Kripke Models of Transfinite Provability Logic

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Abstract

For any ordinal Λ, we can define a polymodal logic GLP_Λ, with a modality $[\xi]$ for each $\xi < \Lambda$. These represent provability predicates of increasing strength. Although GLP_Λ has no non-trivial Kripke frames, Ignatiev showed that indeed one can construct a universal Kripke frame for the variable-free fragment with natural number modalities, denoted GLP_ω^0.

In this paper we show how to extend these constructions for arbitrary Λ. More generally, for each ordinals Θ, Λ we build a Kripke model $\mathfrak{I}_\Lambda^\Theta$ and show that GLP_Λ^0 is sound for this structure. In our notation, Ignatiev's original model becomes $\mathfrak{I}_\omega^{\varepsilon_0}$.

Keywords: proof theory, modal logic, provability logic

1 Introduction

It was Gödel who first suggested interpreting the modal $\Box$ as a provability predicate, which as he observed should satisfy $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ and $\Box\phi \to \Box\Box\phi$. In this way, the Second Incompleteness Theorem could be expressed succinctly as $\Diamond\top \to \Diamond\Box\bot$.

More generally, Löb's axiom $\Box(\Box\phi \to \phi) \to \Box\phi$ is valid for this interpretation, and with this we obtain a complete characterization of the propositional behavior of provability in Peano Arithmetic [11]. The modal logic obtained from Löb's axiom is called GL (for Gödel-Löb) and is rather well-behaved; it is decidable and has finite Kripke models, based on transitive, well-founded frames [10].

Japaridze [5] then suggested extending GL by a sequence of provability modalities $[n]$, for $n < \omega$, where $[n]\phi$ could be interpreted (for example) as ϕ *is derivable using n instances of the ω-rule*. We shall refer to this extension as GLP_ω. GLP_ω turns out to be much more powerful than GL, and indeed Beklemishev has shown how it can be used to perform ordinal analysis of Peano Arithmetic and its natural subtheories [1].

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However, as a modal logic, it is much more ill-behaved than GL. Most notably, over the class of GLP Kripke frames, the formula $[1]\bot$ is valid! This is clearly undesirable. There are ways to get around this, for example using topological semantics. However, Ignatiev in [7] showed how one can still get Kripke frames for the *closed* fragment of GLP_ω, which contains no propositional variables (only $\bot$). This fragment, which we denote GLP_ω^0, is still expressive enough to be used in Beklemishev's ordinal analysis.

Our goal is to extend Ignatiev's construction for GLP_ω^0 to GLP_Λ^0, where Λ is an arbitrary ordinal (or, if one wishes, the class of all ordinals). To do this we build upon known techniques, but dealing with transfinite modalities poses many new challenges. In particular, frames will now have to be much 'deeper' if we wish to obtain non-empty accessibility relations.

Our structures naturally extend the model which was first defined and studied by Ignatiev for GLP_ω^0 in [7], and in our notation becomes $\mathfrak{I}_\omega^{\varepsilon_0}$. Originally, Ignatiev's study was an amalgamate of modal, arithmetical and syntactical methods. In [8] the model was first submitted to a purely modal analysis and [3] built forth on this work. In this paper, we prove soundness and non-triviality of the accessibility relations using purely semantic techniques.

The layout of the paper is as follows. In Section 2 we give a quick overview of the logics GLP_Λ. Section 3 then gives some motivation for the constructions we shall present, and Section 4 reviews some operations on ordinals and the notation we will use.

In Section 5, we introduce our generalized Ignatiev models, denoted $\mathfrak{I}_\Lambda^\Theta$, where Θ, Λ are ordinal parameters. Section 6 then defines some operations on the points of our model, which are called ℓ-*sequences*. With these operations we prove soundness in Section 7.

Finally, Section 8 shows that, for an arbitrarily large ordinal ξ with $\xi < \Lambda$, if Θ is large enough, then $<_\xi$ is non-empty on $\mathfrak{I}_\Lambda^\Theta$. This result is not a full completeness proof, however it is a crucial step; in [4], we shall show how one deduces completeness of GLP_Λ^0 for a Kripke frame $\mathfrak{F}$ from non-triviality of the accessibility relations. The latter result is syntactical and does not depend much on the structure $\mathfrak{F}$.

2   The logic GLP_Λ^0

Let Λ be either an ordinal or the class of all ordinals. Formulas of GLP_Λ^0 are built from $\bot$ using Boolean connectives $\neg, \wedge$ and a modality $[\xi]$ for each $\xi < \Lambda$. As is customary, we use $\langle \xi \rangle$ as a shorthand for $\neg[\xi]\neg$.

Note that there are no propositional variables, as we are concerned here with the *closed fragment* of GLP_Λ.

The logic GLP_Λ^0 (see [2]) is given by the following axioms:

(i) all propositional tautologies,

(ii) $[\xi](\phi \to \psi) \to ([\xi]\phi \to [\xi]\psi)$ for all $\xi < \Lambda$,

(iii) $[\xi]([\xi]\phi \to \phi) \to [\xi]\phi$ for all $\xi < \Lambda$,

(iv) $\langle \xi \rangle \phi \to \langle \zeta \rangle \phi$ for $\zeta < \xi < \Lambda$,

(v) $\langle\zeta\rangle\,\phi \to [\xi]\,\langle\zeta\rangle\,\phi$ for $\zeta < \xi < \Lambda$.

The rules of our logic are Modus Ponens and Necessitation for each modality. Although no full completeness result has yet been published for (hyper)arithmetical interpretations of GLP_Λ with $\Lambda > \omega$ the community is confident that such interpretations exist. One such example would be to interpret $[\alpha]$ as "provable in some base theory using α many nested iterations of the ω rule" in some infinitary calculus. In this paper our focus is on the modal aspects of the logics GLP_Λ only.

