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Rasiowa-Harrop disjunction property

Abstract. We show that there is a purely proof-theoretic proof of the Rasiowa-Harrop disjunction property for the full intuitionistic propositional calculus (**IPC**), via natural deduction, in which commuting conversions are not needed. Such proof is based on a sound and faithful embedding of **IPC** into an atomic polymorphic system. This result strengthens a homologous result for the disjunction property of **IPC** (presented in a recent paper co-authored with Fernando Ferreira) and answers a question then posed by Pierluigi Minari.

Keywords: Rasiowa-Harrop disjunction property, intuitionistic propositional calculus, predicative polymorphism, natural deduction, strong normalization.

1. Introduction

In paper [5], as a corollary of a purely syntactical proof of the faithfulness of the embedding of full intuitionistic propositional calculus **IPC** into the atomic polymorphic system $\mathbf{F_{at}}$, a new syntactical proof of the disjunction property of **IPC** was produced. The major novelty of this alternative proof is the non-necessity of commuting conversions on the natural deduction calculus.

The main goal of the present paper is to answer a question then posed by Pierluigi Minari:

"Is it possible to give a *direct*[1] proof of the Rasiowa-Harrop disjunction property of **IPC** via the faithful embedding of **IPC** into $\mathbf{F_{at}}$ (and the normalization property of the latter)?"

We show that the answer is yes, and somewhat natural, once we consider the atomic polymorphic calculus in the form $\mathbf{F_{at}^{\wedge}}$ (a technical variant of $\mathbf{F_{at}}$).

To make the paper reasonably self-contained we introduce the systems and the properties needed in what follows.

The *atomic polymorphic system* we are going to work with, which we denote by $\mathbf{F_{at}^{\wedge}}$, is the restriction of Jean-Yves Girard system **F** [7, 10] (with

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[1]In this context, *direct* means not using general results apropos intermediate logics (see [8]).

$\to$, $\wedge$, and $\forall$ as primitive connectives) to atomic universal instantiations.[2]

The formulas in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ are defined as the smallest class of expressions that includes the atomic formulas (propositional constants and second-order variables) and is closed under implication, conjunction and second-order universal quantification. In the natural deduction calculus, proofs in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ are built using the following introduction rules:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\to\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\wedge\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}\forall\mathrm{I}$$

where, in the third rule, X does not occur free in any undischarged hypothesis; and the following elimination rules:

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \quad \begin{matrix}\vdots \\ A\end{matrix}}{B}\to\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\wedge\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\wedge\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ \forall X.A\end{matrix}}{A[C/X]}\forall\mathrm{E}$$

where C is an *atomic* formula (free for X in A), and $A[C/X]$ is the result of replacing in A all the free occurrences of X by C. It is the restriction to atomic instantiations in the latter rule that distinguishes $\mathbf{F}_{\mathbf{at}}^{\wedge}$ from $\mathbf{F}$. The impredicative system $\mathbf{F}$ allows, in the $\forall$E rule, the instantiation of X by any (not necessarily atomic) formula of the system.

Since our goal is to study the Rasiowa-Harrop disjunction property in the full intuitionistic propositional calculus **IPC** via the atomic polymorphic system $\mathbf{F}_{\mathbf{at}}^{\wedge}$, we are going to make crucial use of the Russell-Prawitz translation ([9, 11]) from **IPC** into $\mathbf{F}_{\mathbf{at}}^{\wedge}$, that we review below. For each formula A of the full intuitionistic propositional calculus, we define its translation A^* into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ inductively as follows:

$(P)^* :\equiv P$, for P a propositional constant

$(\perp)^* :\equiv \forall X.X$

$(A \to B)^* :\equiv A^* \to B^*$

$(A \wedge B)^* :\equiv A^* \wedge B^*$

$(A \vee B)^* :\equiv \forall X.((A^* \to X) \to ((B^* \to X) \to X))$,

[2] System $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is system $\mathbf{F}_{\mathbf{at}}$ [2, 4] with an extra primitive connective for *conjunction*.

where X is a second-order variable which does not occur in A^* nor in B^*. Note that the Russell-Prawitz translation also allows for the translation of $\wedge$ in terms of $\rightarrow$ and $\forall$. Since our target system $\mathbf{F}^{\wedge}_{\mathbf{at}}$ has $\wedge$ as a primitive symbol the translation can be simplified as above.

The previous translation is, in fact, a sound and faithful embedding of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$, i.e,

$$\Gamma \vdash_{\mathbf{IPC}} A \text{ if and only if } \Gamma^* \vdash_{\mathbf{F}^{\wedge}_{\mathbf{at}}} A^*,$$

where $\Gamma :\equiv A_1, \ldots, A_n$ and $A_1, \ldots, A_n, A$ are formulas in **IPC** and $\Gamma^* :\equiv A_1^*, \ldots, A_n^*$.

The soundness proof can be found in [2, 3] and relies on the phenomenon of *instantiation overflow*. Instantiation overflow ensures that, from formulas of the form

$$\forall X.X$$
$$\forall X.((A \rightarrow X) \rightarrow ((B \rightarrow X) \rightarrow X)),$$

it is possible to deduce in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ (respectively)

$$F$$
$$(A \rightarrow F) \rightarrow ((B \rightarrow F) \rightarrow F),$$

for *any* (not necessarily atomic) formula F. The proof of instantiation overflow is given in [2, 3] and it yields algorithmic methods for obtaining the two kinds of deductions above. For a recent study on instantiation overflow see also [1].

