FOR SOME PROPOSITION

AND SO MANY POSSIBLE WORLDS

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CONTENTS

ABSTRACT

In this thesis, I deal with the notions of a condition holding for some proposition and a proposition being true in a certain number of possible worlds. These notions are called propositional quantifiers and numerical modalizers respectively.

In each chapter, I attempt to dispose of a system. A system consists of: a language; axioms and rules of inference; and an interpretation. To dispose of a system is to prove its decidability and its consistency and completeness for the given interpretation. I shall, in passing, make applications to definability, translatability and other topics.

In Chapter 1, I consider the system S5Q. Its language is that of S5 with Q as a fresh unary operator. Its axioms and rules of inference are those for S5 plus the following special axiom-schemes for Q:

(1) $Q\,A \supset M\,A$

(2) $Q\,A \supset L\,(A \supset B) \vee L\,(A \supset \sim B)$

(3) $L\,(A \equiv B) \supset (Q\,A \supset Q\,B)$

(4) $Q\,A \supset L\,Q\,A$.

'Q A' is interpreted as 'A is true in exactly one possible world.' I dispose of the system by showing that every formula in it is equivalent to one in normal form.

In Chapter 2 I consider the system S5n (n for numerical modalizer). Its language is that of S5 but with the unary operators Q_k for each non-negative integer k. Its axioms and rules are those of S5 plus the following special axiom-schemes for Q_k:

(1) $Q_k A \supset \sim Q_l A$, $l < k$

(2) $Q_k A \equiv \bigvee_{i=o}^{k} Q_i (A \wedge B) \wedge Q_{k-i} (A \wedge \sim B)$.

(3) $L (A \equiv B) \supset (Q_k A \supset Q_k B)$

(4) $Q_k A \supset L Q_k A$

(5) $Q_o A \equiv L \sim A$, $l \geq 0$, $k \geq 1$.

'$Q_k A$' is interpreted as 'A is true in exactly k possible worlds.' I dispose of this system by generalising the normal forms of S5Q.

In the chapters 3-5, I consider three systems which result from adding propositional quantifiers to S5. The first two systems, S5π+ and S5π, contain the usual axioms and rules for quantifiers. The first contains, in addition, the axiom-scheme

$g = (\exists P) (P \wedge (R) (R \supset L (P \supset R)))$. The last, S5π-, results from S5π by restricting the Scheme of Specification, viz., $(P) A (P) \supset A (B)$, B free for P in A (P), to formulas B of the propositional calculus.

To interpret these systems we must specify which propositions the variable P ranges over. For S5π-, we merely require that if p and q are propositions, then (not p) and (p or q) are also propositions. For S5π+, we also require that each possible world be describable i.e. that there be a proposition which is true in that world alone. And for S5π, we require not that each possible world be describable but that there be a proposition which is true in just those possible worlds which are describable.

Again, we dispose of the systems by normal forms. This requires that we eliminate quantifiers and nested occurrences of L by adding new symbols to the language. For S5π+, the operators Q_k suffice. For S5π, the operators Q_k suffice. For S5π, we also require the constant g and a fresh unary operator N. For S5π-, even greater additions are required.

In the last two chapters, 6 and 7, I turn to systems which have essentially the same language as S5n. However, 'Q_k A' is now interpreted as 'A is true in exactly k possible worlds accessible from the given world.' Different conditions on R, the relation of accessibility, lead to different axioms.

In chapter 6 I consider the conditions of reflexivity, symmetry and transitivity, and in Chapter 7 the conditions of being a partial, convergent, total or dense order.

I prove consistency and completeness by the method of maximally consistent systems. The method can yield decidability results, but I do not go into the matter.

I have, as a rule, not given acknowledgements for well-established results or terminology. The main references are at the end of each chapter. Fuller references are in the bibliography.

Chapter 1

THE SYSTEM S5Q

In this chapter I consider a system containing the notion of a proposition being true in exactly one possible world.

§1. The system S5Q

Formation Rules. The wfs of S5Q are defined in the usual way from a set V of (propositional) variables P_1, P_2,..., the binary operator v, the unary operators , L and Q, and parentheses (and). Throughout the thesis I observe some familiar conventions: R and S, with or without numerical subscripts, and P range over variables; A, B, C, D, E and F, with or without subscripts, range over wfs; $\supset$, $\equiv$, M etc. are given standard definitions; each expression is used autonymously, i.e. as a name of itself; and parentheses are added to or omitted from wfs in an obvious way. The wf QA is read as 'A is true in exactly one (possible) world'.

Transformation Rules. The axioms are: all tautologous wfs; Gödel's axiom-schemes for S5. viz., $L\,A \supset A$, $L\,(A \supset B) \supset (L\,A \supset L\,B)$ and four special axiom-schemes for Q, viz.,

1) $Q\,A \supset M\,A$,
2) $Q\,A \supset (L\,(A \supset B) \vee L\,(A \supset \sim B))$,
3) $L\,(A \equiv B) \supset (Q\,A \supset Q\,B)$, and
4) $Q\,A \supset L\,Q\,A$.

The rules of inference are modus ponens (A, $A \supset B/B$) and necessitation ($A/L\,A$).

<u>Semantical Rules</u>. Semantics are based throughout the thesis on Kripke-type structures. A <u>structure</u> is an ordered pair (W, $\emptyset$), where W (worlds) is a non-empty set and $\emptyset$ is a map from V x W into $\{t, f\}$ (truth-values). $\emptyset$ assigns a truth-value to each variable in each world.

With each structure M = (W, $\emptyset$) is associated a unique relation $\models^M_w$ A between wfs and worlds: read as 'A is true in the world w for the structure M'. It is the smallest relation such that:

(i) $\models^M_w P_i$ iff $\emptyset (P_i, w) = t, i = 1, 2, \ldots$

(ii) $\models^M_w \sim A$ iff not $\models^M_w A$

(iii) $\models^M_w A \vee B$ iff $\models^M_w A$ or $\models^M_w B$

(iv) $\models^M_w L A$ iff for all V in W $\models^M_V A$

(v) $\models^M_w Q A$ iff there is exactly one V in W such that $\models^M_V A$.

We now define the notions of validity, being a model and logical consequence. The definitions are general and do not depend upon any particular definition of the relation $\models^M_w$ A. A is <u>valid</u>, $\models$ A, if for all structures M = (W, $\emptyset$) and for all $w \in W$, $\models^M_w$ A. M = (W, $\emptyset$) is a <u>model for</u> A (A <u>has a model</u> M) if for some $w \in W$, $\models^M_w$ A. A is a <u>logical</u> consequence of a set of wfs Δ, $\Delta \models$ A, if for all structures M = (W, $\emptyset$) and for all $w \in W$, $\models^M_w$ A whenever $\models^M_w$ B for all wfs B in Δ.

§ 2. Normal Forms

In this section I show that every wf is provably equivalent to a wf in normal form. (A is (provably) equivalent to B if $\vdash A \equiv B$).

Preliminary Results:

Lemma 1. $\vdash \sim Q\ A \supset L \sim Q\ A$.

Proof. By scheme 4, $\vdash Q\ A \supset L\ Q\ A$. By S5, it follows that $\vdash M\ Q\ A \supset M\ L\ Q\ A$. But by S5 again, $\vdash M\ L\ Q\ A \supset Q\ A$. So by PC (propositional calculus), $\vdash M\ Q\ A \supset Q\ A$. Hence $\vdash \sim Q\ A \supset L \sim Q\ A$.

Lemma 2. (The Equivalence Theorem). Let D be the result of replacing a particular occurrence of A in C by B. Then $\vdash L\ (A \equiv B) \supset (C \equiv D)$.

Proof. First deal with the case when A is the wf C. Then use induction on the length of C. The case when C is of the form QE requires the use of scheme 3 and S5.

Use of the Equivalence Theorem will often be tacit.

Reduction of Degree. The modal operators for S5Q are the symbols L and Q. (Generally, the modal operators for a system are those other than $\vee$ or $\sim$). The modal degree d of a wf is the maximum number of times a modal operator occurs within the scope of modal operators. More precisely, $d\ (P_i) = 0$, $d(\sim A) = d\ A$, $d(A \vee B) = \max\ (d(A),\ d\ (B))$, $d(L\ A) = d\ (Q\ A) = d(A) + 1$. A wf A is non-iterative if $d\ (A) = 0$ or $d(A) = 1$. In this section, we show that each wf is provably equivalent to a non-iterative wf.

First we require a lemma. Let A (C) be the result of replacing each occurrence of B in A(B) by C; and let T be the wf $(P_1 \supset P_1)$ and $\bot$ the wf $\sim T$.

Then:

Lemma 3. If X is a modal operator, $\vdash A(X\ B) \equiv (A\ (T) \wedge X\ B) \vee (A\ (\bot) \wedge \sim X\ B)$

Proof. I shall deal with the case when X is Q. The case when X is L is similar. By scheme (4) and S5, $\vdash Q\ B \supset L\ (Q\ B \equiv T)$. By lemma 2 (the Equivalence Theorem), $\vdash L\ (Q\ B \equiv T) \supset (A(Q\ B) \equiv A(T))$.
It follows by PC that (i) $\vdash Q\ B \supset (A(Q\ B) \equiv A\ (T))$. By lemma 1 and S5, $\vdash \sim QB \supset L\ (Q\ B \equiv \bot)$. By lemma 2, $\vdash L\ (Q\ B \equiv \bot) \supset (A(Q\ B) \equiv A(\bot))$.
It follows by PC that (ii) $\vdash \sim Q\ B \supset (A(Q\ B) \equiv A\ (\bot))$.
Hence from (i) and (ii) by PC,

$$\vdash A\ (Q\ B) \equiv (A(T) \wedge Q\ B) \vee (A\ (\bot) \wedge \sim Q\ B)$$

We can now prove:

Theorem 1. Each wf A is provably equivalent to a non-iterative wf C whose variables are those of A.

Proof: By induction on the modal degree d of A. For $d = 0$ or $d = 1$, let $C = A$. For $d > 1$, apply lemma 3 to all wfs X B of maximum modal degree in A.

Reduction to Normal Form

We now show that every non-iterative wf is equivalent to a wf in normal form. Normal forms are defined as follows: A state-description (sd) in the variables $R_1, R_2, \ldots, R_m$, $m \geq 0$, is a wf $B_1 \wedge B_2 \wedge \ldots \wedge B_m$ where $B_i = R_i$ or $B_i = \sim R_i$ for $i = 1, 2, \ldots, m$. (For $m = o$, the conjunction $B_1 \wedge B_2 \wedge \dots \wedge B_m$ is T). Let $C_1, C_2, \dots, C_n$, $n = 2^m$, be the distinct state descriptions in the variables $R_1, R_2, \dots, R_m$. Then a <u>model description</u> (md) in the variables $R_1, R_2, \dots, R_m$ is a wf.

(*) $C_j \wedge \bigwedge_{i=1}^{n} \alpha_i \, M \, C_i \wedge \bigwedge_{i=1}^{n} \beta_i \, Q \, C_i$, where

(i) each α_i and β_i is $\sim$ or blank

(ii) α_j is blank

(iii) If β_i is blank, then α_i is blank.

First we require a lemma to show that Q can be distributed through disjunction:

<u>Lemma 4.</u>

$\vdash Q\,(A \vee B) \equiv (Q\,A \wedge L\,(B \supset A)) \vee (Q\,B \wedge L\,(A \supset B))$

<u>Proof</u>. By S5, $\vdash L\,(B \supset A) \supset L\,(A \equiv (A \vee B))$.

So by scheme (3), S5 and PC, (i) $\vdash L\,(B \supset A) \supset (Q\,A \equiv Q\,(A \vee B))$. Similarly, (ii) $\vdash L\,(A \supset B) \supset (Q\,B \equiv Q\,(A \vee B))$. Now by scheme (2), $\vdash Q\,(A \vee B) \supset L\,((A \vee B) \supset A) \vee L\,((A \vee B) \supset \sim A)$. But by S5, $\vdash L\,((A \vee B) \supset A) \supset L\,(B \supset A)$ and $\vdash L((A \vee B) \supset \sim A) \supset L\,(A \supset B)$. So by PC, (iii) $\vdash Q\,(A \vee B) \supset L\,(B \supset A) \vee L\,(A \supset B)$. The lemma now follows by PC from (i), (ii) and (iii).

We can now prove:

<u>Theorem 2</u>. Any non-iterative wf is provably equivalent to $\perp$ or a disjunction of model descriptions in the variables of A.

Proof. Suppose QB occurs in A. Then QB is non-iterative and so by PC, B is equivalent to $\perp$ or a disjunction of state-descriptions in the variables of A. In the former case, by schemes (3) and (1) and S5, QB is equivalent to $\perp$. In the latter case, by (3), repeated applications of lemma 4, and PC, QB is equivalent to a truth-functional compound of wfs QC_i and non-iterative wfs LD, where C_i is a sd in the variables of A. But similarly by S5, each non-iterative wf LB is equivalent to a truth-functional compound of wfs MC_i. So A is equivalent to a truth-functional compound of wfs QC_i, MC_i and the variables of A. So by PC, A is equivalent to $\perp$ or a disjunction of wfs (*) which satisfy (i). If (ii) is not satisfied, then (*) is equivalent to $\perp$ by S5. If (iii) is not satisfied, then (*) is equivalent to $\perp$ by scheme (1). Hence by PC, A is equivalent to $\perp$ or a disjunction of md's in the variables of A.

Combining theorems 1 and 2 we obtain the main result:

Theorem 3. (Normal Forms). Any wf A is provably equivalent to $\perp$ or a disjunction of model-descriptions in the variables of A.

§3. S5Q is Characteristic.

We show that S5Q is characteristic, i.e. consistent and complete. For completeness we require the following lemma:

Lemma 1. Each model-description has a model.

Proof. Let the md be (*) of §2. We define M = (W, $\emptyset$) as follows: W is the smallest set such that

(a) $(C_i, 1) \in W$ if α_i is blank.

(b) $(C_i, 2) \in W$ if α_i is blank and β_i is $\sim$; and for $(C_i, \pi) \in W$,

$\emptyset (P_h, (C_i, \pi)) = t$ if P_h is a conjunct of C_i

$= f$ otherwise,

$i = 1, 2, \ldots, n, \quad h = 1, 2, \ldots$.

Clearly $\models^M_{(C_i\pi)} C_g$ iff $i = g$. Now by condition (ii) for (*), $(C_j, 1) \in W$. So $\models^M_{(C_j,1)} C_j$. By (a) and (b), $(C_i, \pi) \in W$ for some π iff α_i is blank. So

$\models^M_{(C_j,1)} \alpha_1 M C_i$. By (a) and (b) and condition (iii) for (*), $(C_i, \pi) \in W$ for exactly one π iff β_i is blank. So $\models^M_{(C_j,1)} \beta_i QC_i$. Hence $\models^M_{(C_j,1)}$ (*) and (*) has a model.

Now we have:

Theorem 1. $\vdash A$ iff $\models A$.

Proof $\Rightarrow$ (Consistency). By a straightforward induction on the length of the proof of A.

$\Leftarrow$ (Completeness). Assume not - $\models A$.

Then by PC, not - $\vdash \sim A \equiv \perp$. So by theorem 2.3,(§2, theorem 3), $\sim A$ is provably equivalent to a disjunction of md's. By lemma 1, any one of these md's has a model. So by consistency, $\sim A$ has a model and A is not valid.

The finite model property and decidability are almost immediate corollaries of previous results. We say that a structure $M = (W,\emptyset)$ is of cardinality c if card (W) = C. Then:

Corollary 1. Let A be a wf with m variables. If A has a model, then it has a model of cardinality $c \leq 2^{m+1}$.

Proof. Suppose that A has a model. Then by theorem 2.3 and consistency, A is equivalent to a disjunction of md's in the m variables of A. But it should be clear from the construction of lemma 1 that any one of these md's has a model of cardinality $c \leq 2^m . 2 = 2^{m+1}$. So by consistency, A has a model of cardinality $c \leq 2^{m+1}$.

Corollary 2. S5Q is decidable.

Proof. By corollary 1 and theorem 1.

Alternatively theorem 2.1 and 2.2 yield a mechanical procedure for determining whether a wf $\sim$ A is equivalent to $\perp$ or a disjunction of md's. In the former case, A is provable; in the latter case, not.

§4. Some Further Systems

1. Other axiomatizations of S5Q. Fairly simple arguments show that the schemes (1) - (4) are independent. However, a more compact though less perspicuous axiomatization may be obtained by replacing schemes (1) and (2) by $QA \supset (L\ (A \supset B) \equiv \sim L\ (A \supset \sim B))$ and (3) and (4) by $L\ (A \equiv B) \supset (Q\ A \supset L\ Q\ B)$. Alternatively, (3) and (4) may be replaced by a rule to infer $A \supset QB$ from $A \supset (L\ (B \supset P) \equiv \sim L\ (B \supset \sim P))$, where each occurrence of a variable in A is within the scope of a modal operator and where P is a variable which does not occur in A or B.

2. The systems S5W, S50 and S5M$_2$.

Suppose we have the following clauses for the unary operators W, O and M_2:

$\models^{M}_{W} WA$ iff $\models^{M}_{W} A$ and for all v in W if $v \neq w$ then

not $- \models^{M}_{v} A$.

$\models^{M}_{W} OA$ iff there is at most one v in W such that $\models^{M}_{v} A$.

$\models^{M}_{W} M_2A$ iff there are at least two distinct v's, v_1 and v_2, in W such that

$\models^{M}_{v_1} A$ and $\models^{M}_{v_2} A$.

Then corresponding characteristic systems require: for S5W the special axioms

$A \supset (MWA \supset WA)$.

$WA \supset A$, $WA \supset L(B \supset A)$ and $L(A \equiv B) \supset (WA \supset WB)$;

for S50

$L \sim A \supset OA$, $OA \supset L(A \supset B) \vee L(A \supset \sim B)$, $L(A \supset B) \supset (OB \supset OA)$ and $OA \supset L\,O\,A$;

and for S5M$_2$

$MA \supset MA$, $M(A \wedge B) \wedge M(A \wedge \sim B) \supset M_2 A$,

$L(A \supset B) \supset (M_2 A \supset M_2 B)$ and $M_2 A \supset L\, M_2 A$.

The four operators Q, W, O and M_2 are interdefinable according to the following valid equivalences: $M\,W\,A \equiv Q\,A$: $(Q\,A \vee L \sim A) \equiv O\,A$; $\sim O\,A \equiv M_2\,A$; and $A \wedge \sim M_2\,A \equiv W\,A$. Completeness for the three systems may be proved by normal forms as for S5Q or by defining Q in terms of the appropriate operator X, deducing the definition of X in terms of Q and the schemes (1) - (4), and then using the completeness of S5Q.

§5. References.