A *Kripke frame*[3] is a structure $\mathfrak{F} = \langle W, \langle R_i \rangle_{i<I} \rangle$, where W is a set and $\langle R_i \rangle_{i<I}$ a family of binary relations on W. To each formula ψ in the closed modal language with modalities $\langle i \rangle$ for $i < I$ we assign a set $\llbracket \psi \rrbracket_{\mathfrak{F}} \subseteq W$ inductively as follows:

$$\llbracket \bot \rrbracket_{\mathfrak{F}} \quad = \varnothing$$

$$\llbracket \neg\phi \rrbracket_{\mathfrak{F}} \quad = W \setminus \llbracket \phi \rrbracket_{\mathfrak{F}}$$

$$\llbracket \phi \wedge \psi \rrbracket_{\mathfrak{F}} = \llbracket \phi \rrbracket_{\mathfrak{F}} \cap \llbracket \psi \rrbracket_{\mathfrak{F}}$$

$$\llbracket \langle i \rangle\, \phi \rrbracket_{\mathfrak{F}} \quad = R_i^{-1}\, \llbracket \phi \rrbracket_{\mathfrak{F}}\,.$$

As always, for a binary relation S on W, if $X \subseteq W$ we denote by $S^{-1}X$ the set $\{ y \in W \mid \exists x{\in}X \; ySx \}$. Often we will write $\langle \mathfrak{F}, x \rangle \Vdash \psi$ instead of $x \in \llbracket \psi \rrbracket_{\mathfrak{F}}$.

It is well-known that GL is sound for $\mathfrak{F}$ whenever R_i^{-1} is well-founded and transitive, in which case we write it $<_i$. However, constructing models of GLP_Λ is substantially more difficult than constructing models of GL, as we shall see.

3 Motivation for our models

The full logic GLP_Λ cannot be sound and complete with respect to any class of Kripke frames. Indeed, let $\mathfrak{F} = \langle W, \langle <_\xi \rangle_{\xi<\lambda} \rangle$ be a polymodal frame.

Then, it is not too hard to check that in $\mathfrak{F}$ we have the following correspondences

(i) Löb's axiom $[\xi]([\xi]\phi \to \phi) \to [\xi]\phi$ is valid if and only if $<_\xi$ is well-founded and transitive,

(ii) the axiom $\langle\zeta\rangle\phi \to \langle\xi\rangle\phi$ for $\xi \le \zeta$ is valid if and only if, whenever $w <_\zeta v$, then $w <_\xi v$, and

(iii) $\langle\xi\rangle\phi \to [\zeta]\langle\xi\rangle\phi$ for $\xi < \zeta$ is valid if, whenever $v <_\zeta w$, $u <_\xi w$ and $\xi < \zeta$, then $u <_\xi v$.

Suppose that for $\xi < \zeta$, there are two worlds such that $w <_\zeta v$. Then from Correspondence (ii) above we see that $w <_\xi v$, while from (iii) this implies that $w <_\xi w$. But this clearly violates (i). Hence if $\mathfrak{F} \models \mathsf{GLP}$, it follows that all accessibility relations (except possibly $<_0$) are empty.

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[3] Since we are restricting to the closed fragment we make no distinction between Kripke frames and Kripke *models*.

However, this does not rule out the possibility that the closed fragments GLP_Λ^0 have Kripke frames for which they are sound and complete. This turned out to be the case for GLP_ω^0 and in the current paper we shall extend this result to GLP_Λ^0, with Λ arbitrary.

More precisely, given ordinals Λ, Θ, we will construct a Kripke frame $\mathfrak{I}_\Lambda^\Theta$ with 'depth' Θ (i.e., the order-type of $<_0$) and 'length' Λ (the set of modalities it interprets). $\mathfrak{I}_\Lambda^\Theta$ validates all frame conditions except for condition (ii). We shall only approximate it in that we require, for $\zeta < \xi$,

$$v <_\xi w \Rightarrow \exists v' <_\zeta w \text{ such that } v' \leftrightarrows_{\mathsf{p}} v.$$

Here p will be a set of parameters and $u' \leftrightarrows_{\mathsf{p}} u$ denotes that u' is p-*bisimilar* to u. The parameters p can be adjusted depending on ϕ in order to validate each instance of the axiom.

One convenient property of the closed fragment is that it is not sensitive to 'branching'. Indeed, consider any Kripke frame $\langle W, < \rangle$ for GL^0. To each $w \in W$ assign an ordinal $o(w)$ as follows: if w is minimal, $o(w) = 0$. Otherwise, $o(w)$ is the supremum of $o(v) + 1$ over all $v < w$.

The map o is well-defined because models of GL are well-founded. Further, because there are no variables, it is easy to check that $o : W \to \Lambda$ (where Λ is a sufficiently large ordinal) is a bisimulation.

Thus to describe the modal logic of W it is enough to describe $o(W)$. We can extend this idea to GLP_Λ; if we have a well-founded frame $\mathfrak{F} = \langle W, \langle <_\xi \rangle_{\xi < \Lambda} \rangle$, we can represent a world w by the sequence $\boldsymbol{o}(w) = \langle o_\xi(w) \rangle_{\xi < \Lambda}$, where o_ξ is defined analogously to o. Thus we can identify elements of our model with sequences of ordinals. It is a priori not clear that this representation suffices also for the polymodal case, and one of the main purposes of this paper is to see that it actually does.

Meanwhile, there are certain conditions these sequences must satisfy. They arise from considering *worms*, which are formulas of the form $\langle \xi_0 \rangle \ldots \langle \xi_n \rangle \top$. In various ways we can see worms as the backbone of the closed fragment of GLP. It is known that each formula of GLP_Λ^0 is equivalent to a Boolean combination of worms.

Given worms A, B and an ordinal ξ, we define $A \prec_\xi B$ if $\vdash B \to \langle \xi \rangle A$. This gives us a well-founded partial order.

