incorporation of additional information.

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Notes

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1. A fuller historical account is in Felsenthal and Machover (1998, Sect. 1.2 and *passim*). The historical material relevant to the present paper is expanded (and corrected in some matters of detail) in Felsenthal and Machover (2005).

2. For an earlier attempt – albeit unsystematic and somewhat crude – to measure voting power, made by Luther Martin, a Maryland delegate to the 1787 Constitutional Convention held in Philadelphia, see Riker (1986) and further discussion in Felsenthal and Machover (2005).

3. Initially he assumes one vote per voter; but then goes on to admit bloc votes of several voters acting together. So the decision rules he considers are, in effect, weighted voting rules with quota set at half, or very slightly over half, of the total weight.

4. A voter may be an individual person or a bloc of several persons who invariably act as one.

5. In general: an assembly with n members has 2^n possible divisions.

6. The theory of more general, non-binary, decision rules, and of voting power under them, is still in its infancy. For some beginnings see Fishburn (1973), Bolger (1986, 1993a, 1993b, 2002), Felsenthal and Machover (1997, 1998), Freixas and Zwicker (2003).

7. At the time of writing, the EU is in its fifth period. According to the 2001 Treaty of Nice, this period will come to an end when the EU is enlarged; or on 1 January 2005 if it is not enlarged by then. The prescriptions of the Treaty of Nice regarding QMV are analysed in Felsenthal and Machover (2001a).

8. The formal definition of *[binary] decision rule* given here is equivalent to that of *simple game* given by Shapley (1962). For historical reasons that will be mentioned in Subsection 2.2, the terminology used in the literature on voting power is infected with the jargon of cooperative game theory. Thus, a decision rule (in the sense to be explained here) is referred to as a *simple game* or *simple voting game* (SVG); a weighted rule is called a *weighted voting game* (WVG); and voters are often called *players*. Any set of voters is referred to as a *coalition* – a particularly confusing usage, which we shall avoid here, because it falsely suggests the connotation of 'coalition' in normal political discourse, as an alliance of persons, parties or states that agree to act together on a more or less regular basis. The set of 'yes' voters in a division whose outcome is positive is referred to as a *winning* coalition; while any other set of voters is called a *losing* coalition – which is another unfortunate usage, because the 'no' voters in a division with negative outcome are also winners in the ordinary sense of this word.

9. This is referred to as the *monotonicity* condition.

10. For a thorough analysis see Taylor and Zwicker (1999). For a simple toy example of a decision rule that cannot be weighted see Felsenthal and Machover (1998, pp. 31–32); for real-life examples see Felsenthal and Machover (2001a).

11. We say ‘victorious’ rather than ‘winning’, in order to avoid confusion with the peculiar game-theoretic sense of the latter term mentioned in note 8. Thus, if the outcome of a division is negative, the ‘no’ camp is victorious and each member of it is successful.

12. Regarding the significance of this assumption, see Subsection 3.1.

13. This way of putting it is due to Barry (1980).

14. Penrose (1952). In (Penrose, 1946) he had used $\psi/2$ rather than ψ itself as measure of voting power. The difference between the two is inessential and does not affect any of the arguments presented by him or by us.

Note that ψ is in fact a *function*, whose value depends on the decision rule and the voter under consideration. When we need to emphasize this, we denote by $\psi_a[\mathcal{W}]$ the value of ψ for voter a under rule $\mathcal{W}$.

In using the symbol ψ here we follow Owen (1995). In Felsenthal and Machover (1998), following other authors, we denote Penrose’s measure by β'; this notation is due to historical reasons that will be mentioned in a moment (see note 18).

15. Another term used instead of ‘decisive’ is *critical*.

16. For an easy proof of the equivalence of the two ways of characterizing ψ, see Felsenthal and Machover (1998, pp. 45–46). There is also an operational way of characterizing ψ. Consider the following thought experiment. You are told that a decision-making council is about to divide on a proposed bill, which will affect your financial position: if the resolution is passed you will gain £1 million; if the resolution is defeated, you will lose £1 million. You know the [binary] decision rule under which the council operates, but you have absolutely no information about the preferences of the council’s members or any other causes that may affect their voting behaviour. If A is the proportion of divisions with positive outcome, then your payoff (expressed in £millions) is 1 with probability A and –1 with probability $1 - A$. Therefore your a priori expected payoff is $2A - 1$.

Now a, one of the council’s members, offers to sell you his voting rights for this particular division at a certain price. Should you be prepared to pay this price?

