Finite-state representations of time

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Abstract. Finite-state methods are applied to the Russell-Wiener notion of time (based on events) and developed into an account of interval relations and temporal propositions. Strings are formed and collected in regular languages and regular relations that are argued to embody temporal relations in their various underspecified guises. The regular relations include retractions that reduce computations by projecting strings down to an appropriate level of granularity, and non-deterministic relations defining notions of partiality within and across such levels.

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1 Introduction

It is a truism that to reason about change, some notion of time is useful to impose order on events. Less clear perhaps is whether or not time is shaped entirely by the events it relates. An event-based notion of time going back to Russell and Wiener ([Ru14,Wi14]) is analyzed in the present work using finite-state methods that extend to interval relations and beyond (e.g. [BE58,Tr58,AF94,BK03,Pr05]). Rather than take for granted some absolute (independent) notion of time (such as the real line $\mathbb{R}$), the basic approach is to (i) form strings from a set Φ encoding events and possibly other things, and (ii) collect the strings in regular languages and regular relations. The partiality of temporal information conveyed in (for instance) everyday speech is, it is argued, more faithfully represented in this manner than through some absolute notion of time. In particular, there is a sense (to be explained below) in which the strings, languages and relations of the present approach embody a wide range of temporal relations that vary in degrees of underspecification and granularity according to the set Φ assumed: the larger Φ is, the finer grained time becomes.

An idea of what the set Φ might be and of how to form strings from it is provided by a concrete example. We can represent a calendar year in terms of its constituent twelve months as the string

$$\mathsf{s}_{mo} \overset{\text{def}}{=} \boxed{\mathsf{Jan}}\boxed{\mathsf{Feb}}\boxed{\mathsf{Mar}}\cdots\boxed{\mathsf{Dec}}$$

of length 12, or, were we also interested in days $\mathsf{d1},\mathsf{d2}\ldots,\mathsf{d31}$, the string

$$\mathsf{s}_{mo,dy} \overset{\text{def}}{=} \boxed{\mathsf{Jan,d1}}\boxed{\mathsf{Jan,d2}}\cdots\boxed{\mathsf{Jan,d31}}\boxed{\mathsf{Feb,d1}}\cdots\boxed{\mathsf{Dec,d31}}$$

of length 365 (for a non-leap year). The alphabet from which s_{mo} and $\mathsf{s}_{mo,dy}$ are formed is the set $Pow(\Phi)$ of subsets of Φ, where $\{\mathsf{Jan},\mathsf{Feb},\ldots,\mathsf{Dec}\} \subseteq \Phi$ in the case of s_{mo}, and $\{\mathsf{Jan},\mathsf{Feb},\ldots,\mathsf{Dec},\mathsf{d1},\mathsf{d2},\ldots,\mathsf{d31}\} \subseteq \Phi$ in the case of $\mathsf{s}_{mo,dy}$. Boxes are drawn instead of the usual curly braces { and } to suggest "snapshots" arranged much like a cartoon/film strip, with time progressing from left to right.

In reducing chronological order to succession within a string of boxes, however, have we imposed a limit on the granularity of moments? Unlike the points in the real line $\mathbb{R}$, a box can split, as $\boxed{\mathsf{Jan}}$ in s_{mo} does to $\boxed{\mathsf{Jan,d1}}\boxed{\mathsf{Jan,d2}}\cdots\boxed{\mathsf{Jan,d31}}$ in $\mathsf{s}_{mo,dy}$, on introducing days $\mathsf{d1}$, $\mathsf{d2}$,..., $\mathsf{d31}$ into the picture. Reversing direction, a system of functions $\pi_X : Pow(\Phi)^* \to Pow(X)^*$ indexed by finite subsets X of Φ is defined below such that

$$\pi_X(\mathsf{s}_{mo,dy}) \;=\; \mathsf{s}_{mo} \quad \text{for} \quad X \stackrel{\text{def}}{=} \{\mathsf{Jan},\, \mathsf{Feb},\ldots,\, \mathsf{Dec}\}.$$

In general, a description s_X of granularity X can be refined to one $\mathsf{s}_{X'}$ of granularity $X' \supseteq X$ provided π_X maps $\mathsf{s}_{X'}$ to s_X. More precisely, if $Fin(\Phi)$ is the set of finite subsets of Φ, and if for $X \subseteq X' \in Fin(\Phi)$, we let $\pi_{X',X}$ be the restriction of π_X to $Pow(X')^*$

$$\pi_{X',X} \;\stackrel{\text{def}}{=}\; \{\langle \mathsf{s}, \pi_X(\mathsf{s})\rangle \mid \mathsf{s} \in Pow(X')^*\} \qquad (\text{for } X \subseteq X' \in Fin(\Phi))$$

then we can collect $Fin(\Phi)$-indexed strings $\{\mathsf{s}_X\}_{X\in Fin(\Phi)}$ such that s_X can be calculated as $\pi_X(\mathsf{s}_{X'})$ for any $X' \supseteq X$ in the *inverse limit of* $\{\pi_{X',X}\}_{X\subseteq X'\in Fin(\Phi)}$

$$\varprojlim \{\pi_{X',X}\}_{X\subseteq X'\in Fin(\Phi)} \;=\; \{\{\mathsf{s}_X\}_{X\in Fin(\Phi)} \;\in \prod_{X\in Fin(\Phi)} Pow(X)^* \mid \\ \mathsf{s}_X \;=\; \pi_X(\mathsf{s}_{X'}) \text{ for all } X \subseteq X' \in Fin(\Phi)\}.$$

It will turn out that the functions $\pi_{X',X}$ can be computed by finite-state transducers ([Fe11]), and that the inverse limit has representations of all Russell-Wiener notions of time based on events encoded in Φ. This includes the real line $\mathbb{R}$, for suitably chosen Φ.[1]

[1] Commenting on the Russell-Wiener construction, S.K. Thomason asserts that it "has the disadvantage that it is difficult to see what assumptions about the temporal relationships among events will ensure that the instants constructed comprise a continuum, isomorphic to the real numbers" (page 85, [Th84]). A case may be made that this defect has since been remedied in [Lü06]. Be that as it may, Thomason considers alternative constructions such as Walker's ([Wa47]) and his own in [Th89], raising the challenge to provide

> a plausible explanation of how finite information-processors like us would come to *think of* time as a continuum ... For that problem, I think, the appropriate mathematical structure is not an event ordering (which models the world's events) but a category ... of finite event orderings (which models observers' developing apprehensions of the worlds's events) (pages 95–96, [Th84]).

The study below of the inverse limit of π_X (over finite subsets X of Φ) is my own attempt to take up this challenge, straying minimally from Russell-Wiener to delimit strings (thereby accommodating the A-series of [McT08] and the Allen relations).

There is, of course, more to temporal relations than Russell-Wiener. Indeed, already the inverse limit above goes further, as we will see below. Beyond that limit, we will consider relations other than π_X to capture the partiality of information. An example is the notion of containment $\sqsupseteq$ that we use to interpret Russell-Wiener over strings in section 2 (before applying the projections π_X to different sets X of events).

1.1 Russell-Wiener and intervals

The Russell-Wiener construction of time from a set E of *events*[2] presupposes a binary relation $\prec$ on E that meets certain properties (to be specified shortly) of *complete* (*temporal*) *precedence*. A $\prec$*-antichain* is a subset of E that has *no* elements e, e' such that $e \prec e'$. An *RW-time* is a $\prec$-antichain that is maximal in that it is *not* a subset of another $\prec$-antichain. Let us write $T_\prec$ for the set of RW-times, and $\bigcirc$ for $\prec$-incomparability on events $e, e' \in E$

$$e \bigcirc e' \quad \stackrel{\text{def}}{\iff} \quad \text{neither } e \prec e' \text{ nor } e' \prec e \ .$$

It is natural to call $\bigcirc$ (*temporal*) *overlap* as $\bigcirc$ holds between events e and e' with an RW-time in common

$$e \bigcirc e' \quad \iff \quad (\exists t \in T_\prec)\ e \in t \text{ and } e' \in t \ .$$

Next, we lift $\prec$ to RW-times $t, t' \in T_\prec$ by existential quantification

$$t \prec_T t' \quad \stackrel{\text{def}}{\iff} \quad (\exists e \in t)(\exists e' \in t')\ e \prec e' \ .$$

The RW-times are linearly ordered by $\prec_T$ provided $\prec$ is "well-behaved" on E. More precisely, let us call $\prec$ *RW*(E) if on E, $\prec$ is irreflexive (never $e \prec e$) and satisfies the implication

$$e \prec e' \text{ and } e'' \prec e''' \quad \text{implies} \quad e \prec e''' \text{ or } e'' \prec e'$$

for all $e, e', e'', e''' \in E$, characteristic of an *interval ordering* (e.g. [Lü06]).