The original axiomatizations of S5Q, S5W and S5O are in Prior's *Egocentric Logic*. The ideas in that paper stimulated me to write this thesis. S5Q was shown to be complete, independently, by Bull, Kaplan and myself. Kaplan and I used essentially the same methods, and I have in places, used his terminology.

Chapter 2

THE SYSTEM S5n

In the last chapter we considered a system which contained the notion of a proposition being true in exactly one world. In this chapter we consider a system which for each non-negative integer K contains the notion of a proposition being true in exactly K worlds.[1]

§1. The System S5n

Formation Rules. The wfs are given by a set V of variables P_1, P_2,..., the binary operator v, the unary operators ~ and L, for each non-negative interger K the unary operator Q_K, and parentheses. For $K \geq 0$, Q_K A is read as 'A is true in exactly K worlds.'

Transformation Rules. The axioms are: all tautologous wfs; Gödel's axiom-schemes for S5; and five special axiom-schemes for Q_K, viz.,

(1) $Q_K A \supset \sim Q_1 A$, $K > 1$,

(2) $Q_K A \equiv \vee_{i=o}^{K} Q_i (A \wedge B) \wedge Q_{K-i} (A \wedge \sim B)$,

(3) $L (A \equiv B) \supset (Q_K A \supset Q_K B)$

(4) $Q_K A \supset L Q_K A$, and

(5) $Q_0 A \equiv L \sim A$, where K, l = 0, 1, 2, ...

The rules of inference are modus ponens and necessitation.

Instead of scheme (5) we could have used $L \sim A$ to abbreviate Q_0 A. For schemes (1)- (4) we could exclude the case K = 0 since this follows from S5 and scheme (5).

[1] For K read κ

Scheme (2) states that A is true in exactly K worlds iff for some $i \leq K$, $A \wedge B$ is true in exactly i worlds and $A \wedge \sim B$ is true in exactly $(K - i)$ worlds.

Semantical Rules. A structure M is defined as for S5Q. $\models^{M}_{W} A$ is defined in the usual way with the following clause for Q_K, $K \geq 0$:

$$\models^{M}_{W} Q_K A \text{ iff there are exactly K v's in W such that } \models^{M}_{v} A.$$

§2. Normal Forms.

We show that each wf has a normal form.

Preliminary Results. We say that T is an orthodox extension of S5 if

(i) it is obtained by adding new unary propositional operators and axioms to S5, and

(ii) for each such operator X, and wfs A and B, $L(A \equiv B) \supset (X A \subset X B)$ and $X A \subset L X A$ are theorems of T.

Now the proof of the Equivalence Theorem in Chapter 1 used schemes (3) and (4), but not schemes (1) and (2), of S5Q. So it should be clear that the Equivalence Theorem holds for any orthodox extension of S5. By schemes (3) and (4), S5n is an orthodox extension and so:

Lemma 1. (Equivalence Theorem). If D is the result of replacing a particular occurrence of A in C by B, then $\vdash_{S5_n} L(A \equiv B) \supset (C \equiv D)$.

Reduction of Degree. Note that the proof of theorem 1.2.1 (Chapter 1, § 2, theorem 1) also uses schemes (3) and (4), but not (1) and (2), of S5Q. To reduce the degree of a wf we use the equivalence $A(XB) \equiv$

~~(A (T) ∧ X B) ∨~~ $(A(\top) \wedge X\, B) \vee (A(\bot) \wedge \sim X\, B)$ of lemma 1.2.3. Now a modal operator or variable occurs on the right-hand side of this equivalence iff it occurs on the left-hand side. So it should be clear that for any orthodox extension of S5 each wf A is provably equivalent to a non-iterative wf B which has the same variables and modal operators as A. Define the <u>degree</u> of a wf A of $S5_n$ (not to be confused with its modal degree) as the least positive integer $l \geq 1$ such that for no $K \geq l$ does Q_K occur in A. Then since $S5_n$ is an orthodox extension it follows that:

<u>Theorem 1</u>. Each wf A of $S5_n$ is provably equivalent to a non-iterative wf B with the same degree and variables as A.

<u>Reduction to Normal Form</u>. We now show that each non-iterative wf has a normal form. Normal forms are defined as follows: Let $C_1, C_2, \ldots, C_n$ be the distinct state-descriptions in the distinct variables $R_1\, R_2 \ldots, R_n$, $m \geq 0$, $n = 2^m$. For $K \geq 1$, let $M_K\, A$ abbreviate $\wedge_{i=1}^{K-1} \sim Q_i A$. $M_K A$ is read as 'A is true in the least K worlds.' Then a <u>model-description</u> (of degree $l \geq 1$) in the variables $R_1, R_2, \ldots, R_m$ is a wf

(*) $C_j \wedge \wedge_{i=1}^{n} \alpha_i\, C_i$ where

(i) each α_i is Q_K for some $K < l$ or M_l, and

(ii) α_j is not Q_o.

First we require a lemma to show that Q_K can be distributed through disjunction:

<u>Lemma 2</u>. For $K \geq 0$,

$$\vdash L\ (A \supset \sim B) \supset Q_K\ (A \vee B) \equiv \bigvee_{i=o}^{K} Q_i\ A \wedge Q_{K-i}\ B$$

<u>Proof</u>. By scheme (2),

$$Q_K\ (A \vee B) \equiv \bigvee_{i=o}^{K} Q_i\ ((A \vee B) \wedge A) \wedge Q_{K-i}\ ((A \vee B) \wedge \sim A).$$

But by S5, $\vdash L\ (((A \vee B) \wedge A) \equiv A)$ and

$\vdash L\ (A \supset \sim B) \supset L\ (((A \vee B) \wedge \sim A) \equiv B)$. The result now follows by the Equivalence Theorem and PC.

We now have:

<u>Theorem 2</u>. Any non-iterative wf A of degree 1 is provably equivalent to or a disjunction of md's of degree 1 in the variables of A.

<u>Proof</u>. By scheme (5), L B is equivalent to $Q_o \sim B$. So by the Equivalence Theorem we may suppose that L does not occur in A. Now suppose that Q_K B occurs in A, $0 \leq K < 1$. Then Q_K B is non-iterative and so by PC, B is equivalent to $\perp$ or a disjunction of sd's in the variables of A. In the former case, by S5 and schemes (5) and (1), Q_K B is equivalent to T if $K = 0$ and to $\perp$ if $K > 0$. In the latter case, by repeated applications of lemma 2, Q_K B is equivalent to a truth-functional compound of wfs $Q_h\ C_i$, where $h \leq K$ and C_i is a sd in the variables of A, i = 1, 2, ..., n. So by the Equivalence Theorem and PC, A is equivalent to $\perp$ or a disjunction of wfs.

$$(**)\ C_j \wedge \bigwedge_{i=1}^{n} \bigwedge_{h=o}^{1-1} \alpha_{ih}\ Q_h\ C_i,$$

where each α_{ih} is blank or $\sim$. Now for each i = 1, 2, ..., n, there are three possibilities:

(a) for some distinct g and h, $\alpha_{ig} = \alpha_{ih}$ = blank;

(b) there is exactly one h such that α_{ih} = blank;

and (c) for each h, $\alpha_{ih} = \sim$; g, h = 1, 2, ..., l = 1.

In case (a) (**) is equivalent to $\perp$ by scheme (1).

In case (b), $\bigwedge_{h=o}^{l-1} \alpha_{ih} Q_h C_i$ is equivalent to $Q_n C_i$ by scheme (1), and by scheme (5) and S5 the whole wf (**) is equivalent to $\perp$ if i = j and h = o.

In case (c), $\bigwedge_{h=o}^{l=1} \alpha_{ih} Q_h C_i$ is equivalent to $M_l C_i$. So by PC, A is equivalent to $\perp$ or a disjunction of md's of degree l in the variables of A.

Combining theorems 1 and 2 we obtain the main result:

Theorem 3. Any wf A of degree l is provably equivalent to $\perp$ or a disjunction of md's of degree l in the variables of A.

§3. $S5_n$ is Characteristic.

As before, we require the following lemma for completeness.

Lemma 1. Each model-description has a model.

Proof. Let the md be (*) of the previous section. We define M = (W, Ø) as follows: W is the set which contains the pairs

(a) $(C_i, \pi) \epsilon$ W for π = 1, 2, ..., K if $\alpha_i = Q_K$

(b) $(C_i, \pi) \epsilon$ W for π = 1, 2, ..., l if $\alpha_i = M_l$; and for $(C_i\ \pi)\ \epsilon$ W,

$\emptyset (P_h, (C_i, \pi))$ = t if P_h is a conjunct of C_i

= f otherwise,

i = 1, 2, ..., n, K = 0, 1, ..., l, h = 1, 2, ...

Clearly, $\models^{M}_{(C_i, \pi)} C_g$ iff i = g. Now by condition (ii) for (*), $(C_j, 1) \epsilon W$.

So $\models^{M}_{(C_j,1)} C_j$.

By (a) and (b) and condition (i) for (*), $(C_i, \pi) \in W$ for exactly K π's iff σ_i is Q_K. So $\models^{M}_{(C_j,1)} \sigma_i C_i$. Hence $\models^{M}_{(C_j,1)}$ (*) and (*) has a model.

We now have:

Theorem 1. $\vdash$ A iff $\models$ A.

Proof. $\Rightarrow$ (Consistency). Straightforward.

$\Leftarrow$ (Completeness). As for S5Q.

We note the following corollaries:

Corollary 1. Let A be a wf with m variables of degree 1. If A has a model, then A has a model of cardinality $C \leq 1.2^m$.

Proof. This should be clear from theorem 3 of § 2, consistency and the construction of M in lemma 1.

Corollary 2. $S5_n$ is decidable.

Proof. As for S5Q.

§4. Some theorems on Definability.

A number-theoretic (nt) relation $Rx_1 x_2 \ldots x_n$ is one defined on the non-negative integers. A nt relation R is simple if there is a number $l \geq 1$ such that for all $a_j > b \geq l$ if $Ra_1, a_2 \ldots a_n$ holds, then $Ra_1 \ldots a_{j-1} ba_{j+1} \ldots a_n$ holds, j = 1, 2, ..., n.

The least l satisfying this condition is called the degree of the relation R. Thus a simple relation fails to distinguish between numbers greater than or equal to its degree. An nt relation R is defined by the wf $A(R_1, \ldots, R_n)$ If for all models $M = (W, \emptyset)$ and for all $w \in W$, $\models^{M}_{w} A(B_1, \ldots, B_n)$ iff

$R_{K_1 \ldots K_n}$, where K_i = Card ($\{w \in W: \models^M_w B_i\}$), i = 1, 2, ..., n. In this section we show that a nt relation is definable (by a wf) iff it is simple.

First we require a lemma on reducing models. Let M = (W, Ø) and M' = (W', Ø') be any two structures. We say that W is identifiable with v in M, $w \sim_M v$, if w, v $\in$ W and for all variables P,

$$\emptyset (P, w) = \emptyset (P, v).$$

M' is a substructure of M if W' $\subseteq$ W and Ø' is the restriction of Ø to V x W'. Let $[w]_M = \{v : v \sim_M w\}$. Then M' is a 1-reduction of M 1, $\geq$ 1, if

(i) M' is a substructure of M

(ii) Whenever Card ($[w]_M$) = K < 1, then $[w]_M \subseteq$ W', and

(iii) whenever Card ($[w]_M$) $\geq$ 1, then

Card ($[w]_M \cap$ W') $\geq$ 1.

Thus the 1-reductions of a structure fail to distinguish between 1 or more truth-functionally identical worlds in the structure. We may now show by an easy induction on the length of A:

Lemma 1. If A is a wf of degree 1 and M' is a 1-reduction of M, then

$\models^{M'}_{w'}$ A iff $\models^M_w$ A for all w' in W' and w in W such that $w' \sim_m w$.

We now come to the main result:

Theorem 1. If R is a nt relation, then R is definable by a wf of degree 1 iff R is a simple relation of degree 1 or less.

Proof. ⇒ Assume that $R_{x_1, x_2 \ldots, x_n}$ is defined by the wf A $(R_1, R_2, \ldots, R_n)$ of degree l. We show that R is a simple relation of degree l or less. Suppose that $a_j > b \geq l$ and $Ra_1, a_2 \ldots a_n$ holds. Define a structure $M = (W, \emptyset)$ as follows:

W is the set of pairs (R_i, π) for $\pi = 1, 2, \ldots, a_i$, $i = 1, 2, \ldots, n$;

$\emptyset (P_h, (R_i, \pi)) = t$ if $h = i$

$= f$ otherwise.

Then clearly Card $(\{w \in W: \models^M_w R_i\}) = a_i$, $i = 1, 2, \ldots, n$. So, by definability, $\models^M_w A(R_1, R_2, \ldots, R_n)$ for all w in W.

Now let $M' = (W', \emptyset')$ be the substructure of M such that

$W' = W - \{(R_j, b + 1), (R_j, b + 2), \ldots, (R_j, a_j)\}$. Then clearly M' is a l-reduction of M. So, by the lemma, $\models^{M'}_{w'} A (R_1, R_2, \ldots, R_n)$ for all w' in W. But clearly,

$$\text{Card } (\{w \in W': \models^{M'}_w R_i\}) \begin{array}{l} = a_i \text{ if } i \neq j \\ = b \text{ otherwise.} \end{array}$$

So by definability, $Ra_1 \ldots a_{j-1}\, b a_{j+1} \ldots a_n$ and R is simple and of degree l or less.

⇐ Assume R is simple and of degree l. We prove by induction on the number of arguments n of R that R is definable by a wf of degree l.

Case 1. $n = 1$. Suppose $R \cap \{0, 1, \ldots, l-1\} = \{a_1, a_2, \ldots, a_m\}$. Then either Rx holds if $x \in \{a_1, a_2, \ldots, a_m\}$, where a_m say is l-1 since R is of degree l, or Rx holds if $x \in \{a_1, a_2, \ldots, a_m\} \cup \{K : K \geq l\}$. In the first case, R is definable by $\bigvee^m_{i=1} Q a_i\, P$, and in the second case by $\bigvee^m_{i=1} Q a_i\, P_1 \vee M_l\, P_1$. In both cases the wfs are of degree l.

Case 2. $n > 1$. Since R is simple, $Rx_1 x_2 \dots x_n$ holds iff

$\bigvee_{k=o}^{l-1} (x_1 = k \wedge Rk\, x_2 \dots x_n) \vee (x_1 > l \wedge Rl\, x_2 \dots x_n)$ holds, where

$x_1 = k$, $Rk\, x_2 \dots x_n$, $x_1 > l$ and $Rl\, x_2 \dots x_n$ are simple and of degree l or less. So by case 1 and the induction hypothesis these relations are definable by $Q_k\, P_1$, $A_k(P_2, \dots, P_n)$, $M_l\, P_1$ and $B(P_2, \dots, P_n)$ of degree l or less respectively. So $Rx_1 x_2 \dots x_n$ is definable by $\bigvee_{k=0}^{l-1} (Q_k\, P \wedge A_k\, (P_2, \dots P_n)) \vee M_l\, P_1 \wedge B\, (P_2, \dots, P_n)$, which is of degree l.

Finally, if B of degree K defines R of degree K, $K < l$, then $B \vee M_l\, (\bot)$ defines R and is of degree l.

We note two corollaries:

Corollary 1. $Q_l\, P$ is not equivalent to a wf B of degree $K \leq l$, $l \geq 1$.

Proof. Assume otherwise. Then B defines $x = l$. So by the theorem $x = l$ is a degree K or less, $K \leq l$. But $x = l$ is of degree $l + 1$.

Corollary 2. The following nt relations are not definable in S5n: $x = y$, $x < y$, x is even, x is prime, $x = y + 1$ etc.

Proof. By theorem 1, since none of these relations is simple.

I have not considered modal systems whose wfs define non-simple nt relations.

§5. Predicate Calculus Analogies.

The operators Q_k (M_k) are analogous to the quantifiers "there are exactly (at least) k individuals such that". In this section we show that $S5_n$ and a part of quantification theory with these quantifiers are intertranslatable.

First we define FC, the first-order functorial calculus with equality:

Formation Rules. Wfs are defined in the usual way from sets F_i of predicate letters, f_1^i, f_2^i, ..., i = 1, 2, ..., the symbol for equality = , the individual variables x_1, x_2, ..., the quantifiers (x_i), the truth-functional connectives $\vee$ and $\sim$, and parentheses.

Transformation Rules. The axioms consist of specification, distribution and vacuous quantification for the quantifiers, i.e.

$(x)\ \emptyset(x) \supset \emptyset(y)$, y free for x in $\emptyset(x)$,

$(x)\ (\emptyset \supset \psi) \supset (x)\ \emptyset \supset (x)\ \psi$ and

$\emptyset \supset (x)\ \emptyset$, x not free in $\emptyset$,

respectively; and for equality, the axiom $x = x$ and the scheme $x = y \supset (\emptyset \supset \psi)$, where ψ is the result of substituting free y for free x in $\emptyset$.

The rules are modus ponens and generalisation $(A/(x)\ A)$.

Semantical Rules. A (quantification) structure $\mathfrak{A}$ is an ordered pair $(D, (\emptyset_i))$, where D is a non-empty set and $(\emptyset_i)$ is a set of maps $\emptyset_i$ from $F_i \times D^i$ into

$\{t, f\}$, i = 1, 2, ... $(D^i = \overbrace{D \times D \times \ldots \times D}^{i})$.

With each structure $\mathfrak{A} = (D, (\emptyset_i))$ is associated a unique relation $\models^{\mathfrak{A}}_{s} \emptyset$ between wfs and infinite sequences of elements in D according to the following rules:

(i) $\models^{\mathfrak{A}}_{S} f^i_j x_{k_1} x_{k_2} \ldots x_{k_n}$ iff $\langle S_{k_1}, S_{k_2}, \ldots, S_{k_n} \rangle$

$$\in \phi_i (f^i_j)$$

(ii) $\models^{\mathfrak{A}}_{S} x_{k_1} = x_{k_2}$ iff $S_{k_1} = S_{k_2}$

(S_k is the k-th term of the sequence S).

(iii) $\models^{\mathfrak{A}}_{S} \sim \phi$ iff not - $\models^{\mathfrak{A}}_{S} \phi$

(iv) $\models^{\mathfrak{A}}_{S} (\phi \vee \psi)$ iff $\models^{\mathfrak{A}}_{S} \phi$ or $\models^{\mathfrak{A}}_{S} \psi$

(V) $\models^{\mathfrak{A}}_{S} (x_i) \phi$ iff for all sequences t such that $t_j = S_j$ for all $j \neq i$, $\models^{\mathfrak{A}}_{t} \phi$.