As first observed by Morriss (1987, p. 226) and proved by Felsenthal and Machover (1998, p. 45), if you use a’s voting rights to vote for the proposed resolution, your expected payoff (in £million) goes up to $2A - 1 + \psi_a$. Therefore you should be prepared to pay for a’s voting rights any price smaller than ψ_a, and refuse to pay any price greater than ψ_a.

17. Others who reinvented some of Penrose’s ideas include Rae (1969) as well as Cubbin and Leech (1983) who reinvented the quantity r; Coleman (1971)who invented two important variants of Penrose’s measure ψ; and Barry (1980) who reinvented both r and ψ. None of these mention Penrose’s work. It also appears that Rae did not know about Banzhaf’s work and that Coleman knew about neither Banzhaf’s nor Rae’s. Barry knew about the Banzhaf index, but misunderstood the reasoning behind it. On the unwitting generalization of the Penrose measure by Steunenberg, Schmidtchen and Koboldt (1999) see below, note 53.

18. In symbols:

$$\beta_a[\mathcal{W}] = \frac{\psi_a[\mathcal{W}]}{\sum_{x \in N} \psi_x[\mathcal{W}]}.$$

where *a* is any voter, *N* is the assembly and *W* is the decision rule.
This procedure of obtaining _ as a relativized form of _ is what mathematicians call 'normalization'.

Dubey and Shapley (1979), who were familiar with both Banzhaf's and Rae's work, but not with Penrose's, reinvented □, which they regarded as 'in many ways more natural' than □, and denoted it by □□. They also commented that the connection between this measure and *r* 'was not noticed for several years' after 1969, not realizing that Penrose had explicitly noted it back in 1946. Thereafter □, alias □′, came to be known as the *absolute Banzhaf index* (as opposed to □, the *relative* Banzhaf index), or as the *Banzhaf measure* or *Banzhaf value*.

19. Note that the values of □ for distinct voters are probabilities of events that are in general not disjoint; therefore their sum is in general not equal to the probability of the union of the corresponding events. For further discussion of the non-additivity of voting power, see Felsenthal and Machover (1998, *§* 7.2). For an analysis of the conditions under which forming a bloc is advantageous to its members, see Felsenthal and Machover (2002).

20. For the theory of TU cooperative games and the Shapley value, see standard books on game theory, e.g. Myerson (1991). Shapley (1953) defined the value named after him.

21. For an early critique, see Luce and Raiffa (1957). For further discussion, pointing out the inconsistency of these axioms with another apparently compelling axiom, see Felsenthal and Machover (1998, pp. 266–277).

22. For this reason, as mentioned in note 8, any set of voters that constitutes the 'yes' camp in a division with positive outcome is called a 'winning coalition'. Any other set of voters is said to be a 'losing coalition'.

23. See Felsenthal and Machover (1998, pp. 200–206) for a discussion of the widespread erroneous view that the Shapley–Shubik index relies on the – highly unrealistic – 'queue' model, according to which all voters queue up in random order to vote for every bill (all possible orders being equi-probable) and in each such queue the entire payoff goes to the so-called pivotal voter.

24. See Felsenthal and Machover (1998, pp. 277–278) for a brief summary of the main differences.

25. See note 18.

26. As an example of such an exceptional case, Coleman (1971, p. 299, n. 3) cites a party convention for nominating a candidate (presumably for the US presidency), '. . . for there are spols [*sic*] to be distributed among those delegations that support the winner… . But this is an unusual case, in which there is a winning candidate, who does have spoils to distribute.'

27. Coleman (1971 p. 271f).

28. The power of member *a* to prevent action is the conditional probability $□_a$ of *a* being decisive – that is, being in a position to change the outcome by changing his or her vote – given that the bill in question is passed; and the power of *a* to initiate action is the conditional probability $□^*_a$ of *a* being decisive, given that the bill is blocked.

If A is the probability of a bill being passed – which Coleman calls 'the power of the collectivity to act' – then $\square_a = 2A\square_a = 2(1 - A)\square^*_a$. Hence $\square_a$ can be obtained from $\square_a$ and $\square^*_a$ jointly, as their harmonic mean: $\square_a = [(\square_a^{-1} + \square^*{}_a^{-1})/2]^{-1}$.

Coleman's resolution of voting power into a negative and a positive component – which the Penrose measure lumps together – is extremely useful in political contexts such as the the CM of the EU, where one component is valued more than the other. See Leech (2002), Leech and Machover (2003), Hosli and Machover (2003).