The proof of faithfulness can be found in [5][3].

The advantage of working in the predicative system $\mathbf{F}^{\wedge}_{\mathbf{at}}$ is that it has (as opposed to **IPC**) no "bad" connectives, i.e., $\perp$ and $\vee$ are absent from the calculus and has no *ad hoc* commuting conversions (see the enjoyable discussions by Girard in Chapter 10 of [7] commenting harshly on what he calls the "defects" of the natural deduction calculus). As we argue in the last section, **IPC** should be seen through the lens of its embedding into $\mathbf{F}^{\wedge}_{\mathbf{at}}$, avoiding this way the "defects" of the system. Two properties of $\mathbf{F}^{\wedge}_{\mathbf{at}}$, we are going to use extensively in the present paper, are the strong normalization

[3]The proof-theoretic proof of faithfulness presented in [5] was given in the context of $\mathbf{F_{at}}$. The proof can be trivially adapted to $\mathbf{F}^{\wedge}_{\mathbf{at}}$, i.e., to the case where conjunction is a primitive symbol.

property for $\beta\eta$-conversions [4][4] and the subformula property for normal proofs [2][5]. We remember that the proper subformulas of a formula of the form $\forall X.A$ are the subformulas of the formulas of the form $A[C/X]$, for C an *atomic* formula free for X in A. As usual, we assume that bound variables in a formula can be freely renamed.

More than the subformula property, in what follows, we are going to need the following stronger result:

PROPOSITION 1.1. *Let Δ be a normal derivation in $\mathbf{F}_{\mathbf{at}}^{\wedge}$. Then*

i) Every formula in Δ is a subformula of the conclusion or of a (undischarged) hypothesis of Δ. [Subformula Property]

ii) If Δ ends in an elimination rule, it has a principal branch*, i.e. a sequence of formulas $A_0, A_1, \ldots, A_n$ such that*

- *A_0 is an (undischarged) hypothesis;*
- *A_n is the conclusion;*
- *for $i = 0, \ldots, n-1$, A_i is the principal premise (i.e. the premise that carries the eliminated symbol) of an elimination rule whose conclusion is A_{i+1}.*

In particular, A_n is a subformula of A_0.

After this introductory section where we presented the goal of the paper, the systems involved and made a quick survey on relevant properties of these systems, the paper is organized as follows: in Section 2, we introduce some definitions relevant to our study e.g. strictly positive subformula or Rasiowa-Harrop formula in both **IPC** and $\mathbf{F}_{\mathbf{at}}^{\wedge}$ frameworks and prove some auxiliary results involving these concepts; in Section 3, we prove that $\mathbf{F}_{\mathbf{at}}^{\wedge}$ enjoys the Rasiowa-Harrop disjunction property and in Section 4 we present the main result of the paper: an alternative proof of the Rasiowa-Harrop disjunction property of **IPC**, via natural deduction, without commuting conversions. We finish (Section 5) with some comments and final considerations.

[4]The strong $\beta\eta$-normalization proof of $\mathbf{F}_{\mathbf{at}}$ presented in [4] generalizes easily to $\mathbf{F}_{\mathbf{at}}^{\wedge}$: a straightforward reducibility clause can be added for conjunction.

[5]Again, paper [2] is about $\mathbf{F}_{\mathbf{at}}$, but the rules for conjunction can be dealt with exactly in the same way (see [7], page 76).

2. Preliminaries

By a *disjunction* in $\mathbf{F}_{\mathbf{at}}^{\wedge}$, denoted by $D^* \vee E^*$ [6], we mean the translation of a disjunction, i.e., a formula of the form $\forall X.((D^* \to X) \to ((E^* \to X) \to X)) :\equiv (D \vee E)^*$ (with D and E formulas in **IPC**).

By a *translated formula* in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ we mean a formula in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ which is the translation (via the Russell-Prawitz translation) of a certain formula in **IPC**.

Easily from the Russell-Prawitz translation we have the following result:

LEMMA 2.1. Let A be a formula in **IPC** and A^* its translation into $\mathbf{F}_{\mathbf{at}}^{\wedge}$. Then

i) A^* has no free variables.

ii) Universal quantifications in A^* always occur in one of the following two specific forms:

  a) $\forall X.X$ or

  b) $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$,

  with D and E formulas in **IPC**. The formulas in *a)* and *b)* are the translations into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ of the subformulas of A: $\perp$ and $D \vee E$ respectively.

iii) If $A^* \equiv B^*$ then $A \equiv B$, where $\equiv$ denotes syntactic equality, B is a formula in **IPC** and B^* is the translation of B into $\mathbf{F}_{\mathbf{at}}^{\wedge}$.