In [6], we study $\boldsymbol{\Omega}(A)$, where

$$\Omega_\xi(A) = \sup_{B \prec_\xi A} \Omega_\xi(B);$$

this gives us a good idea of what sequences may be included in the model. As it turns out, $\boldsymbol{\Omega}(A)$ is a 'local bound' for $\boldsymbol{o}(w)$ (see Definition 5.1).

4   Some ordinal arithmetic

As mentioned in the previous section, a world f in our model will be coded by a sequence that for each ξ tells us the order-type of f with respect $<_\xi$.

These order-types are ordinals and it will be convenient to review some basic properties of ordinals that shall be used throughout this paper. We dedicate this section to this purpose.

We shall simply state the main properties without proof. For further details, we refer the reader to [9]. Ordinals are canonical representatives for well-orders. The first infinite ordinal is as always denoted by ω.

Most operations on natural numbers can be extended to ordinal numbers, like addition, multiplication and exponentiation (see [9]). However, in the realm of ordinal arithmetic things become often more subtle; for example, $1 + \omega = \omega \neq \omega + 1$. Other operations differ considerably from ordinary arithmetic as well.

However, there are also various similarities. In particular we have a form of subtraction available in ordinal arithmetic.

Lemma 4.1

(i) *Given ordinals $\zeta < \xi$, there exists a unique ordinal $\eta = -\zeta + \xi$ such that $\zeta + \eta = \xi$.*

(ii) *Given $\eta > 0$, there exist α, β such that $\eta = \alpha + \omega^\beta$. The value of β is uniquely defined and we denote it $\ell\eta$, the 'last exponent' of η.*

(iii) *Given $\eta > 0$, there exist unique values of α, β such that $\eta = \omega^\alpha + \beta$ and $\beta < \omega^\alpha + \beta$.*

It is convenient to have representation systems for ordinals. One of the most convenient is given by Cantor Normal Forms (CNFs).

Theorem 4.2 (Cantor Normal Form Theorem)
For each ordinal α there are unique ordinals $\alpha_1 \geq \ldots \geq \alpha_n$ such that

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

Another difference between ordinal and ordinary arithmetic is that various increasing functions in ordinal arithmetic have fixpoints where the ordinary counterparts do not. Let us make this precise. We call a function f *increasing* if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$. An ordinal function is called *continuous* if $\bigcup_{\zeta < \xi} f(\zeta) = f(\xi)$ for all limit ordinals [4] ξ. Functions which are both increasing and continuous are called *normal*.

It is not hard to see that each normal function has an unbounded set of fixpoints. For example the first fixpoint of the function $x \mapsto \omega^x$ is

$$\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$$

and is denoted ε_0. Clearly for these fixpoints, CNFs are not too informative as, for example, $\varepsilon_0 = \omega^{\varepsilon_0}$. Here it is convenient to pass to normal forms capable of representing fixed points of the ω-exponential: Veblen Normal Forms (VNF).

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[4] Henceforth we shall write $\lim_{\zeta \to \xi} f(\zeta)$ instead of $\bigcup_{\zeta < \xi} f(\zeta)$.

In his seminal paper [12], Veblen considered for each normal function f its derivative f' that enumerates the fixpoints of f. Taking derivatives can be transfinitely iterated for unbounded f.

Each closed (under taking suprema) unbounded set X is enumerated by a normal function. The derivative X' of a closed unbounded set X is defined to be the set of fixpoints of the function that enumerates X and likewise for transfinite progressions:

$$X_{\alpha+1} := (X_\alpha)';$$
$$X_\lambda \quad := \bigcap_{\alpha<\lambda} X_\alpha \quad \text{for limit } \lambda.$$

By taking $F_0 := \{\omega^\alpha \mid \alpha \in \mathsf{On}\}$ one obtains Veblen's original hierarchy and the φ_α denote the corresponding enumeration functions of the thus obtained F_α.

Beklemishev noted in [2] that in the setting of GLP it is desirable to have $1 \notin F_0$. Thus he considered the progression that started with $F_0^B := \{\omega^{1+\alpha} \mid \alpha \in \mathsf{On}\}$. We denote the corresponding enumeration functions by $\hat\varphi_\alpha$.

In [6] the authors realized that, moreover it is desirable to have 0 in the initial set, whence they departed from $E_0 = \{0\} \cup \{\omega^{1+\alpha} \mid \alpha \in \mathsf{On}\}$. We shall denote the corresponding enumeration functions by e_α.

One readily observes that

$$\begin{aligned}
e_\alpha(0) &= & 0 & \quad\quad \text{for all } \alpha; \\
e_0(1+\beta) &= \varphi_0(1+\beta) &= \hat\varphi_0(\beta) & \quad\quad \text{for all } \beta; \\
e_{1+\alpha}(1+\beta) &= \varphi_{1+\alpha}(\beta) &= \hat\varphi_{1+\alpha}(\beta) & \quad \text{for all } \alpha, \beta.
\end{aligned}$$

Often, we can write an ordinal ω^α in many ways as $\varphi_\xi(\eta)$. However, if we require that $\eta < \varphi_\xi(\eta)$, then both ξ and η are uniquely determined. In other words, for every ordinal α, there exist unique η, ξ such that $\omega^\alpha = \varphi_\xi(\eta)$ and $\eta < \varphi_\xi(\eta)$.

Combining this fact with the CNF Theorem one obtains so-called Veblen Normal Forms for ordinals.

Theorem 4.3 (Veblen Normal Form Theorem) *For all α there exist unique $\alpha_1, \beta_1, \ldots \alpha_n, \beta_n$ ($n \geq 0$) such that*

(i) $\alpha = \varphi_{\alpha_1}(\beta_1) + \ldots + \varphi_{\alpha_n}(\beta_n)$,

(ii) $\varphi_{\alpha_i}(\beta_i) \geq \varphi_{\alpha_{i+1}}(\beta_{i+1})$ *for* $i < n$,

(iii) $\beta_i < \varphi_{\alpha_i}(\beta_i)$ *for* $i \leq n$.

