Typicality of Dynamics and Laws of Nature

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Abstract

Certain results, most famously in classical statistical mechanics and complex systems, but also in quantum mechanics and high-energy physics, yield a coarse-grained stable statistical pattern in the long run. The explanation of these results shares a common structure: the results hold for a 'typical' dynamics, that is, for most of the underlying dynamics. In this paper I argue that the structure of the explanation of these results might shed some light—a different light—on philosophical debates on the laws of nature. In the explanation of such patterns, the specific form of the underlying dynamics is almost irrelevant. The conditions required, given a free state-space evolution, suffice to account for the coarse-grained lawful behaviour. An analysis of such conditions might thus provide a different account of how regular behaviour can occur. This paper focuses on drawing attention to this type of explanation, outlining it in the diverse areas of physics in which it appears, and discussing its limitations and significance in the tractable setting of classical statistical mechanics.

Keywords

Typicality; Statistical Mechanics; Stability; Laws of Nature; Physical Necessity; Non-accidental Regularities.

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Contents

1 Introduction: Laws, stability, and typicality.

It is commonly held that there is no satisfactory philosophical account of the notion of physical necessity. While it can be said that philosophers of science have made some progress in proposing candidate accounts of laws of nature, all of these accounts have major flaws, and in particular, the physical necessity of laws of nature is either postulated or left unexplained.[1] In this paper I propose to look at certain branches of physics that might help to improve our understanding of the source of lawful behaviour, at least within their restricted settings. To this end, I examine results of *stability*, that is, results to the effect that a physical system will evolve into a state that is invariant over time—for instance, the state of equilibrium of a classical gas of particles within a closed environment. More specifically, we will look at the approximate results of *emergent stable behaviour* that are acquired for *a typical underlying dynamics*. A typical dynamics is not the same as an arbitrary dynamics, but it is close: roughly stated, a typical dynamics is supposed to cover most of the dynamics, where 'most' is precisely defined.

In this paper I first want to point out that diverse areas in physics have in common (1) a result of the same type, i.e. a coarse-grained stable statistical pattern, *and* (2) that such a result holds for typical dynamics. The examples that I will cite are from classical statistical mechanics, complex systems, quantum mechanics, and diverse projects in high-energy physics. We will reconstruct in detail the case of classical statistical mechanics, in order to critically assess it. Then, the main aim of this paper is to point out that the structure of the explanation of such results, based on the notion of typicality, can be significant for philosophical debates on laws of nature and physical necessity.

The reason for focusing on the idea of typical dynamics is that such emergent stable patterns are explained almost independently of the specific details of the underlying dynamics. The emergence of a stable pattern does not depend on the specific form of the underlying governing laws, unless the form

[1] I refer to the necessitarian account (Dretske, 1977; Armstrong, 1983; Tooley, 1977), the propensities/dispositional account (Cartwright, 1999; Mumford, 2004; Chakravartty, 2005), the Humean a.k.a. Best System account (Mill, 1884; Lewis, 1999; Earman and Roberts, 2005; Cohen and Callender, 2009), and the primitivist account (Maudlin, 2007; Carroll, 1994).

is "ridiculously special" (Goldstein, 2001, 43). Hence I argue that a dynamical system under suitable conditions, with no specific constraint of dynamical laws (i.e. with no deterministic or indeterministic rules of temporal evolution, usually in the form of differential equations), will exhibit a free state-space evolution which will typically display, in the long run, coarse-grained stable patterns.

Thus, the law-like behaviour at the higher level does not require the postulation of the usual underlying guiding rules of temporal evolution. An analysis of the suitable conditions invoked in each particular setting might help us to understand one way in which regular behaviour can occur—a way hitherto unnoticed in the philosophical literature on laws of nature.[2]

This paper is limited to outlining these results in the diverse areas of physics mentioned, and then discussing their limitations and significance in the setting of classical statistical mechanics. In Section 2 I mention various projects in high-energy physics and quantum mechanics which appeal to the aforementioned dialectics of deriving certain results for most of the underlying dynamics. Then I focus on reconstructing (in Section 3) and critically assessing (in Section 4) the approach of typicality in classical statistical mechanics. In Section 5 I conclude by assessing the philosophical significance that such results may have for philosophical debates on physical necessity.

The suitable conditions aforementioned are standard general constraints that gain a prominent role, for they can be the only modal constraints, that is, the only conditions that play the role of laws. In the literature on complex systems theory it is well known that, besides the underlying laws, the *context* gains an especially prominent role (Frigg and Bishop, 2016). Roughly stated, in the present study it is asked whether sometimes this role is not only prominent but sufficient to account for an otherwise unconstrained motion. [3]

2 Overview of Approaches in High-energy Physics

Let us begin by citing various diverse projects in physics which employ the rationale discussed here. First of all, there are those projects that seek to derive the laws (the standard model interactions) and symmetries of modern physics from what they call a *random dynamics*. According to this hypothesis, *all complex* Lagrangians lead, in the low-energy limit, to the laws of particle physics (Froggatt and

[2]To avoid confusion, a suitable differential equation can always *describe* the state-space evolution of a physical system. When I say that there is no specific constraint of dynamical laws I refer to the ontological claim that there is no governing dynamical law postulated in the theory, i.e. there is no dynamical law that *governs*, or *guides*, the system's evolution.

[3]Besides, the present assessment may influence the plausibility of the physics projects cited: if the explanation of typicality, as we analyse it in the context of classical statistical mechanics, is a successful (or unsuccessful) type of explanation, this tells in favour of (or against) the projects we cite in other fields of physics that employ the same type of explanation.

Nielsen, 1991; Chadha and Nielsen, 1983; Chkareuli et al., 2011). The authors consider a fundamental level displaying a highly complex behaviour. This level is below the current quantum level, for quantum mechanics does not describe a complex dynamics like the one they assume. The random dynamics is thought to inevitably yield the emergence, within some energy limit, of all current symmetries. The limit is the low energy domain, which corresponds to the experimentally accessible energies below 1TeV. Similar research along these lines includes the work of Mukohyama and Uzan (2013), as well as Jacobson and Wall (2010), both of which are concerned specifically with Lorentz symmetry, drawing an analogy with statistical explanations of the second law of thermodynamics (for an attempt to frame such projects in the philosophy of physics literature, see Smeenk and Hoefer, 2015, §4.1 and references therein).

There are also more speculative projects concerning *entropic forces*. Similarly, according to them, the allegedly fundamental interactions, including gravity, are not fundamental but rather emergent, arising from the statistical behaviour of lower-level degrees of freedom. See Verlinde (2011, 2017) or the more elaborated derivation of the Einstein field equations from thermodynamic assumptions given by Jacobson (1995).

For decades there has also existed research on chaotic cosmologies (see e.g. Misner, 1969; Barrow, 1977; Linde, 1983) that assumes an undetermined fundamental chaotic dynamics. Today such research concerns the instants before inflation, where a chaotic dynamics is assumed as a natural default initial state, and it is then investigated how we arrived from that state to the current standard model with broken symmetries and frozen degrees of freedom. A recent example which recurs to this view is Okon and Sudarsky's (2016) attempt to explain dynamically the Past Hypothesis (the universe's very special initial state of low entropy).