We may also add the quantifiers $(\exists_k x)$ and $(\exists_k x)$ to FC with the following clauses:

$\models^{\mathfrak{A}}_{S} (\exists_k x_i) \phi$ iff there are exactly k t's such that $t_j = S_j$ for all $j \neq i$ and $\models^{\mathfrak{A}}_{t} \phi$, and

$\models^{\mathfrak{A}}_{S} (\exists_k x_i) \phi$ iff there are at least k t's such that $t_j = s_j$ for all $j \neq i$ and $\models^{\mathfrak{A}}_{t} \phi$.

As is well known, $(\exists_k x)$ and $(\exists_k x)$ may be defined in FC by the following valid equivalences:

$(\exists_o x) \phi (x) = T$;

for $k > 0$, $(\exists_k x) \emptyset (x) \equiv (\exists y_1) \exists y_2) \ldots (\exists y_k)$

$$\left[\bigwedge_{1 \leq i < j \leq k} \sim (y_i = y_j) \wedge \bigwedge_{i=1}^{k} \emptyset (y_i) \right];$$

and for $k \geq 0$

$$(\exists!_k x) \emptyset (x) \equiv (\exists_k x) \emptyset (x) \wedge \sim (\exists_{k+1} x) \emptyset (x);$$

where $y_1, y_2, \ldots, y_k$ are the first k variables which are not free in $\emptyset (x)$.

If the symbol and axioms for equality are omitted from FC, then the quantifiers $(\exists_k x)$ (or $(\exists!_k x)$) may be added as primitives with appropriate axioms. I do not consider this problem.

If we omit all variables except x $(= x_1)$ and all predicate letters except $f_1^1, f_2^1, \ldots$, then we obtain a theory isomorphic with $S5_n$. For let TA be the result of replacing P_i by $f_i^1\, x$, L by (x), and Q_k by $(\exists!_k x)$. Then we may prove:

<u>Theorem 1</u>. A is valid iff TA is valid.

<u>Proof</u>. Clearly, we may suppose that a quantification structure for TA is an ordered pair $(D, \emptyset_1)$. We may then set up a one-one correspondence $M \rightarrow \mathfrak{Q}_m$ between modal and all such quantification-structures: given $M = (W, \emptyset)$, let $\mathfrak{Q}_m = (W, \emptyset_1)$, where for all w in W, $\emptyset (P_i, w) = \emptyset_1 (f_i^1, W)$, $i = 1, 2, \ldots$. Then we show by an easy induction that for all w in W,

$$\models^{M}_{w} A \text{ iff } \models^{\mathfrak{Q}_m}_{W} TA.$$

The theorem now follows.

Let the axioms for our quantification theory be TA where A is an axiom of S5n and let its rules be modus ponens and generalisation. Then

Theorem 2. A is a theorem iff TA is a theorem.

Proof. By an easy induction on the length of the proof.

From theorems 1 and 2 and the fact that S5n is characteristic, we obtain

Theorem 3. For any wf $\emptyset$ of our quantification theory, $\emptyset$ is a theorem iff $\emptyset$ is valid.

§6. Some Further Systems.

1. Other Axiomatizations of S5n.

As for S5Q, schemes (3) and (4) may be replaced by $L (A \equiv B) \supset (Q_k A \supset L Q_k B)$. The single distribution scheme (2) may be dropped in favour of several simpler schemes, e.g.,

$Q_k A \supset \bigvee_{i=0}^{k} Q_i (A \wedge B)$,

$Q_i (A \wedge B) \wedge Q_{k-i} (A \wedge \sim B) \supset Q_k A$ and

$Q_i (A \wedge B) \wedge Q_k A \supset Q_{k-i} (A \wedge \sim B)$, $i \leq K$.

2. The System S5n with W_k, O_k and M_k as primitive.

Suppose we have the following clauses for the unary operators W_k, O_k and M_k, k = 1, 2, ...:

$\models^M_W W_k A$ iff $\models^M_W A$ and there are exactly k v's in W such that $\models^M_v A$

$\models^M_W O_k A$ iff there are at most k v's in W such that $\models^M_v A$

$\models^{M}_{w}$ M_k A iff there are at least k v's in W such that $\models^{M}_{v}$ A.

The corresponsing characteristic systems require: for S5n with W_k the special axiom-schemes $B \wedge W_k\ A \equiv \vee^{k}_{i=1}\ W_i\ (A \wedge B) \wedge M\ W_{k-i}\ (A \wedge \sim B)$, $L\ (A \equiv B) \supset (W_k\ A \supset W_k\ B)$ and $W_k\ A \supset \sim M\ W_1\ A$, $1 < k$, where '$W_o A$' abbreviates '$L \sim A$';

for S5n with O_k the schemes $O_k\ A \supset O_1\ A$, $k \leq 1$, $O_k\ A \equiv \vee^{k}_{i=o}\ O_i\ (A \wedge B) \wedge O_{k-i}\ (A \wedge \sim B)$, $L\ (A \equiv B) \supset (O_k\ A \supset O_k\ B)$ and $O_k\ A \supset L\ O_k\ A$, where '$O_o A$' abbreviates '$L \sim A$';

and for S5n with M_k the schemes $M_k\ A \supset M_1\ A$, $k \geq 1$, $M_k\ A \equiv \vee^{k}_{i=o}\ M_i (A \wedge B) \wedge M_{k-i}\ (A \wedge \sim B)$, $L\ (A \equiv B) \supset (M_k\ A \supset M_k\ B)$ and $M_k\ A \supset L\ M_k\ A$, where '$M_1\ A$' and '$M_o\ A$' abbreviate '$M\ A$' and 'T' respectively.

The completeness of the systems may be proved by normal forms as for S5n with Q_k. On the other hand, since the operators Q_k, W_k, O_k and M_k are interdefinable, the completeness of one system may be reduced to that of another by means of appropriate definitions.

The new systems can be re-axiomatized in various ways. E.g., the third scheme for the O_k-system may be replaced by $O_i\ (A \wedge B) \wedge O_{k-i}\ (A \wedge \sim B) \supset O_k\ A$, $i \leq k$, and $O_k\ A \supset O_i\ (A \wedge B) \vee O_{k-(i+1)}\ (A \wedge \sim B)$, $i < k$; and the first, second and fourth schemes for the M_k-system may be replaced by the single scheme

$$L\ (A \supset B) \supset (M_k \supset L\ M_1\ A),\ 1 \leq k.$$

3. Some Subsystems

It should be clear from the completeness proof for S5n with Q_k that in proving a wf of degree l, l $\geq$ 1, we need only use axioms of degree l or less. So if we restrict wfs and axioms to those of degree l or less we obtain a characteristic system whose unary operators are L, Q_1, Q_2, ..., Q_{l-1}.

This raises the following problem:

Suppose K is any set of positive integers. What is a characteristic system for wfs A such that Q_k occurs in A iff $k \in K$? A general yet elegant solution to this problem would raise formidable combinatorial difficulties. In case K = {2}, the following axiom-schemes suffice:

$Q_2\, A \supset M\, A$,

$Q_2\, A \wedge Q_2\, (A \vee B) \supset L\, (B \supset A)$,

$Q_2\, A \supset (L\, (A \supset B) \vee L\, (A \supset \sim B) \vee L(A \wedge B \supset C) \vee L\, (A \wedge B \supset \sim C))$,

$Q_2\, (A \vee B) \wedge Q_2\, (C \vee D) \wedge M\, A \wedge M\, B \wedge MC \wedge M\, D \wedge L\, (A \supset \sim B) \wedge L\, (C \supset \sim D) \wedge L\, (A \supset \sim C) \supset Q_2\, (A \vee C)$,

$L\, (A \equiv B) \supset (Q_2\, A \supset Q_2\, B)$ and

$Q_2\, A \supset L\, Q_2\, A$. I omit the proof that the system is characteristic.

Systems with Infinitary Operators

Suppose we have the following clause for the unary operator M_c where c is any infinite cardinal:

$\models^{M}_{w} M_c\, A$ iff there are at least c v's such that $\models^{M}_{v} A$.

Let C be a set of infinite cardinals. Then if we add M_c for $c \in C$ to S5n with M_k we require the following additional schemes:

$M_c A \supset M_d A, d < c,$

$M_c A \equiv M_c (A \wedge B) \vee M_c (A \wedge \sim B),$

$L (A \equiv B) \supset (M_c A \supset M_c B),$

$M_c A \supset L M_c A, c \in C, d \in C \cup \{1, 2, 3, \ldots\}.$

The proof of completeness assumes that for infinite cardinals c and d, $c + d = \max (c, d)$. This may be proved using the axiom of choice.

The system with O_c is similar. The systems with W_c and Q_c are more complicated.

§7. References

Kaplan has constructed and proved the completeness of S5n with M_k and Q_k independently from this author. See his abstract Multiple Possibility. Tarski introduces the quantifiers $(\exists_k x)$ and $(\exists\!\!\exists_k x)$ in his Introduction to Logic. He calls them 'numerical quantifiers.' The operators Q_k and M_k might, by analogy, be called 'numerical modalizers'.

Kaplan uses $\Diamond^k$ for M_k and Δ^k for Q_k. Mostowski uses $(Q_c x)$ for $(\exists_c x)$ where c is an infinite cardinal.

Chapter 3

THE SYSTEM $S5_{\pi+}$

The operators Q_k cannot be defined in terms of L, but they can be defined in terms of L and propositional quantifiers. E.g., $Q_\bullet$ A may be defined as $M A \wedge (P) (L (A \supset P) \vee L (A \supset \sim P))$, where P is not free in A. So in this chapter I consider a system obtained by adding propositional quantifiers to S5.

§1. The System $S5_{\pi+}$

Formation Rules. The wfs are given by a set V of propositional variables P_1, P_2, ..., the binary operator v, the unary operators ~ and L, the quantifiers (P_i) for i = 1, 2,.., and parentheses.

Transformation Rules. The axioms are: all tautologous wfs; Gödel's axiom-schemes for S5; specification, distribution and vacuous quantification, i.e. $(P) A (P) \supset A (B)$, where B is any wf free for P in A (P), $(P) (A \supset B) \supset (P) A \supset (P) B$ and $A \supset (P) A$, where P is not free in A, respectively; and a special axiom $(\exists P_1) (P_1 \wedge (P_2) (P_2 \supset L (P_1 \supset P_2)))$ which we call g.

The rules of inference are modus ponens, necessitation and generalisation (A/(P) A).

Semantical Rules. A structure M is defined in the usual way. However, in the definition of $\models^M_W$ A we require a clause for the quantifier (P). Given a structure M = (W, Ø) we say that the structure M' = (W', Ø') is a

M P structure if $W' = W$ and for all R distinct from P and for all $w \in W$,

$$\emptyset (R, w) = \emptyset' (R, w).$$

Then

$\models^{M}_{w}$ (P) A iff for all M P structures M', $\models^{M'}_{w}$ A.

§2. Normal Forms

We shall show that each wf is provably equivalent to $\perp$ or a disjunction of model-descriptions as defined in Chapter 2.

Preliminary Results.

Lemma 1. (Equivalence Theorem). Suppose that D is the result of replacing a particular free occurence of A in C by a free occurrence of B. Then

$$\vdash L (A \equiv B) \supset C \equiv D.$$

Proof. By induction on the length of C. Use generalisation, distribution and vacuous quantification for the quantifier case.

Lemma 2. (The Barcan Formula)

$$\vdash (P) L A \supset L (P) A.$$

Proof: By specification, $\vdash$ (P) A $\supset$ A. By S5, $\vdash$ M (P) A $\supset$ M A. By Gen, Dis. and Vac. Quant., (i) M (P) A $\supset$ (P) M A is a theorem. Now by S5, $\vdash$ (P) L A $\supset$ L M (P) L A. But by (i) and S5, $\vdash$ L M (P) L A $\supset$ L (P) M L A; and by S5, Gen and Dis, $\vdash$ L (P) M L A $\supset$ L (P) A. So $\vdash$ (P) L A $\supset$ L (P) A.

A wf is modally closed if any occurrence of a variable of the wf is within the scope of L.

Then:

Lemma 3. (Vacuous Modality). If A is modally closed, then $\vdash A \supset L A$

Proof. By induction on the number of quantifiers and truth-functional operators which do not occur within the scope of L. Lemma 2 takes care of the quantifier case.

The Extension $S5\pi+'$. We use the following abbreviations:

$Q A$ for $M A \wedge (P) (L (A \supset P) \vee L (A \supset \sim P))$,

where P is the first variable not free in A;

$T A B$ for $Q A \wedge L (A \supset B)$;

$(\exists_k R) A (R)$ for T if $k = 0$, and for $(\exists R_1) (\exists R_2) \ldots (\exists R_k) \left[\bigwedge_{1 \leq i < j \leq k} \sim L(R_i \equiv R_j) \wedge \bigwedge_{i=1}^{k} A (R_i) \right]$ if $k > 0$, where $R_1, R_2, \ldots, R_k$ are the first k distinct variables not free in A (R);

$(\exists!_k R) A$ for $(\exists_k R) A \wedge \sim (\exists_{k+1} R) A$.

We define an extension $S5\pi+'$ of $S5\pi+$ by adding the unary operators Q_k and the axiom-schemes

$$Q_k A \equiv (\exists!_k R) (T R A), k = 0, 1, 2, \ldots ,$$

The semantical clauses for Q_k are as in the last chapter.

We show that $\pi+$ is characteristic by showing that $\pi+'$ is characteristic. (When no ambiguity can arise, we shall often drop the prefix 'S5').

The Derivation of F.C. We show that if a wf of $\pi+'$ is quantificationally valid, then it is a theorem. We say that a wf of $\pi+'$ is a simple instance of a wf $\emptyset$ of FC if for each predicate letter f of n arguments in $\emptyset$ there is a wf of $\pi+'$ B $(R_1, R_2, \ldots, R_n)$ with exactly n distinct free variables $R_1, R_2, \ldots, R_n$ and with no quantifier (P_i) where x_i occurs in $\emptyset$, such that A is the result of replacing (x_i) in $\emptyset$ by (P_i), $x_i = x_j$ by $L\ (P_i \equiv P_j)$ and $fx_{k_1}\ x_{k_2} \ldots x_{k_n}$ by $A\ (P_{k_1}, P_{k_2}, \ldots, P_{k_n})$. C is an instance of $\emptyset$ if it is the result of replacing free variables in a simple instance A of $\emptyset$ by free wfs. Now

Theorem 1. If $\emptyset$ is a valid wf of FC and C is an instance of $\emptyset$, then C is a theorem of $\pi+'$.

Proof. Suppose C is obtained from a simple instance A of $\emptyset$ as above. Now if $\emptyset$ is valid, then $\emptyset$ has a proof in F.C by completeness. By the rewriting of bound variables, we may suppose that if x_i is free in a wf of the proof and x_i is not free in $\emptyset$ then P_i does not occur in A. We may then replace each wf of the proof by the simple instance in which $B(R_1, R_2, \ldots, R_n)$ replaces f to obtain a proof of A in $S5\pi$. The proof is by induction: all the rules and axioms except the scheme for equality are trivial; for the latter, use lemma 1. Now if A is provable, C is provable by generalisation and specification.

In what follows, use of theorem 1, especially for rewriting bound variables, will often be tacit. Also 1 adopt two conventions: variables exhibited in a proof are distinct; bound variables exhibited in a proof are not free in an exhibited wf unless otherwise indicated.

The Derivation of S5n. We now show how to derive S5n within $\pi+'$. Note that by previous definitions, if A is an instance of $\emptyset$, then $(\exists_k P_i)A$ is an instance of $(\exists_k x_i)\, \emptyset$ and $(\exists_k P_i)$ A is an instance of $(\exists_k x_i)\, \emptyset$.

Lemma 4. The following schemes are theorems of $\pi+'$:

(1) $Q_k\, A \supset \sim Q_1\, A$, $1 < k$,

(2) $Q_k\, A \equiv \vee_{i=o}^{k}\, Q_i\, (A \wedge B) \wedge Q_{k-i}\, (A \wedge \sim B)$,

(3) $L\, (A \equiv B) \supset (Q_k\, A \supset Q_k\, B)$,

(4) $Q_k\, A \supset L\, Q_k\, A$ and

(5) $Q_o\, A \equiv L \sim A$, 1, 1 = 0, 1, 2, ...

Proof.

(1) abbreviates to $(\exists_k P)\, (T\, P\, A) \supset \sim (\exists_1 P)\, (T\, P\, A)$.

But this is an instance of the valid wf:

$(\exists_k x)\, Rxy \supset \sim (\exists_1 x)\, R\, x\, y$.

(2) abbreviates to:

(i) $(\exists_k P)\, T\, P\, A \equiv \vee_{i=o}^{k}\, (\exists_i P)\, TP(A \wedge B) \wedge (\exists_{k-i} P)\, T\, P(A \wedge \sim B)$

We may prove without difficulty:

(ii) $(P)\, [T\, P\, A \equiv T\, P\, (A \wedge B) \vee T\, P\, (A \wedge \sim B)]$.

But (ii) $\supset$ (i) is an instance of the valid wf:

$$(x)\, [R\, x\, y \equiv R\, x\, u \vee R\, x\, v] \supset [(\exists_k x)\, R\, x\, y \equiv \vee_{i=o}^{k}\, (\exists_i x)\, R\, x\, u \wedge (\exists_{k-i} x)\, R\, x\, y].$$

(3) follows by lemma 1

(4) follows by lemma 3

(5) By P.C. it suffices to show

$(\exists P)\, (Q\, P \wedge L\, (P \supset A)) \equiv M\, A$. Now $Q\, P \supset M\, A$ by P C and the definition of Q. Since P is not free in A, it follows by F.C. that

(i) $(\exists P)(Q P \wedge L(P \supset A)) \supset M A$.

Write Q' (P) for $P \wedge (R)\left[R \supset L(P \supset R)\right]$.

Then for the converse we first show that (ii)

$\vdash Q' P \supset Q P$. By FC, $\vdash Q'(P) \supset (R \supset L(P \supset R))$ and $\vdash Q'(P) \supset (\sim R \supset L(P \supset \sim R))$. So by PC and FC, $\vdash Q'(P) \supset (R)\left[L(P \supset R) \vee L(P \supset \sim R)\right]$. But by S5, $\vdash Q'(P) \supset M P$ and (ii) follows.

Now by (ii) and FC, $\vdash A \supset [Q' P \supset (Q P \wedge L(P \supset A))]$. So by FC, $\vdash A \supset [(\exists P) Q' P \supset (\exists P)(Q P \wedge L(P \supset A))]$. But $\vdash (\exists P) Q' P$ by g. So (iii) $\vdash M A \supset (\exists P)(T P A)$ by the Barcan Formula Lemma. (5) now follows from (i) and (iii).

We now obtain:

<u>Theorem 2</u>. If A is a valid wf or a theorem of S5n, then A is a theorem of $\pi+'$.

<u>Proof</u>. From the fact that S5n is characteristic and lemma 4.