REMARK 2.2. Note that Lemma 2.1 impacts on the nesting of universal quantifications in a translated formula. In a translated formula we never have $A(X, Y)$ (with X, Y distinct variables) simultaneously under the scope of the second-order quantifications $\forall X$ and $\forall Y$. Thus, a subformula of a translated formula has no more than one atomic formula (repetitions may occur) obtained by the instantiation of second-order universal quantifications of the translated formula.

PROPOSITION 2.3. *Let $A_0, \dots, A_n$ be a principal branch according to Proposition 1.1, let A_0 be a translated formula in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ and X be a second-order variable. If X occurs free in A_n and $A_n \not\equiv X$, then there are formulas D and E in* **IPC** *such that $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$ (i.e. $(D \vee E)^* \equiv D^* \vee E^*$) is a formula in the principal branch.*

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[6] Note that $\vee$ is not a primitive symbol in $\mathbf{F}_{\mathbf{at}}^{\wedge}$. The abbreviation $D^* \vee E^*$ is introduced for ease of notation.

Proof. By Proposition 1.1, we know that A_n is a subformula of A_0. By the freely renaming of bounded variables in a formula we may assume without loss of generality (see Lemma 2.1 and Remark 2.2) that A_n occurs in A_0. Notice, again by Lemma 2.1 i), that A_n is in A_0 under the scope of a second-order quantification $\forall X$. By Lemma 2.1 ii), we know that the universal quantifications in a translated formula are of one of the following forms: $\forall X.X$ or $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$, with D and E formulas of **IPC**. The second-order universal quantification which has A_n as a subformula can not be $\forall X.X$ because X occurs free in A_n and $A_n \not\equiv X$. Thus, it has to be $\forall X.((D^* \to X) \to ((E^* \to X) \to X)) \equiv (D \vee E)^* \equiv D^* \vee E^*$. Let us argue that $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$ is a formula in the principal branch. Note that, by definition, a principal branch is a sequence of immediate subformulas. If $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$ was not in the principal branch then A_n would not be in the principal branch either because from a formula properly containing $\forall X.((D^* \to X) \to ((E^* \to X) \to X))$, through a single elimination rule we could not obtain a proper subformula of $(D \vee E)^*$. ■

Lemma 2.4. Let $\mathcal{D}$ be a derivation in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ of a formula A from assumptions Γ. Let X be a second-order variable and F be a formula in $\mathbf{F}^{\wedge}_{\mathbf{at}}$. If all universal formulas in $\mathcal{D}$ are subformulas of translated formulas, then $\mathcal{D}$ can be transformed into a (normal) derivation of $A[F/X]$ from $\Gamma[F/X]$.

Proof. The proof is by induction on the number of inferences in $\mathcal{D}$. The base of induction is when $\mathcal{D}$ consists solely of an hypothesis (i.e. A is derived from A). The result is obvious: $A[F/X]$ is derived from $A[F/X]$.

For the implication rules, for the conjunction rules and for the $\forall$E-rule when the conclusion is not obtained by instantiating the bound variable of the universal quantifier by the atomic variable X the result follows immediately by induction hypothesis. Just notice that the rules are still valid when replacing the free occurrences of X by F.

In the case of the $\forall$E-rule of the form $\frac{\forall Y.H}{H[X/Y]}$, note that by hypothesis all the universal formulas in $\mathcal{D}$ are subformulas of translated formulas and so enjoy the property of instantiation overflow. Thus we can replace Y by any formula of the language, in particular by F.

Let us analyse the $\forall$I-rule. We assume (modulo a renaming of variables) that a rule of the form $\frac{H}{\forall X.H}$ does not occur in $\mathcal{D}$. Note that any potential subderivation of $\mathcal{D}$ having $\frac{H}{\forall X.H}$ as its last rule has no free variables X in its undischarged hypothesis.

Similarly, we assume that there is no clash of variables in the sense that a $\forall$I-rule of the form $\frac{H}{\forall Y.H}$ $(Y \not\equiv X)$ occurring in $\mathcal{D}$ is such that Y

does not occur free in F. Thus the result follows immediately by induction hypothesis. ■

In what follows we will need the notion of *Rasiowa-Harrop formula* in both the **IPC** and the $\mathbf{F}_{\mathbf{at}}^{\wedge}$ contexts. For that, we review the notion of *strictly positive subformula* of a formula in **IPC** and introduce a homologous definition in the context of $\mathbf{F}_{\mathbf{at}}^{\wedge}$.

DEFINITION 2.5. Let A be a formula in **IPC**. The set of the *strictly positive subformulas of* A, which we denote by $\mathbf{sp}(A)$, is defined by induction on the complexity of A as follows:

$\mathbf{sp}(P) :\equiv \{P\}$, for P a propositional constant

$\mathbf{sp}(\perp) :\equiv \{\perp\}$

$\mathbf{sp}(D \wedge E) :\equiv \{D \wedge E\} \cup \mathbf{sp}(D) \cup \mathbf{sp}(E)$

$\mathbf{sp}(D \vee E) :\equiv \{D \vee E\} \cup \mathbf{sp}(D) \cup \mathbf{sp}(E)$

$\mathbf{sp}(D \to E) :\equiv \{D \to E\} \cup \mathbf{sp}(E)$.

Note that if B is a subformula of A and $B \notin \mathbf{sp}(A)$ it means that B occurs in the antecedent of some implicative subformula of A.