Finally, a similar dialectics is also found in certain projects in the foundations of quantum mechanics: Valentini's (1991) attempt to derive Born's rule with a quantum analogous of Boltzmann's H-theorem, and Nelson's (1966) attempt to derive the Schrödinger equation by presupposing Brownian motion of classical particles. In the same vicinity, in the foundations of Bohmian mechanics, Goldstein et al. (2010a,b); Goldstein and Tumulka (2010) aim to show that *for typical Hamiltonians* with given eigenvalues all initial state vectors evolve in such a way that the wavefunction will be in thermal equilibrium at most times. For more on these projects, which for reasons of space we can only cite here, see Callender (2007) and references therein.

Needless to say, for obvious reasons (dealing with the early universe, at fundamental sub-quantum levels, etc.) most of these approaches are more speculative than usual standard model physics, and thus have difficulties in delivering empirical predictions. In any case, this is irrelevant for our purpose here, as we are interested in the logical form of their common type of explanation and its significance.

So, how reliable is the typicality approach in the widely discussed setting of classical statistical mechanics? Many worries have been raised both to the typicality approach and its predecessors, such as Boltzmann's H-theorem. Those who raise the worries in statistical mechanics may also be skeptic of the dialectics presented in the aforementioned physical theories, in which the underlying dialectics is the same but is not explicitly discussed.

3 The Typicality Approach in Statistical Mechanics

Let us delve into the typicality approach as it appears in the foundations of classical statistical mechanics. According to this approach, the tendency towards the equilibrium macrostate occurs for initial conditions that are typical, where 'typical' is spelled out in measure-theoretical terms. After stressing that this is insufficient for explaining the tendency towards thermal equilibrium, the typicality of the dynamics has to be included, which again means that it occurs for the overwhelming majority of them, where 'typical' here is spelled out in topological terms.

3.1 Boltzmann's explanation of the second law of thermodynamics

The point of departure is Ludwig Boltzmann's project of understanding the macroscopic properties and laws of thermodynamics in terms of their microconstituents and their laws. This was the main mission of kinetic theory and statistical mechanics. The latter can be said to be the continuation of the former, after introducing irreducible probabilistic distributions not to the microconstituents but to the states of macroscopic entities (to the state of the whole gas). After Boltzmann, plenty of different paths have been pursued in order to obtain a reductive explanation of the laws of thermodynamics (see Uffink (2014); Uffink (2006, Ch.4); Frigg (2008); cf. Albert (2000), Atkins (2007), Sklar (1993, II.3), Filomeno (Forthcoming); for another philosophical assessment of the typicality approach see Lazarovici and Reichert (2015, §2-3)).

In the case of the hard-sphere model of a gas in an isolated container, the macrostate towards which all systems tend is the macrostate in which the gas has spread out all over the box, filling its volume, that is, the 'equilibrium macrostate'.[4] Appealing to combinatoric mathematics, Boltzmann showed that the

[4] We will consider the simple model of hard spheres, which models molecules of a gas closed in a perfectly isolated container. The gas is either ideal or diluted, neglecting long-range forces, with a fixed kinetic energy $T = \frac{p^2}{2m}$. The molecules of the gas, then, are the micro-constituents, and they are modelled not as point-particles but as hard spheres, each with a certain small radius r. The gas molecules interact like billiard balls; they have no effect on one another except when they

equilibrium macrostate is compatible with an overwhelmingly higher number of microstates. Consider a gas composed of n particles with two degrees of freedom each. The state of this system is specified in a $4n$-dimensional phase space Γ by a point x. This point is the microstate, which specifies the position q and momentum p of every particle:
$x = (p_{x_1}, p_{y_1}, p_{x_2}, p_{y_2}, ...p_{x_n}, p_{y_n}, q_{x_1}, q_{y_1}, q_{x_2}, q_{y_2}...q_{x_n}, q_{y_n})$.
The phase space comes endowed with the natural Lebesgue measure μ.[5] The particles obey the laws of classical Hamiltonian mechanics; they define a phase flow ϕ_t that is measure-preserving, which means that for all regions,

$$R \subseteq \Gamma, \mu(R) = \mu(\phi_t(R))$$

which is known as Liouville's theorem. The system is perfectly isolated from the environment, so the energy is conserved. This restricts the motion of the microstate x over a region of Γ that is the energy hypersurface Γ_E, of $4n-1$ dimensions. The Lebesgue measure μ restricted to Γ_E, μ_E, is also invariant. From the macroscopic point of view, the gas is characterised by its macrostates, where the equilibrium macrostate is labeled as M_{eq}. $\Gamma_{M_{eq}}$ is the corresponding macroregion in phase space which contains all $x \in \Gamma_E$ for which the system is in M_{eq}. Macrostates M supervene on microstates; a macrostate is compatible with many different microstates.

The main conclusion of Boltzmann's combinatorial argument is that the measure of Γ_{Meq} with respect to μ_E is overwhelmingly larger than any other macroregion. For the details of the proof, see e.g. Uffink (2006, 4.4), or Boltzmann (1877) himself. In fact, this region occupies almost all the energy hypersurface, as figure 1 conveys.

The entropy is defined as the logarithm of the size of the phase space region of the macrostate:

$$S(M) = k_B \; ln|\Gamma_M|$$

where k_B is the Boltzmann's constant. Given the radical difference between the sizes of the different macro-regions, it was reasonable to think that the non-decrease of entropy stated by the second law and, more generally, the tendency towards equilibrium stated by the 'minus first law' (Brown and Uffink, 2001), will be overwhelmingly more likely to occur. In fact, it follows that

$$S(M_{Eq}) >> S(M_{\neg Eq})$$

where $M_{\neg Eq}$ corresponds to any non-equilibrium macrostate. This would preserve the time-symmetric Newtonian picture of the world while explaining the time-asymmetric behaviour stated by the second

collide. 'Hard' means that the collisions are elastic, i.e. no kinetic energy is transformed into other forms, for instance none is lost in the form of heat. Also assumed is a large number of microscopic constituents, typically of the order of Avogadro's number $N = 10^{23}$ or more.

[5] Justifying the choice of the "natural" measure is problematic; see e.g. (Sklar, 2015, Sect. 4), and Werndl (2013).

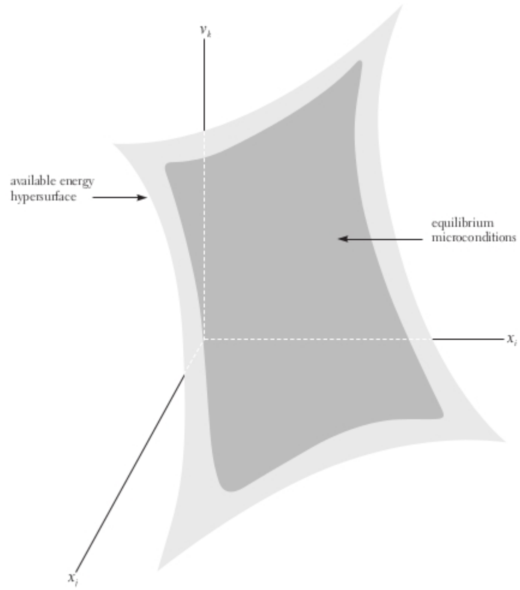

Figure 1: An energy hypersurface, displaying the predominant size of Γ_{Meq}, the region of all the microstates corresponding to the equilibrium macrostate.

law. The second law would not be a strict law but an rather approximation, reflecting the overwhelming likeliness of such behaviour. Hence, there would be no real conflict between reversible microscopic laws and irreversible macroscopic behaviour.

