Tangled modal logic for topological dynamics

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Abstract

The system S4C combines topological and temporal modalities to reason about dynamical systems. Here we consider enriching its language by generalizing the use of the topological operator to its polyadic 'tangled' interpretation, originally introduced by Dawar and Otto in a different context. We provide an axiomatization for the extended system and show that it is sound and complete. It uses a version of the continuity axiom which we call *tangled continuity* and involves the polyadic use of the topological modality. We also show that the resulting system, $\mathsf{S4C}^*$, is more expressive than S4C; specifically, it is better at distinguishing continuous dynamical systems from discontinuous ones. As a corollary we obtain that the tangled continuity axiom cannot be derived from the other axioms, including the standard continuity axiom.

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1. Introduction

Among the many applications of modal logic we may find its topological interpretation [9], useful for spatial reasoning, and its linear-time interpretation [8], used for reasoning about processes. The two can be combined to obtain a logic for reasoning about dynamic topological systems; these are pairs $\langle X, f\rangle$, where $f : X \to X$ is a continuous function. We use the function f to interpret the temporal modality; it then interacts non-trivially with the topological modality due to continuity, which can be expressed by the formula

$$\Diamond f\psi \to f\Diamond\psi.$$

The end result was the modal logic $\mathsf{S4C}$ [2] and its extension, Dynamic Topological Logic [7].

Some time later, in another context, a different extension of the modal logic $\mathsf{S4}$ was proposed [3]. Here, a polyadic modality is introduced which, over the class of finite $\mathsf{S4}$ models, expresses the existence of clusters satisfying a set of formulas. The motivation was that, over the class of finte $\mathsf{S4}$ models, first-order logic and monadic second-order logic are strictly more expressive than the standard modal language $\mathsf{L}_\Diamond$ but equally expressive to the extended language $\mathsf{L}^*_\Diamond$. The language $\mathsf{L}^*_\Diamond$ is also capable of expressing the property of being simulated by a finite transitive model, unlike $\mathsf{L}_\Diamond$ [5].

Following [6], we will present $\mathsf{L}^*_\Diamond$ as a unimodal 'tangled' system where the modal operator admits finite sets of formulas under its scope. We will use L to refer to the language of $\mathsf{S4C}$ and L^* to refer to the respective polyadic extension.

Our goal is to explore the corresponding extension to $\mathsf{S4C}$, which we call $\mathsf{S4C}^*$. Perhaps the two most obvious questions when introducing such an extension are *Is the extension genuinely more expressive?* and *Does it still have a nice axiomatization?* We answer the two questions in the affirmative. Actually, the answer to the first question follows from known results [3, 5] but here we show that the extension is better, specifically, at characterizing continuity. The answer to the second is given by our axiomatization, which we prove to be complete; essential to this axiomatization is a 'tangled' version of the usual continuity axiom.

The paper has the following layout. Section 2 reviews dynamic topological systems and relates topological spaces to $\mathsf{S4}$ Kripke frames. Section 3 defines the tangled closure operator and gives some of its basic properties; we use this to define semantics for our formal language, as described in Section 4. Section 5 then discusses how the tangled closure interacts with continuous functions.

Section 6 defines our axiomatization of $\mathsf{S4C}^*$, and Section 7 gives an overview of our completeness proof, given in detail in Section 8. Finally, Section 9 shows that L^* is more expressive than L, and indeed that the tangled continuity axiom does not follow from the rest of the axioms, including the standard continuity axiom.

2. Dynamic topological systems

Definition 2.1 (topological space)**.** *A* topological space *is a pair* $\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}} \rangle$ *where* $|\mathfrak{X}|$ *is a set and* $\mathcal{T}_{\mathfrak{X}}$ *a family of subsets of* $|\mathfrak{X}|$ *satisfying*

1. $\varnothing, |\mathfrak{X}| \in \mathcal{T}_{\mathfrak{X}}$*;*
2. *if* $U, V \in \mathcal{T}_{\mathfrak{X}}$ *then* $U \cap V \in \mathcal{T}_{\mathfrak{X}}$ *and*
3. *if* $\mathcal{O} \subseteq \mathcal{T}_{\mathfrak{X}}$ *then* $\bigcup \mathcal{O} \in \mathcal{T}_{\mathfrak{X}}$*.*

The elements of $\mathcal{T}_{\mathfrak{X}}$ *are called* open sets*; the complement of an open set is* closed.

Given a set $A \subseteq X_{\mathfrak{X}}$, its *interior*, denoted A°, is defined by

$$A^{\circ} = \bigcup \{U \in \mathcal{T}_{\mathfrak{X}} : U \subseteq A\}\,.$$

Dually, we define the closure $\overline{A}$ as $|\mathfrak{X}| \setminus (|\mathfrak{X}| \setminus A)^{\circ}$; this is the smallest closed set containing A.

Definition 2.2 (dynamic topological systems)**.** *A* dynamic topological system *is a triple* $\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}}, f_{\mathfrak{X}} \rangle$ *where* $\langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}} \rangle$ *is a topological space and* $f_{\mathfrak{X}} : |\mathfrak{X}| \to |\mathfrak{X}|$ *is continuous.*

The modal logic $\mathsf{S4}$ can be interpreted over topological spaces, but it can also be interpreted over the class of transitive, reflexive Kripke frames. These are pairs $\mathfrak{W} = \langle |\mathfrak{W}|, \preccurlyeq_{\mathfrak{W}} \rangle$ where $\preccurlyeq_{\mathfrak{W}}$ is a preorder on the set $|\mathfrak{W}|$; we will omit the subindex and write simply $\preccurlyeq$ unless this may lead to confusion.

It turns out that preorders can be seen as a special case of topological spaces. For $w \in |\mathfrak{W}|$, define

$$\downarrow w = \{v : v \preccurlyeq w\}\,.$$

Then consider the topology $\mathcal{T}_{\preccurlyeq}$ on $|\mathfrak{W}|$ given by setting $U \subseteq A$ to be open if and only if, whenever $w \in U$, we have $\downarrow w \subseteq U$.[1] A topology of this form is

[1] Equivalently, $\mathcal{T}_{\preccurlyeq}$ is the topology generated by the basis of all sets of the form $\downarrow w$.

a *preorder topology*[2]. It is not hard to check that the Kripke semantics given by $\preccurlyeq$ coincide with the topological semantics given by $\mathcal{T}_{\preccurlyeq}$.

An *Aleksandroff space* is a topological space where *arbitrary* intersections of open sets are open; these were introduced in [1]. Aleksandroff spaces and preorder topologies can be identifed using the the following, well-known result:

Theorem 2.1. *Let X be a set and $\mathcal{T}$ a topology on X. Then, $\langle X, \mathcal{T}\rangle$ is an Aleksandroff space if and only if there exists a preorder $\preccurlyeq$ on X such that $\mathcal{T} = \mathcal{T}_{\preccurlyeq}$.*

The preordered version of a dynamic topological system is a *dynamic Kripke frame*; this is a triple $\mathfrak{W} = \langle |\mathfrak{W}|, \preccurlyeq_{\mathfrak{W}}, f_{\mathfrak{W}}\rangle$ where $|\mathfrak{W}|$ is a set, $\preccurlyeq_{\mathfrak{W}}$ a transitive, reflexive binary relation on $|\mathfrak{W}|$ and $f_{\mathfrak{W}} : |\mathfrak{W}| \to |\mathfrak{W}|$ a function such that $w \preccurlyeq v$ implies that $f(w) \preccurlyeq f(v)$. One can check that this monotonicity condition coincides with the continuity of $f_{\mathfrak{W}}$ with respect to the preorder topology.

We will also use the notation

- $w \prec v$ for $w \preccurlyeq v$ but $v \not\preccurlyeq w$ and
- $w \sim v$ for $w \preccurlyeq v$ and $v \preccurlyeq w$.

We define the *depth* of an element $w \in |\mathfrak{W}|$, denoted $\mathrm{dpt}(w)$, as the largest N such that there exist

$$w_0 \prec w_1 \prec ... \prec w_N = w.$$

3. The tangled closure operator

The polyadic extension to the language of $\mathsf{S4C}$ allows us to express the *tangled closure* of a finite family of sets:

Definition 3.1 (Tangled closure)**.** *Let $\langle X, \mathcal{T}\rangle$ be a topological space and $\mathcal{S} \subseteq 2^X$.*

Given $E \subseteq X$, we say $\mathcal{S}$ is tangled *in E if, for all $A \in \mathcal{S}$, $A \cap E$ is dense in E.*

We define the tangled closure *$\mathcal{S}^*$ of $\mathcal{S}$ to be the union of all sets E such that $\mathcal{S}$ is tangled in E.*[3]

[2]More specifically, this is the 'downset' topology; note that many authors use the 'upset' topology instead, generated by sets of the form $\uparrow w$.

[3]For readers familiar with [5], we have that $\mathcal{S}^* = \overline{\mathcal{S}^\natural}$, and similarly $\mathcal{S}^\natural = \mathcal{S}^* \cap \bigcup \mathcal{S}$.

Proposition 3.1 (Properties of the tangled closure)**.** *Let $\mathfrak{X}$ be a topological space and $\mathcal{S} \subseteq 2^{|\mathfrak{X}|}$. Then,*

1. $\bigcap \mathcal{S} \subseteq \mathcal{S}^*$*,*
2. $\mathcal{S}$ *is tangled in* $\mathcal{S}^*$*,*
3. $\mathcal{S}^*$ *is closed and*
4. *if* $\mathcal{S} = \{A\}$*, then* $\mathcal{S}^* = \overline{A}$*.*

Proof. Let $\mathfrak{X}$ and $\mathcal{S}$ be as in the statement of the Proposition.