29. See Brams and Affuso (1976).

30. See beginning of Subsection 1.2 above.

31. Here we are assuming that voting incurs no cost. Otherwise, a voter who is in favour of a bill may not vote for it if the decision rule and other voters' intentions are such that the bill is going to be defeated in any case.

32. For a discussion of the alternative motivations of policy seeking and once seeking in a political context, see Laver and Schofield (1990, Ch. 3).

33. Of course, since (as noted in Subsection 2.1) a binary decision rule is structurally equivalent to a simple cooperative game, some formal tools developed for dealing with cooperative games can usefully be applied in the mathematical treatment of I-power. Examples of such tools are the characteristic function, introduced by von Neumann and Morgenstern (1944), and Owen's (1972) multilinear extensions. However, here the characteristic function is treated as purely conventional enerali device, whereas in cooperative game theory its values are interpreted as the total payoff that a given coalition can obtain.

34. For an example of a theory of I-power that attempts to model binary relations between voters, see Hoede and Bakker (1982). For an example of a theory of I-power that incorporates a eneral model of voters' preferences and states of the world, see Steunenberg et al. (1999).

35. Note that in the terminology of cooperative game theory *any* set of players – in the present case voters – is called a 'coalition'. Thus a coalition exists, at least as a mathematical object, irrespective of whether it is 'formed'.

36. To illustrate this crucial difference between the two notions of voting power, consider an assembly of three voters under two decision rules: unanimity and simple majority. For reasons of symmetry, the P-power of a voter (according to any reasonable measure of P-power) must be 1⁄3 under both rules. But the Penrose measure

71. the only serious candidate for measuring a priori I-power – assigns to each of the three voters a value of 1⁄4 under the unanimity rule and 1⁄2 under the majority rule. Thus, although the *relative* I-power of a voter is again 1⁄3 under either rule, s/he is twice as powerful under the latter rule as under the former.

37. For a detailed discussion see Felsenthal and Machover (1998, Section 6.4 and Ch. 7 *passim*).

38. For a critique of these indices see Felsenthal and Machover (1998, Section 6.4 and Ch. 7 *passim*). The comedy of errors that led to the invention of the Johnston index is described in some detail in Felsenthal and Machover (1998, Section 6.4; 2005).

39. See note 21.

40. In this connection it is worth pointing out that in his empirical study Leech (2002a) shows that in large corporations with 'oceanic' structure – a few very large shareholders and a large number of very small ones – the Banzhaf index of the large shareholders corresponds much better than the S-S index to expert judgment of the real power of these large shareholders.

41. See note 7 and text to that note.

42. Coleman (1973, p. 4), as well as Coleman, Wu and Feld (1977, p. 16) call[s] it 'constitutional power'.

43. This deviation from linearity is referred to as 'degressivity' in EU-speak.

44. A subtler example involves Luxembourg, Denmark and Ireland under the 1981 QMV rule. As the patient reader may verify, the outcome of any possible division of the ten members would have remained unchanged if Luxembourg were to swap her weight with Denmark's or Ireland's. So, although Luxembourg had weight 2 and Denmark and Ireland 3 each, these three member-states must have had exactly the same a priori voting power.

45. This principle, also known as the *Principle of Indifference*, is often attributed to Laplace (1749–1827), but in fact goes back to Jacob Bernoulli's (1654–1705) posthumous classic, *Ars conjectandi* (1713). For a detailed critical discussion see Keynes (1921). The classical principle is in fact a special case of the *Principle of Maximal Information Entropy* of modern information theory; for which see Shannon (1948).

46. See note 17 and text to that note. Thus, Penrose in the Preface to his 1952 booklet: 'In discussing the question of voting I have introduced the assumption of indifference.. . .' Cf. note 45 above. Banzhaf (1966, p. 1316) is more explicit: 'Because *a priori* all voting combinations are equally possible, any objective measure of voting power must treat them as equally significant.' And Coleman (1971, p. 297), making a similar assumption, comments: 'This is appropriate for the analysis of formal power as given by the constitution, that is of organizational rules. It does not, however, provide a basis for behavioral prediction of the collectivity's action, when further information exists about the members.'

47. He also mentions various other objections, of a pragmatic nature, which will be mentioned in Subsection 3.3 below; but brushes them aside as irrelevant or at best of secondary importance compared to his own objection. For a more detailed response to Albert (2003), see Felsenthal and Machover (2003).