The success of this project, however, was threatened in many ways. A number of obstacles have been showing up ever since, such as

· the reversibility objection,
· the recurrence objection,
· the implausibility of the independence assumptions,
· how to interpret of the various probabilities,
· the status of the past hypothesis,
· the validity of the results outside the simplified models studied,
· the failure of the ergodic hypothesis, as it was proved that ergodicity is not sufficient (nor necessary), etc.

The typicality approach arguably helps in clarifying the status of the probabilities (and we believe that many of the other issues have been correctly responded, but we can leave that aside). Our focus on the typicality approach is however concerned with its potential significance for understanding the general phenomenon of *emergence of lawlike behaviour*, and hence, with its potential significance in the debates in philosophy of science about laws of nature.

3.2 Typicality

As with Boltzmann's approach, the main idea of the typicality approach is that a system exhibits entropic behaviour because it is typical for the system to behave in this way. However, as we will see, in contrast to Boltzmann's approach, typicality-based explanations eschew commitment to probabilities.

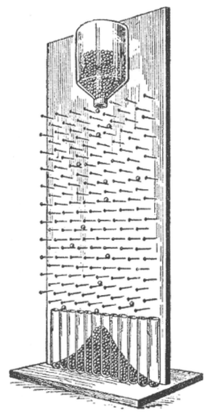

Figure 2: The Galton board.

Maudlin (2011) illustrates the general idea with diverse examples: the toss of a coin, the toss of a die, and the case of a Galton board (depicted in Figure 2). As we already know, the Galton board displays, in the long run, a normal distribution centred in the middle basket. How should we understand the nature of this probability distribution, that we take to be neither subjective nor epistemic? The limiting frequency in the middle basket can be explained in terms of typicality, that is, it occurs because most of the possible initial distributions end up with that result. In other words, the typical behaviour of a ball falling in the Galton board is for it to fall in the centre. And this can be explained by focusing on the typical behaviour of a ball hitting a pin, whereby it is deflected to each side half the time. At the core of this phenomenon lies the law of large numbers: if it is typical to be deflected half the time to the left and half to the right, in the long run (i.e. after the balls have hit a large number of pins), it is expected that the number of turns to the left and to the right will be approximately the same, leading the ball to land approximately in the centre.[6] Typicality, then, is understood as follows: "when some specified

[6]The famous 'law of large numbers' (LLN) can be stated thus:

Theorem 1. ***The Strong Law of Large Numbers, or Borel Strong Law.***
For independent infinite sequences of flips of a fair coin, let B denote the event that the proportion of successes S_n *among the first*

dynamical behaviour (like passing a single pin to the right, or passing successive pins first to the right and then to the left) has the same limiting frequency in a set of initial states that has measure one, that frequency for the dynamical behaviour is typical" (Maudlin, 2011). In this quote, there is no appeal to probabilities but to measure theory. If the set of states that leads to some outcome has measure one, then it can be defined as typical, where the measure is calculated with a flat Lebesgue measure over the appropriate interval.

We can treat the case of a coin toss similarly. Fair coins typically land heads half the time, because most sequences of fair coin tosses, whatever the initial state, lead to that result in the long run (see Figure 3).

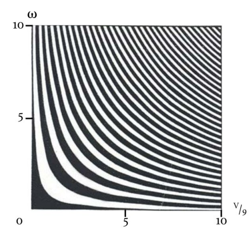

Figure 3: A coin-flip model showing the outputs of heads (black area) and tails in function of the angular speed ω and the vertical velocity V/g (From (Diaconis, 1998))

In the case of the gas in a box, we can describe more precisely the situation as follows. Following Frigg (2009), an element *e* of a set Σ is typical if most members of Σ have property P and *e* is one

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n flips, $\frac{1}{n}S_n$, approaches the limit 1/2 as $n \to \infty$. That is:

$$B := \left\{ x \in \{0,1\}^N : \lim_{n\to\infty} \frac{S_n}{n} = \frac{1}{2} \right\}$$

The probability of the event B is 1, i.e. the set B has Lebesgue measure 1. (Dasgupta, 2011, §3.2)

There is also a *weak* version, which permits a small difference between the expected mean value and the effective outcome. The weak version states that the sample average converges in probability towards the expected value. Following Loeve (1977), where each outcome is X_i, the number of trials n, the sample average $\overline{X_n} = (X_1 + X_2 + ... + X_n)/n$, μ the expected value, and ε a small positive number:

$$\lim_{n\to\infty} P\left(|\overline{X_n} - \mu| > \varepsilon\right) = 0$$

The weak version allows for a certain degree of tolerance for departing from the expected value of a finite random sequence, which is quantified by ε. This version leaves open the possibility that $|\overline{X_n} - \mu| > \varepsilon$ happens an infinite number of times, although at infrequent intervals. Instead, in the strong version, for any $\varepsilon > 0$, the inequality holds for all large enough n (Ross, 2009). In any case, $X_1, X_2, ...X_n$ are assumed to be an infinite sequence of independent and identically distributed integrable random variables with expected value $E(X1) = E(X2) = ... = \mu$.

of them. The element corresponds to a micro-state, the measure employed is the natural Lebesgue measure μ, Σ is the set of all microstates, and P is the property of evolving to equilibrium. 'Most' is thus understood in terms of having measure 1. Conversely, 'atypical' corresponds to having measure 0. (More exactly, the typicality measure is the induced 'microcanonical measure' μ_E restricted to the energy hypersurface.)[7] One of the results of Boltzmann's H-theorem was that, for typical microstates x, the empirical distribution $f_x(q,p)$ will converge to the Maxwell-Boltzmann distribution: $f_x(q,p) \propto e^{-\frac{1}{2}m\beta v^2}$. (For an explanation of the H-theorem as a typicality statement see Lazarovici and Reichert (2015, §3.3).)

More precisely, the definition of typical allows exceptions of subsets of measure zero or even subsets of very small measure. A definition for the crucial notion of 'most' (or 'nearly all') is as follows. Let A_P be the set of elements that exemplify P. Then,

$$\text{“Most of the elements } e \text{ in } \Sigma \text{ exemplify P”} := \frac{\mu(\Sigma \setminus A_P)}{\mu(\Sigma)} < \varepsilon$$

where ε is a small positive number. See Wilhelm (forthcoming) for further details.[8]

Notably, it turns out that one gets a set of frequencies that are typical as an *analytic consequence* of the deterministic dynamics together with a measure over initial states. And "what we in fact believe as a *purely mathematical fact* is that the set of initial states with 50% limiting frequency for deflections to either side is a set of measure one" (Maudlin, 2011, 286, my italics).