Theorem (Russell, Wiener). *If $\prec$ is RW(E), then*

(i) *$\prec_T$ linearly orders $T_\prec$*
(ii) *$\prec$ on E can be recovered from $T_\prec$ as complete precedence*

$$e \prec e' \iff (\forall t, t' \in T_\prec)\ e \in t \text{ and } e' \in t' \text{ implies } t \prec_T t'$$

for all $e, e' \in E$, and
(iii) *the RW-times to which each event belongs form an interval in that*

$$t \cap t'' \subseteq t' \quad \text{whenever} \quad t \prec_T t' \prec_T t''$$

for all $t, t', t'' \in T_\prec$.

[2] For a set whose elements we think of as events (or better yet: event tokens), we write E instead of Φ, but will revert to Φ when generalizing beyond such event tokens.

Focussing on the case $E = \{e, e'\}$ consisting of two events, e and e', we have exactly three RW(E) relations, with RW-times pictured by the three strings

$$\boxed{e,e'} \text{ for } e \bigcirc e', \qquad \boxed{e}\boxed{e'} \text{ for } e \prec e', \qquad \text{and} \qquad \boxed{e'}\boxed{e} \text{ for } e' \prec e$$

of lengths 1, 2 and 2, respectively. Surely, however, between any two events e and e', there are more relations than $\prec$ ($\succ$ and $\bigcirc$) to consider — not to mention strings in $Pow(\{e, e'\})^*$ other than $\boxed{e}\boxed{e'}$, $\boxed{e'}\boxed{e}$ and $\boxed{e,e'}$. Thirteen different interval relations are enumerated in [AF94], describing nine ways for e and e' to overlap, and two ways (each) for e to precede e' (and e' to precede e). See Table 1, where strings are associated with Allen relations according to certain

Table 1. From Russell-Wiener to Allen

RW	**Allen**	$Pow(\{e,e'\})^*$	**Allen**	$Pow(\{e,e'\})^*$	**Allen**	$Pow(\{e,e'\})^*$
$e \bigcirc e'$	e `=` e'	$\boxed{e,e'}$	e `fi` e'	$\boxed{e}\boxed{e,e'}$	e `f` e'	$\boxed{e'}\boxed{e,e'}$
	e `si` e'	$\boxed{e,e'}\boxed{e}$	e `di` e'	$\boxed{e}\boxed{e,e'}\boxed{e}$	e `oi` e'	$\boxed{e'}\boxed{e,e'}\boxed{e}$
	e `s` e'	$\boxed{e,e'}\boxed{e'}$	e `o` e'	$\boxed{e}\boxed{e,e'}\boxed{e'}$	e `d` e'	$\boxed{e'}\boxed{e,e'}\boxed{e'}$
$e \prec e'$	e `m` e'	$\boxed{e}\boxed{e'}$	$e < e'$	$\boxed{e}\boxed{}\boxed{e'}$		
$e' \prec e$	e `mi` e'	$\boxed{e'}\boxed{e}$	$e > e'$	$\boxed{e'}\boxed{}\boxed{e}$		

constructions presented below. Briefly, under these constructions, granularity can be refined by expanding the set of events related by $\prec$ and $\bigcirc$. In particular, it turns out that all thirteen Allen relations between e and e' fall out of the Russell-Wiener construction (RW) applied to an expansion of $\{e, e'\}$ by markers $pre(e)$, $post(e)$, $pre(e')$, $post(e')$, of the past and future of e and e', respectively. That is, RW yields the Allen relations provided that, in McTaggart's terminology [McT08], we first enrich the B-series relation $\prec$ with A-series ingredients for tense. In the case of the Allen relation e `s` e', for instance, we get the string

$$\boxed{pre(e), pre(e')}\boxed{e, e'}\boxed{post(e), e'}\boxed{post(e), post(e')}$$

which the function $\pi_{\{e,e'\}}$ maps to the $Pow(\{e, e'\})^*$-entry

$$\boxed{e, e'}\boxed{e'} = \pi_{\{e,e'\}}(\boxed{pre(e), pre(e')}\boxed{e, e'}\boxed{post(e), e'}\boxed{post(e), post(e')})$$

for e `s` e' in Table 1. The rest of the Allen relations can be obtained similarly. Moreover, the projections π_X point below to many more temporal relations.

1.2 MSO and the alphabet $Pow(\Phi)$

Having suggested $\prec$-incomparability is temporal overlap $\bigcirc$, let us acknowledge cases where this identification is questionable. In the string $\boxed{e}\boxed{e'}\boxed{e}$, for example, we might say e and e' are $\prec$-incomparable but resist the conclusion that they overlap. (The same can be said of days $\mathsf{d}n$ in the string $\mathsf{s}_{mo,dy}$ mentioned above.) In the next section, we map a string s of sets to a triple $\langle E^{\mathsf{s}}, \bigcirc^{\mathsf{s}}, \prec^{\mathsf{s}} \rangle$ that may, depending on s, differentiate $\bigcirc$ from $\prec$-incomparability. The view of strings as models goes back to the classic result tying regular languages over an alphabet Σ to Monadic Second-Order Logic MSO_{Σ} with a binary (successor) relation and a unary relation for each symbol in Σ ([BE58,Tr58]).

Theorem (Büchi, Elgot, Trakhtenbrot). *The regular languages* $\subseteq \Sigma^+$ *are precisely the sets of strings definable in* MSO_{Σ}.

Details about the theorem can be found in [Th97], an instructive point for our purposes being that the MSO variables ranging over unary relations correspond to the elements of the set Φ from which we form the alphabet $Pow(\Phi)$ above. Binding these variables by a quantifier corresponds to using auxiliary symbols in a finite-state computation. What's more, the partial order $\supseteq$ on $Pow(\Phi)$ extends to a natural notion $\mathsf{s} \sqsupseteq \mathsf{s}'$ of s' occurring within s that we will use to define $\bigcirc^{\mathsf{s}}$ and $\prec^{\mathsf{s}}$.

The strings in $Pow(\Phi)^*$ of interest below belong, in general, to $Pow(X)^*$, for some finite subset X of Φ. An alternative to the language $Pow(X)^*$ considered in [Ka05] is to flatten the subsets of X to strings, introducing brackets [and] to enclose elements of X understood to hold at the same period so that, for example, the string $\boxed{e, e'}\boxed{e'}$ of length 2 becomes the string $[\ e\ e'\ ]\ [\ e'\ ]$ of length 7. It is easy to devise a finite-state transducer translating $Pow(X)^*$ to $(X \cup \{[,]\})^*$ in this way. A greater challenge is presented by brackets $[_a$ and $]_a$ decorated with granularity a (e.g., days or months or years) used in the analysis of calendar expressions in [NK09]. The theme in [NK09] of composing finite-state transducers is developed below with granularity a left unspecified — or rather, implicit in the choice of the set X, subsets of which form an alphabet $Pow(X)$.

But what is lost when approximating Φ by its finite subsets $X \in Fin(\Phi)$? The main technical result of the present work describes what can be found in the inverse limit of the projections π_X (for $X \in Fin(\Phi)$). Very briefly, this includes all Russell-Wiener notions of time given by events encoded in Φ, and many more structures besides. In these (additional) structures, the elements of Φ (i.e., MSO variables) are naturally regarded as temporal propositions, or *fluents* for short, generalizing the events that Russell-Wiener restricts to intervals. Intervals and Russell-Wiener are the focus of section 2, where E appears in place of Φ. Section 3 turns to generalizations and the larger matter of approximating fluents, bringing back the notation Φ alongside E.

2 Russell-Wiener in strings

To interpret Russell-Wiener relative to a string s of sets, we define a notion $\sqsupseteq$ of containment between strings such that

(i) $e \bigcirc e'$ holds in s precisely if $\mathsf{s} \sqsupseteq \boxed{e, e'}$, and

(ii) $e \prec e'$ holds in s precisely if s $\sqsupseteq$-contains a string in $\boxed{e}\boxed{}^{*}\boxed{e'}$ but (to ensure disjointness from $\bigcirc$ and its inverse $\succ$) none in

$$\boxed{e,e'} \mid \boxed{e'}\boxed{}^{*}\boxed{e}$$

where $\mid$ is non-deterministic choice (often written $+$).