1. It suffices to show that $\mathcal{S}$ is tangled in $\bigcap \mathcal{S}$; but, clearly, for every $A \in \mathcal{S}$,

$$\begin{aligned} \bigcap \mathcal{S} &\subseteq A \cap \bigcap \mathcal{S} \\ &\subseteq \overline{A \cap \bigcap \mathcal{S}}, \end{aligned}$$

as needed.

2. Let $A \in \mathcal{S}$; we must show that $A \cap \mathcal{S}^*$ is dense in $\mathcal{S}^*$.

Pick $x \in \mathcal{S}^*$. Then, by definition there exists $E \subseteq \mathcal{S}^*$ such that $\mathcal{S}$ is tangled in E. But this implies that

$$x \in \overline{A \cap E} \subseteq \overline{A \cap \mathcal{S}^*}.$$

Since x was arbitrary, we conclude that $\mathcal{S}^* \subseteq \overline{A \cap \mathcal{S}^*}$, as required.

3. Pick $A \in \mathcal{S}$. We have that

$$\begin{aligned} \mathcal{S}^* &\subseteq \overline{A \cap \mathcal{S}^*} \\ &\subseteq \overline{A \cap \overline{\mathcal{S}^*}}; \end{aligned}$$

since $\overline{\mathcal{S}^*}$ is the smallest closed set containing $\mathcal{S}^*$, it follows that

$$\overline{\mathcal{S}^*} \subseteq \overline{A \cap \overline{\mathcal{S}^*}}.$$

But since $A \in \mathcal{S}$ was arbitrary, it follows that $\mathcal{S}$ is tangled in $\overline{\mathcal{S}^*}$, so that $\overline{\mathcal{S}^*} \subseteq \mathcal{S}^*$, i.e., $\mathcal{S}^*$ is closed.

4. Suppose $\mathcal{S} = \{A\}$.

By the first item,

$$A = \bigcap \mathcal{S} \subseteq \mathcal{S}^*;$$

but since $\mathcal{S}^*$ is closed, it follows that $\overline{A} \subseteq \mathcal{S}^*$.

Meanwhile, if E is any set such that $\{A\}$ is tangled in E, then $A \cap E$, and hence A, is dense in E. It follows that $E \subseteq \overline{A}$; since $\mathcal{S}^*$ is the union of all such E, we conclude that $\mathcal{S}^* \subseteq \overline{A}$. □

4. Syntax and semantics

Our grammar is built from a countable set of propositional variables PV; if $p \in \mathsf{PV}$ and $\varphi_0, ..., \varphi_n$ are formulas then

$$p \qquad \varphi_0 \wedge \varphi_1 \qquad \neg\varphi_0 \qquad f\varphi_0 \qquad \Diamond \{\varphi_0, ..., \varphi_n\}$$

are all formulas.

We will write $\Diamond\gamma$ instead of $\Diamond \{\gamma\}$. We also write $\Box\Gamma$ as a shorthand for $\neg\Diamond\neg\Gamma$, where $\neg\Gamma$ is to be understood as $\{\neg\gamma : \gamma \in \Gamma\}$; similarly, $f\Gamma$ denotes the set $\{f\gamma : \gamma \in \Gamma\}$.

A *dynamic topological model* is a dynamic topological system $\mathfrak{X}$ equipped with a valuation

$$[\![\cdot]\!]_{\mathfrak{X}} : \mathsf{L}^* \to 2^{|\mathfrak{X}|}$$

such that

$$\begin{aligned}
[\![\neg\alpha]\!]_{\mathfrak{X}} &= |\mathfrak{X}| \setminus [\![\alpha]\!]_{\mathfrak{X}} \\
[\![\alpha \wedge \beta]\!]_{\mathfrak{X}} &= [\![\alpha]\!]_{\mathfrak{X}} \cap [\![\beta]\!]_{\mathfrak{X}} \\
[\![f\alpha]\!]_{\mathfrak{X}} &= f^{-1} [\![\alpha]\!]_{\mathfrak{X}} \\
[\![\Diamond \{\varphi_0, ..., \varphi_n\}]\!]_{\mathfrak{X}} &= \{[\![\varphi_0]\!]_{\mathfrak{X}}, ..., [\![\varphi_n]\!]_{\mathfrak{X}}\}^* .
\end{aligned}$$

If Φ is a set of formulas, we may also write $[\![\Phi]\!]_{\mathfrak{X}}$ instead of $\{[\![\varphi]\!]_{\mathfrak{X}} : \varphi \in \Phi\}$. The clause for $\Diamond$ then becomes

$$[\![\Diamond\Gamma]\!]_{\mathfrak{X}} = [\![\Gamma]\!]^*_{\mathfrak{X}} .$$

We will denote the monadic counterpart of L^* by L (that is, the fragment of L^* where $\Diamond$ is only applied to singletons) and the f-free fragment by $\mathsf{L}^*_{\Diamond}$. Accordingly, the standard unimodal language will be denoted $\mathsf{L}_{\Diamond}$.

Over the class of Kripke models, the polyadic modality we are employing coincides with that introduced in [3]. There, expressions of the form $\Diamond^*_{\mathbf{p}}\psi$ are introduced, where $\mathbf{p}$ is a finite set of formulas, and then $\Diamond^*(\gamma_0, ..., \gamma_n)$ is suggested as a notational variant; we use the latter version[4]. Then we make the following modifications:

1. Given that over the class of $\mathsf{S4}$ models, the standard $\Diamond\psi$ is equivalent to $\Diamond^*(\psi)$, we drop the monadic modality and use only the polyadic $\Diamond^*$. Note that over models that fail to be reflexive, $\Diamond\psi$ and $\Diamond^*(\psi)$ are not equivalent.
2. Since we no longer need to distinguish between the unary and polyadic operators, we write $\Diamond\Gamma$ instead of $\Diamond^*\Gamma$.

This presentation will allow us to give monadic and polyadic operations a more uniform treatment and make complex formulas easier to read. However, while [3] does not describe the interpretation of $\Diamond^*\Gamma$ in topological terms, over Kripke models it is identical to our interpretation of $\Diamond\Gamma$.

The polyadic operator is also ver close to that used in [5]. More specifically, the formula $\natural\Gamma$ from [5] would become $\bigvee\Gamma \wedge \Diamond\Gamma$ in our notation; alternately one can define $\Diamond\Gamma$ in terms of $\natural\Gamma$ by $\Diamond\natural\Gamma$.

We remark that $\mathsf{L}^*_\Diamond$ is strictly more expressive than the standard modal language $\mathsf{L}_\Diamond$ over the class of transitive, reflexive Kripke models [3, 5].

4.1. Kripke semantics for the tangled closure

We will mainly be dealing with Kripke models in this paper, so it is convenient to characterize the tangled closure operation there. Fortunately, it turns out to be much more transparent than in the case of arbitrary topological models, especially when considering *finite* models.

Suppose that $\mathfrak{W}$ is a dynamic Kripke model and $w \in |\mathfrak{W}|$ satisfies

$$\Diamond\{\gamma_0, ..., \gamma_{n-1}\}.$$

Note that $[\![\gamma_0 \wedge \Diamond\Gamma]\!]_{\mathfrak{W}}$ is dense in $[\![\Diamond\Gamma]\!]_{\mathfrak{W}}$ (because $[\![\Gamma]\!]_{\mathfrak{W}}$ is tangled in $[\![\Gamma]\!]^*_{\mathfrak{W}}$); thus any neighborhood of w contains a point satisfying $\gamma_0 \wedge \Diamond\Gamma$. In particular, we can find such $w_0 \in \downarrow w$ satisfying this formula, i.e., $w_0 \preccurlyeq w$.

[4] Given that the operator is commutative, we may use sets or sequences of formulas indistinctly.

By the same argument, we can then find $w_1 \preccurlyeq w_0$ satisfying $\gamma_1 \wedge \Diamond\Gamma$, since w_0 also satisfies $\Diamond\Gamma$. Continuing in this fashion we can construct a sequence

$$w_0 \succcurlyeq w_1 \succcurlyeq w_2 \succcurlyeq \ldots$$

such that w_i satisfies γ_j if and only if $i \equiv j \pmod{n}$.

Conversely, if such a sequence exists it is evident that it satisfies the fixpoint property for the tangled closure, and thus we have:

Proposition 4.1. *Given a dynamic Kripke model $\langle W, \preccurlyeq, f, [\![\cdot]\!]\rangle$, $w \in W$ and a finite set of formulas Γ, w satisfies $\Diamond\Gamma$ if and only if there is a sequence*

$$w \succcurlyeq w_0 \succcurlyeq w_1 \succcurlyeq \ldots$$

such that each γ_i is satisfied by w_j for infinitely many values of j.

Note that it is not required that the worlds of such a sequence be distinct; in fact, over a finite model, it is necessary that some world is repeated infinitely often. This allows us to obtain an even simpler characterization of $[\![\Diamond\Gamma]\!]$ over finite models.

Recall that a *cluster* in a Kripke frame $\mathfrak{W}$ is a set $C \subseteq |\mathfrak{W}|$ such that, for all $v, w \in C$, $v \sim w$; every world w belongs to a unique cluster, which we denote $[w]$.