<u>Reduction of Quantifier-free wfs</u>. By theorem 2 and the normal form theorem for S5n, we obtain:

<u>Theorem 3</u>. Any quantifier-free wf A of $\pi+'$ of degree 1 is equivalent to or a disjunction of md's of degree 1 in the variables of A.

<u>Elimination of Quantifiers</u>. We now show that each wf $(\exists P)$ D, where D is a md, is equivalent to a quantifier-free wf. The proof requires three lemmas: the first shows that $(\exists P)$ can be distributed through certain conjuncts C of D; the second deals with the case when C is modally closed;

and the third with the case when C is not modally closed.

<u>Lemma 5</u>.(On Incompatible Cases). Suppose there are wfs A_i, $i = 1, 2, \ldots, m$ of A such that whenever P occurs free in A it occurs as part of a free wf of the form $A_i \wedge P$ or $A_i \wedge \sim P$; and suppose that there are wfs B_j, $j = 1, 2, \ldots, n$, such that whenever P occurs free in B it occurs as part of a free wf of the form $B_j \wedge P$ or $B_j \tilde{\wedge} P$. Then if for all i and j, $\vdash A_i \supset \sim B_j$, then $\vdash (\exists P)A \wedge (\exists P)B \equiv (\exists P)(A \wedge B)$.

<u>Proof</u>: $\Leftarrow$ By FC.

$\Rightarrow$ Let R and S be two distinct variables which do not occur in A or B; and let C be the wf $(R \wedge \vee_{i=1}^{m} A_i) \vee (S \wedge \vee_{j=1}^{n} B_j)$. Then $A_i \wedge C$ is provably equivalent to

$A_i \wedge (R \wedge \vee_{i=1}^{m} A_i) \vee A_i \wedge (S \wedge \vee_{j=1}^{n} B_j)$ by PC. But $A_i \wedge (R \wedge \vee_{i=1}^{m} A_i)$ is equivalent to $A_i \wedge R$ by PC, and $A_i \wedge (S \wedge \vee_{j=1}^{n} B_j)$ is equivalent to $\perp$ by PC given $\vdash A_i \supset \sim B_j$ for all j. So $A_i \wedge C$ is equivalent to $A_i \wedge R$. $A_i \wedge \sim C$ is equivalent to $A_i \wedge (\sim R \vee \wedge_{i=1}^{m} \sim A_i) \wedge (\sim S \vee \sim \vee_{j=1}^{n} B_j)$ by PC. But $A_i \wedge \wedge_{i=1}^{m} \sim A_i$ is equivalent to $\perp$ by PC, and $A_i \wedge (\sim S \vee \vee_{j=1}^{n} B_j)$ is equivalent to A_i by PC given $\vdash A_i \supset \sim B_j$ for all j. So $A_i \wedge \sim C$ is equivalent to $A_i \wedge \sim R$. Similarly, $B_j \wedge C$ is equivalent to $B_j \wedge S$ and $B_j \wedge \sim C$ is equivalent to $B_j \wedge \sim S$.

Write A as A (P), B as B (P). Then $\vdash A(R) \supset A(C)$ and $\vdash B(S) \supset B(C)$ by the above paragraph and the equivalence theorem. So by FC, $\vdash (\exists P)A \wedge (\exists P)B \supset (\exists P)(A \wedge B)$.

<u>Lemma 6</u>. The following schemes are theorems of $\pi+'$:

(1) $(\exists P)(Q_m(A \wedge P) \wedge M_n(A \wedge \sim P)) \equiv M_{m+n} A$

(2) $(\exists P)(M_m(A \wedge P) \wedge M_n(A \wedge \sim P)) \equiv M_{m+n} A$

(3) $(\exists P)(Q_m(A \wedge P) \wedge Q_n(A \wedge \sim P)) \equiv Q_{m+n} A$,

$m, n \geq 0$.

<u>Proof</u>: $\Rightarrow$ (1) Let (i) be the wf $Q_m(A \wedge P) \wedge M_n(A \wedge \sim P)$. Now (i) $\supset M_{m+n} A$ is valid. So by theorem 2, $\vdash$ (i) $\supset M_{m+n} A$. So by FC, $\vdash (\exists P)$ (i) $\supset M_{m+n} A$.

(2) and (3). Similarly.

(1). Intuitively, this is obvious. Suppose A is true in the distinct worlds $R_1, R_2, \ldots, R_{m+n}$. Then let P be $R_1 \vee \ldots \vee R_m$. However, the formal proof is a little messy.

The following is a valid wf of S5n:

$$\bigwedge_{1\leq i<j\leq m+n} \sim L(R_i \equiv R_j) \wedge \bigwedge_{i=1}^{m+n} T R_i A \supset \left[\bigwedge_{1\leq i<j\leq m} \sim L(R_i \equiv R_j) \wedge \bigwedge_{i=1}^{m} T R_i (A \wedge \bigvee_{i=1}^{m} R_i)\right] \wedge \sim \left[\bigwedge_{1\leq i<j\leq m+1} \sim L(S_i \equiv S_j) \wedge \bigwedge_{i=1}^{m+1} T S_i (A \wedge \bigvee_{i=1}^{m} R_i)\right]$$

So by theorem 2 it is a theorem. Let B be its antecedent, $C = C(\bigvee_{i=1}^{m} R_i)$ the first conjunct of the consequent and $\sim D = \sim D(\bigvee_{i=1}^{m} R_i)$ the second. (If $m + n = 0$, B is T and if m is 0, C is T and $\bigvee_{i=1}^{m} R$ is $\perp$).

Then it follows by FC that

$\vdash B \supset \sim (\exists S_1) \ldots (\exists S_{m+1}) D$, so $\vdash B \supset (\exists P)\left[C(P) \wedge \sim (\exists S_1) \ldots (\exists S_{m+1}) D(P)\right]$

But $\vdash (\exists R_1) \ldots (\exists R_{m+n}) B \supset (\exists P)\left[(\exists R_1) \ldots (\exists R_m) C(P) \wedge \sim (\exists S_1) \ldots (\exists S_{m+1}) D(P)\right]$.

By FC, $M_k A = \wedge_{i=o}^{k-1} \sim Q_i (A)$ is equivalent to $(\exists_k R)(T R A)$. So this last wf is equivalent to $M_{m+n} A \supset (\exists P) Q_m (A \wedge P)$. Now by S5n and theorem 2,

$\vdash M_{m+n} A \supset (Q_m (A \wedge P) \supset M_n (A \wedge \sim P))$. So by FC,

$$\vdash M_{m+n} A \supset (\exists P) (Q_m (A \wedge P) \wedge M_n (A_n \sim P)).$$

(2) By S5n, $\vdash Q_m (A \wedge P) \supset M_m (A \wedge P)$

So (2) follows from (1).

(3) By S5n, $\vdash \sim M_{m+n+1} A \supset [M_m (A \wedge P) \wedge M_n (A \wedge \sim P) \supset \sim M_{m+1} (A \wedge P) \wedge \sim M_{n+1} (A \wedge P)]$. So (3) follows from (1) by F.C.

Lemma 7. The following schemes are theorems of π+':

(1) $A \supset (\exists P) ((A \wedge P) \wedge Q_1 (A \wedge P))$,

(2) $(\exists P) ((A \wedge P) \wedge Q_m (A \wedge P) \wedge M_n (A \wedge \sim P)) \equiv A \wedge M_{m+n} A$,

(3) $(\exists P) ((A \wedge P) \wedge M_m (A \wedge P) \wedge M_n (A \wedge \sim P)) \equiv A \wedge M_{m+n} A$,

(4) $(\exists P) ((A \wedge P) \wedge Q_m (A \wedge P) \wedge M_n (A \wedge \sim P)) \equiv A \wedge Q_{m+n} A$,

and

(5) $(\exists P)((A \wedge P) \wedge M_m (A \wedge P) \wedge Q_n (A \wedge \sim P)) \equiv A \wedge M_{m+n} A$.

$m > 0,\ n \geq 0.$

Proof.

(1) First we show that (i) $\vdash Q P \supset Q_1 P$.

By PC and S5, $\vdash Q P \supset T PP$. So by FC $\vdash Q P \supset (\exists R) (T R P)$. By FC,

$Q\ P \wedge T\ R\ P$ provably implies $L\ (R \supset P)$, $M\ R$ and $L\ (P \supset R) \vee L\ (P \supset \sim R)$. But by S5, $\vdash L\ (R \supset P) \wedge L\ (P \supset \sim R) \supset \sim M\ R$. So by PC and S5, $\vdash Q\ P \wedge T\ R\ P \supset L\ (P \equiv R)$. It follows by S5 that $\vdash Q\ P \wedge T\ R\ P \wedge T\ S\ P \supset L\ (R \equiv S)$. So by FC, $\vdash Q\ P \supset \sim (\exists_2\ R)\ (T\ R\ P)$.

We now prove (1). By (i) and (ii) in the proof of (5) of lemma 4, $\vdash A \supset [Q'\ P \supset (A \wedge P) \wedge Q_1\ P]$. By F.C, $\vdash A \supset (Q'\ P \supset L\ (P \supset A))$, and by S5 $\vdash L\ (P \supset A) \supset L\ (P \equiv A \wedge P)$. So by the Equivalence Theorem, $\vdash A \supset [Q'\ P \supset (A \wedge P) \wedge Q_1\ (A \wedge P)]$. So by FC, $\vdash A \supset [(\exists\ P)\ Q'\ P \supset (\exists\ P)\ ((A \wedge P) \wedge Q_1\ (A \wedge P))$. But $\vdash (\exists\ P)\ Q'\ P$ by g and so (1) follows.

We can now prove (2) - (4).

⇒ By S5n as for lemma 6.

(2). By (1), $\vdash A \supset (\exists\ P)\ ((A \wedge P) \wedge Q_1\ (A \wedge P))$.

By S5n and theorem 2,

(ii) $\vdash M_{m+n}\ A \wedge Q_1\ (A \wedge P) \supset M_{m+n-1}\ (A \wedge \sim P)$.

So by (i), (ii) and FC,

(iii) $\vdash A \wedge M_{m+n}\ A \supset (\exists\ P)\ ((A \wedge P) \wedge Q_1\ (A \wedge P) \wedge M_{m+n-1}\ (A \wedge \sim P))$

By (1) of lemma 5,

(iv) $\vdash M_{m+n-1}\ (A \wedge \sim P) \supset (\exists\ R)\ (Q_{m-1}\ (A \wedge \sim P \wedge R))$.

By S5n and theorem 2,

(v) $\vdash Q_1\ (A \wedge P) \wedge Q_{m-1}\ (A \wedge \sim P \wedge R) \supset Q_m\ (A \wedge (P \vee R))$.

So by (iii), (iv), (v), PC and FC,

$\vdash A \wedge M_{m+n}\ A \supset (\exists\ P)\ ((A \wedge P) \wedge Q_m\ (A \wedge P))$.

But by S5n and theorem 2 again,

$\vdash M_{m+n} A \supset (Q_m (A \wedge P) \supset M_n (A \wedge \sim P))$.

So $\vdash A \wedge M_{m+n} A \supset (\exists P)((A \wedge P) \wedge Q_m (A \wedge P) \wedge M_n (A \wedge \sim P))$.

(3) and (4) follow from (2) just as in lemma 6....

(2) and (3) follow from (1).

(5) By (2) $\vdash A \wedge M_{m+n} A \supset (\exists P)(P \wedge Q_{n+1} (A \wedge P))$.

By (4), $\vdash P \wedge Q_{n+1} (A \wedge P) \supset (\exists R) \ulcorner P \wedge R \wedge Q_1 (A \wedge P \wedge R) \wedge Q_n (A \wedge P \wedge \sim R) \urcorner$.

But by S5n,

$\vdash A \wedge M_{m+n} A \supset \ulcorner R \wedge Q_n (A \wedge P \wedge \sim R) \supset (\sim P \vee R) \wedge M_m (A \wedge (\sim P \vee R)) \urcorner$

and

$\vdash Q_n (A \wedge (P \wedge \sim R)) \supset Q_n (A \wedge \sim (\sim P \vee R))$.

The result now follows by FC.

Note that in the proofs of (1) - (5) we do not apply necessitation to any wf depending upon g. In other words, (1) - (5) are deduced from g.

We now come to the theorem:

<u>Theorem 4.</u> If D is a model-description of degree 1, then $(\exists P)D$ is equivalent to a quantifier-free wf of π+' of degree 21 whose variables are those of D other than P.

<u>Proof.</u> Let D be the md $C_j \wedge \wedge_{i=1}^{n} \alpha_i C_i$ as in §2 of Chapter 2. If P does not occur in D, then $(\exists P)$ D is equivalent to D and the proof is trivial. Let E be the result of replacing P in D by $\sim P$ and $\sim P$ by P. Clearly E is a md, and by FC and the Equivalence Theorem, $(\exists P)$ D is equivalent $(\exists P)$ E. So we may suppose that P is R_m say and that R_m is a conjunct of C_j.

Now let $D_1, D_2, \ldots, D_{n'}$, $n' = 2^{m-1}$, be the sd's in the variables $R_1, R_2, \ldots, R_{m-1}$. Then D is equivalent to the wf:

$$(*)\ (D_g \wedge P) \wedge \beta_g (D_g \wedge P) \wedge \gamma_g (D_g \wedge \sim P) \wedge \bigwedge_{\substack{h=1\\ h\neq g}}^{n'} \beta_h (D_h \wedge P) \wedge \gamma_h (D_h \wedge \sim P),$$

where

$D_g \wedge P = C_j$, $\quad \beta_h = \alpha_i$ if $D_h \wedge P = C_i$,

$\gamma_h = \alpha_i$ if $D_h \wedge \sim P = C_i$, and $\beta_g \neq Q_o$,

$h = 1, 2, \ldots, n'$ and $i = 1, 2, \ldots, n$.

Now by lemma 6, $(\exists P)$ (*) is equivalent to $(\exists P) \left[(D_g \wedge P) \wedge \beta_g (D_g \wedge P) \wedge \gamma_g (D_g \wedge \sim P)\right] \wedge \bigwedge_{\substack{h=1\\ h\neq g}}^{n'} (\exists P) \left[\beta_h (D_h \wedge P) \wedge \gamma_h (D_h \wedge \sim P)\right]$.

The first conjunct of this wf is equivalent to a quantifier-free wf without P by lemma 7, (2) - (5). The other conjuncts of the wf are equivalent to quantifier free wfs without P by lemma 6, (1) - (3). So $(\exists P)$ D is equivalent to $(\exists P)$ (*), which is equivalent to a quantifier-free wf without P.

We come now to the main result.

The <u>quantificational degree</u> of a wf is the maximum number of times a quantifier occurs within the scope of a quantifier. More precisely, $\partial (P_i) = 0$, $\partial (\sim A) = \partial (A)$, $\partial (A \vee B) = \max (\partial(A), \partial (B))$ and $\partial ((\exists P_i) A) = \partial (A) + 1$. Then:

<u>Theorem 5</u>. (Normal Forms) Suppose A is a wf of $\pi+$ of quantificational degree ∂, then A is provably equivalent to $\perp$ or a disjunction of md's of degree 2^{∂} in the free variables of A.

<u>Proof</u>. By induction on the quantificational degree ∂ of A. If $\partial = 0$, then

use theorem 3. Suppose $\partial > 0$. Then A is a modal compound of variables and wfs of the form $(\exists P)$ B. By the induction hypothesis, B is equivalent to $\perp$ or a disjunction $D_1 \vee D_2 \vee \ldots \vee D_g$ of md's of degree $\leq 2^{\partial-1}$ in the free variables of B. In the former case, $(\exists P)$ B is equivalent to $\perp$. In the latter case $(\exists P)$ B is equivalent to $(\exists P)\, D_1 \vee (\exists P)\, D_2 \vee \ldots \vee (\exists P)\, D_p$. But by theorem 4, each of these disjuncts is equivalent to $\perp$ or a quantifier free wf of $\pi+'$ of degree $2.2^{\partial-1} = 2^{\partial}$ in the free variables of $(\exists P)$ B. So the whole wf A is equivalent to a quantifier-free wf of degree 2^{∂} in the free variables of A. The theorem now follows by another application of theorem 3.

§3. S5π+ is Characteristic.

Lemma 1. Each model description has a model.

Proof. This is lemma 2.3.1.

Theorem 1. For any wf A of $\pi+'$,

$$\models_{\pi+'} A \text{ iff } \vdash A.$$

Proof. $\Rightarrow$ (Consistency). A straight forward exercise. Eg to verify $(P)\, A(P) \supset A(B)$, B free for P in A (P), we show by induction on the length of the wf A (P) that if M' is the M P structure such that $\models^{M'}_{w} P$ iff $\models^{M}_{w} B$ for all $w \in W$ and B is free for P in A (P), then $\models^{M'}_{w} A(P)$ iff $\models^{M}_{w} A(B)$. The reader may also verify that the equivalence $Q_k A \equiv (\exists_k R)(T\, R\, A)$ is valid.

(Completeness). Any wf A of $\pi+'$ is equivalent to a wf of $\pi+$ by the equivalence for Q_k and the Equivalence Theorem. We may then use theorem 2.5, lemma 1 and consistency.

If A is a wf of $\pi+$, then any proof of A in $\pi+'$ may be converted into a proof in $\pi+$ by replacing Q_k A by ($\exists_k$ R) (T R A). So $\vdash_{\pi+'}$ A iff $\vdash_{\pi+}$ A. Hence

<u>Theorem 2</u>. For any wf A of $\pi+$, $\vdash_{\pi+}$ A iff $\models$ A.

<u>Corollary 1</u>. Suppose A is a wf of $\pi+$ with m free variables of quantificational degree ∂. Then if A has a model, A has a model of cardinality $c \leq 2^{\partial} . 2^{m}$.

<u>Proof</u>. By theorem 5 of §2, consistency and corollary 2.3.1.

<u>Corollary 2</u>. S5$\pi+$ is decidable.

<u>Proof</u>. By corollary 1. Alternatively, theorem 2.5 and the preceding theorems and lemmas of §2 yield a mechanical procedure for determining whether a wf $\sim$ A is equivalent to $\perp$ or a disjunction of md's. In the former case, A is a theorem; in the latter case, A is not a theorem.

We note two further corollaries of previous results, one on definability and the other on fragments:

<u>Corollary 3</u>. A nt relation is definable in $\pi+$ iff it is simple.

<u>Proof</u>. With an obvious adaption of definitions from theorem 2.4.1, theorem 2.5 and consistency.

<u>Corollary 4</u>. If A is a wf of S5n, then

$\vdash_{S5n}$ A iff $\vdash_{\pi+'}$ A.

Proof. From the fact that S5n is characteristic, that $\pi+'$ consistent and theorem 2.2.

Corollary 3 shows that very little arithmetic can be developed within $\pi+$. Corollary 4 shows that $\pi+'$ is a conservative extension of $\pi+$. It will be strengthened later in section 5.5.