Note that $\alpha_i \geq \alpha_{i+1}$ does not in general hold in the VNF of α. For example, $\omega^{\varepsilon_0+1} + \varepsilon_0 = \varphi_0(\varphi_{\varphi_0(0)}(0) + \varphi_0(0)) + \varphi_{\varphi_0(0)}(0)$.

5 Generalized Ignatiev models

In this section we will generalize Ignatiev's universal model for GLP_ω^0 to obtain models for GLP_Λ^0, for arbitrary Λ. Our model combines ideas from Ignatiev's construction with some new methods for dealing with limit modalities. The

model is universal in that it validates all theorems and refutes all non-theorems of GLP^0_Λ. It also has some minimality properties that we shall not discuss in this paper.

We use the 'last exponent' operation ℓ described above to define the 'worlds' of our model. They will be (typically infinite) sequences of ordinals which we call ℓ-*sequences*.

Definition 5.1 *Let* Θ, Λ *be ordinals. We define an* ℓ-sequence *(of depth* Θ *and length* Λ*) to be a function*

$$f : \Lambda \to \Theta$$

such that, for every $\zeta < \xi < \Lambda$,

$$\ell f(\zeta) \geq \ell e_{\ell\xi} f(\xi). \tag{1}$$

Note that in ℓ-sequences, for $\xi = \zeta + 1$ we have

$$\ell f(\zeta) \geq \ell e_0 f(\xi) = f(\xi)$$

which is as in the original Ignatiev model for GLP^0_ω. We shall now see that ℓ-sequences can be described either globally, as above, or locally.

Definition 5.2 *Let* Θ, Λ *be ordinals. A function* $f : \Lambda \to \Theta$ *is a* local ℓ-sequence *if and only if, given* $\xi \in (0, \Lambda)$ *there is* $\vartheta < \xi$ *such that*

$$\ell f(\zeta) \geq \ell e_{\ell\xi} f(\xi)$$

for all $\zeta \in [\vartheta, \xi)$.

If one requires equality in the above definition, i.e., $\ell f(\zeta) = \ell e_{\ell\xi} f(\xi)$ then one exactly gets the sequences $\mathbf{\Omega}(A)$ for worms A.

Lemma 5.3 *A function* $f : \Lambda \to \Theta$ *is an* ℓ-sequence *if and only if it is a local* ℓ-sequence.

Proof. Clearly every ℓ-sequence is a local ℓ-sequence.

Now, if f is a local ℓ-sequence, towards a contradiction suppose that it is not an ℓ-sequence, and let $\xi \in (0, \Lambda)$ be least with the property that, for some $\zeta < \xi$,

$$\ell f(\zeta) < \ell e_{\ell\xi} f(\xi). \tag{2}$$

Now pick $\vartheta < \xi$ such that, for all $\zeta' \in [\vartheta, \xi)$,

$$\ell f(\zeta') \geq \ell e_{\ell\xi} f(\xi).$$

Such a ϑ exists, since f is a local ℓ-sequence.

Evidently $\zeta < \vartheta < \xi$, whence by minimality of ξ, $\ell f(\zeta) \geq \ell e_{\ell\vartheta} f(\vartheta)$ and

$$\ell f(\zeta) \geq \ell e_{\ell\vartheta} f(\vartheta) \geq \ell f(\vartheta) \geq \ell e_{\ell\xi} f(\xi).$$

This contradicts (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now rather than considering an ℓ-sequence in isolation, we will be interested in the structure of all ℓ-sequences:

Definition 5.4 *Given ordinals Θ, Λ, define a structure*

$$\mathfrak{I}_\Lambda^\Theta = \left\langle D_\Lambda^\Theta, \langle <_\xi \rangle_{\xi < \Lambda} \right\rangle$$

by setting D_Λ^Θ to be the set of all ℓ-sequences of depth Θ and length Λ. Define $f <_\xi g$ if and only if $f(\zeta) = g(\zeta)$ for all $\zeta < \xi$ and $f(\xi) < g(\xi)$.

One can check that Ignatiev's original model is precisely $\mathfrak{I}_\omega^{\varepsilon_0}$ in our notation. The novelty is that now Λ could be much, much bigger than ω.

6   Operations on ℓ-sequences

$\mathfrak{I}_\Lambda^\Theta$ is not a genuine GLP_Λ frame. However, we shall show that indeed it is a model of GLP_Λ^0. In this section we shall develop some tools which will be useful for proving this fact.

6.1   Simple sequences

A useful elementary notion will be that of *simple sequences*. These are finite increasing sequences of ordinals that only make 'jumps' of the form ω^β. When analizing a formula ψ, it will be easier to extend the modalities appearing in ψ to a simple sequence and treat them, to some extent, as if they appeared in ψ.

Definition 6.1 *A finite sequence of ordinals $\langle \sigma_i \rangle_{i \leq I}$ is* simple *if $\sigma_0 = 0$ and for every $i < I$ there exists β_i such that $\sigma_{i+1} = \sigma_i + \omega^{\beta_i}$.*

Lemma 6.2 *Every finite increasing sequence of ordinals can be extended to a simple sequence.*

Proof. Induction on I. Suppose $\langle \sigma_i \rangle_{i \leq I+1}$ is a finite increasing sequence of ordinals. We assume that there is a simple sequence $\langle \alpha_i \rangle_{i \leq J}$ extending $\langle \sigma_0, ..., \sigma_I \rangle$ with $\alpha_J = \sigma_I$.

Since $\sigma_I < \sigma_{I+1}$, there exists a unique ordinal η such that $\sigma_{I+1} = \sigma_I + \eta$. Write

$$\eta = \sum_{k < K} \omega^{\gamma_k}$$

in Cantor Normal Form.