The correspondent definition in the framework of $\mathbf{F}_{\mathbf{at}}^{\wedge}$ becomes:

DEFINITION 2.6. Let A be a formula in $\mathbf{F}_{\mathbf{at}}^{\wedge}$. The set of the *strictly positive subformulas of* A, which we denote by $\mathbf{sp}(A)$, is defined by induction on the complexity of A as follows:

$\mathbf{sp}(C) :\equiv \{C\}$, for C an atomic formula

$\mathbf{sp}(D \to E) :\equiv \{D \to E\} \cup \mathbf{sp}(E)$

$\mathbf{sp}(D \wedge E) :\equiv \{D \wedge E\} \cup \mathbf{sp}(D) \cup \mathbf{sp}(E)$

$\mathbf{sp}(\forall X.E) :\equiv \{\forall X.E\} \cup \mathbf{sp}(E[C/X])$, for all atomic formulas C free for X in E.

Note that:

i) $A \in \mathbf{sp}(A)$
ii) The strictly positive subformulas of A are in particular subformulas of A.

DEFINITION 2.7. Let A be a formula in **IPC**. We say that A is a *Rasiowa-Harrop formula*[7] (which we denote by RH-formula) if no disjunction, i.e. no formula of the form $D \vee E$ (with D and E formulas in **IPC**), belongs to the set $\mathbf{sp}(A)$.

[7]Rasiowa-Harrop formulas are also known as *Harrop formulas*.

DEFINITION 2.8. Let A be a formula in $\mathbf{F}_{\mathbf{at}}^{\wedge}$. We say that A is a *Rasiowa-Harrop formula* (which we denote by RH-formula) if no disjunction, i.e. no formula of the form $\forall X.((D^* \to X) \to ((E^* \to X) \to X)) :\equiv (D \vee E)^*$ (with D and E formulas in **IPC**) belongs to the set $\mathbf{sp}(A)$.

Equivalently, A is a Rasiowa-Harrop formula if each disjunction (if any) in A occurs only in the antecedent of some implicative subformula of A.

LEMMA 2.9. Let D, E and A be formulas in **IPC**. If $(D \vee E)^* \in \mathbf{sp}(A^*)$ then $D \vee E \in \mathbf{sp}(A)$.[8]

PROOF. By induction on A. If A is a propositional constant or $A \equiv \perp$ the result is trivial (antecedent false, nothing to check). If $A \equiv F \to G$ then $A^* \equiv F^* \to G^*$. By hypothesis $(D \vee E)^* \in \mathbf{sp}(F^* \to G^*) \equiv \{F^* \to G^*\} \cup \mathbf{sp}(G^*)$. Thus $(D \vee E)^* \in \mathbf{sp}(G^*)$. By I.H., $D \vee E \in \mathbf{sp}(G) \subseteq \mathbf{sp}(F \to G)$. Let $A \equiv F \wedge G$. Then $A^* \equiv F^* \wedge G^*$. By hypothesis $(D \vee E)^* \in \mathbf{sp}(F^* \wedge G^*) \equiv \{F^* \wedge G^*\} \cup \mathbf{sp}(F^*) \cup \mathbf{sp}(G^*)$. Therefore $(D \vee E)^* \in \mathbf{sp}(F^*)$ or $(D \vee E)^* \in \mathbf{sp}(G^*)$. So, by I.H., we have that $D \vee E \in \mathbf{sp}(F)$ or $D \vee E \in \mathbf{sp}(G)$. Thus $D \vee E \in \mathbf{sp}(F) \cup \mathbf{sp}(G) \subseteq \mathbf{sp}(F \wedge G)$. Let us analyse the case $A \equiv F \vee G$. Then $A^* \equiv \forall X.((F^* \to X) \to ((G^* \to X) \to X))$. Suppose that $(D \vee E)^* \in \mathbf{sp}(A^*)$. Since $\mathbf{sp}((F \vee G)^*) \equiv \mathbf{sp}(\forall X.((F^* \to X) \to ((G^* \to X) \to X))) \equiv \{\forall X.((F^* \to X) \to ((G^* \to X) \to X))\} \cup \{(F^* \to C) \to ((G^* \to C) \to C) | C \text{ is an atomic formula}\} \cup \{(G^* \to C) \to C | C \text{ is an atomic formula}\} \cup \{C | C \text{ is an atomic formula}\}$, we have that $(D \vee E)^* \equiv (F \vee G)^*$. Thus, by Lemma 2.1, we have $D \vee E \equiv F \vee G$ and we conclude that $D \vee E \in \mathbf{sp}(A)$. ∎

PROPOSITION 2.10. *If A is a Rasiowa-Harrop formula of* **IPC** *then A^* is a Rasiowa-Harrop formula of $\mathbf{F}_{\mathbf{at}}^{\wedge}$.*

PROOF. Suppose that A^* is not a Rasiowa-Harrop formula of $\mathbf{F}_{\mathbf{at}}^{\wedge}$. Take $\forall X.((D^* \to X) \to ((E^* \to X) \to X)) \equiv (D \vee E)^*$ an element of $\mathbf{sp}(A^*)$. By Lemma 2.9, we know that $D \vee E \in \mathbf{sp}(A)$. Thus A is not a Rasiowa-Harrop formula of **IPC**. The proof follows by contraposition. ∎