Lemma 4.1. *If $\mathfrak{W}$ is a finite dynamic Kripke model and $w \in |\mathfrak{W}|$, $w \in [\![\Diamond\Gamma]\!]_{\mathfrak{W}}$ if and only if there is $v \preccurlyeq w$ such that, for all $\gamma \in \Gamma$ there is $u \sim v$ with $u \in [\![\gamma]\!]_{\mathfrak{W}}$.*

Proof. If $w \in [\![\Diamond\Gamma]\!]_{\mathfrak{W}}$, pick $v \preccurlyeq w$ such that v is minimal among all worlds satisfying $\Diamond\Gamma$. Then, $[\![\Gamma]\!]_{\mathfrak{W}}$ is tangled in $[\![\Diamond\Gamma]\!]_{\mathfrak{W}}$, so for every $\gamma \in \Gamma$ there is $u \preccurlyeq v$ with $v \in [\![\gamma]\!]_{\mathfrak{W}}$; by minimality this implies that $u \sim v$.

For the other direction, if there is $v \preccurlyeq w$ such that, for all $\gamma \in \Gamma$ there is $u \sim v$ with $v \in [\![\gamma]\!]_{\mathfrak{W}}$, it is clear that $[\![\Gamma]\!]_{\mathfrak{X}}$ is tangled in $\{w\} \cup [v]$ so $\{w\} \cup [v] \subseteq [\![\Diamond\Gamma]\!]_{\mathfrak{W}}$ and w satisfies $\Diamond\Gamma$. □

Note that the above characterizations do not involve the function $f_{\mathfrak{W}}$ and, indeed, apply to all $\mathsf{S4}$ Kripke models.

5. Tangled continuity

Suppose that $\mathfrak{X}$ is a dynamic topological system. Then, the continuity of $f_{\mathfrak{X}}$ can be expressed by the formula

$$\Diamond f\psi \to f\Diamond\psi,$$

as is shown in [2]. However, it turns out that this is not sufficient to axiomatize the polyadic $\mathsf{S4C}^*$, unless we generalize to a 'tangled' version. Fortunately, it has much the same form as its monadic counterpart.

Below, $f\Gamma$ denotes the set $\{f\gamma : \gamma \in \Gamma\}$.

Lemma 5.1. *Let Γ be a finite set of formulas.*

Then,

$$\Diamond f\Gamma \to f\Diamond\Gamma$$

is valid over the class of dynamic topological models.

Proof. It suffices to show that, given a dynamic topological model $\mathfrak{X}$, the set $\llbracket\Gamma\rrbracket_{\mathfrak{X}}$ is tangled in $f_{\mathfrak{X}}\llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}$; then, whenever x satisfies $\Diamond f\Gamma$, $f_{\mathfrak{X}}(x)$ satisfies $\Diamond\Gamma$ and thus x satisfies $f\Diamond\Gamma$.

So, suppose that $\gamma \in \Gamma$ and $y = f_{\mathfrak{X}}(x)$ for some $x \in \llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}$. Pick any neighborhood U of y; we must show that it contains points in $\llbracket\gamma\rrbracket_{\mathfrak{X}} \cap f_{\mathfrak{X}}\llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}$.

Because $f_{\mathfrak{X}}$ is continuous, there is a neighborhood V of x such that $f_{\mathfrak{X}}V \subseteq U$.

But

$$x \in \overline{\llbracket f\gamma\rrbracket_{\mathfrak{X}} \cap \llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}},$$

so there is $z \in V$ such that $z \in \llbracket f\gamma\rrbracket_{\mathfrak{X}} \cap \llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}$.

Then,

$$f_{\mathfrak{X}}(z) \in \llbracket\gamma\rrbracket_{\mathfrak{X}} \cap f_{\mathfrak{X}}\llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}} \cap U;$$

since U was arbitrary, this shows that $\llbracket\Gamma\rrbracket_{\mathfrak{X}}$ is tangled in $f_{\mathfrak{X}}\llbracket\Diamond f\Gamma\rrbracket_{\mathfrak{X}}$, and hence $y \in \llbracket\Diamond\Gamma\rrbracket_{\mathfrak{X}}$. It follows that x satisfies $f\Diamond\Gamma$, as desired. $\square$

6. Polyadic $\mathsf{S4C}$

Our proposed axiomatization for $\mathsf{S4C}^*$ consists of the following:

Taut All propositional tautologies.

Axioms for $\Diamond$:

- K $\Box(p \to q) \to (\Box p \to \Box q)$
- T $\bigwedge \Gamma \to \Diamond\Gamma$
- $\mathsf{4}$ $\Diamond\Diamond\Gamma \to \Diamond\Gamma$
- Fix $\Diamond\Gamma \to \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge \Diamond\Gamma)$
- Ind Induction for $\Diamond$:

$$p \wedge \Box\Big(p \to \bigwedge_{\gamma\in\Gamma} \Diamond(p \wedge \gamma)\Big) \to \Diamond\Gamma.$$

Axioms for f:

- Neg_f $\neg f p \leftrightarrow f \neg p$
- And_f $f(p \wedge q) \leftrightarrow fp \wedge fq$

Tangled continuity

- TC $\Diamond f\Gamma \to f\Diamond\Gamma$

Rules:

- MP Modus ponens
- Subs Substitution
- N Necessitation for $\Box, f$.

We call the resulting logic $\mathsf{S4C}^*$. We claim that the axiomatizaion is sound for the class of dynamic topological models:

Theorem 6.1. *If φ is a theorem of $\mathsf{S4C}^*$ and $\mathfrak{X}$ is any dynamic topological model, then $[\![\varphi]\!]_{\mathfrak{X}} = |\mathfrak{X}|$.*

Proof. Many of the rules and axioms are standard, but others require more attention. Below, let $\mathfrak{X}$ be any dynamic topological logic and $\varphi \in \mathsf{L}^*$.

Axioms for $\Diamond$:

- T This is valid by Proposition 3.1.1.

4 We have that $[\![\Diamond\Diamond\Gamma]\!]_{\mathfrak{X}} = \{[\![\Diamond\Gamma]\!]\}^*_{\mathfrak{X}}$. By Proposition 3.1.4, the latter is equal to $\overline{[\![\Diamond\Gamma]\!]_{\mathfrak{X}}}$; but then, by Proposition 3.1.3, $[\![\Diamond\Gamma]\!]_{\mathfrak{X}}$ is closed, so that

$$[\![\Diamond\Diamond\Gamma]\!]_{\mathfrak{X}} = \overline{[\![\Diamond\Gamma]\!]_{\mathfrak{X}}} \subseteq [\![\Diamond\Gamma]\!]_{\mathfrak{X}}\,.$$

Fix This axiom expresses the fact that $[\![\Gamma]\!]_{\mathfrak{X}}$ is tangled in $[\![\Diamond\Gamma]\!]_{\mathfrak{X}}$, which follows from Porposition 3.1.2.

Ind If $x \in |\mathfrak{X}|$ satisfies $p \wedge \Box\Big(p \to \bigwedge_{\gamma\in\Gamma} \Diamond(p \wedge \gamma)\Big)$, there is a neighborhood U of x such that, given $\gamma \in \Gamma$,

$$[\![p]\!]_{\mathfrak{X}} \cap U \subseteq \overline{[\![\gamma]\!]_{\mathfrak{X}} \cap [\![p]\!]_{\mathfrak{X}}};$$

since U is open, this implies that

$$[\![p]\!]_{\mathfrak{X}} \cap U \subseteq \overline{[\![\gamma]\!]_{\mathfrak{X}} \cap [\![p]\!]_{\mathfrak{X}} \cap U}.$$

But the latter means that $[\![\Gamma]\!]_{\mathfrak{X}}$ is tangled in $[\![p]\!]_{\mathfrak{X}} \cap U$, and $[\![p]\!]_{\mathfrak{X}} \cap U \subseteq [\![\Gamma]\!]^*_{\mathfrak{X}}$.

But $x \in [\![p]\!]_{\mathfrak{X}} \cap U$, so $x \in [\![\Gamma]\!]^*_{\mathfrak{X}}$, as required.

Axioms for f: These axioms express functionality and are fairly standard.

Tangled continuity: This is Lemma 5.1.

Rules: All rules are standard and we skip them. □

7. A sketch of the completeness proof

In this section, we shall give an informal sketch illustrating the steps we follow throughout the completeness proof. These ideas are developed more formally in subsequent sections, but we believe a quick overview should give the reader a clear notion of the general strategy.

Given a consistent formula φ, our goal is to construct a finite dynamic Kripke model satisfying it. We do this in five steps:

1. First, φ is 'saturated' into a finite set of formulas Φ. We call Φ a *φ-type*. The objective of Φ is to determine the truth values of subformulas of φ. We must also determine the truth value of some additional formulas which we call 'generalized subformulas', but the set of generalized subformulas of φ is always finite.

2. We then extract the 'modal part' of Φ, which we denote $\Phi^{\Diamond\Box}$; these are all formulas in Φ of the form $\Diamond\Gamma$ or $\Box\Gamma$.