§4. The System S5$\pi+^*$

The system $\pi+^*$ is the result of replacing specification in $\pi+$ by Restricted Specification, i.e. (P) A (P) $\supset$ A (B), where B is a wf of PC (the propositional calculus) free for P in A (P). We now show:

Theorem 1. $\vdash_{\pi+}$ A iff $\vdash_{\pi+^*}$ A

Proof. By theorem 3.2, it suffices to prove theorem 3.2 for $\pi+^*$. This calls for a modification of §2. I sketch the details.

Lemmas 1, 2 and 3 do not use Specification and so hold for $\pi+^*$. Theorem 1 holds for $\pi+^*$ as long as C is the result of replacing free variables in a simple instance A of $\emptyset$ by free wfs of PC. So lemma 4 holds as long as A and B are PC wfs. An analysis of Chapter 2 shows that given lemma 4, theorem 2 and hence theorem 3 still hold. Lemma 5 holds as long as the A_i and B_j are PC wfs. Lemma 6 and 7 hold for PC A and so theorem 3 holds.

Theorem 4 now follows.

I have not been able to find a direct derivation of Specification from Restricted Specification.

§5. Predicate and Boolean Analogies.

Let P_+ be the system obtained from FC by dropping = and all non-monadic predicate letters and adding quantifiers and Specification, Distribution, Vacuous Quantification and Generalisation for monadic predicate letters. Thus P_+ is the singularly second-order predicate calculus. Identity may be defined in P_+ by $(f)\ (f\ x \supset f\ y)$. We interpret the f's as ranging over all properties. So the definition of validity in chapter 2.5 may be extended to P_+ in an obvious way.

We now introduce two translations T and T':

Translation T from $\pi+$ into P_+ —

(i) $T\ (P_i) = f_i\ x_1$, $i = 1, 2, \ldots$,

(ii) $T\ (\sim A) = \sim T\ (A)$,

(iii) $T\ ((A \vee B)) = (T\ (A) \vee T(B))$,

(iv) $T\ ((P_i)\ A) = (f_i)\ T\ (A)$, $i = 1, 2, \ldots$, and

(v) $T\ (L\ A) = (x_1)\ T\ (A)$;

Translation T' from P_+ into $\pi+$ —

(i) $T'\ (f_i x_j) = L\ (R_j \supset P_i)$, where the R_j are fresh distinct variables $i, j = 1, 2, \ldots$.

(ii) $T'\ (\sim \phi) = \sim T'\ (\phi)$,

(iii) $T'\ ((\phi \vee \psi)) = (T'\ (\phi) \vee T'\ (\psi))$,

(iv) $T'\ ((f_i)\ \phi) = (P_i)\ T'\ (\phi)$, $i = 1, 2, \ldots$, and

(v) $T'\ ((x_j)\ \phi) = (R_j)\ (Q\ R_j \supset T'\ (\phi))$, $j = 1, 2, \ldots$.

The two translations may be used to derive completeness or decidability results for one system from those for the other. As an example, I prove the completeness of $\pi+$ from the completeness of P_+. First, we require three lemmas:

Lemma 1. If $\models_{\pi+}$ A, then $\models_{P_+}$ T (A).

Proof. By an appropriate induction (cf. theorem 2.5.1).

Lemma 2. $\vdash_{\pi+} Q\ R_1 \supset \left[L\ (R_1 \supset A) \equiv T'\ T\ A \right]$

Proof. By induction on the length of A. This is straight forward once we establish that the following are theorems of $\pi+$:

(i) $Q\ R \supset \left[L\ (R \supset \sim A) \equiv \sim L\ (R \supset A) \right]$,

(ii) $Q\ R \supset \left[L\ (R \supset (B \vee C)) \equiv (L\ (R \supset B) \vee L\ (R \supset C)) \right]$,

(iii) $(R)\ (Q\ R \supset L\ (R \supset A)) \equiv L\ A$.

(iv) $Q\ R \supset \left[L\ (R \supset (P)\ A) \equiv ((P)\ L\ (R \supset A)) \right]$, R not free in A.

If A is a wf whose free variables are $P_{k_1}, P_{k_2}, \ldots, P_{k_n}$, let Γ_A be the wf $Q\ P_{k_1} \wedge Q\ P_{k_2} \wedge \ldots \wedge Q\ P_{k_n}$. Then

Lemma 3. If $\vdash_{P_+} \emptyset$, then $\vdash_{\pi+} \Gamma_{T'\ (\emptyset)} \supset T'\ (\emptyset)$.

Proof. By induction on the length of the proof in $\emptyset$ in P_+. For Specification on individual variables the use of $\Gamma_{T'\ (\emptyset)}$ is essential. The one tricky case is Specification for predicate letters. For this it suffices to show that for any wf $\emptyset\ (x_j)$ of P_+ there is a wf B of $\pi+$ whose free variables are those of $T'\ (\emptyset(x_j))$ but without R_j such that:

$$\vdash_{\pi+} Q\ R_j \supset \left[L\ (R_j \supset B) \equiv T'\ (\emptyset\ (x_j)) \right].$$

This may be proved by induction on the length of $\emptyset\ (x_j)$ or more directly with the help of lemma 2.

Completeness is now straight forward:

Theorem 1. If $\models_{\pi+}$ A, then $\vdash_{\pi+}$ A.

Proof. Assume $\models_{\pi+}$ A. By lemma 1, $\models_{P_+}$ T (A).

By the completeness of P_+, $\vdash_{P_+}$ T (A). By lemma 3, $\vdash_{\pi+} \int_{T'} T (A) \supset T'T A$ i.e. $\vdash_{\pi+}$ Q $R_1 \supset$ T' T (A). By lemma 2, $\vdash_{\pi+}$ Q $R_1 \supset$ L ($R_1 \supset$ A). By π+,

$\vdash_{\pi+}$ A.

In the same way, we may set up a correspondence between π+ (or P_+) and B_+, the first-order theory of Boolean algebras with atoms. B is the first-order theory with binary $\subseteq$ as its sole predicate letter, the usual axioms for a Boolean algebra and the special axiom (x) ($\sim$ (x = $\emptyset$) $\supset$ ($\exists$ y) (q(y) $\wedge$ y $\subseteq$ x)), where q (y) is defined by (z) ((y $\subseteq$ z) $\equiv \sim$ (y $\subseteq \bar{z}$)) and where =, $\emptyset$, 1, $\bar{z}$ etc. are defined in the usual way. For validity, we interpret $\subseteq$ as inclusion in the set of all subsets of a given set.

The relevant translations are:

Translation T from π+ into B_+ -

(i) T (P_i) = (z $\subseteq x_i$), where z is a fresh variable,

(ii) T ($\sim$A) $\sim$ T (A),

(iii) T ((A $\vee$ B)) = (T (A) $\vee$ T (B)),

(iv) T (L A) = (z) (q(z) $\supset$ T (A))

(v) T ((P_i) A) = (x_i) T (A);

Translation T' from B into π+ -

(i) T' ($x_i \subseteq x_j$) = L ($P_i \supset P_j$),

(ii) T' ($\sim \emptyset$) = $\sim$ T' ($\emptyset$),

(iii) T' (($\emptyset \vee \not{\mathcal{L}}$)) = (T' ($\emptyset$) $\vee$ T' ($\not{\mathcal{L}}$)) and

(iv) T' ((x_i) $\emptyset$) = (P_i) T'($\emptyset$).

§6. References

Kaplan has proved the completeness of $\pi+$ (which he calls S5Q) independently from this author but by similar methods. See his abstract S5 with Quantifiable Propositional Variables. The system P is discussed in Church's Introduction, pp. 303-4. The system B is discussed in Tarski's Logic, Semantics, Metamathematics pp. 201-208, especially lemma K, and p. 334 et seq.

Chapter 4

THE SYSTEM S5π

The system of the last chapter contained the special axiom g = $(\exists P_1)(P_1 \wedge (P_2)(P_2 \supset L(P_1 \supset P_2)))$. In this chapter we consider the system which results from dropping this axiom.

§1. The System S5π

Formation Rules. As for S5π+

Transformation Rules. The axioms are: all tautologous wfs; Gödel's axiom schemes for S5; and specification, distribution and vacuous quantification for the quantifiers.

The rules of inference are *modus ponens*, necessitation and generalisation.

Semantical Rules. We redefine the notion of a structure. A *structure* M is an ordered triple (W, I, $\emptyset$) where W is a non-empty set, I is an empty or infinite subset of W, and $\emptyset$ is a map from V x W into $\{t, f\}$ such that for any w in I there are an infinite number of v's in I such that $w \sim_M v$ i.e. for all variables P, $\emptyset(P, w) = \emptyset(P, v)$.

We then define M P structure, truth etc. as for S5π+.

Why do we need I and the conditions on I and $\emptyset$ in the definition above? Let us say that a world w of W is *describable* if there is a proposition true in W and W alone; otherwise w is *indescribable*. Now the proposition g is true iff the actual world is describable. So Lg is true iff every world is describable. Now in S5π we do not assume Lg i.e. that every world is describable, so we need to allow for a set I of indescribable worlds. Now suppose A is true in a finite number of indescribable worlds $w_1, w_2, \ldots, w_n$, $n > 0$. Then $A \wedge \sim g$ is true in $w_1, w_2, \ldots, w_n$ and in no other worlds. Let us assume that for any two distinct worlds there is a proposition true in one of them but false in the other. Then if $n > 1$ there are propositions R_i, $i = 2, 3, \ldots, n$, such that R_i is true in w_1 but false in w_i. But then $A \wedge g \wedge \bigwedge_{i=2}^{n} R_i$ (or $A \wedge \sim g$ if $n = 1$) is true in w_1 and w_1 alone. So w_1 is describable. This is a contradiction. It follows that if A is true in some undescribable worlds then it is true in an infinite number of indescribable worlds, and this is what the conditions on I and Ø guarantee.

§2. Normal Forms.

We shall show that each wf is provably equivalent to $\perp$ or a disjunction of quantifier-free model-descriptions of an appropriate sort. The argument is complicated somewhat by the absence of the axiom g.

Preliminary Results.

Since lemmas 1, 2 and 3 of section 3.2 do not use the axiom g, they also hold for S5π.

The Extension S5π'.

The extension π' of π is obtained by adding the unary operators Q_k, $k = 0, 1\ldots$, and N and the constant g, and the following axiom-schemes

$Q_k \equiv (\exists_k R)(T\,R\,A)$, $k = 0, 1\ldots$,

$g \equiv (\exists P_1)(P_1 \wedge Q\,P_1)$, and

$NA \equiv M(\sim g \wedge A)$.

The new definition of g is equivalent to the old and so should cause no confusion.

The corresponding semantical clauses are; $\models^{M}_{w} Q_k A$ iff there are exactly k v's in W-I such that $\models^{M}_{v} A$,

$\models^{M}_{w} g$ iff $w \in W - I$

$\models^{M}_{w} NA$ iff there is a v in I such that $\models^{M}_{v} A$.

The Derivation of FC. Theorem 3.2.1 does not use the axiom g and so holds for π.

The Derivation of S5n. In the proof of lemma 3.2.4 we only use the axiom g for (5) ⇒ i.e. $\vdash Q_o A \supset L \sim A$. Now in the proof that S5n is characteristic we only use this result to help eliminate L A by $Q_o \sim A$ and to rule out the combination $A \wedge Q_o A$. So in place of theorem 3.2.2 we have:

Theorem 1. If A is a valid wf (or theorem) of S5n not containing L or any variable not within the scope of a modal operator, then $\vdash_{\pi'} A$.

We also have:

Lemma 1. The following are theorems of S5π:

(1) $N A \equiv N (A \wedge B) \vee N (A \wedge \sim B)$

(2) $M (g \wedge A) \equiv \sim Q_o A$

(3) $Q_k (g \wedge A) \equiv Q_k A, k \geq 0$.

Proof.

(1) By S5, $\vdash M (\sim g \wedge A) \equiv M (\sim g \wedge A \wedge B) \vee M (\sim g \wedge A \wedge \sim B)$. But this abbreviates to (1).

(2) $\Rightarrow A \supset (\exists P)(Q\,P \wedge L\,(P \supset A))$ is deduced from g at the end of the proof of (5) of lemma 3.2.4. So $\vdash_{S5\pi} A \wedge g \supset \sim Q_o\,(A)$. So by the Barcan formula lemma, $\vdash_{S5\pi} M\,(A \wedge g) \supset \sim Q_o\,A$.

$\Leftarrow$ Clearly, $\vdash Q\,P \wedge L\,(P \supset A) \supset (M\,P \supset M\,(g \wedge A))$. But $\vdash Q\,P \supset M\,P$. So $\vdash Q\,P \wedge L\,(P \supset A) \supset M\,(g \wedge A)$. So by F C, $\vdash \sim Q_o\,(A) \supset M(g \wedge A)$.

(3) By theorem 1, $\vdash Q_o\,(\sim g \wedge A) \supset (Q_k\,(g \wedge A) \equiv Q_k\,A)$. By (2) $\vdash \sim Q_o\,(\sim g \wedge A) \supset M\,(g \wedge \sim g \wedge A)$. So by S5, $\vdash Q_o\,(\sim g \wedge A)$. Hence $\vdash Q_k\,(g \wedge A) \equiv Q_k\,A$.

Reduction of Quantifier-free wfs.

First we redefine the notion of a md to allow for the occurrence of g and N. Suppose $C_1, C_2, \ldots, C_n$ are the distinct state-descriptions in the variables $R_1, R_2, \ldots, R_m$, $m \geq 0$, $n = 2^m$. Then a model-description of degree $l > 0$ in the variables $R_1, R_2, \ldots, R_m$ is a wf.

(*) $\alpha\, g \wedge C_j \wedge \Lambda_{i=1}^{n}\, \beta_i\, C_i \wedge \Lambda_{i=1}^{n}\, \gamma_i\, C_i$, where

(i) α is blank or $\sim$, β_i is Q_k for some $k < l$ or M_l and γ_i is N or $\sim N$,

(ii) If α is blank, then β_j is not Q_o, and

(iii) If α is $\sim$, then γ_j is N.

We now have:

Theorem 2. Any quantifier-free wf A of π' of degree l is equivalent to or a disjunction of md's of degree l in the variables of A.

Proof. Consider the conditions $\vdash X\,A \supset L\,X\,A$ and $\vdash L\,(A \equiv B) \supset (X\,A \supset X\,B)$. By the Barcan formula lemma, N and Q_k satisfy these conditions. So by the beginning of section 2.2, A is equivalent to a non-iterative wf B of degree 1 in the variables of A.

By (1) of lemma 1 and S5,

$\vdash N\,(A \vee B) \equiv NA \vee NB$ and $\vdash \sim N\bot$. By (2) of lemma 1, $\vdash LA \equiv \sim N \sim A \wedge Q_o \sim A$. By theorem 1, the distribution results in lemma 2.2.2 hold. So B is equivalent to a truth-functional compound of the variables of A, g, $Q_k\,(g \wedge C)$, $Q_k\,(\sim g \wedge C)$, $N\,(g \wedge C)$ and $N\,(\sim g \wedge C)$, where C is a sd in the variables. By (3) of lemma 1, $Q_k\,(g \wedge C)$ is equivalent to $Q_k\,(C)$. So $Q_k\,(\sim g \wedge C)$ is equivalent to T if k = 0 and to $\bot$ if k > 0. By S5, $N\,(g \wedge C)$ is equivalent to $\bot$ and $N\,(\sim g \wedge C)$ to $N\,(C)$.

So B is equivalent to $\bot$ or a disjunction of wfs (*) which satisfy condition (i). By (2) of lemma 1, $\vdash g \wedge C_j \supset \sim Q_o\,C_j$. So (*) satisfies condition (ii). By S5, (*) satisfies condition (iii).

Elimination of Quantifiers. Lemma 3.2.5 (On Incompatible Cases) does not use the axiom g and so holds for π. Lemmas 3.2.5-6 are replaced by the following four lemmas.

Lemma 2. The following schemes are theorems:

(1) $(\exists P)\,[Q_m\,(A \wedge P) \wedge M_n\,(A \wedge \sim P)] \equiv M_{m+n}A$,

(2) $(\exists P)\,[M_m\,(A \wedge P) \wedge M_n\,(A \wedge \sim P)] \equiv M_{m+n}\,A$, and

(3) $(\exists P)\,[Q_m\,(A \wedge P) \wedge Q_n\,(A \wedge \sim P)] \equiv Q_{m+n}\,A$.

Proof. As in lemma 3.2.6.

Lemma 3. The following schemes are theorems:

(1) $(\exists P) \ulcorner N(A \wedge P) \wedge N(A \wedge \sim P) \urcorner \equiv N A$,

(2) $(\exists P) \ulcorner N(A \wedge P) \wedge \sim N(A \wedge \sim P) \urcorner \equiv N A$, and

(3) $(\exists P) \ulcorner \sim N(A \wedge P) \wedge \sim N(A \wedge \sim P) \urcorner \equiv \sim N A$.

Proof:

$\Rightarrow$ (1), (2) and (3). By (1) of lemma 1.

(1) By the definition of g, $\vdash \sim g \supset \sim (E P) \ulcorner P \wedge (R) (L (P \supset R) \vee L (P \supset \sim R)) \urcorner$. So by F C and S5, $\vdash \sim g \supset (P) (P \supset (\exists R) (M (P \wedge \sim R) \wedge M (P \wedge R)))$. So $\vdash A \wedge \sim g \supset (\exists R) (M (A \wedge \sim g \wedge R) \wedge M (A \wedge \sim g \wedge R))$. So by the Barcan Formula lemma and definition of N, $\vdash N A \supset (E R) (N(A \wedge R) \wedge N (A \wedge \sim R))$.

(2) By S5, $\vdash M (\sim g \wedge A) \supset M (\sim g \wedge A \wedge T) \wedge \sim M (\sim g \wedge A \wedge \sim T)$ So by the definition of N, $\vdash N A \supset N (A \wedge T) \wedge \sim N (A \wedge \sim T)$. So by F C, $\vdash N A \supset (\exists P) (N (A \wedge P) \wedge \sim N (A \wedge \sim P))$.

(3) By (1) of lemma 1, $\vdash \sim NA \supset \sim N (A \wedge P) \wedge \sim N (A \wedge \sim P)$. (3) now follows by FC.

Lemma 4. The following schemes are theorems:

(1) $(\exists P) \ulcorner g \wedge (A \wedge P) \wedge Q_m (A \wedge P) \wedge M_n (A \wedge \sim P) \urcorner \equiv g \wedge A \wedge M_{m+n} A$,

(2) $(\exists P) \ulcorner g \wedge (A \wedge P) \wedge M_m (A \wedge P) \wedge M_n (A \wedge \sim P) \urcorner \equiv g \wedge A \wedge M_{m+n} A$,

(3) $(\exists P) \ulcorner g \wedge (A \wedge P) \wedge Q_m (A \wedge P) \wedge Q_n (A \wedge \sim P) \urcorner \equiv g \wedge A \wedge Q_{m+n} A$

(4) $(\exists P) \ulcorner g \wedge (A \wedge P) \wedge M_m (A \wedge P) \wedge Q_n (A \wedge \sim P) \urcorner \equiv g \wedge A \wedge M_{m+n} A$.