48. According to PIR, each point of an infinite probability space must be assigned probability 0. In the discrete case, this would imply that the whole space must also have probability 0, which is absurd. In the case of a continuous random variable X, PIR is usually interpreted as assigning to X a uniform distribution in some finite interval. But if we transform X to $Y = f(X)$, where f is a one-to-one continuous function, then by the same token Y should also have a uniform distribution. If f is not linear, this leads to contradiction.

49. See also more recent discussion by Howson (2000, pp. 81–86).

50. Other objections they direct specifically against the Banzhaf index are due to the conceptual confusion, discussed in Subsection 2.2, of regarding this index as one of P-power, based on cooperative game theory. This leads them to the fallacy

discussed in Subsection 1.3, of treating the Banzhaf index as an additive quantity. For a detailed discussion of these errors of Garrett and Tsebelis see Felsenthal and Machover (2001b; 2002).

51. For responses to Garrett and Tsebelis see Holler and Widgr´en (1999), Lane and Berg (1999), Felsenthal and Machover (2001b); for a response to Moberg see Hosli and Machover (2003).

52. Thus, for example, Owen (1982) provides a model that incorporates alliances; Steunenberg et al. (1999) provide a theoretical framework that includes the interplay of preferences. Note however that applying such theories in reality to obtain reliable figures for actual voting power is easier said than done. The required empirical data about voters' interactions and preferences, even where they can in principle be quantified, are rarely measurable with sufficient accuracy.

53. The theories mentioned in note 52 pass this test. The case of Steunenberg et al. (1999) is particularly remarkable. According to oral communication by these authors, they did not set out with the intention of eneralizing Penrose's measure of a priori voting power, nor were they aware that their theory of so-called 'strategic' voting power constituted such a generalization – as was proved subsequently by Felsenthal and Machover (2001b). This provides an independent theoretical argument in favour of the theory of strategic voting power, as well as a further confirmation of the Penrose approach.

54. See in this connection the paper by Leech (2003).

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Tables

Table 1: EUCM – QMV weights and quota, first five periods

Country	1958	1973	1981	1986	1995
Germany	4	10	10	10	10
Italy	4	10	10	10	10
France	4	10	10	10	10
Netherlands	2	5	5	5	5
Belgium	2	5	5	5	5
Luxembourg	1	2	2	2	2
UK		10	10	10	10
Denmark		3	3	3	3
Ireland		3	3	3	3
Greece			5	5	5
Spain				8	8
Portugal				5	5
Sweden					4
Austria					4
Finland					3
Total	17	58	63	76	87
Quota	12	41	45	54	62
Least #	3	5	5	7	8
Quota %	70.59	70.69	71.43	71.05	71.26

Notes The penultimate row gives the least number of members whose total weight equals or exceeds the quota. The last row gives the quota as percentage of the total weight.

Table 2: EUCM – Penrose measure (□) under QMV, first five periods

Country	1958	1973	1981	1986	1995
Germany	0.3125	0.2070	0.1953	0.1396	0.1129
Italy	0.3125	0.2070	0.1953	0.1396	0.1129
France	0.3125	0.2070	0.1953	0.1396	0.1129
Netherlands	0.1875	0.1133	0.1016	0.0723	0.0594
Belgium	0.1875	0.1133	0.1016	0.0723	0.0594
Luxembourg	0.0000	0.0195	0.0508	0.0195	0.0229
UK		0.2070	0.1953	0.1396	0.1129
Denmark		0.0820	0.0508	0.0498	0.0363
Ireland		0.0820	0.0508	0.0498	0.0363
Greece			0.1016	0.0723	0.0594
Spain				0.1182	0.0934
Portugal				0.0723	0.0594
Sweden					0.0484
Austria					0.0484
Finland					0.0363
Total	1.3125	1.2383	1.2383	1.0850	1.0110

Note Totals are subject to rounding errors.

Table 3: EUCM – Banzhaf index (_) under QMV, first five periods

Country	1958	1973	1981	1986	1995
Germany	0.238	0.167	0.158	0.129	0.112
Italy	0.238	0.167	0.158	0.129	0.112
France	0.238	0.167	0.158	0.129	0.112
Netherlands	0.143	0.091	0.082	0.067	0.059
Belgium	0.143	0.091	0.082	0.067	0.059
Luxembourg	0.000	0.016	0.041	0.018	0.023
UK		0.167	0.158	0.129	0.112
Denmark		0.066	0.041	0.046	0.036
Ireland		0.066	0.041	0.046	0.036
Greece			0.082	0.067	0.059
Spain				0.109	0.092
Portugal				0.067	0.059
Sweden					0.048
Austria					0.048
Finland					0.036
Total	1.000	1.000	1.000	1.000	1.000

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