Containment can be decomposed into two relations: *subsumption* $\unrhd$, defined as componentwise inclusion $\supseteq$ between strings of the same length

$$\alpha_1 \cdots \alpha_n \;\unrhd\; \alpha'_1 \cdots \alpha'_m \quad \stackrel{\text{def}}{\iff} \quad n = m \text{ and } \alpha_i \supseteq \alpha'_i \text{ for } 1 \leq i \leq n,$$

and (to compare strings of different lengths) a map *unpad* that strips off initial and final empty boxes $\Box$

$$\mathit{unpad}(\mathsf{s}) \;\stackrel{\text{def}}{=}\; \begin{cases} \mathit{unpad}(\mathsf{s}') & \text{if } \mathsf{s} = \Box\mathsf{s}' \text{ or else if } \mathsf{s} = \mathsf{s}'\Box \\ \mathsf{s} & \text{otherwise} \end{cases}$$

so that, for example,

$$\mathit{unpad}(\boxed{}^{n}\boxed{e}\boxed{}\boxed{e,e'}\boxed{}^{m}) = \boxed{e}\boxed{}\boxed{e,e'} \quad \text{for all } n, m \geq 0.$$

We compose subsumption $\unrhd$ with *unpad*-equivalence for *containment* $\sqsupseteq$

$$\mathsf{s} \sqsupseteq \mathsf{s}' \quad \stackrel{\text{def}}{\iff} \quad (\exists \mathsf{s}'')\ \mathsf{s} \unrhd \mathsf{s}'' \text{ and } \mathit{unpad}(\mathsf{s}'') = \mathit{unpad}(\mathsf{s}')\ .$$

Thus, if s contains s' (e.g. if s is s'), then so do $\mathit{unpad}(\mathsf{s})$ and $\mathsf{s}''\mathsf{s}$ and $\mathsf{s}\mathsf{s}''$ for all s''. Now, we let $\mathbb{E}(\mathsf{s})$ be the triple $\langle E^{\mathsf{s}}, \bigcirc^{\mathsf{s}}, \prec^{\mathsf{s}} \rangle$ defined as follows. We say e s-*overlaps* e' precisely if e and e' share a box in s

$$e \bigcirc^{\mathsf{s}} e' \quad \stackrel{\text{def}}{\iff} \quad \mathsf{s} \sqsupseteq \boxed{e,e'}$$

and put into E^{s} each e such that $e \bigcirc^{\mathsf{s}} e$

$$E^{\mathsf{s}} \;\stackrel{\text{def}}{=}\; \{e \mid \mathsf{s} \sqsupseteq \boxed{e}\}.$$

For $\prec^{\mathsf{s}}$, let us extend $\sqsupseteq$ to languages L, conceived as disjunctions, agreeing that s *contains* L if s contains some element of L

$$\mathsf{s} \sqsupseteq L \quad \stackrel{\text{def}}{\iff} \quad (\exists \mathsf{s}' \in L)\ \mathsf{s} \sqsupseteq \mathsf{s}'$$

so that $\mathsf{s} \sqsupseteq \mathsf{s}'$ iff $\mathsf{s} \sqsupseteq \{\mathsf{s}'\}$. We then say e s-*precedes* e' if e occurs in s to the left of e' but never in the same box as e' or to the right of e'

$$e \prec^{\mathsf{s}} e' \quad \stackrel{\text{def}}{\iff} \quad \mathsf{s} \sqsupseteq \boxed{e}\boxed{}^{*}\boxed{e'} \text{ and not } \mathsf{s} \sqsupseteq \boxed{e,e'} \mid \boxed{e'}\boxed{}^{*}\boxed{e}.$$

Table 2. Axioms for a (RWK) event structure $\langle E, \bigcirc, \prec\rangle$

(A_1)	$e \bigcirc e$ (i.e. $\bigcirc$ is reflexive)
(A_2)	$e \bigcirc e'$ implies $e' \bigcirc e$
(A_3)	$e \prec e'$ implies not $e \bigcirc e'$
(A_4)	$e \prec e'$ and $e' \bigcirc e''$ and $e'' \prec e'''$ implies $e \prec e'''$
(A_5)	$e \prec e'$ or $e' \prec e$ or $e \bigcirc e'$

2.1 RW strings delimited

Following [KR93], let us call a triple $\langle E, \bigcirc, \prec\rangle$ a *(Russell-Wiener-Kamp) event structure* if it satisfies (A_1) to (A_5) in Table 2. Note that

$$\prec \text{ is RW}(E) \iff \langle E, \prec\text{-incomparability}, \prec\rangle \text{ is an event-structure}$$

and in every event structure $\langle E, \bigcirc, \prec\rangle$, $\bigcirc$ is $\prec$-incomparability. Given a string s of sets, the truth of (A_1) and (A_2) relative to $\bigcirc = \bigcirc^{\mathsf{s}}$ follows from $\boxed{e,e} = \boxed{e}$ and $\boxed{e,e'} = \boxed{e',e}$, Interpreting $\prec$ as $\prec^{\mathsf{s}}$ also validates (A_3) and (A_4). This leaves (A_5), a counter-example to which is provided by the string $\boxed{e}\boxed{e'}\boxed{e}$. With this in mind, we define an element $e \in E^{\mathsf{s}}$ to be an s-*interval* if for $\mathsf{s} = \alpha_1 \cdots \alpha_n$,

$$e \in \alpha_i \cap \alpha_j \text{ and } i \leq k \leq j \quad \text{implies} \quad e \in \alpha_k$$

for all integers i, j, k from 1 to n. A string s is said to be *RW* if every $e \in E^{\mathsf{s}}$ is an s-interval. If s is RW, then

$$e \prec^{\mathsf{s}} e' \iff \mathsf{s} \sqsupseteq \boxed{e}\boxed{}^{*}\boxed{e'} \text{ and not } \mathsf{s} \sqsupseteq \boxed{e,e'}$$

and moreover, recalling that $\mathbb{E}(\mathsf{s})$ is $\langle E^{\mathsf{s}}, \bigcirc^{\mathsf{s}}, \prec^{\mathsf{s}}\rangle$,

Proposition 1. *If* s *is RW, then* $\mathbb{E}(\mathsf{s})$ *is an event structure.*

As a string s need not be RW, it is useful to define the subset $I(\mathsf{s})$ of E^{s} consisting of s-intervals

$$I(\mathsf{s}) \stackrel{\text{def}}{=} \{e \in E^{\mathsf{s}} \mid e \text{ is an } \mathsf{s}\text{-interval}\} .$$

For example, $I(\boxed{e}\boxed{e'}\boxed{e}) = \{e'\}$. Next, for any set X, we define the function ρ_X on strings (of sets) to componentwise intersect with X

$$\rho_X(\alpha_1 \cdots \alpha_n) \stackrel{\text{def}}{=} (\alpha_1 \cap X) \cdots (\alpha_n \cap X)$$

(throwing out non-X's from each box) so that, for instance, if $\hat{\mathsf{s}}$ is $\boxed{e}\boxed{e'}\boxed{e}$, $\rho_{I(\hat{\mathsf{s}})}(\hat{\mathsf{s}}) = \boxed{}\boxed{e'}\boxed{}$. In general, $\rho_{I(\mathsf{s})}(\mathsf{s})$ is RW for every string s of sets; consequently, $\mathbb{E}(\rho_{I(\mathsf{s})}(\mathsf{s}))$ is an event structure.

Apart from whittling a string s down to some substring $\mathsf{s}' \sqsubseteq \mathsf{s}$ of it (such as $\rho_{I(\mathsf{s})}(\mathsf{s})$), it is useful for some purposes, to fatten s up to a string that $\sqsupseteq$-contains it. Take, for instance, the thirteen strings over $Pow(\{e, e'\})$ in Table 1 associated with the various Allen relations between intervals e and e'. To capture all thirteen strings as representations of Russell-Wiener notions $T_{\prec}$ of time, we need to get around the $\subseteq$-maximality requirement on the sets in $T_{\bigcirc}$. One way is for each $e \in E$, to adjoin fresh "events" $pre(e)$ and $post(e)$ into an expansion $E_{\pm}$ of E

$$E_{\pm} \stackrel{\text{def}}{=} E \cup \{pre(e) \mid e \in E\} \cup \{post(e) \mid e \in E\}$$

and delimit occurrences of e in a string by $pre(e)$ to the left and by $post(e)$ to the right. For instance, $\boxed{e}\boxed{}\boxed{e'}$ becomes $\boxed{e, pre(e')}\boxed{post(e), pre(e')}\boxed{post(e), e'}$ while $\boxed{e}\boxed{e,e'}\boxed{e'}$ becomes $\boxed{e, pre(e')}\boxed{e,e'}\boxed{e', post(e)}$. On RW strings, $pre(e)$ and $post(e)$ negate e, whilst preserving RW-ness. More precisely, let us call a string $\mathsf{s} = \alpha_1\alpha_2\cdots\alpha_n$ *E-delimited* if for all $e \in E$ and $i \in \{1, 2, \ldots, n\}$,

$$pre(e) \in \alpha_i \iff \mathsf{s} \sqsupseteq \boxed{e} \text{ but } \alpha_1 \cdots \alpha_i \not\sqsupseteq \boxed{e}$$

and

$$post(e) \in \alpha_i \iff \mathsf{s} \sqsupseteq \boxed{e} \text{ but } \alpha_i \cdots \alpha_n \not\sqsupseteq \boxed{e} \ .$$

It is immediate that there is a unique E-delimited string $\mathsf{s}' \in Pow(E_{\pm})^*$ such that $\rho_E(\mathsf{s}') = \mathsf{s}$ — namely, $\alpha'_1 \cdots \alpha'_n$ where α'_i is defined to be α_i unioned with

$$\{pre(e) \mid e \in (\bigcup_{j=i+1}^{n} \alpha_j) - \bigcup_{j=1}^{i} \alpha_j\} \ \cup \ \{post(e) \mid e \in (\bigcup_{j=1}^{i-1} \alpha_j) - \bigcup_{j=i}^{n} \alpha_j\}$$

(for $1 \leq i \leq n$). Henceforth, we write $\mathsf{s}_{\pm}$ for this E-delimited string.