LEMMA 2.11. The elimination rules of $\mathbf{F}_{\mathbf{at}}^{\wedge}$ are such that:

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[8] Note that a general result for arbitrary formulas is not valid: $B^* \in \mathbf{sp}(A^*)$ not even implies that B is a subformula of A. Just take $B \equiv P$, with P a propositional constant in **IPC**, and $A \equiv \perp$. We have that $P^* \in \mathbf{sp}(\perp^*)$ but P is not a subformula of $\perp$. Moreover, $B^* \in \mathbf{sp}(A^*)$ and B a subformula of A still does not imply that $B \in \mathbf{sp}(A)$. Just take $A \equiv R \to (P \vee Q)$ and $B \equiv R$, with P, Q and R distinct propositional constants in **IPC**. We have that $R^* \equiv R \in \mathbf{sp}((P^* \to R) \to ((Q^* \to R) \to R)) \subseteq \mathbf{sp}((R \to (P \vee Q))^*)$ but $R \notin \mathbf{sp}(R \to (P \vee Q))$.

i) The conclusion is a strictly positive subformula of the principal premise.

ii) If the principal premise is a RH-formula then the conclusion is a RH-formula.

PROOF. i) By Definition 2.6, we have that $\mathbf{sp}(B) \subseteq \mathbf{sp}(A \to B)$, $\mathbf{sp}(A) \subseteq \mathbf{sp}(A \wedge B)$, $\mathbf{sp}(B) \subseteq \mathbf{sp}(A \wedge B)$ and $\mathbf{sp}(A[C/X]) \subseteq \mathbf{sp}(\forall X.A)$ for C an atomic formula. The result follows because a formula belongs to its **sp** class.

ii) Immediate, by definition of RH-formulas, noticing again that $\mathbf{sp}(B) \subseteq \mathbf{sp}(A \to B)$, $\mathbf{sp}(A), \mathbf{sp}(B) \subseteq \mathbf{sp}(A \wedge B)$ and $\mathbf{sp}(A[C/X]) \subseteq \mathbf{sp}(\forall X.A)$ for C an atomic formula. ■

PROPOSITION 2.12. *Let $A_0, A_1, \ldots, A_n$ be a principal branch according to Proposition 1.1 (in particular, A_0 is the undischarged hypothesis and A_n is the conclusion). Then*

i) $\mathbf{sp}(A_n) \subseteq \mathbf{sp}(A_0)$. *In particular,* $A_n \in \mathbf{sp}(A_0)$.

ii) If A_0 is a RH-formula of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ then A_n is a RH-formula of $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

PROOF. By induction on n, applying Lemma 2.11 and its proof. ■

3. Rasiowa-Harrop disjunction property of $\mathbf{F}^{\wedge}_{\mathbf{at}}$

Having in view to present, in the next section, a purely proof-theoretic proof of the Rasiowa-Harrop disjunction property of **IPC**, via natural deduction without the need of commuting conversions, we prove below the following Rasiowa-Harrop disjunction property of $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

THEOREM 3.1 (Rasiowa-Harrop disjunction property of $\mathbf{F}^{\wedge}_{\mathbf{at}}$). *Let A, B, D be the translations into the language of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ of given formulas of* **IPC**. *Let D be a RH-formula of $\mathbf{F}^{\wedge}_{\mathbf{at}}$.*

If $D \vdash_{\mathbf{F}^{\wedge}_{\mathbf{at}}} A \vee B$ then $D \vdash_{\mathbf{F}^{\wedge}_{\mathbf{at}}} A$ or $D \vdash_{\mathbf{F}^{\wedge}_{\mathbf{at}}} B$.

PROOF. Suppose that $D \vdash_{\mathbf{F}^{\wedge}_{\mathbf{at}}} A \vee B$. Since $\mathbf{F}^{\wedge}_{\mathbf{at}}$ enjoys the (strong) normalization property, take a normal derivation of $A \vee B$ from D in $\mathbf{F}^{\wedge}_{\mathbf{at}}$:

$$\begin{array}{c} D \\ \vdots \\ \forall X.((A \to X) \to ((B \to X) \to X)) \end{array}$$

The last rule must be an introduction rule. Note that if it was an elimination rule, the principal branch would have a RH-formula (D) as undischarged hypothesis and a conclusion ($A \vee B$) which is not a RH-formula, contradicting Proposition 2.12-ii).

Thus we have

$$\dfrac{\begin{array}{c} D \\ \vdots \\ (A \to X) \to ((B \to X) \to X) \end{array}}{\forall X.((A \to X) \to ((B \to X) \to X))}\ \forall\mathrm{I}$$

Again the penultimate rule has to be an introduction rule. If it was an elimination rule, by Proposition 2.3, the principal branch, with D as undischarged hypothesis and $(A \to X) \to ((B \to X) \to X)$ as conclusion, would have a formula of the form $\forall X.((F \to X) \to ((G \to X) \to X)) \equiv F \vee G$, with F and G translated formulas in $\mathbf{F}^{\wedge}_{\mathbf{at}}$.[9] Thinking in the principal branch $D, \ldots, F \vee G$, we would have a RH-formula D as undischarged hypothesis but a non RH-formula $F \vee G$ as the conclusion, contradicting Proposition 2.12.