Then define, for each $k \leq K$,

$$\beta_k = \sigma_I + \sum_{i < k} \omega^{\gamma_i}.$$

Finally, setting

$$\boldsymbol{\delta} = \langle \alpha_0, \alpha_1, ..., \alpha_J, \beta_1, \beta_2, ..., \beta_K \rangle$$

gives us the desired simple extension of $\boldsymbol{\sigma}$. $\square$

6.2 Approximations of ℓ-sequences

Given a formula ϕ and an ℓ-sequence $f \in D_\Lambda^\Theta$ with $\langle \mathfrak{I}_\Lambda^\Theta, f \rangle \Vdash \phi$, there is a sense in which every ℓ-sequence g that is 'close enough' to f also satisfies ϕ. To make this precise, we will define $\langle p, \boldsymbol{\sigma} \rangle$-*approximations* of f.

Below, given an ordinal ξ in Veblen Normal Form $\sum_{i<I} \varphi_{\alpha_i}(\beta_i)$, we define the *width* of ξ recursively as the maximal sum-size in the VNF of ξ:

$$\mathrm{wdt}(\xi) := \max(\{I\} \cup \{\mathrm{wdt}(\beta_i) : i < I\}).$$

Similarly, the *height* of ξ is defined as the maximal number of nested φ's in the VNF of ξ:

$$\mathrm{hgt}(\xi) := 1 + \max_{i<I} \mathrm{hgt}(\beta_i).$$

Both the width and height of 0 are stipulated to be zero.

Note that α_i will not be used in computing the height or width of ξ; these are seen as atomic symbols and will take on only finitely many possible values.

More specifically, we say α is a *subindex* of ξ if, when writing

$$\xi = \sum_{j<J} \varphi_{\alpha_j}(\beta_j)$$

in Veblen Normal Form, we have that either $\alpha = \alpha_j$ for some $j < J$ or α is, inductively, a subindex of some β_j.

Definition 6.3 *Given a natural number p and a finite sequence of ordinals $\boldsymbol{\sigma} = \langle \sigma_0, ..., \sigma_I \rangle$, we say β is a $\langle p, \boldsymbol{\sigma} \rangle$-approximation of α if*

(i) *$\beta < \alpha$,*

(ii) *$\mathrm{wdt}(\beta)$ and $\mathrm{hgt}(\beta)$ are both at most p,*

(iii) *every subindex of β is of the form $\ell \sigma_i$.*

Clearly, for fixed α, p and $\boldsymbol{\sigma}$ one can only make finitely many syntactical expressions of nested width and height with subindices in $\boldsymbol{\sigma}$. Thus, there are only finitely many $\langle p, \boldsymbol{\sigma} \rangle$-approximations of a given α, and hence there is a maximum one: we denote it by $\lfloor \alpha \rfloor_{\boldsymbol{\sigma}}^p$. It will be convenient to stipulate $\lfloor 0 \rfloor_{\boldsymbol{\sigma}}^p = -1$. Clearly $\lfloor \alpha \rfloor_{\boldsymbol{\sigma}}^p$ is weakly monotone in all of its arguments.

We are not interested in approximating only ordinals, but rather entire ℓ-sequences:

Definition 6.4 *Let $\boldsymbol{\sigma}$ be a simple sequence $\langle 0 = \sigma_0, \ldots, \sigma_I \rangle$. We extend the use of $\lfloor \cdot \rfloor_{\boldsymbol{\sigma}}^p$ to sequences $f : \Lambda \to \Theta$ as follows:*

$$\lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\xi) = \begin{cases} 0 & \text{for } \xi > \sigma_I \\[2mm] \lfloor f(\sigma_I) \rfloor_{\boldsymbol{\sigma}}^p + 1 & \text{for } \xi = I \\[2mm] \lfloor f(\sigma_i) \rfloor_{\boldsymbol{\sigma}}^p + 1 + e_{\ell \sigma_{i+1}} \lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\sigma_{i+1}) & \text{for } \xi = \sigma_i \text{ with } i < I \\[2mm] e_{\ell \sigma_{i+1}} \lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\sigma_{i+1}) & \text{for } \sigma_i < \xi < \sigma_{i+1} \end{cases}$$

Note that it is nearly never the case that $\lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\xi) = \lfloor f(\xi) \rfloor_{\boldsymbol{\sigma}}^p$; however as we will see later in Lemma 6.8, they cannot be too different. Before this, we observe that this operation always produces ℓ-sequences.

Lemma 6.5 *Given any* $f : \Lambda \to \Theta$ *and parameters* $p, \boldsymbol{\sigma}$, $g = \lfloor f \rfloor_{\boldsymbol{\sigma}}^p$ *is an* ℓ-*sequence.*

Further, it has the property that for all $i \leq I$

$$\ell g(\sigma_i) = \ell e_{\ell \sigma_{i+1}} g(\sigma_{i+1}). \tag{3}$$

Proof. Using Lemma 5.3 it suffices to see that g is a local ℓ-sequence. To see this, we make a few case distinctions.

$g(\xi) = 0$. Note that this covers the case where $\xi > \sigma_I$. In this case, the inequality $\ell g(\zeta) \geq e_{\ell \xi} g(\xi)$ holds trivially for all $\zeta < \xi$ since the right-hand side is zero.

$g(\xi) > 0$ **and** $\xi = \sigma_i$ **for some** i.
   Then, $\ell g(\zeta) = \ell e_{\ell \xi} g(\xi)$, for all $\zeta \in [\sigma_i, \xi)$.