With this we construct a cluster $\mathcal{C}$ satisfying some $\Diamond\Gamma_0 \in \Phi^{\Diamond\Box}$, i.e. a set of φ-types such that, for each $\gamma \in \Gamma_0$, there is some $\Psi \in \mathcal{C}$ with $\gamma \in \Psi$. In doing so, we must make sure that none of the formulas of the form $\Box\Delta \in \Phi^{\Diamond\Box}$ become false. Looking at the semantics of $\Box\Delta$, this means that at least one $\delta \in \Delta$ must be true in all $\Psi \in \mathcal{C}$.

3. Each φ-type $\Psi \in \mathcal{C}$ contains enough information to determine which formulas would be satisfied by its temporal successor (or, more properly, which formulas would be satisfied by the temporal successor of a world w satisfying Ψ). Thus we can 'unwind' $\mathcal{C}$ into a sequence

$$\mathcal{C} = \mathcal{D}_0, \mathcal{D}_1, ..., \mathcal{D}_n,$$

where n depends on the temporal nesting depth of φ and such that there are functions $g_i : \mathcal{D}_i \to \mathcal{D}_{i+1}$ which can be used to interpret the temporal modality. We think of these as a sequence of clusters in a larger dynamic Kripke model.

4. Because we have already satisfied $\Diamond\Gamma_0$, we can assume inductively that each

$$\Diamond\Sigma \in \Phi^{\Diamond\Box} \setminus \{\Diamond\Gamma_0\} = \Theta$$

can be satisfied in some Kripke model $\mathfrak{V}_\Sigma$, given that Θ has less formulas of the form $\Diamond\Gamma$ than $\Phi^{\Diamond\Box}$. For this we must choose Γ_0 appropriately (and, more generally, $\mathcal{C}$ may have to satisfy several formulas of the form $\Diamond\Gamma \in \Phi^{\Diamond\Box}$ at once).

Meanwhile, if $\Diamond\Sigma \in \Psi \in \mathcal{D}_i$ for some $i > 0$, once again we can assume inductively (this time by induction on *temporal depth*) that there is a model $\mathfrak{V}_\Sigma$ satisfying $\Diamond\Sigma$.

Thus we can 'paste' all of the models $\mathfrak{V}_\Sigma$ onto the clusters $\mathcal{D}_0, ...\mathcal{D}_n$ to obtain a model $\mathfrak{W}$ satisfying $\Phi^{\Diamond\Box}$. For this we will use the operation $\oplus$ given by Definition 8.7.

5. Finally, we add a root w_0 satisfying Φ to $\mathfrak{W}$. The end result is a dynamic Kripke model which satisfies φ.

Having done this we conclude that any consistent formula is satisfiable, or dually, that any valid formula is derivable.

8. Completeness

In this section we will carry out our completenss proof in full detail. Before constructing genuine models we will construct 'typed models', which are essentially models which include additional syntactic information.

We define a *type* to be a finite set of formulas. For a type Φ, define $\mathrm{gsub}(\Phi)$ as the smallest set such that

- $\Phi \subseteq \mathrm{gsub}(\Phi)$
- if $\varphi \in \mathrm{gsub}(\Phi)$ and ψ is a subformula of φ, then $\psi \in \mathrm{gsub}(\Phi)$
- if $\psi \in \mathrm{gsub}(\Phi)$ does not begin with a negation, then $\neg\psi \in \mathrm{gsub}(\Phi)$
- if $f\varphi \in \mathrm{gsub}(\Phi)$ and ψ is a subformula of φ, then $f\psi \in \mathrm{gsub}(\Phi)$ and
- if $f\Box\Gamma \in \mathrm{gsub}(\Phi)$ then $\Box f\Gamma \in \mathrm{gsub}(\Phi)$ and $\Box f\Box\Gamma \in \mathrm{gsub}(\Phi)$.

It is to be understood that all $\gamma \in \Gamma$ are subformulas of $\Diamond\Gamma$. Note that $\mathrm{gsub}(\Phi)$ is finite provided that Φ is finite.

Definition 8.1 (Saturation)**.** *Say a set of formulas* Φ *is* saturated *if , whenever* $\varphi \in \Phi$ *and* $\psi \in \mathrm{gsub}(\Phi)$ *does not begin with a negation, either* $\psi \in \Phi$ *or* $\neg\psi \in \Phi$*.*

Ψ *is a* saturation *of* Φ *if* Ψ *is saturated and* $\Psi \subseteq \mathrm{gsub}(\Phi)$*. We denote the set of saturations of* Φ *by* $\mathrm{sat}(\Phi)$*, and the set of consistent saturations of* Φ *by* $\mathrm{cons}(\Phi)$*.*

In what follows and throughout the text we use $\Theta \vdash \varphi$ merely as a shorthand for $\vdash \bigwedge \Theta \to \varphi$.

Lemma 8.1. *If* Φ *is a consistent type, then* Φ *has a consistent saturation.*

Futher,

$$\Phi \vdash \bigvee_{\Psi \in \mathrm{cons}(\Phi)} \bigwedge \Psi.$$

Proof. By propositional reasoning we have that

$$\Phi \vdash \bigvee_{\Psi \in \mathrm{sat}(\Phi)} \bigwedge \Psi;$$

if Ψ is inconsistent, $\vdash \neg \bigwedge \Psi$, so again by propositional reasoning we can remove all such disjuncts and obtain

$$\Phi \vdash \bigvee_{\Psi \in \mathrm{cons}(\Phi)} \bigwedge \Psi.$$

But then $\mathrm{cons}(\Phi)$ must be non-empty, otherwise it would follow that Φ is inconsistent. $\square$

We define the *duration* of a formula φ as the nesting depth of f occurring in φ; if Φ is a finite set of formulas, the duration of Φ is the greatest duration of a formula $\varphi \in \Phi$.

Definition 8.2 (Unwinding). *Given a type Φ of duration I, we define its 'unwinding' to be the sequence of types* $\mathrm{uw}(\Phi) = \left\langle \Phi^f_i \right\rangle_{i<I}$ *with $\Phi^f_0 = \Phi$ and*

$$\Phi^f_{i+1} = \left\{ \varphi : f\varphi \in \Phi^f_i \right\}.$$

Clearly, $\Phi^f_{i+1} = (\Phi^f_i)^f_1$. The general idea is that if a world w in a dynamic topological model $\mathfrak{X}$ satisfies Φ, then $f^i_{\mathfrak{X}}(x)$ satisfies Φ^f_i.

Lemma 8.2 (Unwinding lemma). *If Φ is a saturated consistent type with duration I and $i \leq I$, then Φ^f_i is a saturated consistent type with duration $I - i$.*

Proof. Suppose that Φ is saturated and consistent.

To see that Φ^f_i is saturated for all i, we proceed by induction on i, assuming inductively that Φ^f_i is saturated and consistent.

Let us first check that Φ^f_{i+1} is saturated. Suppose that $\neg\varphi \notin \Phi^f_{i+1}$. Then, $f\neg\varphi \notin \Phi^f_i$, and hence $\neg f\varphi \notin \Phi^f_i$ (since $\vdash \neg f\varphi \to f\neg\varphi$ and Φ^f_i is saturated and consistent, therefore closed under derivability). It follows that $f\varphi \in \Phi^f_i$ and thus $\varphi \in \Phi^f_{i+1}$.

Now, if Φ^f_{i+1} were inconsistent, we would have $\vdash \neg \bigwedge \Phi^f_{i+1}$, so that $\vdash f\neg \bigwedge \Phi^f_{i+1}$ (by necessitation for f).

Applying Neg_f and And_f, we also have that $\vdash \neg \bigwedge f\Phi^f_{i+1}$. But $f\Phi^f_{i+1} \subseteq \Phi^f_i$, which would make Φ^f_i inconsistent, contradicting our induction hypothesis. $\square$

Given a set of formulas Φ, we write $\Phi^{\Diamond}$ for the set of formulas in Φ of the form $\Diamond\Gamma$, $\Phi^{\Box}$ for the subset of formulas of the form $\Box\Gamma$ and $\Phi^{\Diamond\Box}$ for the set of formulas of Φ either of the form $\Diamond\Gamma$ or $\Box\Gamma$. We say a type Φ is *modal* if $\Phi = \Phi^{\Diamond\Box}$, and *saturated modal* if $\Phi = \Psi^{\Diamond\Box}$ for some saturated Ψ.

In order to define typed models, it is useful to introduce the notion of a *task.*

Definition 8.3. *Say a type of the form*

$$\Sigma = \{\Diamond\Gamma\} \cup \{\Box\Delta : \Delta \in \mathcal{D}\}$$

is a task.