4 The Typicality of the Dynamics

So far in the typicality approach, no explicit mention of the dynamics has been made. Yet it has been argued that for the typicality explanation to suffice, something else must be verified, i.e. that the results

[7]The typicality measure is not a probability measure (see e.g. Werndl, 2013). Yet, a link can be established, namely that a typicality measure $\mu(x)$ implies that $P(x) \sim 1$. A suggestion for understanding this link is to interpret the probability P as an objective probability "deriving from ranges in suitably structured initial state spaces" (Rosenthal, 2009, 2012). Thus, the probability of an event x consists in the proportion of initial states that lead to x within the space of all possible initial states. Typicality statements seem to ground this notion of deterministic probability (the first to suggest an interpretation of probabilities in deterministic systems was, as far as we know, Batterman 1992). Poincaré's method of arbitrary functions lies at the core of Rosenthal's proposal, and thus can be considered to ground typicality statements, as also suggested in (Filomeno, 2019b).

[8]A minor terminological clarification: for the sake of clarity we (and the authors we cite) talk of 'typical states', 'typical trajectories', etc. although strictly speaking what is defined as typical in a typicality explanation is a property P that is shared by the elements of the set (e.g. the microstates, trajectories, or whatever else). Again, see Wilhelm (forthcoming) for a clear discussion around typicality explanations.

hold for most variations of the actual Newtonian dynamics. Before analyzing how the typicality of dynamics is defined in §4.2, let us pause to analyse an important aspect regarding how Newtonian dynamics has been treated in statistical mechanics.

4.1 A preliminary remark on the actual Newtonian dynamics

Several assumptions are made in kinetic theory and statistical mechanics, varying depending on the philosophical approach and the system under study. The deterministic laws of classical mechanics are assumed. Now, the assumption that will become relevant concerns the appearance of randomness: in spite of the guiding laws being deterministic Newtonian laws, there is "*an assumption about the erratic nature of the dynamics*" (Uffink, 2006, 5). To quote Sklar (1993), one of the chief assumptions of the Maxwell–Boltzmann programme, and an assumption that runs right through to the ergodic theory of modern physics, is that at some level of description a condition of *independence* must be met "for the theory to properly explain why the correct values of state parameters of gas systems can be obtained from taking the average values of mechanical properties of the individual particles of the system".

There have been various strategies for underpinning this requirement of independence: the assumption of equal initial probabilities, the postulation of dynamical properties as in the ergodic hypothesis (which we can informally phrase as stating that, over long periods of time, the time spent by a system in some region of the phase space of microstates with the same energy is proportional to the volume of this region; for a formal definition see below fn 14), or rerandomization posits such as hypotheses of molecular chaos, e.g. the Stoßzahlansatz in Boltzmann's theory (the assumption about the lack of correlation of the particles' velocities before they collide). The point is that each of these introduces a condition of randomness or independence at different locations in either the gas model employed or in the method for calculating the properties of the gas (e.g. temperature, pressure, or entropy). Hence a central question in the foundations of statistical mechanics: how can such posits about the erratic wandering nature of the dynamics coexist with the deterministic guiding laws?

This apparent randomness is relevant for our purpose, as will become clearer in the next subsections §4.2 and §4.3. This apparent randomness is due to the randomizing effect of the actual dynamics, a phenomenon which can be understood by appealing to the relation between determinism and chaos and then between chaos and randomness. Leaving aside the discussions on the definition of these notions, it will suffice for us to understand 'chaos' through its most famous feature, namely high sensitivity to initial conditions, and randomness in terms of a lack of correlations between a state and its past states.[9] As to the relation between deterministic classical mechanics and chaos, we can easily

[9] 'Randomness' so understood can be embedded in the framework of the Ergodic Hierarchy: a hierarchy of dynamical

visualize classical mechanics' high sensitivity to initial conditions in the hard-sphere model of figure 4: the outcome direction after the collision of one sphere with the convex spherical surface of another sphere is highly sensitive. Then the appearance of randomness can be approached via such chaotic property of the Newtonian dynamics, since it washes out the correlations with past states, yielding a random-looking evolution. As it is known in the dynamical systems literature, "interactions in the domain $\Gamma_{M_{ab}}$ will be *so convoluted as to appear uniformly smeared out* in Γ_{M_b}. It is therefore reasonable that the future behaviour of the system, as far as macrostates go, will be *unaffected by their past history*" (Lebowitz, 1999, my italics).[10] [11]

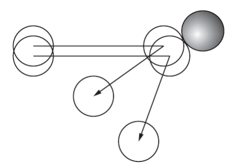

Figure 4: The collision of two classical circular particles is highly sensitive to initial conditions.

Having remarked on this randomizing effect of the actual dynamics, let us spell now out a typicality approach which explicitly includes a specification of the features that the dynamics must verify.

4.2 A typicality explanation explicitly including the dynamics

Frigg (2009, 5) correctly remarks that *typical states do not automatically attract trajectories.* In other words, before we were implicitly granting that the underlying dynamics leads to equilibrium, but this need not be so. That is, it does not lie in the "nature" of atypical states to evolve into typical ones. It could perfectly well be the case that a phase flow leads to anti-thermodynamic behaviour. Analogously, Frigg

properties, where the strongest property is the maximum degree of randomness, the Bernoulli property (in which there is no correlation with *any* its past states), and the weakest property is the weakest degree of randomness, the ergodic property (Berkovitz et al., 2006). 'Chaos' can also be defined within such a framework: Werndl (2009) defines it in terms of strong mixing; Belot and Earman (1997), define it in terms of the higher Kolmogorov level; while Berkovitz et al. (2006) advocate seeing it as a matter of degree, quantified according to the position within the hierarchy. For any of these definitions, chaos is a sufficiently strong randomization property, for it will be stronger than ergodicity (which is in turn stronger than ϵ-ergodicity, which will be the required condition for thermodynamic behaviour, as explained later). For definitions of these notions and their interrelation, see e.g. Frigg et al. (2016).

[10] Here, M is the system's macrostate, Γ the phase space, Γ_M the phase space region corresponding to M, M_a the system's initial macrostate, and M_b its later macrostate. $\Gamma_{M_{ab}}$ is the region of Γ_{M_b} that came via Γ_{M_a} (the set of microstates within Γ_{M_b} that are on trajectories that come from Γ_{M_a}).

[11] One of the many details I omit for the sake of space is that the independence assumptions can be of different sorts. One example is Maxwell's assumption, which posits independence in the components of the velocity vector of each gas particle.

points out that the approach to equilibrium will not occur just because the corresponding microstates are more numerous. The disparity of sizes of the macroregion of equilibrium and the other regions does not by itself imply that an entropy decrease will be atypical. Thus, definitions such as those offered by Goldstein and Lebowitz (2004, 57) are unsatisfactory; the dynamics must be explicitly included in the definition of typicality.

This amounts to go one step further and ask whether the 2^{nd} law holds typically in the space of possible Hamiltonians (relative to a measure of typicality over such space, as we will see later), that is, explore whether most possible dynamics, subject to some general constraints, would lead the system to exhibit thermodynamic behaviour.