Proposition 2. *For every finite set E, there is a finite-state transducer that computes the map $\mathsf{s} \mapsto \mathsf{s}_{\pm}$ from $Pow(E)^*$ to $Pow(E_{\pm})^*$.*

If s is RW, then so is $\mathsf{s}_{\pm}$, making $\mathbb{E}(\mathsf{s}_{\pm})$ an event structure in which moreover, each of the 13 Allen relations on E^{s} can be expressed via $\bigcirc^{\mathsf{s}_{\pm}}$, *pre*, *post* and Boolean connectives — e.g.

$$\begin{array}{lcl} e\ \mathtt{d}^{\mathsf{s}}\ e' & \iff & pre(e) \bigcirc^{\mathsf{s}_{\pm}} e' \text{ and } post(e) \bigcirc^{\mathsf{s}_{\pm}} e' \\ e\ \mathtt{m}^{\mathsf{s}}\ e' & \iff & e \bigcirc^{\mathsf{s}_{\pm}} pre(e') \text{ and } post(e) \bigcirc^{\mathsf{s}_{\pm}} e' \text{ and} \\ & & \text{neither } e \bigcirc^{\mathsf{s}_{\pm}} e' \text{ nor } post(e) \bigcirc^{\mathsf{s}_{\pm}} pre(e') \end{array}$$

for all $e, e' \in E^{\mathsf{s}}$.

Having modified a string s to obtain an event structure capturing Russell-Wiener and Allen, we turn next to string functions that preserve the event structure encoded, bringing out what is essential to the encoding. Consider, for example, $\boxed{e}^{+}\boxed{e'}^{+}$, all strings of which encode the same event structure. The *block compression* $bc(\mathsf{s})$ *of* a string s reduces all adjacent identical boxes $\alpha\alpha^n$ to one α

$$bc(\mathsf{s}) \stackrel{\text{def}}{=} \begin{cases} bc(\alpha\mathsf{s}') & \text{if } \mathsf{s} = \alpha\alpha\mathsf{s}' \\ \alpha bc(\alpha'\mathsf{s}') & \text{if } \mathsf{s} = \alpha\alpha'\mathsf{s}' \text{ with } \alpha \neq \alpha' \\ \mathsf{s} & \text{otherwise} \end{cases}$$

so that, for example,

$$\mathit{bc}(\mathsf{s}) = \boxed{e}\boxed{e'} \quad \text{for every } \mathsf{s} \in \boxed{e}^+ \boxed{e'}^+ \ .$$

The map bc is a regular relation, and implements the slogan "no time without change" [KR93] (page 674). Clearly, bc does *not* alter the event structure $\mathbb{E}(\mathsf{s})$ represented by a string s

$$\mathbb{E}(\mathit{bc}(\mathsf{s})) = \mathbb{E}(\mathsf{s}) \ .$$

Neither does unpadding, which suggests defining a function π that unpads after (or equivalently: before) block compression

$$\pi(\mathsf{s}) \stackrel{\text{def}}{=} \mathit{unpad}(\mathit{bc}(\mathsf{s})) \quad [= \mathit{bc}(\mathit{unpad}(\mathsf{s}))]$$

so that, for example,

$$\pi(\mathsf{s}) = \boxed{e}\boxed{e'} \quad \text{for every } \mathsf{s} \in \boxed{}^* \boxed{e}^+ \boxed{e'}^+ \boxed{}^* \ .$$

On delimited strings $\mathsf{s}_\pm$, π captures what is essential for representing event structures.

Proposition 3. *For RW strings* s *and* $\mathsf{s}' \in \mathit{Pow}(E)^*$, *the following four conditions,* (a) *to* (d), *are equivalent*

(a) $\mathit{bc}(\mathsf{s}) = \mathit{bc}(\mathsf{s}')$
(b) $\mathbb{E}(\mathsf{s}_\pm) = \mathbb{E}(\mathsf{s}'_\pm)$
(c) $\mathit{bc}(\mathsf{s}_\pm) = \mathit{bc}(\mathsf{s}'_\pm)$
(d) $\pi(\mathsf{s}_\pm) = \pi(\mathsf{s}'_\pm)$.

As $\mathit{bc}(\mathit{bc}(\mathsf{s}_\pm)) = \mathit{bc}(\mathsf{s}_\pm) = \pi(\mathsf{s}_\pm)$, we have from Proposition 3

Corollary 4. *For RW strings* $\mathsf{s} \in \mathit{Pow}(E)^*$,

$$\mathbb{E}(\mathsf{s}_\pm) \ = \ \mathbb{E}(\mathit{bc}(\mathsf{s}_\pm)) \ = \ \mathbb{E}(\pi(\mathsf{s}_\pm)).$$

2.2 The projections π_X

Next, we compose the functions ρ_X (picking out X-elements) and π for the projection π_X mapping a string s of sets to

$$\pi_X(\mathsf{s}) \ \stackrel{\text{def}}{=} \ \pi(\rho_X(\mathsf{s})) \ = \ \mathit{unpad}(\mathit{bc}(\rho_X(\mathsf{s}))) \ .$$

Recalling the strings

$$\begin{array}{rcl} \mathsf{s}_{mo} & \stackrel{\text{def}}{=} & \boxed{\mathsf{Jan}}\boxed{\mathsf{Feb}}\cdots\boxed{\mathsf{Dec}} \\ \mathsf{s}_{mo,dy} & \stackrel{\text{def}}{=} & \boxed{\mathsf{Jan,d1}}\boxed{\mathsf{Jan,d2}}\cdots\boxed{\mathsf{Dec,d31}} \end{array}$$

we have for $mo \stackrel{\text{def}}{=} \{\mathsf{Jan}, \mathsf{Feb}, \ldots \mathsf{Dec}\}$,

$$\begin{aligned}
\rho_{mo}(\mathsf{s}_{mo,dy}) &= \boxed{\mathsf{Jan}}^{31}\boxed{\mathsf{Feb}}^{28}\cdots\boxed{\mathsf{Dec}}^{31} \\
\pi_{mo}(\mathsf{s}_{mo.dy}) &= \mathit{unpad}(bc(\boxed{\mathsf{Jan}}^{31}\boxed{\mathsf{Feb}}^{28}\cdots\boxed{\mathsf{Dec}}^{31})) \\
&= \mathsf{s}_{mo} \qquad \text{(as promised).}
\end{aligned}$$

Also, for the months $m \in mo$, $\pi_{\{m\}}(\mathsf{s}_{mo,dy}) = \boxed{m}$, whereas for the days $d \in \{\mathsf{d1}, \ldots, \mathsf{d31}\}$, $\pi_{\{d\}}(\mathsf{s}_{mo,dy}) \neq \boxed{d}$. In general, the set $I(\mathsf{s})$ of s-intervals consists of every e such that s becomes $\boxed{e}$ under $\pi_{\{e\}}$

$$e \in I(\mathsf{s}) \iff \pi_{\{e\}}(\mathsf{s}) = \boxed{e}\ .$$

Hence, we may look for strings representing RW(E)-relations in the language $\bigcap_{e \in E} \pi^{-1}_{\{e\}}\boxed{e}$ of strings s where each $e \in E$ is an s-interval. Since π preserves the event structure encoded by an RW-string, we can reduce $\bigcap_{e \in E} \pi^{-1}_{\{e\}}\boxed{e}$ to its π_E-image which we shall call $\mathcal{L}_\pi(E)$

$$\mathcal{L}_\pi(E) \stackrel{\text{def}}{=} \{\pi_E(\mathsf{s}) \mid \mathsf{s} \in \mathit{Pow}(E)^* \text{ and } (\forall e \in E)\ \pi_{\{e\}}(\mathsf{s}) = \boxed{e}\}.$$

In fact, $\mathcal{L}_\pi(E)$ may go beyond RW(E) because RW-times are (unlike boxes in $\mathcal{L}_\pi(E)$) required to be $\subseteq$-maximal. For example, for $E = \{e, e'\}$, we get not only the three RW(E)-strings $\boxed{e, e'}$, $\boxed{e}\boxed{e'}$ and $\boxed{e'}\boxed{e}$, but all 13 strings in Table 1 (one per interval relation in [AF94]), which can be divided up as follows. Put the 9 ways for e and e' to overlap (according to Allen) in

$$\begin{aligned}
\text{Allen}(e \bigcirc e') &\stackrel{\text{def}}{=} (\epsilon \mid \boxed{e} \mid \boxed{e'})\boxed{e, e'}(\epsilon \mid \boxed{e} \mid \boxed{e'}) \\
&= \boxed{e, e'} \mid \boxed{e, e'}\boxed{e} \mid \boxed{e, e'}\boxed{e'} \mid \cdots \mid \boxed{e'}\boxed{e, e'}\boxed{e'}
\end{aligned}$$