A set $\mathcal{C}$ of saturations of Σ realizes Σ *if*

- $\Gamma \subseteq \bigcup \mathcal{C}$*;*
- *if $\Psi, \Theta \in \mathcal{C}$ then $\Psi^{\Diamond\Box} = \Theta^{\Diamond\Box}$ and*
- *for all $\Delta \in \mathcal{D}$, $\neg\Delta \not\subseteq \bigcup \mathcal{C}$.*

With this we can now define our typed models:

Definition 8.4 (typed model)**.** *We define a* typed model *of duration I as a structure*

$$\mathfrak{w} = \langle |\mathfrak{w}|, \preccurlyeq_{\mathfrak{w}}, f_{\mathfrak{w}}, t_{\mathfrak{w}} \rangle$$

where $\langle |\mathfrak{w}|, \preccurlyeq_{\mathfrak{w}} \rangle$ is a finite preorder with $|\mathfrak{w}| \neq \varnothing$ and

1. $|\mathfrak{w}| = \coprod_{i \leq I} |\mathfrak{w}|_i$*, with each $|\mathfrak{w}|_i$ open*
2. *$f_{\mathfrak{w}} : |\mathfrak{w}| \to |\mathfrak{w}|$ is continuous, for $i < I$, $f_{\mathfrak{w}}|\mathfrak{w}|_i \subseteq |\mathfrak{w}|_{i+1}$, and $f_{\mathfrak{w}} \upharpoonright |\mathfrak{w}|_I$ is the identity*
3. *$t_{\mathfrak{w}}$ is a function assigning a consistent saturated type to each $w \in |\mathfrak{w}|$*
4. *for all $w \in |\mathfrak{w}|$,*
$$t_{\mathfrak{w}}(f_{\mathfrak{w}}(w)) = (t_{\mathfrak{w}}(w))^f_1$$
5. *for every $w \in |\mathfrak{w}|$ and $\Diamond\Gamma \in t_{\mathfrak{w}}(w)$ we have that either $\{\Diamond\Gamma\} \cup t^{\Box}_{\mathfrak{w}}(w)$ is realized by $t_{\mathfrak{w}}([w])$ or else[5] there is $v \prec_{\mathfrak{w}} w$ such that $\Diamond\Gamma \in t_{\mathfrak{w}}(v)$*
6. *if $\Box\Delta \in t_{\mathfrak{w}}(w)$, $\Diamond\neg\Delta$ is not realized by $t_{\mathfrak{w}}([w])$ and, for all $v \preccurlyeq_{\mathfrak{w}} w$, $\Box\Delta \in t_{\mathfrak{w}}(w)$.*

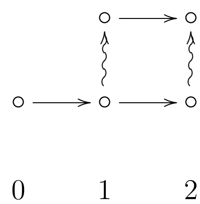

Figure 1: A fairly simple typed model $\mathfrak{w}$. Each circle is a world which would be assigned a specific type, not shown in the diagram. Squiggly arrows indicate $\preccurlyeq$, straigth arrows f. Note that f is the identity on $|\mathfrak{w}|_2$ (by definition), so there are implicit reflexive arrows on the worlds at time 2. There are also implicit reflexive $\preccurlyeq$-arrows on all worlds.

Note that we do not require that any specific $|\mathfrak{w}|_i$ be non-empty; this will be essential since many times we will wish for typed models to 'begin late'. Note, however, that if $|\mathfrak{w}|_i$ is non-empty, then so is $|\mathfrak{w}|_{i+1}$.

A typed model $\mathfrak{w}$ *begins* at J if J is the least value for which $|\mathfrak{w}|_J \neq \varnothing$.

A *rooted typed model* is a typed model $\mathfrak{w}$ of duration I beginning at J with a designated world $0_{\mathfrak{w}} \in |\mathfrak{w}|_J$ such that, for all $i \in [J, I]$ and $w \in |\mathfrak{w}|_i$, $w \preccurlyeq f^i_{\mathfrak{w}}(0_{\mathfrak{w}})$.

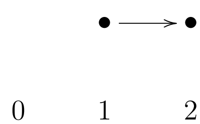

Figure 2: An even simpler typed model $\mathfrak{v}$. Note that there are no worlds at time zero, but this is fine since functionality is not violated.

A typed model gives rise to a model $\dot{\mathfrak{w}}$ in the obvious way, by setting

$$[\![p]\!]_{\dot{\mathfrak{w}}} = \{w \in |\mathfrak{w}| : p \in t_{\mathfrak{w}}(w)\}\,.$$

We then have that:

Lemma 8.3. *If $\mathfrak{w}$ is a typed model, $w \in |\mathfrak{w}|$ and φ is any formula, $\varphi \in t_{\mathfrak{w}}(w)$ implies that $w \in [\![\varphi]\!]_{\dot{\mathfrak{w}}}$.*

Proof. We omit the proof, which proceeds by a standard induction on formulas. □

[5] If $V \subseteq |\mathfrak{w}|$, $t_{\mathfrak{w}}(V)$ denotes the set of all $t_{\mathfrak{w}}(v)$ with $v \in V$.

So in order to show that a formula is satisfiable, it suffices to construct a typed model that satisfies it.

Often we will want to construct a typed models from smaller pieces. Here we define the basic operation we will use to do this, and establish the conditions that the pieces must satisfy.

Definition 8.5 (Cluster path)**.** *A* cluster path *of duration I is a sequence $\vec{\mathcal{P}} = \langle \mathcal{P}_i \rangle_{i \le I}$ of saturated, consistent types such that if $\Psi \in \mathcal{P}_i$ then $\Psi_1^f \in \mathcal{P}_{i+1}$ and all elements of $\mathcal{P}_{i+1}$ are of this form.*

$$\{\Phi, \Psi\} \longrightarrow \left\{\Phi_1^f, \Psi_1^f\right\} \longrightarrow \left\{\Phi_2^f, \Psi_2^f\right\}$$

$$0 \qquad\qquad\qquad 1 \qquad\qquad\qquad 2$$

Figure 3: A cluster path. The moment in time is indicated by the numbers at the bottom.

Given a set of types $\mathcal{G}$, we extend the use of the unwinding operator uw and define uw$(\mathcal{G})$ to be the cluster path $\vec{\mathcal{P}}$ with $\mathcal{P}_i = \left\{\Psi_i^f : \Psi \in \mathcal{G}\right\}$.

Cluster paths behave well with respect to realizability provided that their initial cluster does, in the sense described below:

Definition 8.6 (Correctness)**.** *A cluster $\mathcal{G}$ is* correct if

1. *for any $\Psi, \Theta \in \mathcal{G}$, $\Psi^{\Diamond\Box} = \Theta^{\Diamond\Box}$ and*
2. *for any $\Box\Delta \in \bigcup \mathcal{P}_0$, $\neg\Delta \not\subseteq \bigcup \mathcal{G}$, i.e. $\mathcal{G}$ does not realize $\Diamond\neg\Delta$.*

A cluster path $\vec{\mathcal{P}}$ is correct *if all $\mathcal{P}_i$ are correct.*

Lemma 8.4. *Suppose $\vec{\mathcal{P}}$ is a cluster path such that $\mathcal{P}_0$ is correct and each $\Psi \in \mathcal{P}_0$ is saturated and consistent.*

Then, $\vec{\mathcal{P}}$ is correct as well.

Proof. First we must show that $\Psi^{\Diamond\Box}$ is constant on each $\mathcal{P}_i$.

We do this by induction on i. Suppose that $\Box\Delta \in \Phi_{i+1}^f \in \mathcal{P}_{i+1}$ and choose any $\Psi_{i+1}^f \in \mathcal{P}_{i+1}$; we must show that $\Box\Delta \in \Psi_{i+1}^f$ as well.

By definition $f\Box\Delta \in \Phi_i^f$, and since Φ_i^f is saturated and consistent (Lemma 8.2), $\Box f\Box\Delta \in \Phi_i^f$, from which it follows inductively that $\Box f\Box\Delta \in \Psi_i^f$ and thus $f\Box\Delta \in \Psi_i^f$. But then $\Box\Delta \in \Psi_{i+1}^f$, as desired.

Next we show that, given $\Box\Delta \in \Psi \in \mathcal{P}_{i+1}$, $\Diamond\neg\Delta$ is not realized by $\mathcal{P}_{i+1}$.

Assume that $\Psi = \Phi^f_1$, with $\Phi \in \mathcal{P}_i$.

Towards a contradiction, suppose that for each $\delta \in \Delta$ there is $\Psi_\delta \in \mathcal{P}_{i+1}$ such that $\neg\delta \in \Psi_\delta$. We can write $\Psi_\delta = (\Phi_\delta)^f_1$ for some $\Phi_\delta \in \mathcal{P}_i$.

Then, for each δ, $f\Box\Delta \in \Phi_\delta$, so $\Box f\Delta \in \Phi_\delta$, again because Φ_δ is closed under derivability within $\mathrm{gsub}(\Phi_\delta)$.

Meanwhile, $f\neg\delta \in \Phi_\delta$, so that $\neg f\delta \in \Phi_\delta$ as well. But this shows that $\mathcal{P}_i$ realizes $\Diamond\neg f\Delta$, contradicting the assumption that $\mathcal{P}_i$ does not realize any incorrect tasks.