$g(\xi) > 0$ **and** $\sigma_i < \xi < \sigma_{i+1}$ **for some** i. We first claim that $\ell \xi < \ell \sigma_{i+1}$. Indeed, since $\boldsymbol{\sigma}$ is simple, we have that

$$\sigma_{i+1} = \sigma_i + \omega^{\ell \sigma_{i+1}}.$$

Meanwhile, since $\xi > \sigma_i$ we can write

$$\xi = \sigma_i + \sum_{j \leq J} \omega^{\beta_j}.$$

Now, clearly if $\beta_J = \ell \xi$ were greater or equal to $\ell \sigma_{i+1}$, we would have $\xi \geq \sigma_{i+1}$, contrary to our assumption. In particular, note that this implies $\ell \sigma_{i+1} > 0$.
   But then we know that $e_{\ell \sigma_{i+1}} g(\sigma_{i+1})$ is a fixpoint of $e_{\ell \xi}$, and thus for all $\zeta \in [\sigma_i, \xi)$,

$$\ell g(\zeta) = \ell e_{\ell \sigma_{i+1}} g(\sigma_{i+1})$$

$$= \ell e_{\ell \xi} e_{\ell \sigma_{i+1}} g(\sigma_{i+1})$$

$$= \ell e_{\ell \xi} g(\xi).$$

This covers all cases and shows that g is an ℓ-sequence satisfying (3), as desired. $\square$

Lemma 6.6 *If* $\boldsymbol{\sigma} = \langle \sigma_i \rangle_{i \leq I}$ *is a simple sequence,* $f \in D_\Lambda^\Theta$ *and* $\xi < \Lambda$, *then* $\mathrm{wdt}(\lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\xi))$ *and* $\mathrm{hgt}(\lfloor f \rfloor_{\boldsymbol{\sigma}}^{\overline{p}}(\xi))$ *are both at most*[5] $p + I$.

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[5] Actually, $\mathrm{wdt}(\lfloor f \rfloor_{\boldsymbol{\sigma}}^p(\xi)) \leq p + 1$, but it seems more convenient to bound the height and width uniformly.

Proof. One can check easily that $\mathrm{wdt}(\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)) \le \mathrm{wdt}(\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_{i+1})) + 1$ and

$$\mathrm{hgt}(\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)) \le \mathrm{hgt}(\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_{i+1})) + 1.$$

Thus the width and height of all terms is bounded by $I + p$; intermediate terms (i.e., $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\xi)$ for $\sigma_i < \xi < \sigma_{i+1}$) obviously have width and height bounded by that of $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)$. □

The following simple observation will be quite useful later:

Lemma 6.7 *If $\alpha < \xi$, then*

(i) *If $\beta \le \ell\xi$, then $\alpha + \omega^\beta \le \xi$;*

(ii) *If $\beta < \ell\xi$, then $\alpha + \omega^\beta < \xi$.*

Proof. By observations on the Cantor normal form of ξ. □

Lemma 6.8 *For every $f \in D^\Theta_\Lambda$ and $i \le I$,*

$$\lfloor f(\sigma_i) \rfloor^p_{\boldsymbol{\sigma}} < \lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i) \le f(\sigma_i).$$

Moreover, if $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i) < f(\sigma_i)$ then $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_j) < f(\sigma_j)$ for all $j < i$.

Proof. That $\lfloor f(\sigma_i) \rfloor^p_{\boldsymbol{\sigma}} < \lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)$ is obvious from the definition of $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)$, since it is always of the form

$$\lfloor f(\sigma_i) \rfloor^p_{\boldsymbol{\sigma}} + \omega^\rho \tag{4}$$

for some ordinal ρ. In particular we see $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i) > 0$.

To see the other inequality, we use backwards induction on i; clearly

$$\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_I) \le f(\sigma_I),$$

since $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_I) = \lfloor f(\sigma_I) \rfloor^p_{\boldsymbol{\sigma}} + 1$ and $\lfloor f(\sigma_I) \rfloor^p_{\boldsymbol{\sigma}} < f(\sigma_I)$.

Now, assume inductively that

$$\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_{i+1}) \le f(\sigma_{i+1}),$$

and once again write $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i)$ in the form (4).

First we note that the function ℓe_α is increasing independently of α: if $\alpha = 0$ it is the identity; otherwise, $\ell e_\alpha = e_\alpha$, which is a normal function.

Thus we have that, if $\sigma_{i+1} = \sigma_i + \omega^\alpha$,

$$\rho = \ell e_\alpha \lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_{i+1}) \overset{\mathrm{IH}}{\le} \ell e_\alpha f(\sigma_{i+1}) = \ell f(\sigma_i),$$

where the last equality is by Lemma 6.5.

In either case we get $\rho \le \ell f(\sigma_i)$ so by Lemma 6.7.1,

$$\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i) = \lfloor f(\sigma_i) \rfloor^p_{\boldsymbol{\sigma}} + \omega^\rho \le f(\sigma_i).$$

Moreover, if $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_i) < f(\sigma_i)$ then we use Lemma 6.7.2 to conclude $\lfloor f \rfloor^p_{\boldsymbol{\sigma}}(\sigma_{i+1}) < f(\sigma_{i+1})$. □

6.3   Concatenation

Definition 6.9 *Given sequences $f, g : \Lambda \to \Theta$, we define their λ-concatenation $f \overset{\lambda}{*} g : \Lambda \to \Theta$ by*

$$f \overset{\lambda}{*} g(\xi) = \begin{cases} f(\xi) & \text{if } \xi < \lambda \\ g(\xi) & \text{otherwise.} \end{cases}$$

Lemma 6.10 *If $f, g \in D_\Lambda^\Theta$ and $g(\lambda) \leq f(\lambda)$, then $f \overset{\lambda}{*} g$ is an ℓ-sequence.*

If, further, $g(\lambda) < f(\lambda)$, then $f \overset{\lambda}{*} g <_\lambda f$.