(where ϵ is the empty string), and the 2 ways for e to precede e' in

$$\text{Allen}(e \prec e') \stackrel{\text{def}}{=} \boxed{e}\boxed{e'} \mid \boxed{e}\boxed{}\boxed{e'}$$

and similarly for $e' \prec e$. All together,

$$\mathcal{L}_\pi(\{e, e'\}) = \text{Allen}(e \prec e') \mid \text{Allen}(e \bigcirc e') \mid \text{Allen}(e' \prec e)$$

and in fact,

Proposition 5. *For every $\mathsf{s} \in \mathit{Pow}(E)^*$ and all s-intervals e and e',*

$$\begin{aligned}
e \bigcirc^{\mathsf{s}} e' &\iff \pi_{\{e,e'\}}(\mathsf{s}) \in \text{Allen}(e \bigcirc e') \\
e \prec^{\mathsf{s}} e' &\iff \pi_{\{e,e'\}}(\mathsf{s}) \in \text{Allen}(e \prec e')
\end{aligned}$$

and in accordance with axiom (A_5) *in Table 2,*

$$\pi_{\{e,e'\}}(\mathsf{s}) \in \text{Allen}(e \bigcirc e') \mid \text{Allen}(e \prec e') \mid \text{Allen}(e' \prec e).$$

3 Fluents and bounded alternations

The projections π_X apply to strings whether or not they are RW, pointing to a notion more general than an event e — namely, a fluent (or temporal formula) φ. A set Φ of fluents is interpreted relative to a *linear Φ-model*, a triple $\mathbf{M} = \langle T, <, v\rangle$ where $<$ linearly orders T and $v \subseteq T \times \Phi$. (In the terminology of Kripke semantics, $\langle T, <\rangle$ is a frame, the elements of T are possible worlds, and v is a Φ-valuation.) For simplicity, we henceforth shorten "linear Φ-model" to "Φ-model." Now, a string $\mathsf{s} = \alpha_1 \cdots \alpha_n \in Pow(\Phi)^+$ can be identified with the Φ-model $\langle T_{\mathsf{s}}, <_{\mathsf{s}}, v_{\mathsf{s}}\rangle$ where T_{s} is $\{1, \ldots, n\}$, $<_{\mathsf{s}}$ is the restriction of the usual order on integers to T_{s}, and v_{s} is the set of pairs $\langle i, \varphi\rangle$ such that $\varphi \in \alpha_i$. Generalizing from a string s to a Φ-model $\mathbf{M} = \langle T, <, v\rangle$, let us agree that

(i) φ is a $\mathbf{M}$-*interval* if for all $t, t', t'' \in T$,

$$v(t, \varphi) \text{ and } v(t', \varphi) \text{ and } t < t'' < t' \quad \text{implies} \quad v(t'', \varphi)$$

(ii) $\mathbf{M}$ is *RW* if every $\varphi \in \Phi$ is a $\mathbf{M}$-interval.

Φ-models $\mathbf{M}$ can be related to the event structures $\mathbf{E} = \langle E, \bigcirc, \prec\rangle$ from the previous section through maps between these two kinds of models, switching between Φ and E as required. Given a Φ-model $\mathbf{M}$, let $\mathbb{E}(\mathbf{M})$ be $\langle \Phi, \bigcirc_{\mathbf{M}}, \prec_{\mathbf{M}}\rangle$ where

$$\varphi \ \bigcirc_{\mathbf{M}} \ \varphi' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists t)\ v(t, \varphi) \text{ and } v(t, \varphi')$$
$$\varphi \ \prec_{\mathbf{M}} \ \varphi' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\forall t, t' \text{ s.t. } v(t, \varphi) \text{ and } v(t', \varphi'))\ t < t'$$

for all $\varphi, \varphi' \in \Phi$. Clearly, if $\mathbf{M}$ is RW, then $\mathbb{E}(\mathbf{M})$ is an event structure.

Russell-Wiener specifies how to go the opposite direction: given a triple $\mathbf{E} = \langle E, \bigcirc, \prec\rangle$, let $\mathbb{M}(\mathbf{E})$ be $\langle T_{\prec}, \prec_T, \ni_{\prec}\rangle$ where $T_{\prec}$ is the set of $\subseteq$-maximal $\prec$-antichains, and

$$t \prec_T t' \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad (\exists e \in t)(\exists e' \in t')\ e \prec e'$$

for $t, t' \in T_{\prec}$, and

$$\ni_{\prec} \ \stackrel{\text{def}}{=} \ \{\langle t, e\rangle \in T_{\prec} \times E \mid e \in t\}\ .$$

The point of Russell-Wiener is that

$$\text{given an event structure } \mathbf{E},\ \mathbb{M}(\mathbf{E}) \text{ is RW and } \mathbf{E} = \mathbb{E}(\mathbb{M}(\mathbf{E}))$$

(a restatement of the theorem in §1.1 above). But were we to start with a Φ-model $\mathbf{M}$, what do we get with $\mathbb{M}(\mathbb{E}(\mathbf{M}))$? In general, $\mathbb{M}(\mathbb{E}(\mathbf{M}))$ differs from $\mathbf{M}$. If $\mathbf{M}$ is a delimited string $\mathsf{s}_{\pm}$ where s is RW, we can say more (identifying a string with its Φ-model).

Proposition 6. *If* s *is RW, then so is* $\mathsf{s}_{\pm}$ *and*

$$\pi(\mathsf{s}_{\pm}) \ = \ \mathbb{M}(\mathbb{E}(\mathsf{s}_{\pm})) \ = \ bc(\mathsf{s}_{\pm})\ .$$

The key to Proposition 6 is that for $\mathsf{s}_{\pm} = \alpha'_1 \cdots \alpha'_n$, exactly one of e, $pre(e)$ and $post(e)$ belongs to α'_i

$$|\{e, pre(e), post(e)\} \cap \alpha'_i| \ = \ 1$$

for all $e \in E^{\mathsf{s}}$ and $1 \leq i \leq n$, provided s is RW.

RW or not, strings can be glued together to form infinite Φ-models via the inverse limit of the projections π_X indexed by the family $Fin(\Phi)$ of finite subsets X of Φ

$$\{\{\mathsf{s}_X\}_{X \in Fin(\Phi)} \ \in \prod_{X \in Fin(\Phi)} Pow(X)^* \mid \mathsf{s}_X \ = \ \pi_X(\mathsf{s}_{X'}) \text{ for all } X \subseteq X' \in Fin(\Phi)\}$$

which we denote $\mathfrak{IL}(\Phi)$. The aim of the present section is to explore precisely how $\mathfrak{IL}(\Phi)$ might represent Φ-models, beyond the projection of strings $\mathsf{s} \in Pow(\Phi)^*$ as $Fin(\Phi)$-indexed approximations $\{\pi_X(\mathsf{s})\}_{X \in Fin(\Phi)} \in \mathfrak{IL}(\Phi)$, which we can form for RW and non-RW strings s alike. The basic tool is the concept of a φ-alternation, used below to flesh out the notion of time implicit in $\mathfrak{IL}(\Phi)$.

3.1 Defining, bounding and counting φ-alternations

Fix a Φ-model $\mathbf{M} = \langle T, <, v \rangle$, and let $\varphi \in \Phi$. A *φ-alternation* in $\mathbf{M}$ is an n-tuple $\langle t_1, \ldots, t_n \rangle$ such that $t_1 < t_2 < \cdots < t_n$ and for $1 \leq i < n$,

$$v(t_i, \varphi) \iff \text{not } v(t_{i+1}, \varphi)$$

in which case the φ-alternation is said to be *from* t_n and *have length* $n-1$. For any (finite) string $\mathsf{s} \in Pow(\Phi)^+$, a φ-alternation in s has length at most the length of $\pi_{\{\varphi\}}(\mathsf{s})$ plus 1 (because π unpads). More precisely, the longest φ-alternation in s has length one less than the length of the string $bc(\rho_{\{\varphi\}}(\mathsf{s}))$ provided $\mathsf{s} \sqsupseteq \boxed{\varphi}$.

A Φ-model $\mathbf{M}$ is *alternation-bounded* (a.b.) if for all $\varphi \in \Phi$, there is an integer m such that all φ-alternations in $\mathbf{M}$ have length $< m$. Clearly, $\mathbf{M}$ must be a.b. if it is to be represented in $\mathfrak{IL}(\Phi)$, as a fluent φ with unbounded alternations would rule out any finite approximation $\mathsf{s}_{\{\varphi\}}$ for $\mathbf{M}$. Note also that not only is every string in $Pow(\Phi)^*$ a.b., but every RW Φ-model is a.b. (finite or not).