Thus we have

$$\dfrac{\dfrac{\begin{array}{c} D \qquad\qquad [A \to X]^u \\ \vdots \\ (B \to X) \to X \end{array}}{(A \to X) \to ((B \to X) \to X)}\ {\to}\mathrm{I}^u}{\forall X.((A \to X) \to ((B \to X) \to X))}\ \forall\mathrm{I}$$

Let us argue that again the last rule can not be an elimination rule. If it was an elimination rule, the principal branch would be of the form i) $D, \ldots, (B \to X) \to X$ or of the form ii) $A \to X, \ldots, (B \to X) \to X$.

Again, by Propositions 2.3 and 2.12, case i) does not occur.

Case ii) does not occur either, because the formula following $A \to X$ in the principal branch would be X and the principal branch would stop there (never reaching $(B \to X) \to X$). Note that a second-order variable can not be a principal premise in an elimination rule.

Thus we have

[9] Easily we could argue that $F \equiv A$ and $G \equiv B$. But it is not needed in the argument.

$$\dfrac{\dfrac{\dfrac{\begin{array}{ccc} D & [A \to X]^u & [B \to X]^v \\ & \vdots & \\ & \mathrm{X} & \end{array}}{(B \to X) \to X}\,{\to}\mathrm{I}^v}{(A \to X) \to ((B \to X) \to X)}\,{\to}\mathrm{I}^u}{\forall X.((A \to X) \to ((B \to X) \to X))}\,\forall\mathrm{I}$$

Since X is a variable, the last rule has to be an elimination rule.

The possible cases (for the last rule) are: i) $\frac{\forall X.X}{X}$, ii) $\frac{H \wedge X}{X}$, iii) $\frac{X \wedge H}{X}$, and iv) $\frac{H \to X \quad H}{X}$, with H a formula in $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

In case i) we have a proof of the form

$$\begin{array}{ccc} D & A \to X & B \to X \\ & \vdots & \\ & \dfrac{\forall X.X}{X}\,\forall\mathrm{E} & \end{array}$$

Note that D and potentially $A \to X$ and $B \to X$ are the only possible undischarged hypotheses.

Thus we have the following derivation in $\mathbf{F}^{\wedge}_{\mathbf{at}}$

$$\begin{array}{ccc} D & A \to X & B \to X \\ & \vdots & \\ & \begin{array}{c}\forall X.X \\ \hline\hline A\end{array}\ \text{i.o.} & \end{array}$$

where the double line hides the proof in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ that exists by instantiation overflow.

Therefore, by the (strong) normalization property of $\mathbf{F}^{\wedge}_{\mathbf{at}}$, there is a normal proof of A (in $\mathbf{F}^{\wedge}_{\mathbf{at}}$) from the hypothesis D and possibly $A \to X$ and $B \to X$. By the subformula property we know that any universal formula in such a proof is a subformula of a translated formula. Thus, by Lemma 2.4 (taking $F :\equiv X \to X$) we conclude that there is a normal derivation in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ of A from D. Note that X does not occur free in D nor in A.

We argue that cases ii) and iii) never occur. Notice that, by Proposition 1.1, we would have a principal branch of the form $D, \ldots, H \wedge X, X$ (case $X \wedge H$ is similar). [$A \to X$ or $B \to X$ can not be the undischarged hypothesis on the top of the principal branch because such branches would have just two formulas $A \to X, X$ or $B \to X, X$.] By Proposition 2.3, the principal branch would have a formula of the form $F \vee G$, with F and G translated formulas in $\mathbf{F}^{\wedge}_{\mathbf{at}}$. Contradiction by Proposition 2.12, because the principal branch $D, \ldots, F \vee G$ would have a RH-formula as (undischarged) hypothesis and a non RH-formula as conclusion.[10]

Let us analyse case iv). We know that the principal branch has to start with the undischarged hypothesis $A \to X$, $B \to X$ or D. If the principal branch starts with $A \to X$ we have that the proof has the form

$$\dfrac{A \to X \qquad \begin{array}{c} A \to X \qquad B \to X \qquad D \\ \vdots \\ A \end{array}}{X}$$

(with $H \equiv A$)

But then we have a normal proof of the form

$$\begin{array}{c} A \to X \qquad B \to X \qquad D \\ \vdots \\ A \end{array}$$

which we already saw (applying Lemma 2.4) that yields a proof of A from D.

If the principal branch starts with $B \to X$ the analysis is entirely similar replacing A by B and we obtain a proof in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ of B from D.

The principal branch can not start with D, because if it was the case, by Proposition 2.3, the principal branch $D, \ldots, H \to X$ would have a formula of the form $F \vee G$ with F and G translated formulas in $\mathbf{F}^{\wedge}_{\mathbf{at}}$. Contradiction by Proposition 2.12.