The scope of this generalization will affect its significance. For instance, it is easier (and in fact Frigg and Werndl (2012) provide some alleged numerical support) to generalize the results in the restricted class of Hamiltonians that describe gases. This would be significant for the project in statistical mechanics of explaining the macroscopic patterns of thermodynamics. Now, as we will discuss later, in this paper we also want to point out that the more these results are generalized in the dynamics space, the more they approach the goal we are aiming at, namely, a typicality explanation of coarse-grained regular behaviour. We want to focus attention on a typicality explanation in which the results would hold almost irrespective of the underlying dynamics. Then, such an explanation could be phrased without postulating any 'dynamics', that is: the results would hold irrespective of the specific physical system's state-space *trajectory*.

The idea amounts to the following. Let a generic dynamical system $(\Gamma, \Sigma, \mu, \phi_t)$ be defined as a probability space $[\Gamma, \Sigma, \mu]$ and a transformation ϕ_t of it.[12] ϕ_t has the generic form of an ordinary differential equation: $\frac{d}{dt}x = f(x,t)$; which is autonomous of the independent variable t which represents time, where x is the position of the system in the n-dimensional state-space and n is the number of degrees of freedom. Then, informally, the idea amounts to seek whether not only an inverse-square guiding law $f(x,t) \propto r^{-2}$ yields evolution to the Maxwell-Boltzmann distribution, but that also a, say, inverse-cube law $f(x,t) \propto r^{-3}$ or an inverse-fifth law $f(x,t) \propto r^{-5}$ would yield the same result in the long term.

If the results were sufficiently typical (sufficiently generic or sufficiently wide in scope), stable behaviour would obtain for almost any underlying dynamics; i.e. it would obtain almost independently of how the dynamics is described. Supported by Maudlin's diagnosis above (p. 10), we would be faced

[12] As previously defined, Γ is a set of elements, interpreted as a state-space (e.g., phase space in classical mechanics); Σ is a σ-algebra of measurable subsets of Γ; and there is a probability measure μ on Σ as usually defined. In our classical case, the natural Lebesgue measure.

with a non-causal mathematical explanation of some coarse-grained, in the long run, "necessary" behaviour. Notably, the necessity involved would not be physical necessity, but rather something like "mathematical" or "statistical necessity" (along this line, see Filomeno, 2019b).

Thus, together with the statement of the previous subsection that equilibrium states are typical, the Hamiltonian must also be typical. As defined in Section 3, take Γ_E to be the hypersurface of $6n-1$ dimensions of the phase space Γ in which the energy is conserved. Taking M_{eq} to be the equilibrium macrostate, take Γ_{Meq} to be the macroregion consisting of all $x \in \Gamma_E$ for which the macroscopic variables assume the values characteristic for M_E. Then, the underlying idea of the typicality of dynamics in statistical mechanics is expressed by Goldstein (2001, 43, my italics):

> Γ_E consists almost entirely of phase points in the equilibrium macrostate Γ_{Meq}, with ridiculously few exceptions whose totality has volume of order $10^{-10^{20}}$ relative to that of Γ_E. For a non-equilibrium phase point x of energy E, the Hamiltonian dynamics governing the motion x(t) would have to be ridiculously special to avoid reasonably quickly carrying x(t) into Γ_{Meq} and keeping it there for an extremely long time—unless, of course, x itself were *ridiculously special*

Frigg and Werndl (2012) formalize which conditions a dynamics has to meet to be typical. A first step is to restrict the discussion to the obtaining of *epsilon-ergodicity*, because they argue that this property is sufficient for thermodynamic behaviour (Vranas, 1998) and (Frigg and Werndl, 2011).[13] The proposal that the relevant Hamiltonians being ergodic explains thermodynamic behaviour was originally proposed by Boltzmann, but had since then been subjected to numerous criticisms and has accordingly been left aside by the philosophical and scientific community.[14] ϵ-ergodicity is meant to be a relaxed version of strict ergodicity. It does not require that $\mu_E(B) = 0$, allowing sets of initial conditions to be on non-ergodic solutions with respect to M_{eq}, only if the sets are of small size; that

[13]In this subsection we will follow the path of these authors for the sake of the exposition, but our main goal does not really depend on its specific actual success.

[14] Informally, a dynamical system is ergodic if and only if the proportion of time spent in a region A equals the measure of A, that is, the time average is equal to the phase average. Formally, a dynamical system $(\Gamma_E, \mu_E, \phi_t)$ is ergodic if and only if for any measurable subset A and any microstate $x \in \Gamma_E$,

$$T_A(x) = \mu_E(A)$$

except for a set B with $\mu(B) = 0$; where $T_A(x)$ is defined as the time-average of a solution originating in $x \in \Gamma_E$ relative to the measurable set A thus:

$$T_A = \lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_A(\Phi_\tau(x))d\tau$$

where $\chi(A) = 1$ for $x \in A$ and 0 otherwise. Considering B as the set of microstates which lie on non-ergodic solutions with respect to M_{eq}, it can also be said that system is ergodic iff $\mu_E(B) = 0$.

is: $\mu_E(B) \leq \epsilon$ (with ϵ a small number). It is proved in (Frigg and Werndl, 2012, 9) that, for ϵ-ergodic systems, initial conditions that lie on thermodynamic solutions are typical (cf. Vranas 1998). This is what they label as 'm-typical' ('m' standing for 'measure'). Then an explicit typicality measure of the dynamics is included. Thus their complete argument has this form (Frigg and Werndl, 2012, 6, 14):

Premise 1: The macrostate structure of the gas is such that equilibrium states are typical in Γ_E.
Premise 2: The Hamiltonian of the gas is typical for the class of all relevant Hamiltonians.
Conclusion: Typical initial conditions lie on solutions exhibiting thermodynamic behaviour.

The task, then, is to ascertain the truth of premise 2, granting their proof in which the relevant property P is being ϵ-ergodic. The Hamiltonian has to be typical with respect to a certain measure. The measure, for a dynamical trajectory, has to be a topological measure; thus, it can be said that it has to be 'topology-typical', or 't-typical'.

The conclusion of Frigg and Werndl (2012), I advance, is the following: the Hamiltonians are t-typical with respect to the so-called Whitney topology, where this is proved for the hard-sphere model and for the Lennard-Jones potentials, a subset of potentials which suffices for a wide class of realistic gases. Here we will also cite recent results that support a wider class of potentials.

Topological typicality of the Hamiltonians. To measure the typicality of the Hamiltonians in its dynamics space, instead of the Lebesgue measure we must employ a topological notion, namely that of 'generic' (aka 'comeagre'). A set is generic if and only if its complement is a countable union of nowhere-dense sets. Intuitively, the idea is to provide a measure able to inform us whether the other sets are scarce.[15] As we are searching for the typicality of the Hamiltonians, we seek to claim something like the following:

TypDyn: ϵ-ergodic Hamiltonians are comeagre/generic in the entire class of gas Hamiltonians G.

TypDyn is a refined version of premise 2 above. Unfortunately, the class G is too big. G is not easily definable, so Frigg and Werndl restrict their claim to something narrower: first, to the hard-sphere model, then, to the sub-class L of smooth Hamiltonians that are small perturbations of the so-called Lennard-

[15] In an equivalent rephrasing, a property is defined as *generic* if and only if it holds for a countable intersection of open dense subsets (Wiggins, 2003, 162). In slight more detail, a property of a vector field is said to be C^k *generic* if and only if the set of vector fields possessing that property contains a residual subset in the C^k topology, where a residual subset contains the intersection of a countable number of sets, each of which is open and dense in the topological space. k denotes the degree of differentiability ($0 \leq k < \infty$).