Next, given an a.b. Φ-model $\mathbf{M}$ and a fluent $\varphi \in \Phi$, we define the *φ-alternation count of* $\mathbf{M}$ *at* a time t to be

$$\sharp_{\mathbf{M}}(t)(\varphi) \ \stackrel{\text{def}}{=} \ n \text{ such that there is a } \varphi\text{-alternation in } \mathbf{M} \text{ from } t \text{ of length } n \text{ but none of length } n+1.$$

For example, if a string s is RW and $\mathsf{s}_{\pm} = \alpha'_1 \cdots \alpha'_n$,

$$\sharp_{\mathsf{s}}(i)(e) = \begin{cases} 0 & \text{if } pre(e) \in \alpha'_i \text{ or } e \in \alpha'_i \cap \alpha'_0 \\ 1 & \text{if } (e \in \alpha'_i \text{ and } pre(e) \in \alpha'_0) \text{ or } (post(e) \in \alpha'_i \text{ and } e \in \alpha'_0) \\ 2 & \text{otherwise — i.e., } post(e) \in \alpha'_i \text{ and } pre(e) \in \alpha'_0 \end{cases}$$

for $1 \leq i \leq n$ and $e \in E^{\mathsf{s}}$.

3.2 The Φ-model $\mathbf{M}_\sharp$ and its approximations in $\mathfrak{IL}(\Phi)$

Given an a.b. Φ-model $\mathbf{M} = \langle T, <, v\rangle$, we can abstract $t \in T$ and $\varphi \in \Phi$ from $\sharp_{\mathbf{M}}(t)(\varphi)$ for a function $\sharp_{\mathbf{M}} : T \to (\Phi \to \mathbb{N})$ characterizing a time t by the function $\sharp_{\mathbf{M}}(t) : \Phi \to \mathbb{N}$ counting φ-alternations in $\mathbf{M}$. We apply the function $\sharp_{\mathbf{M}}$ to $\mathbf{M}$ for the triple $\mathbf{M}_\sharp \overset{\text{def}}{=} \langle T_\sharp, <_\sharp, v_\sharp\rangle$ where

$$\begin{aligned} T_\sharp &\overset{\text{def}}{=} \{\sharp_{\mathbf{M}}(t) \mid t \in T\} \\ <_\sharp &\overset{\text{def}}{=} \{\langle \sharp_{\mathbf{M}}(t), \sharp_{\mathbf{M}}(t')\rangle \mid t < t'\} \\ v_\sharp &\overset{\text{def}}{=} \{\langle \sharp_{\mathbf{M}}(t), \varphi\rangle \mid v(t,\varphi)\} \ . \end{aligned}$$

Observe that

$$t < t' \text{ and } \sharp_{\mathbf{M}}(t) = \sharp_{\mathbf{M}}(t_1) \text{ and } \sharp_{\mathbf{M}}(t') = \sharp_{\mathbf{M}}(t_1') \quad \text{implies} \quad t_1 < t_1'$$

and whenever $\sharp_{\mathbf{M}}(t)(\varphi) = \sharp_{\mathbf{M}}(t')(\varphi)$,

$$v(t,\varphi) \iff v(t',\varphi)$$

whence

Proposition 7. *For any a.b. Φ-model $\mathbf{M}$, $\mathbf{M}_\sharp$ is a Φ-model, and*

$$\begin{aligned} \sharp_{\mathbf{M}}(t) <_\sharp \sharp_{\mathbf{M}}(t') &\iff t < t' \\ v_\sharp(\sharp_{\mathbf{M}}(t), \varphi) &\iff v(t,\varphi) \end{aligned}$$

for all $t, t' \in T$ and $\varphi \in \Phi$.

By Proposition 7, $\sharp_{\mathbf{M}}$ is an isomorphism $\mathbf{M} \cong \mathbf{M}_\sharp$ provided it is one-to-one (injective): $(\forall t, t' \in T \text{ s.t. } t \neq t')(\exists \varphi \in \Phi)\ \sharp_{\mathbf{M}}(t)(\varphi) \neq \sharp_{\mathbf{M}}(t')(\varphi)$. Let us say $\mathbf{M}$ is *coarse* if for all $t, t' \in T$ such that $t < t'$, there exists $\varphi \in \Phi$ and a $t'' \in T$ separating t from t' in that $t < t'' \leq t'$ and

$$v(t,\varphi) \iff \text{not } v(t'',\varphi)$$

(making either $\langle t, t'\rangle$ or $\langle t, t'', t'\rangle$ a φ-alternation). Any a.b. Φ-model $\mathbf{M}$ is coarse exactly if $\sharp_{\mathbf{M}}$ is one-to-one over T.

Next, we approximate $\mathbf{M}_\sharp$ in $\mathfrak{IL}(\Phi)$. To simplify notation, we will often suppress the subscript $\mathbf{M}$, with $\mathbf{M}$ understood fixed in the background. For any finite subset $X \subseteq \Phi$, we define an equivalence $\approx_X$ on $T_\sharp$ by restricting the functions (in $T_\sharp$) to X

$$f \approx_X f' \quad \overset{\text{def}}{\iff} \quad (\forall \varphi \in X)\ f(\varphi) = f'(\varphi)$$

for all $f, f' \in T_\sharp$. Because each $\approx_X$-equivalence class is determined by one of finitely many functions from X to $\{0, 1, \ldots, m\}$ where m is the least upper bound on the lengths of all φ-alternations for $\varphi \in X$, there are only finitely many $\approx_X$-equivalence classes $[f]^{\approx_X}$. Let us order the $\approx_X$-equivalence classes pointwise according to

$$[f]^{\approx_X} <_X [f']^{\approx_X} \quad \overset{\text{def}}{\iff} \quad (\forall \varphi \in X)\ f(\varphi) \leq f'(\varphi) \text{ and } (\exists \varphi \in X)\ f(\varphi) < f'(\varphi)$$

and arrange them in $<_X$-increasing order

$$[f_1]^{\approx_X} <_X [f_2]^{\approx_X} <_X \cdots <_X [f_k]^{\approx_X}.$$

We then map each f_i to the set α_i of fluents $\varphi \in X$ that $v_\sharp$-hold at f_i

$$\alpha_i \overset{\text{def}}{=} \{\varphi \in X \mid v_\sharp(f_i, \varphi)\}$$

and unpad $\alpha_1 \cdots \alpha_k$ to obtain the $Pow(X)^*$-string

$$\mathsf{s}_X^{\mathbf{M}} \overset{\text{def}}{=} unpad(\alpha_1 \cdots \alpha_k)$$

that pictures $\mathbf{M}_\sharp$ up to granularity X.

Proposition 8. *For any a.b. Φ-model $\mathbf{M}$,*

$$\{\mathsf{s}_X^{\mathbf{M}}\}_{X \in Fin(\Phi)} \in \mathfrak{IL}(\Phi).$$

3.3 Times implicit in elements of $\mathfrak{IL}(\Phi)$

Reversing the direction of Proposition 8, given $\mathbf{s} = \{\mathsf{s}_X\}_{X \in Fin(\Phi)} \in \mathfrak{IL}(\Phi)$, let us pick out functions from Φ to $\mathbb{N}$ that can serve as times described by $\mathbf{s}$. The idea is that for such a function f, its restriction $f_X : X \to \mathbb{N}$ to $X \in Fin(\Phi)$ is roughly the alternation count $\sharp_{\mathsf{s}_X}(i)$ at some time i in s_X. This is "rough" because unpadding in π necessitates adjustments to X and to $\sharp_{\mathsf{s}_X}$ depending on whether or not a fluent $\varphi \in \Phi$ occurs in the first box of every $X \cup \{\varphi\}$-approximation $\mathsf{s}_{X \cup \{\varphi\}}$. We define

$$left(\mathbf{s}) \overset{\text{def}}{=} \{\varphi \in \Phi \mid (\forall \varphi' \in \Phi)\ \mathsf{s}_{\{\varphi,\varphi'\}} \trianglerighteq \boxed{\varphi}\boxed{}^*\}$$

and say $f : \Phi \to \mathbb{N}$ is an $\mathbf{s}$-*time* if for all $X \in Fin(\Phi)$, there exists $Y \in Fin(\Phi)$ such that $X \subseteq Y$ and for some $i \in dom(\sharp_{\mathsf{s}_X})$, we have for every $\varphi \in Y$,

$$f(\varphi) = \begin{cases} \sharp_{\mathsf{s}_Y}(i)(\varphi) + 1 & \text{if } \varphi \in \alpha_1 \text{ and } \varphi \notin left(\mathbf{s}) \\ \sharp_{\mathsf{s}_Y}(i)(\varphi) & \text{otherwise} \end{cases}$$

where $\mathsf{s}_Y = \alpha_1 \cdots \alpha_n$ (so $1 \leq i \leq n$). Now, let $T_\mathbf{s}$ be the set of $\mathbf{s}$-times, and let $\mathbf{M_s}$ be the triple $\langle T_\mathbf{s}, <_\mathbf{s}, v_\mathbf{s}\rangle$ where $<_\mathbf{s}$ consists of pairs $\langle f, f'\rangle \in T_\mathbf{s} \times T_\mathbf{s}$ ordered pointwise

$$(\forall \varphi \in \Phi)\ f(\varphi) \leq f'(\varphi) \text{ and } (\exists \varphi \in \Phi)\ f(\varphi) < f'(\varphi)$$

while $v_\mathbf{s}$ consists of pairs $\langle f, \varphi\rangle \in T_\mathbf{s} \times \Phi$ such that

$$(f(\varphi) \text{ is even and } \varphi \in left(\mathbf{s})) \text{ or } (f(\varphi) \text{ is odd and } \varphi \notin left(\mathbf{s}))$$

(recalling that alternation counts start at 0).