This finishes the proof. ∎

4. Rasiowa-Harrop disjunction property of IPC

We are now able to present our main result: an alternative purely proof-theoretic proof, via natural deduction, of the Rasiowa-Harrop disjunction property of **IPC**. This alternative strategy is cemented on the Rasiowa-Harrop disjunction property of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ proved in the previous section and takes advantage of the sound and faithful embedding of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$. The main

[10] Another way of reaching a contradiction would be to observe that an easy consequence of Proposition 2.3 is that A_n (resulting from eliminations, in a principal branch, from $F \vee G$) could never be of the form $H \wedge X$ nor $X \wedge H$.

interest of reasoning via $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is that the system has only "good" connectives[11] – and has no commuting conversions[12].

THEOREM 4.1 (Rasiowa-Harrop disjunction property of **IPC**). *Let D be a RH-formula of* **IPC** *and let A and B be formulas of* **IPC**.

If $D \vdash_{\mathbf{IPC}} A \vee B$ then $D \vdash_{\mathbf{IPC}} A$ or $D \vdash_{\mathbf{IPC}} B$.[13]

PROOF. Suppose that $D \vdash_{\mathbf{IPC}} A \vee B$. Since the Russell-Prawitz translation of the full intuitionistic propositional calculus into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is sound, we have $D^* \vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} (A \vee B)^*$, i.e., $D^* \vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} \forall X.((A^* \to X) \to ((B^* \to X) \to X))$, which, according to our notation, is also written as $D^* \vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} A^* \vee B^*$. Since, by hypothesis, D is a RH-formula of **IPC**, applying Proposition 2.10, we know that D^* is a RH-formula of $\mathbf{F}_{\mathbf{at}}^{\wedge}$. By the Rasiowa-Harrop disjunction property of $\mathbf{F}_{\mathbf{at}}^{\wedge}$ (Theorem 3.1), we have $D^* \vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} A^*$ or $D^* \vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} B^*$. Since the embedding of **IPC** into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is faithful, we conclude that $D \vdash_{\mathbf{IPC}} A$ or $D \vdash_{\mathbf{IPC}} B$. ■

5. Final comments

1) The reason we worked with an atomic polymorphic system which directly deals with conjunction ($\mathbf{F}_{\mathbf{at}}^{\wedge}$ instead of $\mathbf{F}_{\mathbf{at}}$) is because with the "natural" definition of RH-formulas in the atomic polymorphic context (as presented in Definition 2.8) the RH-disjunction property is valid in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ but not in $\mathbf{F}_{\mathbf{at}}$.

The proof that the RH-disjunction property is valid in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ was presented in Theorem 3.1. To argue that the RH-disjunction property is not valid in $\mathbf{F}_{\mathbf{at}}$ it is enough to present a counterexample. Let P, Q and R be distinct propositional constants in **IPC**. Let D be the following formula in $\mathbf{F}_{\mathbf{at}}$: $((P \vee Q) \wedge R)^*$, i.e., $D :\equiv \forall X.(((P \vee Q)^* \to (R^* \to X)) \to X)$[14]. We can prove that $D \vdash_{\mathbf{F}_{\mathbf{at}}} P^* \vee Q^*$, but $D \nvdash_{\mathbf{F}_{\mathbf{at}}} P^*$ nor $D \nvdash_{\mathbf{F}_{\mathbf{at}}} Q^*$. Moreover, since

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[11] We avoid this way the "bad" elimination rules (for $\perp$ and $\vee$) of **IPC** pointed by Girard in [7], page 74.

[12] Note that, in the framework of **IPC**, in order to ensure that normal proofs enjoy the subformula property, the natural deduction calculus has to be enriched with some *ad hoc* conversions, the so called commuting (or permutative) conversions. See Girard's criticism of such conversions on [7], page 80. System $\mathbf{F}_{\mathbf{at}}^{\wedge}$ does not suffer from the previous defects pointed by Girard: $\mathbf{F}_{\mathbf{at}}^{\wedge}$ has no commuting conversions, no "bad" connectives and normal proofs still enjoy the subformula property.

[13] We are not losing generality considering a single RH-formula D instead of a tuple of RH-formulas $\Gamma \equiv D_1, \ldots, D_n$. Just note that $D_1 \wedge \ldots \wedge D_n$ is a RH-formula and $D_1 \wedge \ldots \wedge D_n \vdash_{\mathbf{IPC}} D_i$ (for $i \in \{1, \ldots, n\}$) and $\Gamma \vdash_{\mathbf{IPC}} D_1 \wedge \ldots \wedge D_n$.

[14] The Russell-Prawitz translation of $A \wedge B$ into $\mathbf{F}_{\mathbf{at}}$ is defined by $(A \wedge B)^* :\equiv \forall X.((A^* \to (B^* \to X)) \to X)$, where X does not occur free in A^* nor in B^*. See [3, 4] for the

$$\begin{aligned}
\mathbf{sp}(D) &\equiv \mathbf{sp}(((P \vee Q) \wedge R)^*) \\
&\equiv \mathbf{sp}(\forall X.(((P \vee Q)^* \to (R^* \to X)) \to X)) \\
&\equiv \{\forall X.(((P \vee Q)^* \to (R^* \to X)) \to X)\} \cup \\
&\quad \cup \{((P \vee Q)^* \to (R^* \to C)) \to C\} \cup \{C\},
\end{aligned}$$

for all atomic formula C,
no element of the set $\mathbf{sp}(D)$ is a disjunction. Thus, we conclude that D is a RH-formula in $\mathbf{F_{at}}$.[15]

2) In the Russell-Prawitz translation of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$, instead of the translation of disjunction presented in Section 1, we could have used the alternative translation: $(A \vee B)^* :\equiv \forall X.(((A^* \to X) \wedge (B^* \to X)) \to X)$, where X is a second-order variable which does not occur in A^* nor in B^*.