Jones potential. This constraint, the authors argue, is quite acceptable given their aim of explaining the thermodynamic behaviour of gases. Other potentials have also been studied, as we explain below. Thus, first we restrict the Hamiltonians to the smooth ones, namely, those with a fixed kinetic energy $T(p,q) = \frac{p^2}{2m}$. In the hard-sphere model (see fn 4 in p. 5), the potential V describes the abrupt repulsive forces of elastic collisions between two particles thus: $V(r) = \infty$ for $d < r$, and 0 otherwise; where d is the distance between the particles and r was the radius. In this model, numerical simulations and analytical results support the claim that the motion is ergodic (as originally conjectured by Boltzmann) for almost all parameter values. Sinai (1970) aimed to prove the assumption implicit in Boltzmann and Gibbs that, in the hard-sphere model, the trajectories of the particles are erratic paths wandering freely over the energy surface and spending equal times in equal hyperareas of this surface.[16] Then he proved that the hard-sphere gas is unstable and that this suffices to guarantee ergodicity and mixing – see Frigg and Werndl (2012) for details.

Then a potential $V(p,q)$ which has been thoroughly studied is the so-called Lennard-Jones potential, the most notorious and used to describe intermolecular interactions:[17]

> The Lennard-Jones potential is important because there is good evidence that the interaction between many real gas molecules is accurately described by that potential at least to a good degree of approximation. Hence, whatever potentials G comprises, many real gases cluster in a subclass of G, namely L, and so knowing how the members of L behave tells us a lot about how real gases behave. (Frigg and Werndl, 2012, 11)

After that, the choice of the Whitney topology comes from a physically natural way of saying that two Hamiltonians are close; namely, when the difference between the Hamiltonians themselves as well as all their derivatives is small (Frigg and Werndl, 2012, 10). The goal then is to confirm what we summarize as:

TypDyn2: ϵ-ergodic Hamiltonians are generic in the sub-class L of smooth gas Hamiltonians with a

[16] The proof is limited to 3 spheres; since then many proofs for a larger number of spheres have been sought, for references see Spohn (2012, 149).

[17] Specifically, this potential has the form:

$$V = 4\alpha\left(\left(\frac{\rho}{r}\right)^{12} - \left(\frac{\rho}{r}\right)^{6}\right)$$

where α describes the depth of the potential well, r the distance between two particles, and ρ is the distance at which the inter-particle potential is 0. The potential of the entire system is obtained by summing over all two-particle interactions. The r^{-12} term describes the repulsion forces at short ranges, The r^{-6} term describes the attraction forces at long ranges (van der Waals force or dispersion force).

Lennard-Jones potential with respect to the Whitney topology.

Now, to prove that TypDyn2 is the case, they show that the sub-class L is generic by referring to numerical simulations in which Hamiltonians that are ϵ-ergodic for the energy values defined are typical in L. In particular, Ford (1973) improved Sinai's results, extending the proof to systems that have attractive as well as repulsive interparticle forces. For further supportive numerical simulations see the references cited in Frigg and Werndl (2012, 12-14). For still other types of potentials see the references cited in Frigg and Werndl (2012, 18) and the more recent Gallagher et al. (2012) and Pulvirenti et al. (2014).

Before these results, there could have been pessimism about the typicality of dynamics in gases, due to the Kolmogorov–Arnold–Moser (KAM) theorem, which predicts nonergodic regions of stability. These are finite regions in phase space in which there are trajectories confined to perpetual quasiperiodic motion. In spite of the KAM theorem, the simulations of Ford (1973) provide evidence that, as the energies and densities of particles are varied (so even in the allegedly problematic cases of lower energies or higher densities), the expected entropic behaviour obtains without evidence of KAM regions. The KAM regions might be perfectly negligible, given that none has been found in the simulations and "nothing known precludes their being so small as to be physically irrelevant" (Ford, 1973, 1).

At the end of the day, the numerical simulations suggest that the Lennard-Jones gas, which has repulsive as well as attractive forces, exhibits thermodynamic behaviour. These results confirm that this type of gases conforms to the properties of ergodicity that were originally demanded by Boltzmann (and Gibbs), and that we demand on behalf of a typicality account that seems to hold for most of the dynamics.

The scope of the results. The class of Hamiltonians under consideration is not the entire set G, but rather a subset, albeit an important one, i.e. the subset L of Lennard-Jones potentials, together with other potentials, and with the also widely used potential of the hard-sphere model. Frigg and Werndl argue that this is acceptable because L is the most relevant subset for realistic gases and suffices for an approximate proof. While it can be thought that this is quite satisfactory for the long-standing issue in the foundations of statistical mechanics of explaining the tendency of gases towards equilibrium, for our purpose it would be welcome to extend the range of dynamics as far as we can. For ultimately we are exploring to what extent *lawful behaviour can emerge for an almost arbitrary, free state space trajectory under minimal constraints*. The fewer constraints we invoke, the wider the scope of the typicality explanation. Thus, for our purpose, we do not need to stop when typicality is proven for a wide class of *realistic* dynamics of gases. We would also like to know whether typicality is proven for the *unrealistic* dynamics.

Regarding the remaining wide and varied class of unrealistic dynamics, we could try to find further positive or negative results about the typicality of their properties. An exploration of typical properties of the dynamics space has been one of the main tasks in the history of dynamical systems, in the area known as topological dynamics. We want to conclude this section by surveying what are, to our knowledge, its most relevant results.

4.3 Further exploration of the dynamics space

In topological dynamics we can find results regarding the typical properties of trajectories, for instance, random-looking behaviour such as ergodicity, mixing and the like. As has often been pointed out, and as we have suggested in § 4.1, a dynamics can be random-looking in virtue of its chaotic properties. And such kind of property seems to be the required property, as for instance Frigg (following Dürr and Maudlin) says:

> the Galton Board seems to exhibit random behaviour. Why is this? Dürr's and Maudlin's answer is that the board appears random because random-looking trajectories are typical in the sense that the set of those initial conditions that give rise to nonrandom-looking trajectories has measure zero in the set of all possible initial conditions, and this is so because the board's dynamics is chaotic (Dürr, 1998, sec.2). Translating this idea into the context of SM suggests that the relevant property P is being chaotic. (Frigg, 2009, 1004)

Then, in this section we will outline some key results about whether or not typical trajectories are chaotic, or mixing, or ergodic, or other related properties. We first cite classic results for and against the genericity of chaos, which will be superseded because their domain is too constrained. Then we cite further results concerning wider domains of possible trajectories.

4.3.1 Classic but excessively constrained results.

There are methods that allow us to rule out the possibility of closed orbits, and thus of periodic motion: the methods of index theory, the existence of Lyapunov functions, and gradient systems (see any textbook on dynamical systems, e.g., Strogatz 1994). However, these desirable results have to be studied case by case, analysing each dynamical equation, and they say nothing about how frequently such periodic orbits are ruled out. On the negative side, the Poincaré-Bendixson theorem proves the existence of periodic orbits, which is incompatible with the existence of chaotic motion (Strogatz, 1994, 210). However, this result has only been proven for vector fields on a plane, that is, for 2 degrees of freedom.