Proposition 9. *For every $\mathbf{s} \in \mathfrak{IL}(\Phi)$, $\mathbf{M_s}$ is an a.b. Φ-model and its $\mathfrak{IL}(\Phi)$-approximation described by Proposition 8 is $\mathbf{s}$.*

3.4 The real line from $\hat{\mathbf{s}}$-times and back to event structures

Proposition 9 raises the question: given an a.b. Φ-model $\mathbf{M}$, does applying the transformation $\mathbf{M} \mapsto \{\mathsf{s}_X^{\mathbf{M}}\}_{X \in Fin(\Phi)}$ from §3.2 followed by $\mathbf{s} \mapsto \mathbf{M_s}$ from §3.3 return a Φ-model isomorphic to $\mathbf{M}$? Not necessarily. For two reasons. First of all, $\mathbf{M}_\sharp$ need not be isomorphic to $\mathbf{M}$ (i.e. $\mathbf{M}$ may not be coarse), and nothing in $\mathbf{M}$ beyond $\mathbf{M}_\sharp$ is relevant to the $\mathfrak{IL}(\Phi)$-element obtained in Proposition 8. Secondly, even if $\mathbf{M} \cong \mathbf{M}_\sharp$, the set $T_\sharp$ of times in $\mathbf{M}_\sharp$ need not include every $\mathbf{s}^{\mathbf{M}}$-time. For example, let Φ be the set $\mathbb{Q}$ of rational numbers, and $\hat{\mathbf{s}}$ be the $Fin(\mathbb{Q})$-indexed string $\{\hat{\mathsf{s}}_X\}_{X \in Fin(\mathbb{Q})}$ such that for any finite sequence $q_1 < q_2 < \cdots < q_n$ in $\mathbb{Q}$,

$$\hat{\mathsf{s}}_{\{q_1, q_2, \ldots, q_n\}} \stackrel{\text{def}}{=} \boxed{q_1}\boxed{q_1, q_2} \cdots \boxed{q_1, q_2, \ldots, q_n}.$$

We can mimic the construction of the real line $\mathbb{R}$ via Dedekind cuts, equating a real number r with the $\hat{\mathbf{s}}$-time $f_r : \mathbb{Q} \to \{0, 1\}$ mapping $q \in \mathbb{Q}$ to

$$f_r(q) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } r < q \\ 1 & \text{otherwise.} \end{cases}$$

For any set T of real numbers that contains $\mathbb{Q}$, the triple $\langle T, <_T, v_T \rangle$ consisting of the restriction $<_T$ of the usual ordering to T and the $\mathbb{Q}$-valuation

$$v_T \stackrel{\text{def}}{=} \{\langle r, q \rangle \in T \times \mathbb{Q} \mid q \leq r\}$$

is a coarse $\mathbb{Q}$-model with $\mathfrak{IL}(\mathbb{Q})$-projection $\hat{\mathbf{s}}$. In fact, we can throw $-\infty$ into T, as the constant function returning 1 is an $\hat{\mathbf{s}}$-time. (The constant map 0 is not an $\hat{\mathbf{s}}$-time, because π unpads.)

Shifting to event structures, however, Russell-Wiener coincides with the transformation $\mathbf{s} \mapsto \mathbf{M_s}$ described in Proposition 9, for a suitable choice of $\mathbf{s}$.

Proposition 10. *For any event structure $\mathbf{E} = \langle E, \bigcirc, \prec \rangle$, there is a $Fin(E)$-indexed string $\mathbf{s} = \{\mathsf{s}_X^{\prec}\}_{X \in Fin(E)}$ in $\mathfrak{IL}(\Phi)$ such that*

$$\prec \;=\; \{\langle e, e' \rangle \in E \times E \mid \mathsf{s}^{\prec}_{\{e,e'\}} \in Allen(e \prec e')\}$$

(recalling that in an event structure, $\bigcirc$ is fixed to $\prec$-incomparability) and

$$\mathbb{M}(\mathbf{E}) \;\cong\; \mathbf{M_s}.$$

Given a finite subset X of E, we construct $\mathsf{s}_X^{\prec}$ to approximate the set $T_\prec$ of RW-times of $\prec$ on E as follows. Let $\approx_X$ be the equivalence on $T_\prec$ that holds between RW-times t and t' precisely if for all $e \in X$,

(i) $e \in t \iff e \in t'$, and
(ii) $(\exists e' \in t)\ e' \prec e \iff (\exists e' \in t')\ e' \prec e$, and
(iii) $(\exists e' \in t)\ e \prec e' \iff (\exists e' \in t')\ e \prec e'$.

Note that (i) says t and t' do not differ on e, (ii) on $pre(e)$

$$pre(e) \in t \iff pre(e) \in t'$$

and (iii) on $post(e)$

$$post(e) \in t \iff post(e) \in t' \ .$$

The number of $\approx_X$-equivalence classes cannot exceed the number $3^{|X|}$ of functions mapping $e \in X$ to one of $e, pre(e)$ and $post(e)$. Whether or not $T_\prec$ is isomorphic to the real line $\mathbb{R}$, the relation $\prec_X$ on $\approx_X$-equivalence classes U, U' given by

$$U \prec_X U' \ \stackrel{\text{def}}{\iff} \ (\exists t \in U)(\exists t' \in U')\ t \prec t'$$

is discrete (for X finite), and can be coded as a string $\hat{\mathsf{s}}$ over the alphabet $Pow(X)$. The X-approximation $\mathsf{s}_X^{\prec}$ is then just $unpad(\hat{\mathsf{s}})$. The construction is a slight variant of that described in Proposition 8 from an a.b. Φ-model (as opposed to an event structure).

4 Conclusion

Temporal progression is reduced above to succession within a string, following established practice in temporal logic (e.g. [Em90]). More distinctive features of the present work include

(i) the link with Russell and Wiener for a relational conception of time based on events, making intervals conceptually prior to points
(ii) the use of projections π_X to relate various strings, with granularity refined (or coarsened) by expanding (or reducing) X
(iii) the suspension of the requirement that the temporal extent of an event $e \in X$ be an interval, effectively stepping from an event e to a fluent φ that describes a type, rather than a particular occurrence or token.

Given a $Fin(\Phi)$-indexed string $\mathbf{s}$ in the inverse limit $\mathfrak{IL}(\Phi)$ of $\{\pi_X\}_{X \in Fin(\Phi)}$, an $\mathbf{s}$-time (or $\mathbf{s}$-possible world) counts φ-alternations, for $\varphi \in \Phi$. Restricting Φ to a finite set X bounds granularity below the infinite precision of real numbers, arbitrarily small increments in which lead to Sorites chains/arguments problematic for the vagueness of natural language (e.g. [Fe10]). The presentation of real numbers as $\hat{\mathbf{s}}$-times in §3.4 illustrates how far short finite strings may fall of the full range of Kripke models — or to put it the other way round, how much arbitrary Kripke models overshoot the bounded temporal granularity of ordinary natural language statements. Be that as it may, the projections π_X knit together different strings, and we close by reflecting on these projections, with an eye to variations and generalizations of π_X.

4.1 Reducible languages and formulas

A useful service for a function between strings to perform is to simplify certain notions of interest, as the projections π_X do for event structures. More precisely, given an alphabet Σ, and a function $f : \Sigma^* \to \Sigma^*$ between strings over Σ, a language $L \subseteq \Sigma^*$ *is f-reducible* if

$$\mathsf{s} \in L \iff f(\mathsf{s}) \in f[L] \tag{1}$$

(for all $\mathsf{s} \in \Sigma^*$) where $f[L]$ is the f-image $\{f(\mathsf{s}) \mid \mathsf{s} \in L\}$ of L. Whereas the left-to-right direction of (1) always holds, the right-to-left is a genuine constraint, reducing the cost of checking membership in L insofar as the f-image $f[L]$ of L is smaller than L and the computational cost of f can be ignored. This is the case, for instance, where Σ is $Pow(\Phi)$ and for some $e \in \Phi$,

(i) L is the set of strings s over Σ such that e is an s-interval, and
(ii) f is $\pi_{\{e\}}$, making $f[L]$ the singleton set consisting of the string $\boxed{e}$.

As to the computational cost of f, this cost can be can minimized by working with strings at the level of f — e.g. over the alphabet $Pow(X)$ for a reduced set $X \in Fin(\Phi)$. Recalling that $I(\mathsf{s})$ denotes the set of s-intervals, we can generalize the equation

$$I(\mathsf{s}) \;=\; \{e \in \Phi \mid \pi_{\{e\}}(\mathsf{s}) = \boxed{e}\}$$

using the π_X-image $\mathcal{L}_\pi(X)$ of $\bigcap_{e\in X} \pi^{-1}_{\{e\}}\boxed{e}$ (adapted from §2.2). to assert the π_X-reducibility of the set of strings s such that each $e \in X$ is an s-interval.