We conclude that $\mathcal{P}_{i+1}$ cannot have realized an incorrect task, hence it is correct, as desired. $\Box$

Definition 8.7. *Let $\vec{\mathcal{P}}$ be a cluster path of duration D and $\vec{\mathfrak{v}} = \langle \mathfrak{v}^n \rangle_{n<N}$ a sequence of typed models of duration D.*

Define a structure $\mathfrak{w} = \vec{\mathcal{P}} \oplus \vec{\mathfrak{v}}$ by setting[6]

- *for $i < D$,*
$$|\mathfrak{w}|_i = \mathcal{P}_i \cup \coprod_{n<N} |\mathfrak{v}^n|_i;$$
- *$w \preccurlyeq_{\mathfrak{w}} v$ if either $w \in \mathcal{P}_i$ and $v \in |\mathfrak{w}|_i$ or $w, v \in |\mathfrak{v}^n|$ and $w \preccurlyeq_{\mathfrak{v}^n} v$;*
- $f_{\mathfrak{w}}(w) = \begin{cases} w^f_1 & \text{if} \quad w \in \mathcal{P}_i \\ f_{\mathfrak{v}^n}(w) & \text{if} \quad w \in |\mathfrak{v}^n| \end{cases}$
- $t_{\mathfrak{w}}(u) = \begin{cases} u & \text{if} \quad u \in \mathcal{P}_i \\ t_{\mathfrak{v}^n}(u) & \text{if} \quad u \in |\mathfrak{v}^n|. \end{cases}$

Not all constructions of the form $\vec{\mathcal{P}} \oplus \vec{\mathfrak{v}}$ yield typed models; for this they must satisfy the following condition:

Definition 8.8 (coherence)**.** *Let $\vec{\mathcal{P}}$ be a cluster path and $\vec{\mathfrak{v}} = \langle \mathfrak{v}^n \rangle_{n<N}$ a sequence of rooted typed models. Denote the root of $\mathfrak{v}^n$ by 0_n.*

The pair $\left\langle \vec{\mathcal{P}}, \vec{\mathfrak{v}} \right\rangle$ is coherent *if*

1. *$\vec{\mathcal{P}}$ is correct,*
2. *whenever $\Diamond\Gamma \in \bigcup \mathcal{P}_i$, either $\Gamma \subseteq \bigcup \mathcal{P}_i$ or there is $n < N$ such that $\mathfrak{v}^n$ begins on i and $\Diamond\Gamma \in t_{\mathfrak{v}^n}(0_n)$,*

[6]Note that we are generalizing the use of $\oplus$ from [4].

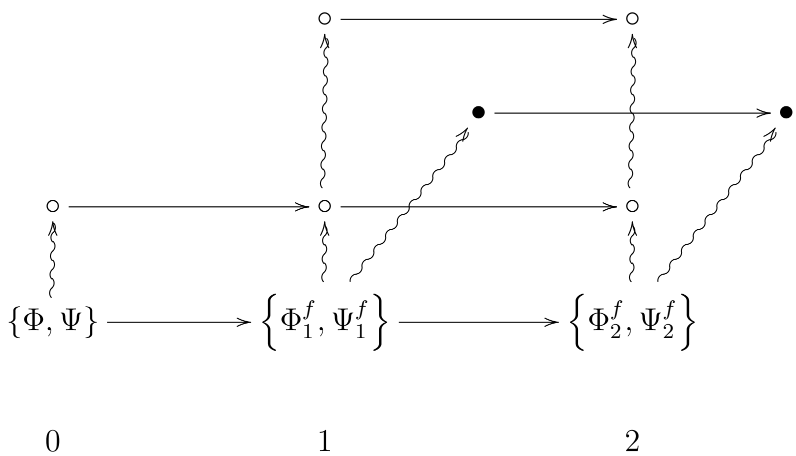

Figure 4: The typed model $\left\langle \vec{\mathcal{P}} \oplus \{\mathfrak{w}, \mathfrak{v}\} \right\rangle$. The elements are as defined in Figures 1,2,3.

3. *if* $\mathfrak{v}^n$ *begins at* J *and* $\Box\Delta \in \bigcup \mathcal{P}_J$*, then* $\Box\Delta \in t_{\mathfrak{v}^n}(0_n)$*.*

The notion of coherence is useful because of the following:

Lemma 8.5. *Let* $\vec{\mathcal{P}}$ *be a cluster path and* $\vec{\mathfrak{v}}$ *a sequence of rooted typed models. Then,* $\vec{\mathcal{P}} \oplus \vec{\mathfrak{v}}$ *is a typed model whenever* $\left\langle \vec{\mathcal{P}}, \vec{\mathfrak{v}} \right\rangle$ *is coherent.*

Proof. We will not give a full proof; mainly one must check that $\mathfrak{w} = \vec{\mathcal{P}} \oplus \vec{\mathfrak{v}}$ satisfies Definition 8.4.

The most involved property to verify is Definition 8.4.6, particularly when $w \in \mathcal{P}_i$ for some i.

Suppose, then, that $w \in \mathcal{P}_i$.

First we must check that, if $\Box\Delta \in t_{\mathfrak{w}}(w)$, $t_{\mathfrak{w}}([w])$ does not realize $\Diamond\neg\Delta$. However, this follows directly from Lemma 8.4.

Next we show that, if $v \preccurlyeq w$ and $\Box\Delta \in t_{\mathfrak{w}}(w)$, then $\Box\Delta \in t_{\mathfrak{w}}(v)$.

If v is also an element of $\mathcal{P}_i$, we can use Lemma 8.4 once again to see that $\Box\Delta \in t_{\mathfrak{w}}(v)$, since $\mathcal{P}_i$ is correct.

Otherwise, $v \in |\mathfrak{v}^n|_i$ for some n. Suppose that $\mathfrak{v}^n$ begins at time J.

Here we use induction on i, or more precisely $i - J$. The case for $i = J$ is already covered by Definition 8.5.3, since $v \preccurlyeq 0_n$ and $\Box\Delta \in t_{\mathfrak{w}}(0_n)$.

For the inductive step, suppose that $\Box\Delta \in \Psi \in \mathcal{P}_{i+1}$. Then, because $\vec{\mathcal{P}}$ is a cluster path we have that $\Psi = \Phi_1^f$ for some $\Phi \in \mathcal{P}_i$, and hence $f\Box\Delta \in \Phi$. This implies that $\Box f\Box\Delta \in \Phi$, so every world in $|\mathfrak{v}^n|_i$ satisfies $\Box f\Box\Delta$ (by induction hypothesis); in particular, the root[7] r of $|\vec{\mathfrak{v}}|_i$ satisfies this formula, and thus also $f\Box\Delta$. It follows that the root $f_{\mathfrak{v}^n}(r)$ of $|\vec{\mathfrak{v}}|_{i+1}$ satisfies $\Box\Delta$; but then $v \preccurlyeq f_{\mathfrak{v}^n}(r)$, and since $\mathfrak{v}^n$ is a typed model, $\Box\Delta \in t_{\mathfrak{w}}(r)$, which is what we needed.

We will skip the other properties, which are more straightforward. $\Box$

From here on, our goal is to show that consistent types have typed models. Whenever possible, we will isolate reasoning that can be done within S4 to take advantage of the familiar completeness result for this logic; an example of this is the following lemma.

Lemma 8.6. *If $\mathcal{G}$ is a finite set of types and ψ is any formula, then*

$$\begin{aligned} \vdash \psi \wedge \Box(\psi \to \textstyle\bigvee_{\Gamma\in\mathcal{G}} \bigwedge_{\gamma\in\Gamma} \Diamond\gamma) \\ \to \textstyle\bigvee_{\Gamma\in\mathcal{G}} \Diamond(\psi \wedge \Box(\psi \to \bigwedge_{\gamma\in\Gamma} \Diamond\gamma)). \end{aligned}$$

Proof. Without loss of generality we can assume that all formulas are in the basic modal language, considering all polyadic occurrences of $\Diamond\Gamma$ as propositional variables. Since S4 is complete for finite preordered models, it suffices to show that the formula is valid over this class.

Suppose that $\mathfrak{W}$ is a finite S4 model satisfying

$$\psi \wedge \Box(\psi \to \bigvee_{\Gamma\in\mathcal{G}} \bigwedge_{\gamma\in\Gamma} \Diamond\gamma)$$

on some world w.

Now pick $v \preccurlyeq w$ which is minimal amongst all worlds satisfying ψ. Since v satisfies $\bigvee_{\Gamma\in\mathcal{G}} \bigwedge_{\gamma\in\Gamma} \Diamond\gamma$, v satisfies $\bigwedge_{\gamma\in\Gamma_0} \Diamond\gamma$ for some $\Gamma_0 \in \mathcal{G}$. Because v is minimal, if $u \preccurlyeq v$ also satisfies ψ, then $u \sim v$ and hence u satisfies $\bigwedge_{\gamma\in\Gamma_0} \Diamond\gamma$.

But this shows that v satisfies $\psi \wedge \Box(\psi \to \bigwedge_{\gamma\in\Gamma_0} \Diamond\gamma)$, so w satisfies

$$\bigvee_{\Gamma\in\mathcal{G}} \Diamond(\psi \wedge \Box(\psi \to \bigwedge_{\gamma\in\Gamma} \Diamond\gamma)),$$

as desired. $\Box$

[7]This is where we use the fact that $\mathfrak{v}^n$ is rooted.

Lemma 8.7. *If Θ is a modal type and*

$$\Theta \vdash \bigvee_{\Box\Gamma\in\Theta} \bigwedge_{\gamma\in\Gamma} \Diamond(\neg\gamma \wedge \bigwedge \Theta),$$

then Θ is inconsistent.[8]

Proof. Let $\theta = \bigwedge \Theta$.

If

$$\theta \vdash \bigvee_{\Box\Gamma\in\Theta} \bigwedge_{\gamma\in\Gamma} \Diamond(\neg\gamma \wedge \theta),$$

then

$$\theta \vdash \theta \wedge \Box(\theta \to \bigvee_{\Box\Gamma\in\Theta} \bigwedge_{\gamma\in\Gamma} \Diamond(\neg\gamma \wedge \theta)),$$

which by Lemma 8.6 implies that

$$\theta \vdash \bigvee_{\Box\Gamma\in\Theta} \Diamond(\theta \wedge \Box(\theta \to \bigwedge_{\gamma\in\Gamma} \Diamond(\neg\gamma \wedge \theta))).$$

But then we can apply $\mathsf{Ind}(\Gamma, \theta)$ to get

$$\theta \vdash \bigvee_{\Box\Gamma\in\Theta} \Diamond\Diamond\neg\Gamma.$$

On the other hand, $\Diamond\Diamond\neg\Gamma$ is provably equivalent to $\Diamond\neg\Gamma$, which in turn is provably equivalent to $\neg\Box\Gamma$[9], showing that Θ is inconsistent. $\Box$

One important step in our proof is checking that consistent tasks are realizable. The notion of *covering* between sequences of types will be a useful tool in establishing this.