Proof. Immediate from the definition of $f \overset{\lambda}{*} g$ and Lemma 5.3. $\qquad\qquad\square$

7   Soundness

The sequence $\lfloor f \rfloor_{\boldsymbol{\sigma}}^p$ does not satisfy the same formulas of the modal language as f, but it does satisfy the same formulas that are 'simple enough'. To see this we extend the notion of *n-bisimulation* to the slightly more general notion of $\langle p, \boldsymbol{\sigma} \rangle$-*bisimulation*:

Definition 7.1 *Given $f, g \in D_\Lambda^\Theta$, a sequence of ordinals $\boldsymbol{\sigma}$ and $p < \omega$, we say f is $\langle p, \boldsymbol{\sigma} \rangle$-bisimilar to g (in symbols, $f \leftrightarroweq_{\boldsymbol{\sigma}}^p g$) by induction on p as follows:*

For $p = 0$, any two ℓ-sequences are $\langle p, \boldsymbol{\sigma} \rangle$-bisimilar.

For $p = q + 1$, $f \leftrightarroweq_{\boldsymbol{\sigma}}^p g$ if and only if, for every ξ of the form σ_i:

Forth. Whenever $f' <_\xi f$, there is $g' <_\xi g$ with $f' \leftrightarroweq_{\boldsymbol{\sigma}}^q g'$.

Back. Whenever $g' <_\xi g$, there is $f' <_\xi f$ with $f' \leftrightarroweq_{\boldsymbol{\sigma}}^q g'$.

The following lemma is standard in modal logic.

Lemma 7.2 *If $f \leftrightarroweq_{\boldsymbol{\sigma}}^p g$, then f and g validate the same formulas ψ of modal depth p where all the modalities in ψ are among $\boldsymbol{\sigma}$.*

There is a close relation between $\langle p, \boldsymbol{\sigma} \rangle$-approximation and $\langle p, \boldsymbol{\sigma} \rangle$-bisimulation.

The following lemma will be quite useful in making this precise. Say that two ℓ-sequences are $\langle p, \langle \sigma_k \rangle_{k \leq I} \rangle$-*close* if, for all $k \leq I$,

(i) $\lfloor f(\sigma_k) \rfloor_{\boldsymbol{\sigma}}^p < g(\sigma_k)$ and

(ii) $\lfloor g(\sigma_k) \rfloor_{\boldsymbol{\sigma}}^p < f(\sigma_k)$.

We will write $f \sim_{\boldsymbol{\sigma}}^p g$.

Lemma 7.3 *Let g, f, f' be ℓ-sequences and $\langle p, \langle \sigma_k \rangle_{k \leq I} \rangle$ be parameters.*

Suppose that for some $i \leq I$, $f' <_{\sigma_i} f$, and $f \sim_{\boldsymbol{\sigma}}^{p+I} g$.

For $j \leq i$, let

$$g_j = g \overset{\sigma_j}{*} \lfloor f' \rfloor_{\boldsymbol{\sigma}}^p.$$

Then, we have that g_j is an ℓ-sequence, $g_j <_{\sigma_j} g$ and $f' \sim_{\boldsymbol{\sigma}}^p g_j$.

Proof. To see that g_j is an ℓ-sequence, by Lemmas 6.10 and 6.5, it suffices to show that $g_j(\sigma_j) = \lfloor f' \rfloor_{\boldsymbol{\sigma}}^p(\sigma_j) < g(\sigma_j)$. By Lemma 6.8 we see $\lfloor f' \rfloor_{\boldsymbol{\sigma}}^p(\sigma_i) \leq f'(\sigma_i)$, and since $f' <_{\sigma_i} f$ we have that $g_j(\sigma_i) < f(\sigma_i)$. Thus, by Lemma

6.8 again, we see that also $g_j(\sigma_j) < f(\sigma_j)$. Meanwhile, by Lemma 6.6, the height and width of $g_j(\sigma_j)$ are bounded by $p+I$, so that $g_j(\sigma_j)$ is a $\langle p+I, \boldsymbol{\sigma}\rangle$-approximation of $f(\sigma_j)$ and thus $g_j(\sigma_j) \le \lfloor f(\sigma_j)\rfloor^{p+I}_{\boldsymbol{\sigma}}$. Now, by assumption $\lfloor f(\sigma_j)\rfloor^{p+I}_{\boldsymbol{\sigma}} < g(\sigma_j)$, so $g_j(\sigma_j) < g(\sigma_j)$ as required and g' is indeed an ℓ-sequence.

As $g_j(\xi) = g(\xi)$ for $\xi < \sigma_j$ we also conclude that $g_j <_{\sigma_j} g$. Thus, it remains to see that 1 and 2 hold for g_j and f'.

For $k < j$ we see that

$$\lfloor g_j(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} \le \lfloor g_j(\sigma_k)\rfloor^{p+I}_{\boldsymbol{\sigma}} < f(\sigma_k) = f'(\sigma_k),$$

and by a symmetric argument, $\lfloor f'(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} < g_j(\sigma_k)$.

For $k \ge j$ we use Lemma 6.8 to obtain

$$\lfloor f'(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} < \lfloor f'\rfloor^p_{\boldsymbol{\sigma}}(\sigma_k) = g_j(\sigma_k)$$

and

$$\lfloor g_j(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} = \lfloor \lfloor f'\rfloor^p_{\boldsymbol{\sigma}}(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} \le \lfloor f'(\sigma_k)\rfloor^p_{\boldsymbol{\sigma}} < f'(\sigma_k).$$

$\square$

Lemma 7.4 *Let* $\boldsymbol{\sigma} = \langle 0 = \sigma_0, \ldots, \sigma_I\rangle$ *be a simple sequence. If* $f, g \in D^\Theta_\Lambda$ *are such that* $f \sim^{Ip}_{\boldsymbol{\sigma}} g$, *then* $g \leftrightarrows^p_{\boldsymbol{\sigma}} f$.

Proof. We prove the claim by induction on p. By symmetry it is enough to consider the 'forth' condition.