Proof. The corresponding results (1) - (4) of lemma 3.2.7 are deduced within $S5_\pi$ from (1) of lemma 3.2.2 which is deduced from g. So we obtain the above results from right to left. From left to right is trivial from lemma 2.

Lemma 5. The following schemes are theorems:

(1) $(\exists P)\ [\sim g \wedge (A \wedge P) \wedge N\ (A \wedge P) \wedge N\ (A \wedge \sim P)] \equiv \sim g \wedge A \wedge N\ A,$

(2) $(\exists P)\ [\sim g \wedge (A \wedge P) \wedge N\ (A \wedge P) \wedge \sim N\ (A \wedge \sim P)] \equiv \sim g \wedge A \wedge N\ A.$

Proof: $\Rightarrow$ (1) and (2). By (1) and (2) of lemma 3.

$\Leftarrow$ (1) By (1) of lemma 3, $\vdash N\ A \supset (\exists P)\ (N\ (A \wedge P) \wedge N\ (A \wedge \sim P)).$ But the consequent provably implies $(\exists P)\ (P \wedge N\ (A \wedge P) \wedge N\ (A \wedge \sim P)) \vee (\exists P)\ (\sim P \wedge N(A \wedge P) \wedge N\ (A \wedge \sim P))$. But the last disjunct provably implies $(\exists P)\ (P \wedge N\ (A \wedge P) \wedge N\ (A \wedge \sim P))$.

(1) $\Leftarrow$ now follows.

(2) By S5, $\vdash \sim g \wedge A \wedge NA \supset \sim g \wedge (A \wedge T) \wedge N\ (A \wedge T) \wedge \sim N\ (A \wedge \sim T)$. (2) $\Leftarrow$ Now follows by FC.

We can now eliminate the existential prefix from md's.

Theorem 3. If D is a md of degree 1 of π', then $(\exists P)$ D is equivalent to a quantifier-free wf of π' of degree 21 whose variables are those of D other than P.

Proof. Suppose D is (*) as above. If P does not occur in D, then $(\exists P)$ D is equivalent to D. So we may suppose that $m > 0$ and that P is R_m. Now $(\exists P)$ D is equivalent to $(\exists P)$ D' where D' results from D by writing $\sim P$ for P, P for $\sim P$. So we may also suppose that P is a conjunct of C_j.

Let $D_1, D_2, \ldots, D_{n'}$, $n' = 2^{m-1}$, be the sd's in the variables R_1, $R_2, \ldots, R_{m-1}$. For $i = 1, 2, \ldots, n'$, let i_1 (i_2) be such that C_{i_1} is $D_i \wedge P$ (C_{i_2} is $D_i \wedge \sim P$) and suppose that $j = h_1$. First suppose α is blank. Then D is equivalent to the wf

$$(**)\ B \wedge \bigwedge_{\substack{i=1\\ i \neq h}}^{n'} \beta_{i_1} (D_i \wedge P) \wedge \beta_{i_2} (D_i \wedge \sim P) \wedge \bigwedge_{i=1}^{n'} \gamma_{i_1} (D_i \wedge P) \wedge$$

$$\gamma_{i_2} (D_i \wedge \sim P), \text{ where B is } g \wedge C_j \wedge \beta_{h_1} (D_h \wedge P) \wedge \beta_{h_2} (D_h \wedge \sim P).$$

Now by lemma 1, Q_k (C) is equivalent to Q_k ($g \wedge C$) and N (C) is equivalent to N ($\sim g \wedge C$). So by lemma 3.2.5 (On Incompatible Cases), ($\exists$ P) D is equivalent to

$$(\exists P)\ B \wedge \bigwedge_{\substack{i=1\\ i \neq h}}^{n'} (\exists P) \left[\beta_{i_1} (D_i \wedge P) \wedge \beta_{i_2} (D_i \wedge \sim P)\right] \wedge$$

$$\bigwedge_{i=1}^{n'} (\exists P) \left[\gamma_{i_1} (D_i \wedge P) \wedge \gamma_{i_2} (D_i \wedge \sim P)\right].$$

We wish to show that each conjunct of this wf is equivalent to a quantifier-free wf of π' of degree 21 whose variables are those of ($\exists$ P) D. This holds for ($\exists$ P) B by lemma 4. It holds for the second group of conjuncts by lemma 2, and for the third group of conjuncts by lemma 3. If α is $\sim$, the argument is similar. Let $B = \sim g \wedge C_j \wedge \gamma_{h_1} (D_h \wedge P) \wedge \gamma_{h_2}$ $(D_h \wedge \sim P)$, redefine the other conjuncts accordingly and use lemma 5 in place of lemma 4.

We now come to the main result:

<u>Theorem 4</u>. (Normal Forms). Suppose A is a wf of π of quantificational

degree d. Then A is equivalent to $\perp$ or a disjunction of md's of degree 2^d in the free variables of A.

Proof. From theorems 3 and 2 as in the proof of theorem 3.2.4.

§3. $S5_\pi$ is Characteristic

Lemma 1. Each md has a model.

Proof. Let the md be (*) of the last section. We define $M = (W, I, \emptyset)$ as follows:

W consists of

(a) (C_i, π) for $\pi = 1, 2, \ldots, k$ if $\beta_i = Q_k$

(b) (C_i, π) for $\pi = 1, 2, \ldots, l$, if $\beta_i = M_l$, and

(c) (C_i, π) for $\pi = 0, -1, -2, \ldots$ if $\gamma_i = N$;

I is the set of (C_i, π) in W such that π is not positive;

and for $(C_i, \pi) \in W$,

$$\emptyset(P_h, (C_i, \pi)) = t \text{ if } P_i \text{ is a conjunct of } C_i$$

$$= f \text{ otherwise,}$$

where $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, l$ and $h = 1, 2, \ldots$.

If σ is blank let w be $(C_j, 1)$ and if σ is $\sim$ let W be $(C_j, 0)$. By conditions (ii) and (iii) on (*), $w \in W$. The construction then ensures that $\models^M_w$ (*).

Theorem 1. For any wf A of π',

$\models_{\pi'}$ A iff $\vdash$ A.

Proof. ⇒ (Consistency). A straightforward exercise. Eg we show that $g \equiv (\exists P)(P \wedge Q P)$ as follows. Suppose $\models^{M}_{w} g$, where $M = (W, I, \emptyset)$. Then $w \in W - I$. Let $M' = (W, I, \emptyset)$ be the M P structure such that $\emptyset'(P, v) = t$ iff $v = w$. Then we may easily show that $\models^{M'}_{w} P \wedge Q P$. So $\models^{M}_{w} (\exists P)(P \wedge Q P)$. Now suppose not $\models^{M}_{w} g$. Take any M P structure $M' = (w, I, \emptyset')$ such that $\models^{M'}_{w} P$. Then we may find an M' R structure $M'' = (W, I, \emptyset'')$ such that not - $\models^{M''}_{w} \ulcorner L(P \supset R) \equiv \sim L(P \supset \sim R)\urcorner$. For since $w \in I$, there is an infinite set U, of v's in I such that $\emptyset'(P, v) = t$. But then we can define $\emptyset''$ so that for an infinite number of v's in U, $\emptyset''(R, v) = t$ and for an infinite number of v's in U, $\emptyset''(R, V) = f$

⇐ (Completeness). From theorem 2.4 consistency and lemma 1.

By the Equivalence Theorem, any proof in π' may be converted to a proof of π by replacing Q_k, g and N by their respective definitions. So for any wf A of π, $\vdash_{\pi} A$ iff $\vdash_{\pi'} A$. Hence

Theorem 2. For any wf A of π,

$\vdash_{\pi} A$ iff $\models_{\pi} A$.

The following two corollaries are obtained in the usual manner.

Corollary 1. Suppose A is a wf of π with m free variables of quantificational degree d. Then if A has a model, then A has a model $M = (W, I, \emptyset)$ such that Card $(W - I) \leq 2^d \cdot 2^m$ and Card $(I) \leq \aleph_0$.

Corollary 2. π is decidable.

Let S5ng be the system obtained from S5n by adding the constant g and replacing axiom scheme (5), viz $Q_o A \equiv L \sim A$, by $Q_o A \equiv L \sim (A \wedge g)$. Then since (1) to (3) of lemma 1 hold for S5ng, we may easily show:

Theorem 3. For any wf A of S5ng,

$\vdash_{S5ng} A$ iff $\models A$.

Combining theorems 1 and 3, we obtain:

Corollary 4. For any wf A of S5ng,

$\vdash_{S5ng} A$ iff $\vdash_{\pi'} A$.

Thus S5ng is the quantifier-free fragment of π' just as S5n is the quantifier-free fragment of $\pi+'$.

§4. The System S5π*

The system π^* is the result of replacing Specification in π by Restricted Specification and adding the axiom $h = (\exists P_3)\ L\ [P_3 \equiv g]$ i.e. $(\exists P_3)\ L\ [P_3 \equiv (\exists P_1)\ (P_1 \wedge (P_2)\ (P_2 \supset L\ (P_1 \supset P_2)))]$.

Thus h states that there is a proposition which says that the world is describable.

We shall show that π^* is deductively equivalent to (has the same theorems as) π. But first we require:

Lemma 1.

$\vdash_{\pi^*} (P)\ A\ (P) \supset A\ (B)$, where B is free for P in A (P) and is a truth-functional compound of variables and g

Proof. Let B = B (g), let R be a variable not occurring in A (P) or B, and let C (R) be (P) A (P) $\supset$ A (B(R)). We wish to prove $\vdash_{\pi^*} C\ (g)$. B (R) is a wf of PC. So by Restricted Specification (i) $\vdash_{\pi^*} C\ (R)$. By the Equivalence Theorem,

(ii) $\vdash_{\pi^*} L\ (R \equiv g) \supset (C\ (R) \equiv C\ (g))$. So by PC from (i) and (ii), $\vdash L\ (R \equiv g) \supset C(g)$. So by FC, $\vdash (\exists\ R)\ L\ (R \equiv g) \supset C(g)$. $\vdash C(g)$ now follows by the axiom h.

We now come to the theorem:

Theorem 1. $\vdash_{\pi^*} A$ iff $\vdash_{\pi} A$.

Proof. By theorem 3.2, it suffices to prove theorem 3.2 for π^*. As in §3.4, an examination of §2 shows that Specification may be restricted to truth-functional compounds of variables and g as specified in lemma 1 above. In fact, only the use of the lemma on Incompatible Cases for theorem 2.3 calls for an application of the above lemma.

It is easy to show that h is a theorem of $\pi +^*$. For by the axiom g and Necessitation, $\vdash_{\pi +^*} L\ (g \equiv T)$; so by the Equivalence Theorem it suffices to prove $\vdash_{\pi +^*} (\exists\ P)\ L \ulcorner P \equiv T \urcorner$, which is easy by S5 and FC. So we may use the above theorem to show that $\pi +^*$ is deductively equivalent to $\pi +$.

§5. Strong Completeness.

Let Δ be a set of wfs. We say A is deducible from Δ, $\Delta \vdash A$, if there are wfs $B_1, B_2, \ldots, B_m$ in Δ such that $\vdash B_1 \wedge B_2 \wedge \ldots \wedge B_m \supset A$. In section §3 we showed that π was complete, i.e. that if $\models A$ then $\vdash A$. In this section we show that π' and hence π are strongly complete, i.e. that if $\Delta \models A$ then $\Delta \vdash A$. (In this section, unless otherwise stated, $\vdash$ is relativized to π' and the wfs are of π'.)

A set of wfs Δ is a system if whenever $\Delta \vdash A$ then $A \in \Delta$. Δ is consistent if not $\Delta \vdash \bot$; Δ is complete if for all wfs A, $\Delta \vdash A$ or

$\Delta \vdash \sim A$; and Δ is maximally consistent (mc) if Δ is consistent and complete. The proofs of the next three theorems need no rehearsing:

Theorem 1 (Deduction Theorem). If $\Delta, A \vdash B$, then $\Delta \vdash A \supset B$.

Theorem 2. (Lindenbaum's Lemma) Every consistent set is contained in a maximally consistent system.

Theorem 3. If Δ is a mc system, then

(1) $\sim A \in \Delta$ iff not $A \in \Delta$ and

(2) $A \vee B \in \Delta$ iff $A \in \Delta$ or $B \in \Delta$.

Use of theorems 1 and 3 will often be tacit. We now show that every mc system has a model. First some definitions. A state-sequence S is an infinite sequence of wfs $A_1, A_2, \ldots$ such that $A_i = P_i$ or $A_i = \sim P_i$ for $i = 1, 2, \ldots$, For $n \geq 1$, let $S^n = A_1 \wedge A_2 \wedge \ldots \wedge A_n$. Then S subsumes a wf C if for some n, $S^n = C$.

Let Δ be an mc system fixed for the following discussion. With each wf A we associate a denumerable cardinal $\eta(A)$ as follows:

(a) $\eta(A) = K$ if $Q_k A \in \Delta$,

(b) $\eta(A) = \aleph_0$ if $\sim Q_k A \in \Delta$ for all $k = 0, 1, 2, \ldots$.

By the theoremhood of $Q_k A \supset \sim Q_l A$, $k < l$, and the consistency of Δ, η is a well-defined function. It may be extended to state-sequences S in the following way:

(c) $\eta(S) = K$ if there is an m such that for all $n \geq m$, $\eta(S^n) = K$.

(d) $\eta(S) = \aleph_0$ otherwise.

We also define a map σ from wfs into $\{0, 1\}$ as follows:

(a) $\sigma(A) = 0$ if $NA \in \Delta$

(b) $\sigma(A) = 1$ otherwise.

It is extended to state-sequences S in the following way:

(c) $\sigma(S) = 0$ if there is an m such that for all $n \geq m, \sigma(S^n) = 0$.

(d) $\sigma(S) = 1$ otherwise.

We now prove:

Lemma 1. Let C be a state-description in the variables $P_1, P_2, \ldots, P_m$. Then

(1) If $\eta(C) = K$, then $\sum_{S \text{ subsumes } C} \eta(S) = K$.

(2) If $\eta(C) = \aleph_0$. then there is an S which subsumes C such that $\eta(S) = \aleph_0$

(3) If $\sigma(C) = 0$, then there is an S such that S subsumes C and $\sigma(S) = 0$

(4) If $\sigma(C) = 1$, then for each S which subsumes C, $\sigma(S) = 1$.

Proof.

(1) The proof of (1) is easier to see than to state. Suppose that A is any sd in the variables $P_1, P_2, \ldots, P_g$, $g \geq 1$. For $h \geq g$, let A^h be the set of sd's in $P_1, P_2, \ldots, P_h$ which contain A as a conjunct. Then by repeated applications of the distributive law for Q_k (scheme (2) of S5n), and by theorem 3,

(i) $\sum_{B \in A^h} \eta(B) = \eta(A)$

For $n \geq m$, let $p_n = \text{Card}\{B \in C^n : \eta(B) \neq 0\}$. By (i), if $\eta(A) \neq 0$ then either $\eta(A \wedge P_{g+1}) \neq 0$ or $\eta(A \wedge \sim P_{g+1}) \neq 0$. So for $n' \geq n$, $p_{n'} \geq p_n$.

Suppose there are $n_1, n_2, \ldots, n_{k+1}$ such that $n_8 < n_2 < \ldots < n_{k+1}$ and $p_{n_1} < p_{n_2} < \ldots < p_{n_{k+1}}$. Then $p_{mk+1} > K$. So $\sum_{B \epsilon C} \frac{\eta(B)}{n_{k+1}} > K$, contrary to (i). If follows that there is an n such that for all $n' \geq n$, $p_{n'} = p_n$. Let n_o be the least such n.

Assume $D \epsilon C^{n_o}$ and $n \geq n_o$. Suppose $\eta(D) = 0$. Then if $E \epsilon D^n$, $\eta(E) = 0$ by (i). So if S subsumes D, $\eta(S) = 0$. Suppose $\eta(D) = 1 \neq 0$. Now since p_{n_o} is maximum it follows by (i) that there is an E in D^n such that $\eta(E) = 1$ and for all F distinct from E in D^n, $\eta(F) = 0$. Let S be the sequence such that $S^{n_o} = D$ and for $n > n_o$, $\eta(S^n) = 1$. It should be clear that $\eta(S) = 1$ and that for any T distinct from S which subsumes D, $\eta(T) = 0$.

Now S subsumes C iff S subsumes some member of C^{n_o}. So it follows by the paragraph above that $\sum_{S \text{ subsumes } C} \eta(S) = \sum_{D \epsilon C^{n_o}} \eta(D)$. But the last term = $\eta(C) = k$ by (i).

(2) It follows by (i) that if $\eta(A) = \aleph_o$, then either $\eta(A \wedge P_{g+1}) = \aleph_o$ or $\eta(A \wedge \sim P_{g+1}) = \aleph_o$ So we may construct a sequence S such that $S^m = C$ and for $n > m$, $\eta(S^n) = \aleph_o$.

(3) Clearly it suffices to show that if $\sigma(C) = 0$ then $\sigma(C \wedge P) = 0$ or $\sigma(C \wedge P) = 0$ or $\sigma(C \wedge \sim P) = 0$. But this follows by theorem 3 and (1) of lemma 2.1.

(4) Clearly, it suffices to show that if $\sigma(C) = 1$ then $\sigma(C \wedge P) = 1$ and $\sigma(C \wedge \sim P) = 1$. But again, this follows by theorem 3 and (i) of lemma 2.

We now have:

Lemma 2. Every maximally consistent systen Δ has a model.

Proof. We define $M = (W, I, \emptyset)$ as follows:

W consists of

(i) all (S, π) for $0 < \pi \leq \eta(S)$ and

(ii) all (S, π) for $\pi \leq 0$ if $\sigma(S) = 0$; let I be the set of all (S, π) in W such that π is not positive; and for all (S, π) in W, let $\emptyset(P_h, (S, \pi)) =$

t if P_h is a term of S

= f otherwise,

where π is any integer and h any positive integer.

By the completeness of Δ, there is a state-sequence $S_o = A_1, A_2, \ldots$ such that $A_i \in \Delta$, $i = 1, 2, \ldots$. Let w be $(S_o, 0)$ if $\sigma(S) = 0$ and be $(S_o, 1)$ if $\sigma(S) = 1$. Then given $\vdash_{\pi'} M A \equiv \sim Q_o A \vee N A$, we may easily show that $w \in W$.