Definition 8.9. *Suppose that $\vec{\Sigma} = \langle \Sigma_i \rangle_{i<I}$ and $\vec{\Theta} = \langle \Theta_j \rangle_{j<J}$ are finite sequences of types.*

A choice function *on $\vec{\Sigma}$ is a sequence $\langle \sigma_i \rangle_{i<I}$ such that $\sigma_i \in \Sigma_i$ for all $i < I$.*

We say that $\vec{\Sigma}$ covers *$\vec{\Theta}$ if, given any choice function $\vec{\sigma}$ on $\vec{\Sigma}$, there is $j < J$ such that, for every $\theta \in \Theta_j$, there is $i < I$ for which $\sigma_i \to \theta$ is a substitution instance of a propositional tautology.*

[8] We use $\Theta \vdash \varphi$ merely as a shorthand for $\vdash \bigwedge \Theta \to \varphi$.

[9] Here one needs to use the fact that $\Diamond\Gamma \leftrightarrow \Diamond\neg\neg\Gamma$ is derivable, which can be shown by checking that each satisfies the other's fixpoint property.

Lemma 8.8. *If $\vec{\Sigma}$ covers Θ, then*

$$\mathsf{S4} \vdash \bigwedge_{i<I} \Diamond \bigvee_{\sigma\in\Sigma_i} \sigma \to \bigvee_{j<J} \bigwedge_{\theta\in\Theta_j} \Diamond\theta.$$

Proof. Once again we can, without loss of generality, assume that all formulas are in the basic modal language. It suffices to show that the formula is valid over the class of finite $\mathsf{S4}$ models.

Let $\mathfrak{W}$ be any $\mathsf{S4}$ model and suppose $w \in |\mathfrak{W}|$ satisfies

$$\bigwedge_{i<I} \Diamond \bigvee_{\sigma\in\Sigma_i} \sigma.$$

Then, for every $i < I$ there is $v_i \preccurlyeq w$ such that v_i satisfies $\bigvee_{\sigma\in\Sigma_i} \sigma$, and hence there is some $\sigma_i \in \Sigma_i$ such that v_i satisfies σ_i. This gives us a choice function $\vec{\sigma}$.

Now, because $\vec{\Sigma}$ covers $\vec{\Theta}$, there is $j < J$ such that for all $\theta \in \Theta_j$ there is $i < I$ such that $\sigma_i \to \theta$ is a tautology, hence v_i satisfies θ. It follows that w satisfies $\Diamond\theta$, and since $\theta \in \Theta_j$ was arbitrary we have that w satisfies $\bigwedge_{\theta\in\Theta_j} \Diamond\theta$, so it satisfies

$$\bigvee_{j<J} \bigwedge_{\theta\in\Theta_j} \Diamond\theta,$$

as desired. □

Lemma 8.9. *If a task Σ is unrealizable, then it is inconsistent.*

Proof. Let

$$\Sigma = \{\Diamond\Gamma\} \cup \{\Box\Delta : \Delta \in \mathcal{D}\}.$$

For each $\gamma \in \Gamma$ let

$$\Sigma_\gamma = \{\gamma\} \cup \{\Box\Delta : \Delta \in \mathcal{D}\}$$

and let $\mathcal{C}_\gamma$ be the set of all formulas of the form $\bigwedge \Theta$, where Θ is a consistent saturation of Σ_γ.

Let $\mathcal{D}^\neg = \{\neg\Theta : \Theta \in \mathcal{D}\}$.

We claim that if Σ is unrealizable, then $\langle \mathcal{C}_\gamma : \gamma \in \Gamma\rangle$ covers $\mathcal{D}^\neg$; indeed, suppose otherwise. Then, there exist $\langle \Theta_\gamma : \gamma \in \Gamma\rangle$ such that $\bigwedge \Theta_\gamma \in \mathcal{C}_\gamma$ and for every $\Delta \in \mathcal{D}$ there is $\delta \in \Delta$ such that $\bigwedge \Theta_\gamma \to \neg\delta$ is not a tautology for any γ, which in particular implies that $\neg\delta \notin \Theta_\gamma$.

Then, $\langle \Theta_\gamma : \gamma \in \Gamma \rangle$ clearly realizes Σ, contradicting our assumption.

Thus $\langle \mathcal{C}_\gamma : \gamma \in \Gamma \rangle$ covers $\mathcal{D}^\neg$. Note that for all $\gamma \in \Gamma$,

$$\Sigma \vdash \Diamond \bigvee_{\theta \in \mathcal{C}_\gamma} \theta$$

(by the axiom Fix and Lemma 8.1), so using Lemma 8.8 we see that

$$\Sigma \vdash \bigvee_{\Delta \in \mathcal{D}} \bigwedge_{\delta \in \Delta} \Diamond(\neg\delta \wedge \bigwedge \Sigma),$$

which by Lemma 8.7 implies that Σ is inconsistent. □

An important part of constructing our models is picking the tasks to realize in the appropriate order. The following result characterizes those tasks which must be realized first.

Lemma 8.10. *Suppose a modal type Ψ is saturated consistent, and let $\mathcal{C}$ be the set of all Γ such that $\Diamond\Gamma \in \Psi$ with the property that*

$$\vdash \Diamond\Gamma \wedge \bigwedge \Psi^{\Box} \rightarrow \bigwedge \Psi^{\Diamond\Box}$$

(so that, in a sense, Ψ is minimal amongst all types satisfying $\Diamond\Gamma$).

Then, there is a set of types $\mathcal{G}$ simultaneously realizing all $\Diamond\Gamma \in \bigcup \mathcal{C}$.

Proof. Let Ψ, $\mathcal{C}$ be as in the statement of the Lemma.

Let $\mathcal{D}$ be the set of all Δ with $\Box\Delta \in \Psi$.

First we note that, for $\Sigma \in \mathcal{C}$ we have that

$$\vdash \Diamond\Sigma \wedge \bigwedge_{\Delta \in \mathcal{D}} \Box\Delta \rightarrow \bigwedge_{\Gamma \in \mathcal{C}} \Diamond\Gamma \tag{1}$$

(the right-hand conjunction follows from $\bigwedge \Psi^{\Diamond\Box}$).

We now claim that

$$\vdash \bigwedge_{\Gamma \in \mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta \in \mathcal{D}} \Box\Delta \rightarrow \Diamond \bigcup \mathcal{C}.$$

It suffices to check that the antecedent validates the induction schema for $\Diamond \bigcup \mathcal{C}$; but if $\gamma \in \bigcup \mathcal{C}$, then $\gamma \in \Sigma$ for some $\Sigma \in \mathcal{C}$, hence

$$\vdash \Diamond\Sigma \wedge \bigwedge_{\Delta \in \mathcal{D}} \Box\Delta \rightarrow \Diamond(\gamma \wedge \Diamond\Sigma \wedge \bigwedge_{\Delta \in \mathcal{D}} \Box\Delta),$$

and together with (1) this implies that

$$\vdash \Diamond\Sigma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta \to \Diamond(\gamma \wedge \bigwedge_{\Gamma\in\mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta).$$

But then

$$\vdash \bigwedge_{\Gamma\in\mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta \to \bigwedge_{\gamma\in\bigcup\mathcal{C}} \Diamond(\gamma \wedge \bigwedge_{\Gamma\in\mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta),$$

which by necessitation for $\Box$ followed by

$$\mathsf{Ind}\left(\mathcal{C}; \bigwedge_{\Gamma\in\mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta\right)$$

shows that

$$\bigwedge_{\Gamma\in\mathcal{C}} \Diamond\Gamma \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta \to \Diamond\bigcup\mathcal{C}$$

is derivable.

This implies that

$$\Diamond\bigcup\mathcal{C} \wedge \bigwedge_{\Delta\in\mathcal{D}} \Box\Delta$$

is consistent (otherwise Ψ would be inconsistent), and by Lemma 8.9 it is also realizable, as desired. ☐

Lemma 8.11. *If a modal type Ψ is consistent, it is satisfiable.*

Proof. Let Ψ be a consistent modal type. We can assume that it is saturated, by extending it if necessary using Lemma 8.1.

Let $\mathcal{C}$ be the set of all Γ such that $\Diamond\Gamma \in \Psi$ and there is no consistent saturated type Θ with $\Diamond\Gamma \in \Theta^{\Diamond} \subsetneq \Psi^{\Diamond}$ and $\Psi^{\Box} \subseteq \Theta$. By Lemma 8.10, there is a set of types $\mathcal{G}$ simultaneously realizing all tasks from $\mathcal{C}$. Set $\vec{\mathcal{P}} = \mathrm{uw}(\mathcal{G})$.