3) In the well-known book [7], Girard defends that the elimination rules for $\perp$ and $\vee$ are not as natural and well-behaved as the other (natural deduction) inference rules of **IPC**. He argues that since the presence of such rules gives rise to problems and "boring complications" (e.g. the need for commuting conversions in order to have the subformula property in normal derivations) "one tends to think that natural deduction should be modified to correct such atrocities". In recent years [3, 4] we have suggested that system $\mathbf{F_{at}}$ (which embeds **IPC** and has very well-behaved rules) provides a natural framework for full intuitionistic propositional logic. Nowadays the author believes that system $\mathbf{F}^{\wedge}_{\mathbf{at}}$ is even more adequate and appealing for studying **IPC**. $\mathbf{F}^{\wedge}_{\mathbf{at}}$ keeps the advantages of $\mathbf{F_{at}}$ – no bad connectives, no commuting conversions, strong normalization property, subformula property, sound and faithful embedding of **IPC** into the system – and has an advantage over $\mathbf{F_{at}}$ – equality of proofs for $\beta\eta$-conversions. In $\mathbf{F}^{\wedge}_{\mathbf{at}}$ (as opposed to $\mathbf{F_{at}}$) we have that $\beta\eta$-conversions of **IPC** translate into $\beta\eta$-conversions of the atomic polymorphic system[16] and, as a consequence, from the strong normalization of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ considering $\beta\eta$-conversions we can derive a proof of the strong normalization of **IPC** with respect to $\beta\eta$-conversions (see [6]). The present paper (see comment 1) above) reinforce the idea that $\mathbf{F}^{\wedge}_{\mathbf{at}}$ is

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translation of **IPC** into $\mathbf{F_{at}}$ and for a proof that the universal formula that results from the translation of conjunction also enjoys instantiation overflow. Note that, conjunction is not a primitive symbol in $\mathbf{F_{at}}$, reason why the simpler translation of conjunction presented in Section 1 in the context of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ can not be adopted in $\mathbf{F_{at}}$.

[15] Note that the example above does not provide a counterexample for the RH-disjunction property in the framework of $\mathbf{F}^{\wedge}_{\mathbf{at}}$: the formula $D \equiv ((P \vee Q) \wedge R)^* \equiv (P \vee Q)^* \wedge R^*$ is not a RH-formula in $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

[16] In $\mathbf{F_{at}}$ such result is not valid precisely due to the η-conversion for conjunction.

really convenient for studying structural proof-theoretic properties of **IPC**: $\mathbf{F}_{\mathbf{at}}^{\wedge}$ enjoys the Rasiowa-Harrop disjunction property and from such proof an alternative proof of the Rasiowa-Harrop disjunction property in the context of **IPC** can be derived.

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References

[1] Dinis, B., and G. Ferreira, 'Instantiation overflow', *Reports on Mathematical Logic*, 51 (2016), 15–33.

[2] Ferreira, F., 'Comments on predicative logic', *Journal of Philosophical Logic*, 35 (2006), 1–8.

[3] Ferreira, F., and G. Ferreira, 'Commuting conversions vs. the standard conversions of the "good" connectives', *Studia Logica*, 92 (2009), 63–84.

[4] Ferreira, F., and G. Ferreira, 'Atomic polymorphism', *The Journal of Symbolic Logic*, 78 (2013), 260–274.

[5] Ferreira, F., and G. Ferreira, 'The faithfulness of $\mathbf{F}_{\mathbf{at}}$: a proof-theoretic proof', *Studia Logica*, 103(6) (2015), 1303–1311.

[6] Ferreira, G., 'Eta-conversions of **IPC** implemented in atomic **F**'. To appear in *Logic Jnl IGPL*, published online July 1, 2016. doi:10.1093/jigpal/jzw035 (16 pages).

[7] Girard, J.-Y., Y. Lafont, and P. Taylor, *Proofs and Types*, Cambridge University Press, 1989.

[8] Minari, P., and A. Wronski, 'The property (HD) in intermediate logics. A partial solution of a problem of H. Ono', *Reports on Mathematical Logic*, 22 (1988), 21–25.

[9] Prawitz, D., *Natural Deduction*, Almkvist & Wiksell, Stockholm, 1965. Reprinted, with a new preface, in Dover Publications, 2006.

[10] Reynolds, J. C., 'Towards a theory of type structure', *Lecture Notes in Computer Science*, Colloque sur la programmation, B. Robinet, editor, Springer, 19 (1974), 408–425.

[11] Russell, B., *Principles of Mathematics*, George Allen and Unwin, London, 1903 (2nd edition 1937).

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