Now if C is an sd in the variables $P_1, P_2, \ldots, P_m$, then $\models^{M}_{(S,\pi)} C$ iff S subsumes C. So for each variable P_h, $\models^{M}_{w} P_h$ iff $P_h \in \Delta$. By (1) and (2) of lemma 1, $\models^{M}_{w} Q_k C$ iff $Q_k C \in \Delta$. And by (3) and (4) of lemma 1, $\models^{M}_{w} N C$ iff $N C \in A$.

Now suppose that a wf $A \in A$. By the consistency of Δ and the theorem on normal forms, A is equivalent to a disjunction of md's. By (2) of theorem 2, one of these md's $D \in \Delta$. So by the above paragraph, $\models^{M}_{w} D$. But D provably implies A. So by the consistency of π', $\models^{M}_{w} A$.

We now have:

Theorem 4. If $\Delta \models A$, then $\Delta \vdash_{\pi'} A$.

Proof. Suppose not $\Delta \vdash_{\pi'} A$. Then by PC and the Deduction Theorem, $\Delta' = \Delta \cup \{\sim A\}$ is consistent. So by Theorem 2, Δ' is contained in an mc system Δ^u. But then by lemma 2, Δ^u has a model. So not $\Delta \models A$.

Familiarly from theorem 4 we obtain

Theorem 5. If the members of Δ and A are wfs of π, then if $\Delta \models A$ then $\Delta \vdash_{\pi} A$.

By similar reasoning we could have proved strong completeness for π+ and S5n. Alternatively, we could derive the result for π+ (or π+') from theorem 5 by considering deductions from Lg (I omit details); the result for S5n then follows from S5n being the quantifier-free fragment of π+'.

§ 6. Predicate and Boolean Analogies.

Let P be the system obtained from P+ of section 3.5 by dropping all individual variables but x_1. Thus P is the uniform singulary second-order predicate calculus.

We now introduce a one-one translation T from π onto P:

(i) $T(P_i) = f_i\, x_1$,

(ii) $T(\sim A) = \sim T(A)$,

(iii) $T((A \vee B)) = (T(A) \vee T(B))$,

(iv) $T(L\, A) = (x_1)\, T\, A$ and

(v) $T((P_i)A) = (f_i)\, T(A)$.

Then a straightforward argument by induction establishes:

Theorem 1. $\vdash_{\pi} A$ iff $\vdash_{P} T(A)$.

Since each $\emptyset$ of P = T A for some A, the decidability of P follows from the decidability of π.

Theorem 1 suggests the following semantics for P: identify f_i with P_i; let the structures for P coincide with those for π; and then define $\models^M_w$ for P so that $\models^M_w \emptyset$ iff $\models^M_w T^{-1} (\emptyset)$. Given theorem 1 and that π is characteristic, it immediately follows that P is also characteristic.

The translation from π into Boolean Algebra is more difficult. Let B be the first-order theory obtained from B+ of section 3.5 by replacing its special axiom by $(\exists x)(y)(y \subseteq x \equiv (u)(u \subseteq y \supset u = 0 \vee (\exists v)(q(v) \wedge v \subseteq u))$. B is the elementary theory of separable Boolean Algebras.

The map T' from B into π is the same as the map T' in section 3.5 from B+ into π+. To map π into B, it suffices to show that for each wf L A of π there is a wf $\emptyset$ of B such that T' $(\emptyset)$ is equivalent to LA. To this end, we now define for each variable R of π the following map S_R in π:

(i) $S_R (P_i) = L (R \supset P_i)$,

(ii) $S_R (\sim A) = (S) (M S \wedge L (S \supset \sim S_s (A))$,

where S is a variable distinct from R and not free in A.

(iii) $S_R (A \wedge B) = S_R (A) \wedge S_R (B)$,

(iv) $S_R (L A) = L \sim R \vee (R) (S_R (A))$

(v) $S_R ((P_i) A) = (P_i) S_R (A)$.

Because of the simplicity of (ii), we suppose that wfs are written with $\wedge$ rather than $\vee$ as a primitive.

We now prove:

Lemma 1. $\vdash_{\pi} L\ (R \supset A) \equiv S_R\ (A)$, as long as R does not occur in A.

Proof. By induction on the length of A.

(1) $A = P_i$. By PC, $\vdash_{\pi} L\ (R \supset P_i) \equiv L\ (R \supset P_i)$.

But $S_R\ (A) = L\ (R \supset P_i)$.

(2) $A = \sim B$. We first prove

(i) $\vdash L\ (R \supset \sim B) \equiv (S)\ (M\ S \wedge L\ (S \supset R) \supset \sim L\ (S \supset B))$

$\Rightarrow$ is straightforward by S5. For $\Leftarrow$, we see that each wf in the following list provably implies its successor:

$\sim L\ (R \supset \sim B)$,

$M\ (R \wedge B)$, by S5,

$M\ (R \wedge B) \wedge L\ (R \wedge B \supset R) \wedge L\ (R \wedge B \supset B)$, by S5,

$(\exists\ S)\ (M\ S \wedge L\ (S \supset R) \wedge L\ (S \supset B))$ where S is distinct from R and not free in B, by FC, and finally $\sim (S)\ (M\ S \wedge L\ (S \supset R) \supset \sim L\ (S \supset B))$, by FC.

Hence $\Leftarrow$ by PC.

By IH, $\vdash L\ (S \supset B) \equiv S_S\ (B)$ So by the Equivalence Theorem and (i),

$\vdash L\ (R \supset \sim B) \equiv (S)\ (M\ S \wedge L\ (S \supset R) \supset \sim S_S\ (B)) = S_R\ (\sim B)$.

(3) $A = B_1 \wedge B_2$. By S5,

$\vdash L\ (R \supset B_1 \wedge B_2) \equiv L\ (R \supset B_1) \wedge L\ (R \supset B_2)$.

By the I.H.,

$\vdash L\ (R \supset B_i) \equiv S_R\ (B_i)$, $i = 1, 2$.

So $L\ (R \supset B_1 \wedge B_2)$ is equivalent to

$S_R\ (B_1) \wedge S_R\ (B_2) = S_R\ (B_1 \wedge B_2)$.

(4) $A = L\ B$. By S5, $\vdash L\ (R \supset L\ B) \equiv L \sim R \vee L\ B$.

But by S5, FC and (Restricted) Specification, L B is equivalent to (R) L (R $\supset$ B), which, by I.H., is equivalent to (R) S_R (B). So L (C $\supset$ L B) is equivalent to S_R (L B).

(5) A = (P_i) B. By FC, $\vdash$ (R $\supset$ (P_i) B) $\equiv$ (P_i) (R $\supset$ B) if P_i is distinct from C. So by S5,

$\vdash$ L (R $\supset$ (P_i) B) $\equiv$ L (P_i) (R $\supset$ B). But by the Barcan Formula Lemma, L (P_i) (R $\supset$ B) is equivalent to (P_i) L (R $\supset$ B), which, by I.H, is equivalent to (P_i) S_R (B) = S_R ((P_i) B).

For each variable y of B, we now set up a translation T_y from B into π as follows:

(i) T_y (P_i) = $y \subseteq x_i$,

(ii) T_y ($\sim$ A) = (z) ($z \neq 0 \wedge z \subseteq y \supset \sim T_y$ (A)),

where y is not free in A and distinct from z.

(iii) T_y (A $\wedge$ B) = T_y (A) $\wedge$ T_y (B),

(iv) T_y (L A) = [y = $\emptyset$ $\vee$ ($\exists$ y) ((z) ($z \subseteq y$) $\wedge$ T_y A)]

(v) T_y ((P_i) A) = (x_i) T_y (A).

An easy induction on the length of A establishes:

<u>Lemma 2</u>. $\vdash_{\pi}$ T' T_{x_i} (A) $\equiv$ S_{P_i} (A) as long as P_i does not occur in A.

We now obtain the required result:

<u>Theorem 1</u>. For each wf L A of π there is a wf $\emptyset$ of B such that $\vdash_{\pi}$ L A $\equiv$ T' $\emptyset$.

<u>Proof</u>. Let B = (P_i) T_{x_i} (A), where P_i does not occur in A. For T' B = (P_i) T' T_{x_i} (A) is equivalent to (P_i) S_{P_i} (A), by lemma 2, which is equivalent to (P_i) L (P_i $\supset$ A), by lemma 1, which is equivalent to L A by π.

As before, we may use the translations T and T' to derive completeness of decidability results for one system from the corresponding results for the other system. I omit details.

§7. An Alternative Semantics for π

Call the previous semantics for π semantics$_1$, the present semantics semantics$_2$. A structure$_2$ is a triple (W, I, $\emptyset$); where W is non-empty, $I \subseteq W$ and $\emptyset$ is a map from V x W into {t, f}. There are no restrictions on I or $\emptyset$. We define $\models^{M}_{2W}$ by the standard clauses. However, we redefine the notion of an MP-structure. A structure$_2$ M' = (W', I', $\emptyset$') is an M P-structure, M = (W, I, $\emptyset$), if there is a map f from W' onto W such that

(i) for each w in W-I and v in W', f(v) = w iff v = w,

(ii) for each w in I there are at most two v's in I' such that f (v) = w,

and (iii) for all R distinct from P and v in W', $\emptyset$ (R, v) = $\emptyset$ (R, f (v)).

Thus each quantifier enables a world in I to "split" into two.

We now show that validity$_1$ and validity$_2$ coincide. Given a structure, M = (W, I, $\emptyset$), the structure$_2$ M* = (W*, I*, $\emptyset$*), called the shrinkage of M, is the structure such that

(i) W* - I* = W - I,

(ii) $I^* = I/_{\sim M}$ i.e. the set of equivalence classes with respect to $\sim$ M, and

(iii) for each variable P

$\emptyset$* (P, w) = $\emptyset$ (P, w) for $w \in W - I$, and

$\emptyset^*$ (P, [w]) = $\emptyset$ (P_9 w) for w ϵ I.

Clearly, M* is properly defined.

Let g be the map from W onto W* such that for w ϵ W - I, g (w) = W and for w ϵ I, g(w) = [w]. Then:

Lemma 1. For each w in W, $\models^{M}_{1^w}$ A iff $\models^{M^*}_{2g(w)}$ A.

Proof. By induction on the length of A. The one tricky case is when A is of the form (P) B. Here it suffices to note two facts: (1) If M' is an MP - $structure_1$, then M'* is an M* $P\text{-}structure_2$; and (2) each M* $P\text{-}structure_2$ is isomorphic to a $structure_2$ M'* for some M P-stricture M'.

Since each $structure_2$ is isomorphic to M* for some $structure_1$ M, lemma 1 implies:

Theorem 1. $\models_1$ A iff $\models_2$ A.

In terms of $semantics_2$, π has the finite model property, whereas for $semantics_1$ we can do no better than corollary 3.1.

It should be clear that any semantics along the lines of $semantics_2$ will do just as well as long as each world in I is allowed to split into two. Thus we could require that each world in I must split into two; or we could allow for each world in I to split into any number of worlds.

Further semantics for π will be developed in the next chapter.

§8. References

Bull has proved completeness for π by semantic tableaux in On Modal Logic with Propositional Quantifiers. However, his semantics are different from my own and he does not prove decidability. The system B is discussed in Kreisel's and Krivine's Elements of Mathematical Logic, chapter 4, section 6.

Chapter 5

THE SYSTEM $S5_{\pi-}$

In this chapter we consider the system which results from replacing Specification in π by Restricted Specification.

§1. The System $S5_{\pi-}$

Formation Rules. As for π.

Transformation Rules. As for π but with Specification replaced by Restricted Specification, viz., $(P)\ A\ (P) > A\ (B)$, where B is a wf of PC free for P in A(P).

Semantical Rules. We redefine the notion of a structure. A structure M is an ordered triple (W, P, ϕ), where W is a non-empty set, P (propositions) is a field of subsets of W, i.e. a set of subsets of W closed under complementation in W and finite union, and ϕ is a map from V into P.

Given a structure $M = (W, P, \phi)$, then a structure $M' = (W', P', \phi')$ is an MP-structure if $W' = W$, $P' = P$ and for all R distinct from P, $\phi'(R) = \phi\ (R)$.

We then define $\models^{M}_{w} A$ in the usual way. Of course, the first clause must now be:

(i) $\models^{M}_{w} P_i$ iff $w \in \phi(P_i)$, $i = 1,2,\ldots$.

For convenience, I have slightly altered the definition of the assignment function $\emptyset$. It now assigns a proposition, i.e. set of worlds, to each variable. The set of propositions is given by P. Restricted Specification merely requires that P be closed under union and complementation.

§2. $S5_{\pi-}$ is Characteristic.

By suitably extending the language of π-, we may eliminate all quantifiers and occurrences of L and thereby prove that π- is characteristic and decidable.

The working-out is extremely lengthy and I shall merely indicate the required extension π-' of π-. π-' is obtained from π' of the last chapter by adding the constants e and f, the unary operators E and F and the axiom-schemes:

$$e \equiv (\exists P)\left[P \wedge L\,(P \supset \sim g) \wedge (R)\,(L(R \supset \sim g) \wedge L\,(P \supset R) \supset L\,(R \supset P))\right].$$

$$f \equiv \sim e \wedge (\exists P)\,(P \wedge L\,(P \supset \sim g))$$

$$EA \equiv M\,(A \wedge e)$$

$$FA \equiv M\,(A \wedge f).$$

e states that there is a true minimal proposition which is true in indescribable worlds only. f states that there is a true proposition, but no true minimal proposition, which is true in indescribable worlds only.

It should be clear that $\pi-$ corresponds to B-, the first-order theory of Boolean algebra. However, I know of no direct way of translating $\pi-$ into B-.

The semantics for $\pi-$ allow for a slightly odd possibility. There may be two distinct worlds w and v in W but no proposition p in P such that $w \in p$ and not $(v \in p)$. In other words, some worlds may not be distinguishable. In such a case, a proposition p which satisfies QP may be a set of indistinguishable worlds. We may avoid such possibilities as follows. Let $M = (W, P, \emptyset)$ be a structure. Then for $w, v \in W$, let us say $w \sim_M v$ if for all p in P, $w \in p$ iff $v \in p$. We may easily show that $\sim_M$ is an equivalence relation. Let $M/\sim = (W', P', \emptyset')$, where $W' = W/\sim$, $P' = \{P/\sim : p \in P\}$ and for all variables P, $\emptyset'(P) = \emptyset(P)/\sim$. In $M/\sim$ each world has been identified with a set of indistinguishable worlds. We may then easily show that $M/\sim$ is a structure and that:

<u>Lemma</u>. $\models^{M}_{w} A$ if $\models^{M/\sim}_{[w]_{\sim M}} A$.

Hence validity reduces to validity for all structures $M/\sim$.

<u>§3. The Systems π and $\pi+$ Reconsidered</u>.

In terms of $\pi-$ we can give new semantical accounts of π and $\pi+$.

<u>The System π</u>. I give two new interpretations of π.

1. Let us say that a structure $M = (W, P, \mu)$ is <u>separable</u> if there

is a p in P which is the union of all the atoms in P. (q is an atom in P if q is non-empty and for all r in P either $q \subseteq r$ or $q \subseteq \bar{r}$). We say M satisfies A (A is valid in M) if for all w in M, $\models^{M}_{W}$ A. Then:

Theorem 1. $\vdash_{\pi}$ A iff A is satisfied by all separable M.

Proof. The result follows from the following equivalences:

(i) $\models h \supset A$ iff A is satisfied by all separable M

(ii) $\models h \supset A$ iff $\vdash_{\pi-} h \supset A$

(iii) $\vdash_{\pi-} h \supset A$ iff $\vdash_{\pi^*} A$

(iv) $\vdash_{\pi^*} A$ iff $\vdash_{\pi} A$

(i) follows from the fact that $\models^{M}_{W} h$ iff M is separable, (ii) from the fact that π is characteristic, (iii) from the easily proved Deduction Theorem for $\pi-$ (where the discharged assumption formula is modally closed), and (iv) from theorem 4.4.1.

Let us say that M satisfies a system S if M satisfies each theorem of S. Then M satisfies π^* iff M satisfies h i.e. iff M is separable. So by (iv),

Theorem 2. M satisfies π iff M is separable.

Theorem 2 has two interesting corollaries.

Corollary 1. $\pi-$ is not deductively equivalent to π.

Proof. Let $W = I^+ \cup I^-$, where I^+ is the set of positive integers and I^- is the set of negative integers. Let P consist of the finite subsets of I^+ and the complements in W of finite subsets of I^+. Then P is a field. But I^+, the union of the atoms in W, is not itself an element of P. So the structure $M = (W, P, \emptyset)$, is not separable. Now M satisfies $\pi-$, but by theorem 2, M does not satisfy π.

We say a structure $M = (W, P, \emptyset)$ is closed if for all wfs A there is a p in P such that $p = \{w : w \in W \; \& \; \models^{M}_{w} A\}$.

Corollary 2. M is closed iff M is separable.

Proof. Specification in π may be replaced by the less general scheme :

$(P) \sim L\,(P \equiv A) \supset \sim L\,(A \equiv A)$, P not free in A, which is equivalent to the scheme $(\exists P)\, L\,(P \equiv A)$, P not free in A. But M satisfies this latter scheme iff M is closed. So by theorem 2, M is closed iff M is separable.

2. A function f from V into the power set $\mathcal{P}(W)$, where W is non-empty, is called a Bull-evaluation if it satisfies the following conditions:

(a) If A and A' are similar (alphabetic variants), then $f(A) = f(A')$

(b) $f(\sim A) = W - f(A)$

(c) $f(A \vee B) = f(A) \cup f(B)$

(d) $f(L\,A) = W$ if $f(A) = W$

$= 0$ otherwise

(e) $f((P_i)\, A\, (P_i))$ = the intersection of all sets $f(A(B))$ where B is free for P_i in $A\,(P_i)$.

Thus in Bull-evaluations the variables range over wfs. We say $\models_{B} A$ iff $f(A) = W$ for all Bull-evaluations f.

We may establish by an easy induction that if $\vdash_{\pi} A$, then $\models_{B} A$. To prove completeness, we need to be able to convert structures M into Bull-evaluations. So given $M = (W, P, \emptyset)$, let f_M be a function from wfs into $\mathcal{P}(W)$ defined by $f_M(A) = \{w : w \in W \; \& \; \models^{M}_{w} A\}$. Let us say that a structure M is <u>spaced</u> if for each p in P and for each k there is a $l \geq k$ such that $\emptyset(P_l) = p$. Then

<u>Lemma 1</u>. If M is a spaced separable structure, then f_M is a Bull-evaluation.

<u>Proof</u>. (e) is the one tricky case.

(A) Suppose $w \in f((P_i)\, A(P_i))$ i.e.

$\models^{M}_{w} (P_i)\, A(P_i)$. By theorem 2, M satisfies Specification. So $\models^{M}_{w} A(B)$ for each wf B free for P_i in $A(P_i)$, and w belongs to the required intersection.