Proposition 11. *For every subset X of Φ and every $\mathsf{s} \in Pow(\Phi)^*$,*

$$X \subseteq I(\mathsf{s}) \iff \pi_X(\mathsf{s}) \;\in\; \mathcal{L}_\pi(X)\ .$$

Whereas $\bigcap_{e\in X} \pi^{-1}_{\{e\}}\boxed{e}$ is infinite, the language $\mathcal{L}_\pi(X)$ is finite for any finite X. Focusing on the case $X = \{e, e'\}$, Proposition 5 from §2.2 says the relations of overlap $\bigcirc$ and precedence $\prec$ between e and e' are $\pi_{\{e,e'\}}$-reducible. We are stepping here from a language L to a condition (or formula) φ characterizing membership in L, with the understanding that *φ is f-reducible* if L is f-reducible. The finest $\pi_{\{e,e'\}}$-reducible relations between intervals e and e' are the 13 Allen relations, each given in Table 1 by a singleton set $\pi_{\{e,e'\}}[L]$ — e.g.

$$\mathsf{s} \models e = e' \iff \pi_{\{e,e'\}}(\mathsf{s}) = \boxed{e, e'}\ .$$

We can refine f-reducible notions, where f is defined by a cascade of regular relations (as π_X is from $\rho_X; bc; unpad$)[3] with each successive relation reducing the input, by dropping some of the relations from the cascade. For example, let bc_X be bc after ρ_X

$$bc_X(\mathsf{s}) \stackrel{\text{def}}{=} bc(\rho_X(\mathsf{s}))$$

[3] Such cascades are employed in [NK09] over a different alphabet.

to get four times as many bc_X-reducible languages than π_X-reducible languages inasmuch as

$$\mathit{bc}[L] \ = \ (\square \mid \epsilon)\pi[L](\square \mid \epsilon) \ .$$

Thus, not only do we quadruple the Allen relations between s-intervals e and e', but taking the case $X = \{e\}$, we can (for instance) $\mathit{bc}_{\{e\}}$-reduce the notion that e is *left-bounded in* s, understood to mean: s is a non-empty string and e is not in the first symbol of s. Note that

$$\mathsf{s} \models \text{left-bounded}(e) \iff \mathit{bc}_{\{e\}}(\mathsf{s}) \in (\boxed{}\boxed{e})^*(\square \mid \epsilon) \ .$$

The failure of left-bounded(e) to be $\pi_{\{e\}}$-reducible suggests that the function *unpad* (separating bc from π) abstracts away information about boundedness. The matter is not so straightforward, however, as the equivalence

$$\mathsf{s} \models \text{left-bounded}(e) \iff \pi_{\{pre(e)\}}(\mathsf{s}_\pm) = \boxed{pre(e)}$$

provides a different function f for which left-bounded(e) is f-reducible — viz., the composition $\cdot_\pm ; \pi_{\{pre(e)\}}$ of the delimiting map $\mathsf{s} \mapsto \mathsf{s}_\pm$ followed by $\pi_{\{pre(e)\}}$.

4.2 Superpositions and retractions

Structuring the symbols of the alphabet as sets simplifies many of the finite-state constructions of present interest, including the projections ρ_X that pick out relevant variables, and a binary operation, superposition & ([Fe04]), that is useful for merging strings together. For two strings of the same length over the alphabet $Pow(\Phi)$, *superposition* forms the componentwise union

$$\alpha_1 \cdots \alpha_n \ \& \ \alpha'_1 \cdots \alpha'_n \stackrel{\text{def}}{=} (\alpha_1 \cup \alpha'_1) \cdots (\alpha_n \cup \alpha'_n)$$

for $\alpha_i, \alpha'_i \subseteq \Phi$. To illustrate,

$$\boxed{e, e'} = \boxed{e} \ \& \ \boxed{e'} \qquad \boxed{e}\boxed{e'} = \boxed{e}\boxed{} \ \& \ \boxed{}\boxed{e'} \qquad \boxed{e'}\boxed{e} = \boxed{e'}\boxed{} \ \& \ \boxed{}\boxed{e} \ .$$

The relation of subsumption (mentioned in §2 above) is the natural notion of containment derived from &

$$\mathsf{s} \trianglerighteq \mathsf{s}' \iff \mathsf{s} \text{ and } \mathsf{s}' \text{ have the same length, and } \mathsf{s} = \mathsf{s} \ \& \ \mathsf{s}' \ .$$

We generalize the superposition operation & from strings of the same length to languages over the alphabet $Pow(\Phi)$, collecting superpositions $\mathsf{s} \ \& \ \mathsf{s}'$ of strings s and s' of the same length from languages L and L' (respectively) in the *superposition*

$$L \ \& \ L' \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \{\mathsf{s} \ \& \ \mathsf{s}' \mid \mathsf{s} \in L \cap Pow(\Phi)^n \text{ and } \mathsf{s}' \in L' \cap Pow(\Phi)^n\}.$$

Next, we modify & further, relative to a given function f on strings. Writing $f^{-1}L$ for the inverse f-image of a language L

$$f^{-1}L \stackrel{\text{def}}{=} \{\mathsf{s} \in Pow(\Phi)^* \mid f(\mathsf{s}) \in L\} \ ,$$

let us form

$$\begin{aligned} L^f &\stackrel{\text{def}}{=} f^{-1}f[L] \\ &= \{\mathsf{s} \in Pow(\Phi)^* \mid (\exists \mathsf{s}' \in L)\ f(\mathsf{s}) = f(\mathsf{s}')\} \end{aligned}$$

which we refer to as the *f-closure of* L (as $L \subseteq L^f = L^{f^f}$ and the operation preserves $\subseteq$-inclusion: $L \subseteq L'$ implies $L^f \subseteq L'^f$). L is f-reducible precisely if its f-closure L^f is a subset of L — i.e., as it is always the case that $L \subseteq L^f$,

$$L \text{ is } f\text{-reducible} \iff L^f = L \ .$$

Now, the *f-superposition* $L \&_f L'$ of languages L and L' is the f-image of the superposition of the f-closures of L and L'

$$L \&_f L' \stackrel{\text{def}}{=} f[L^f \ \& \ L'^f] \ .$$

Taking f to be π, we can reconstruct the sets $\mathcal{L}_\pi(\{e_1, \ldots, e_n\})$ (mentioned in Proposition 11) by mapping a finite sequence $e_1 \cdots e_n$ in Φ to a language $\mathcal{E}(e_1 \cdots e_n)$ by induction on n as follows: $\mathcal{E}(e_1) \stackrel{\text{def}}{=} \boxed{e_1}$ and

$$\mathcal{E}(e_1 \cdots e_{n+1}) \stackrel{\text{def}}{=} \mathcal{E}(e_1 \cdots e_n) \ \&_\pi \ \boxed{e_{n+1}} \qquad \text{for } n \geq 1.$$

Proposition 12. *For every finite subset* $\{e_1, \ldots, e_n\}$ *of* Φ,

$$\mathcal{E}(e_1 \cdots e_n) = \mathcal{L}_\pi(\{e_1, \ldots, e_n\}).$$

For some purposes, however, the operation $\&_\pi$ is too blunt an instrument. To illustrate the finer control $\&_{bc}$ affords over $\&_\pi$, let us reformulate the example from [NK09] of the 12 months of year 2008 in our framework as

$$\begin{aligned} &\boxed{\mathsf{y2008}} \ \&_{bc} \ \boxed{\mathsf{Jan}}\boxed{\mathsf{Feb}}\boxed{\mathsf{Mar}} \cdots \boxed{\mathsf{Dec}} \\ = &\boxed{\mathsf{y2008,Jan}}\boxed{\mathsf{y2008,Feb}}\boxed{\mathsf{y2008,Mar}} \cdots \boxed{\mathsf{y2008,Dec}} \end{aligned}$$

(noting the equation fails if the subscript bc is replaced by π).

Given a function f such that $f; f = f$ and a subset X of Φ, let us call the composition $f_X \stackrel{\text{def}}{=} \rho_X; f$ of ρ_X with f a *retraction* if f_X preserves the structure $\&_f$ introduces

$$f_X(\mathsf{s} \ \&_f \ \mathsf{s}') = f_X(\mathsf{s}) \ \&_f \ f_X(\mathsf{s}')$$

(where a string s is, as usual, conflated with the language $\{\mathsf{s}\}$). Clearly, π_X and bc_X are retractions. It is also immediate that

Proposition 13. *For all functions* $f, g, h : \Sigma^* \to \Sigma^*$, *if* $h = g; f$ *where* $g; g = g$ *then every* h*-reducible language is* g*-reducible.*

Of particular interest for retractions is the case where $g = \rho_X$. By Proposition 13, f_X-reducible languages are ρ_X-reducible.

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