Thus, we suppose that $f \sim^{I(p+1)}_{\boldsymbol{\sigma}} g$, and $f' <_{\sigma_i} f$. We must find $g' <_{\sigma_i} g$ such that $g' \leftrightarrows^p_{\boldsymbol{\sigma}} f'$.

But by Lemma 7.3, $g' = g \overset{\sigma_i}{*} \lfloor f'\rfloor^{Ip}_{\boldsymbol{\sigma}}$ satisfies $g' <_{\sigma_i} g$ and $g' \sim^p_{\boldsymbol{\sigma}} f'$. By induction hypothesis we can conclude that also $g' \leftrightarrows^p_{\boldsymbol{\sigma}} f'$, as required. $\square$

Theorem 7.5 (Soundness) GLP^0_Λ *is sound for* $\mathfrak{I}^\Theta_\Lambda$.

Proof. That each of the modalities satisfy the GL axioms is a consequence of the well-foundedness and transitivity of $<_\xi$.

Let us see that the axiom $\langle\zeta\rangle \phi \to [\xi]\langle\zeta\rangle \phi$, for $\xi > \zeta$, is valid. Thus, suppose f satisfies $\langle\zeta\rangle \phi$, so that there is $g <_\zeta f$ which satisfies ϕ. Then, if $h <_\xi f$ with $\zeta < \xi$, we have that $h(\eta) = f(\eta)$ for all $\eta \le \zeta$, so it is also the case that $g(\zeta) < h(\zeta)$ and hence h satisfies $\langle\zeta\rangle \phi$. Since h was arbitrary we conclude that f satisfies $[\xi]\langle\zeta\rangle \phi$.

The validity of any instance of $\psi = \langle\xi\rangle \phi \to \langle\zeta\rangle \phi$ follows from Lemma 7.2. Let $\boldsymbol{\sigma}$ be a simple saturation of all the ordinals appearing in ψ so that $\xi = \sigma_i$ and $\zeta = \sigma_j$ for some $j \le i$. Let p be the modal depth of ψ. If for some ℓ-sequence f we have that $\langle\mathfrak{I}^\Theta_\Lambda, f\rangle \Vdash \langle\sigma_i\rangle \phi$, then there is some $f' <_{\sigma_i} f$ such that $\langle\mathfrak{I}^\Theta_\Lambda, f'\rangle \Vdash \phi$. Now we note that $f \sim^{Ip+I}_{\boldsymbol{\sigma}} f$ and apply Lemma 7.3 to see that for $g = f \overset{\sigma_j}{*} \lfloor f'\rfloor^{Ip}_{\boldsymbol{\sigma}}$, we have that $g \sim^{Ip}_{\boldsymbol{\sigma}} f'$ and $g <_{\sigma_j} f$. By Lemmata 7.4 and 7.2 we see that $\langle\mathfrak{I}^\Theta_\Lambda, g\rangle \Vdash \phi$ and thus $\langle\mathfrak{I}^\Theta_\Lambda, f\rangle \Vdash \langle\sigma_j\rangle\phi$, as required. $\square$

8   Non-triviality: the first step towards completeness

In this section we will show that, for arbitrary λ, if $\lambda < \Lambda$, then the relation $<_\lambda$ is non-empty on $\mathfrak{I}_\Lambda^\Theta$, provided Θ is large enough. For this, it suffices to find an ℓ-sequence f with $f(\lambda) > 0$.

This, of course, shows that our structures $\mathfrak{I}_\Lambda^\Theta$ indeed give interesting models of GLP_Λ^0. However, as we shall prove in the upcoming [6], it implies much more; for indeed, if Λ is a limit ordinal and $\mathfrak{F}$ is a Kripke frame such that, for all $\lambda < \Lambda$, $<_\lambda$ is non-empty, it immediately follows that GLP_Λ^0 is complete for $\mathfrak{F}$. Moreover, this can be shown by purely syntactical methods and does not depend substantially on the structure of $\mathfrak{F}$.

However, said syntactical considerations require some care and lie beyond the scope of the current paper. For now, we shall limit ourselves to establishing non-triviality of the accessibility relations.

Lemma 8.1 *Let α, ϑ be ordinals and let the CNF of α be given by*

$$\alpha = \omega^{\alpha_0} + ... + \omega^{\alpha_N}.$$

Then, there exists an ℓ-sequence f with $f(0) = e_{\alpha_0} \dots e_{\alpha_N}(\vartheta)$ and $f(\alpha) = \vartheta$.

Proof. Let $\alpha' = \omega^{\alpha_0} + ... + \omega^{\alpha_{N-1}}$. By induction on α, there is an ℓ-sequence f' with $f'(\alpha') = e_{\alpha_N}(\vartheta)$ and $f'(0) = e_{\alpha_0} \dots e_{\alpha_N}(\vartheta)$. Consider f given by

$$f(\gamma) = \begin{cases} f'(\gamma) & \text{if } \gamma \leq \alpha', \\ e_{\alpha_N}(\vartheta) & \text{if } \gamma \in (\alpha', \alpha), \\ \vartheta & \text{if } \gamma = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

It is very easy to check that $f : \Lambda \to e_{\alpha_0} \dots e_{\alpha_N}(\vartheta)$ is an ℓ-sequence with all the desired properties for any $\Lambda \geq \alpha$. $\qquad\square$

Corollary 8.2 *Let Λ, Θ be ordinals, and write*

$$\Lambda = \omega^{\alpha_0} + \dots + \omega^{\alpha_N}.$$

GLP_Λ^0 *is sound for $\mathfrak{I}_\Lambda^\Theta$ independently of Θ. If, further, we have that*

$$\Theta > e_{\alpha_0} \dots e_{\alpha_N}(1),$$

then $<_\lambda$ is non-empty for all $\lambda < \Lambda$.

Proof. Immediate from Theorem 7.5 and Lemma 8.1. $\qquad\square$

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