Similarly, the literature on statistical mechanics has been critical of ergodicity, mainly on the grounds of the KAM theorem. Yet, a careful look at the theorem shows that it can hardly be taken to represent the *general* behaviour of classical mechanical systems; see above (§4.2 p.17), and Frigg (2009, 1005-6), Frigg and Werndl (2011, §5), Berkovitz et al. (2006, §4.1), and Sklar (1993, 174-5).[18]

4.3.2 Generic properties of the dynamics space

Non-integrability. Moving to the more general framework of topological dynamics, we find a result that stands for the typicality of chaotic behaviour in (Markus and Meyer, 1974, §3). They showed that, in the space of all normalized and infinitely differentiable Hamiltonian systems on a compact symplectic manifold, most of the Hamiltonians are non-integrable, a property which is assumed to imply chaos.

Structural stability. Other results concern the notion of 'structural stability', indirectly related to chaos. Roughly stated, a dynamical system (whether it be a continuous vector field or a discrete map) is said to be structurally stable if nearby systems have qualitatively the same dynamics. Smale (1966) proved that structural stability is *not* a generic property for n-dimensional diffeomorphisms when $n \geq 2$, or n-dimensional vector fields when $n \geq 3$ (Wiggins, 2003, 164), (Ruelle, 1989, 43 fn36). The connection between instability and the exponential divergence of nearby trajectories characteristic of chaos is straightforward.

Hyperbolicity. Another result shows that hyperbolic behaviour is generic: the Kupka-Smale theorem. It proves that the hyperbolicity of all periodic points holds for a set of C^r diffeomorphisms that is a dense set in the C^1 topology (Katok and Hasselblatt (1997, 289,292,295) and Ruelle (1989, 46)). 'Hyperbolicity' refers to the presence of expanding and contracting directions of the derivative. The presence of these directions produces exponential behaviour of trajectories on some set and, in the end, the stretching and folding gives rise to complex long-term behaviour. This effectively makes the dynamics appear random.

Ergodicity as a generic property. Another positive result comes from Oxtoby and Ulam (1941), who discovered that ergodicity is generic for measure-preserving homeomorphisms on all compact manifolds. More specifically, they discovered that the set of dynamical flows that are not ergodic belongs to the first category in the set of measure-preserving generalized dynamical flows.

[18]See also Frigg and Werndl (2011, §6) for a critique of the limited significance of the result by (Markus and Meyer, 1974, §4) against the typicality of ergodicity.

Extension of the results for ergodicity and weak mixing. Further, these results have since been extended. The genericity of ergodicity has been extended to automorphisms: Halmos (1944a) shows that ergodicity is generic in the weak topology in the space of all automorphisms. Finally, a further extension of the latter result is that the genericity of the stronger condition of weak mixing has also been proved, both in homeomorphisms and in automorphisms (Katok and Stepin, 1970; Halmos, 1944b).

A discussion of these and other results can be found in (Alpern and Prasad, 2001). It is beyond the scope and available space of this paper to properly discuss the applicability of these results: the technical notions that appear in the different results impose different constraints, e.g. the results applying to homeomorphisms or to automorphisms, or the scope being restricted to compact manifolds. Thus, the scope of the possible physical systems described requires further discussion (again, see Alpern and Prasad, 2001 for discussion). Then, these results can be studied case by case in different physical theories such as classical statistical mechanics, but also in any of the theories cited in Section 2. Be that as it may, in what follows we will discuss the type of conditions involved and the corresponding significance that this approach can have.

5 Discussion: the Role of the Constraints

We have focused on and assessed a type of explanation, the typicality approach in statistical mechanics which, as we have pointed out, is common in other areas of physics. The motivation for our analysis has been that the typicality approach (1) can be considered an explanation of coarse-grained *stable* behaviour, and (2) such stable, law-like, behaviour is explained almost independently of the specific details of the underlying dynamics.

This is familiar in the literature on scientific explanation, where it has been studied how, in diverse settings, the details of the underlying level are irrelevant (see e.g. Batterman, 2001; Strevens, 2008, 2013; Rohwer and Rice, 2013; Batterman, 2018, 2019.)The typicality approach looks like a more general yet less fine-grained explanation in comparison with the renormalization group explanations of the universality of critical phenomena, in which most of the details of the micro-level turn out to be irrelevant. Renormalization group techniques take into account the space of possible Hamiltonians, and in the procedure of coarse-graining and removing irrelevant micro-level details they show that for all Hamiltonians the system's state-space trajectory converges to a fixed point. The typicality approach, as we have seen, can be considered a *mathematical explanation* or, more specifically, a *statistical explanation*, of this sort. This quote by Strevens (2003, 62) clearly illustrates the underlying idea:

> The value of a [...] probability may come out the same on many different, competing stories about

fundamental physics. The probability of heads on a tossed coin, for example, is one half in Newtonian physics, quantum physics, and the physics of medieval impetus theory.

One might doubt that the typicality approach does provide a proper explanation. For a defence of the legitimacy of typicality explanations see Wilhelm (forthcoming).[19] We believe that a typicality explanation, although probably not the best explanation, it fulfils central desiderata that one would expect from an explanation. At least in our particular case study it seems that, *if* the kind of explanation we propose were successful,[20] we would have gained some insight on the existence of regularities in the world.

Of course, in the cases presented, some specific dynamical law is assumed. The interesting point is that, as we have argued, this does not imply that some specific dynamical law *must* be assumed. In classical statistical mechanics, the facts
(i) that the results are supposed to hold for most of the dynamics, and
(ii) that the relevant typical dynamical property is some randomizing property,
independently support that, merely by imposing prior general constraints, *a free evolution of a point in phase space will display, in the long run, coarse-grained stable patterns.*

Then, the process that has been studied in this paper yields a further reduction of degrees of freedom that occurs in the long run in coarse-grained levels. Now, the significance of the typicality of dynamics is conditional on the prior general constraints that in each case are imposed. For the only law-like assumptions are found in such constraints. We refer for instance to the *boundary conditions*, or to the *kinematical conditions* (spelled out below). A paradigmatic example of a condition that has been implicitly presupposed, both in the results for the actual dynamics in §4.2 as well as in some of the results that widened the range of the dynamics in §4.3, is the principle of *conservation of energy*. (For Hamiltonian dynamical systems are closed measure-preserving systems in which energy is conserved.) In general, the kind of constraints to which we refer are familiar in different physical theories. The point is that they are not the usual differential equations dictating the time-evolution of physical systems, but rather conditions which are still 'lawful' (or 'modal', or 'nomic') in that they constrain the possibility space. An illustrative standard example is that of a marble rolling on a bowl: its motion is constrained to the surface of the bowl. A constraint such as the bowl is different from the guiding dynamics, but it delimits the possible degrees of freedom.[21]

[19]See Lazarovici and Reichert (2015, §6) for a complementary philosophical assessment of the typicality approach in statistical mechanics, also in connection to the notion of law of nature. See (*ibidem*, §5) for further discussion of common objections to the typicality approach.

[20]In this paper we only propose such kind of explanation, of course we are far from defending its actual success.