Now we will find a set of models $\vec{\mathfrak{v}}$ such that $\left\langle \vec{\mathcal{P}}, \vec{\mathfrak{v}} \right\rangle$ is coherent, and then apply Lemma 8.5 to build a typed model satisfying Ψ. The elements of $\vec{\mathfrak{v}}$ will be of the form $\langle \mathfrak{v}^{\Gamma,i} : \Diamond\Gamma \in \bigcup\mathcal{P}_i \rangle$, where $\mathfrak{v}^{\Gamma,i}$ has the property that $|\mathfrak{v}^{\Gamma,i}|_i$ satisfies $\Diamond\Gamma$.

Let $\Diamond\Gamma \in \bigcup\mathcal{P}_i$. We will consider two cases: when $i = 0$ and when it is not.

If $i = 0$, we have that if $\Gamma \in \mathcal{C}$, then $\Diamond\Gamma$ is already realized by $\mathcal{P}_0$. So we can assume $\Gamma \notin \mathcal{C}$, and hence there is a saturated, consistent modal type Ψ_Γ extending $\{\Diamond\Gamma\} \cup \Psi^\Box$ and such that $\Psi_\Gamma^\Diamond \subsetneq \Psi^\Diamond$; by induction on $\#\Psi^\Diamond$ we have a rooted typed model $\mathfrak{v}^{\Gamma,0}$ whose root satisfies Ψ_Γ.

Otherwise, we have that $i > 0$. In this case, we can disregard the first elements in $\vec{\mathcal{P}}$ and consider $\langle \mathcal{P}_j \rangle_{i \le j < I}$. This is a cluster path of duration $I - i$; by induction on I, given $\Theta \in \mathcal{P}_i$ and a task

$$\Sigma = \{\Diamond\Gamma\} \cup \Theta^\Box \subseteq \Theta,$$

there is a rooted typed model $\mathfrak{v}^{\Gamma,i}$ of depth $I - i$ satisfying Σ, which we may consider as a typed model of duration I beginning on i.

It is clear that in all cases, $|\mathfrak{v}^{\Gamma,i}|_i$ contains a world satisfying $\Diamond\Gamma$, so that $\left\langle \vec{\mathcal{P}}, \vec{\mathfrak{v}} \right\rangle$ is coherent; thus by Lemma 8.5, $\vec{\mathcal{P}} \oplus \vec{\mathfrak{v}}$ is a typed model satisfying Ψ, as claimed. Note that it can be rooted by designating any $\Phi \in \mathcal{P}_0$ as a root. $\Box$

Theorem 8.1. $\mathsf{S4C}^*$ *is complete for interpretations on finite* $\mathsf{S4}$*-models.*

Proof. If a formula φ is consistent, then by Lemma 8.1, $\{\varphi\}$ has a consistent saturation Φ; it follows that $\Phi^{\Diamond\Box}$ is consistent, so by Lemma 8.11 it is satisfiable in some finite typed model $\mathfrak{w}$. By Lemma 8.5, $\mathrm{uw}(\{\Phi\}) \oplus \{\mathfrak{w}\}$ is a typed model for φ, which by Lemma 8.3 implies that φ is satisfiable. $\Box$

9. Independence and Expressivity

In this section we will show that the tangled continuity axiom is not derivable from the other axioms of $\mathsf{S4C}^*$ (including Cont), as well as showing that L^* not only is more expressive than L (which has been proven in [3, 5]), but also that it can distinguish discontinuities in cases when L cannot. This means that it gives a sharper characterization of the class of all dynamic topological models within the class of all models based on a topological space and a (possibly discontinuous) function.

An essential tool for this will be the following model:

Definition 9.1 (The model $\mathfrak{B}$). *Any ordinal ξ can be written in the form $\omega \cdot \zeta + n$, where $n < \omega$ is uniquely defined; say ξ is* even *if n is even, ξ is odd otherwise.*

We then define the Kripke model $\mathfrak{B}$ as the ordinal ω_1 with the usual ordering and the identity function as $f_\mathfrak{B}$, then define $[\![p]\!]_\mathfrak{B}$ to be the set of even ordinals below ω_1; all other valuations are empty.

For each formula $\Box\gamma \in \mathsf{L}^*$, there is a unique ordinal $\zeta_\gamma \leq \omega_1$ such that $\xi \in |\mathfrak{B}|$ satisfies $\Box\gamma$ if and only if $\xi < \zeta_\gamma$.

Let α be the supremum of all $\zeta_\gamma < \omega_1$. Since L^* is countable, $\alpha < \omega_1$.[10]

Definition 9.2 (The models $\mathfrak{A}$ and $\mathfrak{C}$)**.** *We define another* $\mathsf{S4}$ *model* $\mathfrak{A}$ *consisting of a single cluster with two points and all valuations empty; that is,* $|\mathfrak{A}| = \{a, b\}$ *with* $a \sim_{\mathfrak{A}} b$*.*

We then define a discontinuous dynamic topological model $\mathfrak{C}$ *by setting* $|\mathfrak{C}| = |\mathfrak{A}| \cup |\mathfrak{B}|$ *and*

$$f_{\mathfrak{C}}(x) = \begin{cases} \alpha & \text{if} \quad x = a, \\ \alpha + 1 & \text{if} \quad x = b, \\ x & \text{if} \quad x \in |\mathfrak{B}|. \end{cases}$$

We keep the valuations as in $\mathfrak{A}$ *and* $\mathfrak{B}$*.*

Two important facts about $\mathfrak{C}$. First, $f_{\mathfrak{C}}$ is discontinuous at a since $b \preccurlyeq_{\mathfrak{C}} a$ but it is not the case that $f_{\mathfrak{C}}(b) \preccurlyeq f_{\mathfrak{C}}(a)$. Next, because $f_{\mathfrak{C}}$ is the identity on $|\mathfrak{B}|$, given $\xi \in |\mathfrak{B}|$ and a formula $\varphi \in \mathsf{L}^*$, ξ satisfies φ if and only if ξ satisfies the formula φ' obtained by removing all occurrences of f in φ.

With this in mind, we claim the following:

Lemma 9.1. $\mathfrak{C} \models \mathsf{Cont}$*.*

Proof. Let $\varphi = \Diamond f\psi \to f\Diamond\psi$ be an instance of Cont.

One can see that $f_{\mathfrak{C}}$ is continuous everywhere except at a, so evey other point satisfies φ.

Let us check that a satisfies φ as well.

Suppose, then, that a satisfies $\Diamond f\psi$. Then either a satisfies $f\psi$, in which case it also satisfies $f\Diamond\psi$, or b does.

If b satisfies $f\psi$, then $\alpha + 1$ satisfies ψ and thus $\Diamond\psi$. But α and $\alpha + 1$ satisfy the same formulas of the form $\Diamond\delta$ (by construction), so α also satisfies $\Diamond\psi$, as desired. ∎

Thus L "cannot tell" that $f_{\mathfrak{C}}$ is discontinuous. However, let us see that L^* *can*:

[10] If we replace L^* by L, α becomes ω; this can be seen by checking that $\langle \mathfrak{B}, \omega \rangle$ is k-bisimilar to $\langle \mathfrak{B}, n \rangle$ for n large enough. We do not expect the situation to be much different for L^* but the actual value of α is unimportant.

Lemma 9.2. $\mathfrak{C} \not\models \Diamond f \{p, \neg p\} \to f \Diamond \{p, \neg p\}$*, which is an instance of* TC*.*

Proof. Clearly, $a \in [\![fp]\!]_{\mathfrak{C}}$ and $b \in [\![f\neg p]\!]_{\mathfrak{C}}$. Since $a \sim b$, it follows that $a, b \in [\![\Diamond \{fp, f\neg p\}]\!]_{\mathfrak{C}}$.

Meanwhile, $\alpha \notin [\![fp]\!]_{\mathfrak{C}}$, since ω_1 is well-founded so that there can be no infinite sequence

$$x_0 \geq x_1 \geq x_2 \geq \ldots$$

with x_i satisfying p if i is even, $\neg p$ if i is odd, as in the Kripke interpretation of $\Diamond \{p, \neg p\}$.

Hence a does not satisfy $f \Diamond \{p, \neg p\}$, as claimed. □

From the above considerations we can immediately draw two conclusions:

Theorem 9.1. TC *is not derivable in* $(\mathsf{S4C}^* - \mathsf{TC}) + \mathsf{Cont}$*.*

Proof. It is easy to check that $\mathfrak{C}$ satisfies all other axioms of $\mathsf{S4C}^*$ and as we have seen it satisfies Cont, but not TC. □

Corollary 9.1. L^* *is more expressive than* L *over the class of possibly discontinous dynamic topological models.*[11]

Proof. We claim that there can be no formula equivalent to

$$\varphi = \Diamond f \{p, \neg p\} \to f \Diamond \{p, \neg p\}\,.$$

Indeed, suppose $\varphi^{\mathsf{L}} \in \mathsf{L}$ were equivalent to φ. Because S4C is complete [2], it would follow that $\mathsf{S4C} \vdash \varphi^{\mathsf{L}}$, given that φ is valid over the class of dtm's. But this is impossible, since we have shown that S4C is sound for $\mathfrak{C}$ but $\mathfrak{C} \not\models \varphi$. □

Corollary 9.1 is not new: it has already been proven in [3, 5], even for the purely topological fragment. However, our construction is interesting in that it shows L^* is more expressive than L in a sense that is directly related to continuity, and thus that it is better at capturing the class of dynamic topological models.

[11] Defined like dynamic topological models, except without the continuity condition.

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