[21]And it is possible to also constrain the range of parameters of the system, to constrain the range of quantitative forms

A classification of constraints and assessment of their import in complex dynamical systems is (Hooker, 2013). An especially basic type of constraint are the holonomic constraints, which means that they can be expressed purely *geometrically*, so that they are *independent of the behaviour of the system*. They can be written as some function of the space-time geometry, satisfying an equation of the form $f(r_1, r_2, ..., r_n, t) = 0$, where the r_i are the system coordinates and t is time (Hooker, 2013, 19).[22] The distinction that we are interested in highlighting, between constraints and the guiding dynamics, is often neglected in modelling physical systems, because *in practice* it is not necessary to differentiate the constraints from the dynamics (what Hooker calls "the interaction dynamics"). Hence constraints and dynamical laws are conflated in the Lagrangian formalism: the dynamical law that *describes* the motion—a differential equation—encompasses the constraints, it "compresses constraint and interaction information into a flow and in that sense suppresses the explicit details of the interactions and constraints" (Hooker, 2013, 15).

The resulting picture becomes clearer after characterizing the space of the *kinematically and dynamically possible models*. The kinematically possible is a space of functions that represent histories of the system. Define T_s as the class of all the kinematically possible trajectories in state-space. The idea is that T_s delineates the subspace of "metaphysical possibilities consistent with the theory's basic ontological assumptions" (Pooley, 2013, 12). This is generally defined by specifying the independent and dependent variables of the theory and the degree of smoothness of candidate functions, as well as any boundary conditions that they must satisfy (Belot, 2011, 5).[23] In the framework of classical mechanics, consider a model of gravitating point particles with distinct masses, as in (Belot, 2011, 7). A point in the space of the kinematically possible models of the theory assigns to each of the particles a worldline in spacetime, without worrying about whether the worldlines of each particle jointly satisfy the Newtonian laws of motion. There is then a subset of the kinematically possible models called the set of dynamically possible models. This is a space of solutions which is a $6N$-dimensional submanifold whose points correspond to the motions of particles that obey Newton's laws. Such a submanifold constrains us too much, to a single set of laws, i.e. the actual ones. In more detail, Pooley (2013, 12) (see also ibidem, 40) says:

> "In a coordinate-dependent formulation of Newtonian theory like that so far considered,

its dynamics can take. The effect of such constraints appears as constraints on variables (Hooker, 2013, 2).

[22] More precisely, "A constraint is holonomic if its local (momentary, nearby) differential form is integrable to yield a global constraint relation that is purely a matter of space-time geometry and independent of the system's dynamical states, e.g. the frictionless bowl as constraint for the rolling marble" (Hooker, 2013, 7, fn9).

[23] The whole space of metaphysical possibilities is thought to be too wide: it allows, for instance, that any function from points in space to points in space qualifies as a possible trajectory; thus, T_s follows from assumptions aimed at excluding certain behaviour, i.e. certain conceptual possibilities not ruled out by logic.

> the [kinematically possible models] might be sets of *inextendible smooth* curves in $\mathbb{R}^4$ which are nowhere tangent to surfaces of constant t (where $(t, \vec{x}) \in \mathbb{R}^4$). The models assign to the curves various parameters (m, ...). Under the intended interpretation, the curves represent possible trajectories of material particles, described with respect to a canonical coordinate system, and the parameters represent various dynamically relevant particle properties, such as mass. The space of [dynamically possible models] consists of those sets of curves that satisfy the standard form of Newton's equations"

The constraints that determine the kinematically possible, then, are *the only modal constraints*, the only conditions that play the role of laws.

An example ubiquitous in all of physics are the global continuous symmetries (which any dynamical law obeys), which are related to the conservation of some quantity. The literature on symmetries abounds in discussions on their status, but a standard view on symmetries is to take them as meta-laws or, quoting Wigner (1960), 'super-principles'. Notice that the symmetries can be exhibited *by the geometrical structure of space* in physical theories (by its topological, affine, and metric structure). In fact, Wigner highlighted that symmetry principles are grounded in the stable properties of the defining structure of spacetime (Martin, 2003, 50). According to such a geometrical interpretation, the symmetries of the laws are interpreted as symmetries of spacetime itself; they codify "the geometrical structure of the physical world" (Brading and Castellani, 2013, §5). To give an example, think of the homogeneity of space, assumed "in the physical description of the world since the beginning of modern science" (ibidem, §2.1). Another example is the time-translation invariance, related with the aforementioned conservation of energy via Noether's theorems.[24] [25]

Thus, if in a certain setting the explanation based on the typicality of dynamics is correct, I want to conclude with the moral that, in *any* account of laws of nature (such as those cited in footnote 1 or any other account), in order to explain the high degree of order (the ubiquity and extreme stability of certain regularities) at least in such setting, one does not need to ontologically commit to the usual governing dynamical laws. Instead, one can focus on accounting for conditions of a more general type, the constraints discussed in this section, which would suffice to explain the emergence of law-like behaviour. So, when the primitivist postulates (without much argument, as far as I know) laws of nature, she can appeal to the present moral to at least attenuate the suspicion that the (according to some,

[24] However, even if it is a paradigmatic example, we should keep in mind that it might not be true, according to general relativity (see e.g. Maudlin et al., 2019).

[25] I recur to the same examples in an analogous discussion in (Filomeno, 2019b, §4.2.3). It is interesting to note that both approaches converge at this point, while one starts from a result in chaos theory (Poincaré's method of arbitrary functions) while here we have started from the "neo-Boltzmannian" typicality explanation in statistical mechanics.

mysterious) primitive entity postulated looks like a contrived, highly specific, set of governing differential equations.[26] The deflationist (a Humean, say), according to what I have suggested, is justified in dispensing with some modal entities—the dynamical governing laws—although I do not see how she could dispense with all modal notions, as long as she aims to explain the extremely regular and apparently non-accidental patterns of the Humean mosaic. Some kind of justification should be given by the Humean to the remaining general modal constraints.[27] The propensity theorist might also be interested in picking up our conclusion, in order to suggest that those minimal constraints are to be found in dispositional properties. Although what we have proposed could also be seen as a reductive explanation of such propensities.

To recapitulate, in this paper I have:
(i) highlighted the common structure of explanation in certain projects in different fields of physics (§1-2);
(ii) explicitly seen what is needed for such typicality explanation to be successful in classical statistical mechanics (§3-4); and
(iii) argued that, in order to account for the emerging law-like behaviour, the specific form of the underlying dynamical laws is irrelevant, while the constraints gain a prominent role (§5). The fact that in certain contexts a free state-space evolution yields in the long run a stable reduction of further degrees of freedom supports the thesis that stable patterns can occur without needing to postulate—to ontologically commit to—the usual set of governing dynamical equations. The constraints that characterize the context, i.e. that define the kinematically possible states, are left as the only modal constraints. Hence, this shifts the attention to the ontological nature of such kinematical constraints.

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[26] In (Filomeno, 2016) I argue against a widespread satisfaction among philosophers and physicists primitivists about laws, which hope that a future Theory of Everything will be self-explanatory.

[27] I defend the need of the Humean account of laws to say something more in (Filomeno, 2019a). Then, the present study, as well as my Filomeno (2019b), helps the Humean, but is still insufficient.

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