(B) Suppose $w \notin f((P_i)\, A(P_i))$. Then for some MP structure $M' = (W, P, \emptyset')$, not - $\models^{M'}_{w} A(P_i)$. Let k be the greatest number such that P_{k-1} is free in $A(P_i)$. Then since M is spaced there is an $l \geq 1$ such that $\emptyset'(P_i) = \emptyset(P_l)$. It follows that not - $\models^{M}_{w} A(P_l)$. So w does not belong to the required intersection.

We now have:

<u>Theorem 3</u>. If $\models_{B} A$, then $\vdash A$.

<u>Proof</u>. Suppose not - $\vdash A$. An examination of section 4.3 shows that there is a separable structure $M = (W, P, \emptyset)$ with countable P such that for some $w \in W$, not-$\models^{M}_{w} A$. Let K be the greatest number such that P_{K-1} is free in A; let $p_1, p_2, \ldots$ be an enumeration of the elements of P; and let $M' = (W, P, \emptyset')$ where

(i) $\emptyset'(P_j) = \emptyset(P_j)$ for $j < k$

(ii) $\emptyset'(P_j) = p_i$ if j is some power of the i-th prime,

(iii) $\emptyset(P_j) = \emptyset(P_i)$ otherwise.

Now, clearly, M' is spaced and not $-\models^{M'}_{w} A$. So $f_{M'}$ is a Bull-evaluation by lemma 1 and $f_{M'} A = \{ w: w \in W \;\&\; \models^{M}_{w} A\} \neq W$.

To sum up: we have given four different interpretations of π. The first (section 4.1) distinguished between describable and indescribable worlds and gave an ad hoc rule for the quantifier. The second (section 4.7) accounted for the quantifier in terms of splits. The third let the variables range over all propositinns in a separable structure. And the fourth let the variables range over all wfs. The third account is, to my mind, the most philosophically interesting.

The System π+. Let us say that a structure M = (W, P, Ø) is atomistic if each w in W belongs to an atom in P. Then:

Theorem 4. $\vdash_{\pi+}$ A iff A is satisfied by all atomistic structures.

Theorem 5. M satisfies π+ iff M is atomistic.

Proofs. Similar to theorems 1 and 2, but with $\pi+^*$ instead of π^* and g instead of h.

Theorem 5 also has some interesting corollaries:

Corollary 1. π+ is not deductively equivalent to π.

Proof. Let $W = J \cup I^+$, where J is the set of rationals r such that $0 \leq r < 1$. Let P consist of the finite unions of intervals $\{r: a \leq r < b\}$, where a, $b \in W$, and the complements in W of such finite unions. Then M = (W, P, Ø) is separable, since I^+, the union of all atoms, $= W - \{r : 0 \leq r < 1\}$; but M is not atomistic. So the corollary follows by theorem 5.

Corollary 2. P is not deductively equivalent to the uniform fragment (i.e. with one variable x_1) of P+.

Proof. By section 3.5, $\vdash_{\pi+}$ A iff $\vdash_{P+}$ T A. By theorem 4.6.1, $\vdash_{\pi}$ A iff $\vdash_{P}$ T A. So the result follows by corollary 1.

The explanation for this curious result is that in P+ we can define identity by (f) (f x ⊃ f y) and so prove Tg, but that this is not possible in P.

§4. Some Further Systems.

1. Kripke's System. This is given at the end of Kripke's A Completeness Theorem in Modal Logic. It is equivalent to the result of adding to π the axioms N T ∨ M_k T for k = 1, 2, ... A structure M = (W, I, ∅) satisfies the system iff W is infinite and a wf is a theorem of the system iff it is satisfied by all structures M with infinite W. The system is not finitely axiomatizable. For suppose Δ were a suitable finite set of axioms to add to π. Then Δ ∪ {∼ N T} would be consistent and so have a finite model M = (W, O, ∅) by corollary 3.3.1.

Kripke's own remarks on the system seem to be incorrect.

2. Extensions of π+. For each extension of π+ i.e. system obtained by adding new axioms to π+, we can find a condition on the cardinality of W so that A is a theorem of the system iff it is satisfied by all structures whose W satisfy the condition. E.g., if the extension is {M_1 T, M_2 T, ...} the condition is that W be infinite. If the extension is {Q_2 T, Q_4 T, ...} the condition is that W contain an even number of worlds.

3. A System Between π and π+. Let us say that a wf A is closed if each variable of A is either within the scope of L or a quantifier for that

variable. Now it follows with the help of normal forms that for $\pi+$,

(i) If A is closed, then $\vdash A \supset L\,A$.

(i) does not hold for π since $g \supset L\,g$ is not a theorem of π. However, if we add $g \supset L\,g$ as an axiom to π we obtain the weakest extension of π which does satisfy (i). Again, this follows with the help of normal forms. Since $g \supset L\,g$ is equivalent in π to $L\,g \vee L \sim g$, a wf is a theorem of the system iff it is satisfied by all structures $M = (W, I, \emptyset)$ in which either I or W - I is empty.

By adapting the normal forms for S5π, we may show that all the systems considered in this section are decidable.

<u>§5. Some Theorems on Fragments</u>

In this section we extend our results on quantifier-free fragments of $\pi+'$.

Let $\pi''-$ be the result of adding to π the unary operators Q_k with the axiom-schemes $Q_k\,A \equiv (\exists_k\,P)\,(T\,P\,A) \wedge \sim M\,(A \wedge \sim g)$, $k = 0, 1, 2 \ldots$ Then

<u>Theorem 1</u>: For each wf A of S5n, $\vdash_{S5n} A$ iff $\vdash_{\pi''-} A$.

<u>Proof</u> (A). We may easily verify that axiom-schemes (1), (2) and (5) of S5n are theorems of $\pi''-$ for all wfs of PC and that schemes (3) and (4) of S5n are theorems $\pi''-$ for all wfs whatsoever. But as is clear from section 2.2, this is all that is required to derive S5n.

(B). If a wf A of S5n is a theorem of $\pi''-$, it is valid in $\pi+'$ and so provable in S5n.

Note that scheme (5) of S5n is a theorem of $\pi''-$ only because of the strengthened definition of Q_k above.

An <u>extension</u> of π'' is the result of adding new axioms to π''. An extension S of π'' is a <u>conservative extension</u> of S5n (S5Q) if for each wf A of S5n (S5Q) $\vdash_{S5n(S5Q)} A$ iff $\vdash_S$ A. A system S is <u>contained</u> in a system T is each theorem of S is a theorem of T. The next theorem characterizes the extensions of π'' which are conservative extensions of S5n (or S5Q):

<u>Theorem 2</u>. An extension S of π'' is a conservative extension of S5n (S5Q) iff S is contained in $\pi+'$.

<u>Proof</u>: $\Leftarrow$ From theorem 1 and corollary 3.3.4.

$\Rightarrow$ Assume S is an extension of $\pi-''$ not contained in $\pi+$. Then there is an axiom A of S which is not a theorem of $\pi+$. So if B is the closure of A,

(i) $\vdash_S$ B, and

(ii) not $-\vdash_{\pi+'}$ B.

By theorem 3.2.4, B is equivalent (in $\pi+'$) to a wf of the form $V_{k\epsilon K} Q_k T$ or a wf of the form $V_{k\epsilon K} Q_k T \vee M_l T$, where K is a (possibly empty) set of integers j such that $1 \leq j < l$. In the first case, B implies $\sim Q_l T$ (in $\pi+'$). In the second case, there is j, $1 \leq j < k$, such that $j \not\epsilon K$, otherwise $\vdash_{\pi+'}$ B, contrary to (ii); so B implies $\sim Q_j T$. In both cases, there is an $i \geq 1$ such that B implies $\sim Q_i T$.

We now define a wf A_i of S5Q for each i = 1, 2, ... A_i = $\Lambda_{j=1}^{i} Q_1 C_j \wedge \Lambda_{j=i+1}^{n} Q_o C_j$, where $C_1, C_2, \ldots, C_n$ are the distinct sd's in the variables $P_1, P_2, \ldots, P_m$ and where m is the least integer such that $2^m \geq i$. Clearly, A_i implies $Q_i T$ and so $\sim Q_i T$ implies $\sim A_i$.

Since B implies $\sim A_i$ (in π+'), $\vdash_S B \wedge L\, g \supset \sim A_i$ and so $\vdash_S B \supset (\sim L\, g \vee \sim A_i)$. But $\vdash_S A_i \supset L\, g$. So $\vdash_S B \supset \sim A_i$ and by (i), $\vdash_S \sim A_i$. But $\sim A_i$ is not a theorem of S5Q (or S5n), since A_i has a model. So S is not a conservative extension of S5Q (or S5n).

In the light of theorem 2, it might be thought that no proper extension of π+' (i.e. one not contained in π+') is a conservative extension of S5 and that no proper extension of S5n is a conservative extension of S5Q. However, neither of these results hold. To fault the first, add $M_k\, T$ for any $k > 1$ to π+'; and to fault the second, add $M_2\, A \vee M_2 \sim A$ to S5n. Then by the methods of section 2.4 we may show that the resulting systems are conservative extensions of S5 and S5Q respectively.

There are many other applications of our methods. E.g., we may use them to provethe semantical analogues of Scrogg's results on extensions of S5.

§6. References

The interpretation in which variables range over wfs is Bull's and is given in <u>On Modal Logic with Propositional Variables</u>.

Tarski proved the decidability of B-, presumably by the elimination of quantifiers, in 1949. However, as far as I know, the proof has not been published.

Chapter 6

THE SYSTEMS M_n, B_n AND $S4^o_n$

In Chapter 2 we considered S5 with the operators Q_k. In this chapter we consider some weaker systems with these operators.

§1. The Systems

The system M_n is defined as follows:

Formation Rules. The wfs are given by the set V of variables, the binary operator v and the unary operators $\sim$, L, and M_k, k = 1,2, ..., M_o A abbreviates $A \supset A$ and Q_k A, k = 0, 1, 2, ..., abbreviates $M_k A \wedge \sim M_{k+1} A$. Thus M_k, rather than Q_k, is now primitive. The use of M_n for an operator and the name of a system should cause no confusion.

Transformation Rules. The axioms are: all tautologous wfs; the schemes $L A \supset A$ and $L (A \supset B) \supset (L A \supset L B)$; and the schemes

(1) $M_k A \supset M_l A$, $l < k$

(2) $M_k A \equiv V^k_{i=o} M_i (A \wedge B) \wedge M_{k-i} (A \wedge \sim B)$,

(3) $L (A \supset B) \supset (M_k A \supset M_k B)$, and

(4) $M_1 A \equiv M A$, k, l = 1, 2, ...

The system B_n is obtained from M_n by adding the axiom-scheme

(5) $A \supset L M A$.

The system $S4^o_n$ is obtained by adding the schemes:

(6) $M M_k A \supset M_k A$, $k \geq 1$; and

(7) $M_m (B \wedge Q_l (B \wedge M_k (A \wedge M_l B))) \supset M_{k.n} A$, where $n = qt((m-1), l) + 1$. (qt (x, y) is the quotient on dividing x by y).

We may define the general notion of a system as follows. A system S is a set of wfs such that

(i) if $\vdash_{M_n} A$, then $A \in S$, and

(ii) if $A, A \supset B \in S$, then $B \in S$.

Semantical Rules. We redefine the notion of a structure. A structure M is an ordered triple (W, R, ∅) where W is a non-empty set, R is a relation defined on W, and ∅ is a map from V x W into {t, f}. 'wRv' may be read as 'v is accessible from w.'

With each structure M = (W, R, ∅) we associate a unique relation $\models^M_w A$. It is the smallest relation such that:

(i) $\models^M_w P_i$ iff $\emptyset(P_i, w) = t$, $i = 1, 2, \ldots$.

(ii) $\models^M_w \sim A$ iff not - $\models^M_w A$

(iii) $\models^M_w (A \vee B)$ iff $\models^M_w A$ or $\models^M_w B$

(iv) $\models^M_w L\,A$ iff for all v st $w R v$, $\models^M_v A$

(v) $\models^M_w M_i\,A$ iff there are at least i distinct worlds $v_1, v_2, \ldots, v_i$ st $w R v_j$ and

$\models^M_{v_j} A$, $j = 1, 2, \ldots, i$, $i = 1, 2, \ldots$.

We redefine the following conditions on R:

R is reflexive if for all $w \in W$, $w R w$

R is symmetric if for all $w, v \in W$, $w R v \Rightarrow v R w$

R is transitive if for all $w, v, u \in W$, $w R v$ and $v R u \Rightarrow w R u$.

We say A is valid in M = (W, R, ∅), $\models^M A$, if for all $w \in W$, $\models^M_w A$.

Validity is then defined as follows:

$\models_{M}$ A if $\models^{M}$ A for all structures M ($=$ (W, R, $\emptyset$)) such that R is reflexive

$\models_{B_n}$ A if $\models^{M}$ A for all structures M such that R is reflexive and symmetric

$\models_{S4o_n}$ A if $\models^{M}$ A for all structures M such that R is reflexive and transitive.

§2. Maximally Consistent Systems

We shall prove completeness for the systems by means of mc systems. For future use we shall state many results in general form.

Let S be a system. Then an S-system or system of S is one which includes S. In conformity with standard notation, we write '$\vdash_S$ A' for 'A $\in$ S'. Let Δ be a set of wfs. Then $\Delta \vdash_S$ A if there are wfs A_1, A_2, ..., $A_M \in \Delta$ such that $\vdash (A_1 \supset \ldots \supset (A_m \supset A)\ldots)$. Δ is S-consistent if not $\Delta \vdash_S \perp$. Δ is S-complete if for every wf A either $\Delta \vdash$ A or $\Delta \vdash \sim$ A. Δ is maximally consistent (mc) in S if Δ is S-consistent and S-complete.

I remind the reader of the following theorems.

Theorem 1. (Deduction Theorem). If $\Delta \cup \{A\} \vdash_S$ B, then $\Delta \vdash_S (A \supset B)$.

Theorem 2. (Lindenbaum's Lemma). Every S-consistent set is contained in an mc system in S.

Theorem 3. If Δ is an mc system in S, then $\sim$ A $\in \Delta$ iff not A $\in \Delta$ and (A $\vee$ B) $\in \Delta$ iff A $\in \Delta$ or B $\in \Delta$.

We now turn to some modal features of our systems. A system S is <u>normal</u> if whenever A $\in$ S, L A $\in$ S. Let S be a consistent normal system. Then we let W_S be the set of mc systems in S. By theorem 2, W_S is non-empty. For i = 1,2,..., we define the relations R_i as follows:

for w, v $\in W_S$, w R_i v if for any wf A whenever A $\in$ v M_i A $\in$ w. The relations R_i play a crucial role in what follows.

First we note two trivial lemmas:

<u>Lemma 1</u>. w R_i v iff $\{A: \sim M_i \sim A \in w\} \subseteq v$.

<u>Proof</u>. Straightforward given the Equivalence Theorem, which follows with the help of axiom-scheme (3).

<u>Lemma 2</u>. If w R_i v, then w R_j v, $j < i$.

<u>Proof</u>. By scheme (1).

Use of theorem 3 and lemmas 1 and 2 will often be tacit.

The next result is fundamental:

<u>Lemma 3</u>. M_i A $\in$ w iff there are at least i distinct ordered pairs $(\mathbf{V}_j, \alpha_j)$, st w R $\alpha_j \mathbf{V}_j$ and A $\in \mathbf{V}_j$, $\alpha_j \geq 1$, j = 1, 2, ..., i.

<u>Proof</u>.

$\Rightarrow$ By induction on i.

<u>Basis</u> i = 1. Assume M_1 A $\in$ w. Let $\Gamma = \{A: L A \in w\}$. Suppose $\Gamma \cup \{A\}$ is not consistent. Then, by PC and the Deduction Theorem, there are wfs $A_1, \ldots, A_n \in \Gamma$ st $\vdash_S A_1 \supset (A_2 \supset \ldots \supset (A_n \supset \sim A))$. So by the system M, $\vdash_S L A_1 \supset (L A_2 \supset \ldots \supset (L A_n \supset L \sim A)$.) Hence L $\sim$ A $\in$ w. So by axiom-scheme (4), $\sim M_1$ A $\in$ w, contrary to the consistency of w. So $\Gamma \cup \{A\}$ is consistent and by Lindenbaum's Lemma $\Gamma \cup \{A\}$ is contained

in a mc S-system v. So by lemma 1 and scheme (4), $w\ R_1\ v$.

Inductive Case. Assume that the lemma holds for $i < k$. (The Induction Hypothesis).

Assume $M_k\ A \in w$. We distinguish two cases:

(A) There is a wf B and numbers p, $q < k$ such that $p + q > k$ and $M_p\ (A \wedge B)$, $M_q\ (A \wedge \sim B) \in w$. By I H, there are p distinct pairs (v_j, α_j) st $w\ R\ \alpha_j\ v_j$ and $(A \wedge B) \in v_j$, $j = 1, 2, \ldots, p$, and there are q distinct pairs (u_h, β_h) st $w\ R\ \beta_h\ u_n$ and $(A \wedge \sim B) \in u_h$, $h = 1, 2, \ldots, q$. Since $B \in v_j$ and $\sim B \in v_h$, $v_j \neq v_h$ for any j or h. So there are at least $p + q \geq k$ pairs, viz (v_j, α_j), (u_h, β_h) which satisfy the condition in the consequent of the lemma.

(B) There is no wf B and there are no numbers p, $q < k$ as in (A). By scheme (2) and theorem 3, for each wf B there is an i st $M_i\ (A \wedge B)$, $M_{k-i}\ (A \wedge \sim B) \in w$. So for each wf B, either (a) $M_k\ (A \wedge B) \in w$ or (b) $M_k\ (A \wedge \sim B) \in w$. In case (a), $\sim M\ (A \wedge \sim B) \in w$. For otherwise, $M_1\ (A \wedge \sim B) \in w$ by scheme (4), $M_{k-1}\ (A \wedge B) \in w$ by scheme (1), and so (A) would be satisfied after all. Similarly, in case (b), $\sim M\ (A \wedge B) \in w$.

Now let $\Gamma = \{A: \sim M_k \sim A \in w\}$.

Now suppose $\Gamma \cup \{A\}$ is inconsistent. Then there are wfs $A_1, \ldots, A_m \in \Gamma$ such that (i) $\vdash_S (A_1 \wedge \ldots \wedge A_m) \supset \sim A$. By the paragraph above, either (a) $\vdash_w L\ (A \supset A_i)$ for $i = 1, 2, \ldots, m$ or (b) $\vdash_w L\ (A \supset \sim A_i)$ for some $i = 1, 2, \ldots, m$. In case (a), $L\ (A \supset (A_1 \wedge \ldots \wedge A_m)) \in w$. But then by M and (i), w is inconsistent; a contradiction. In case (b), by scheme (3) $\sim M_k\ A \in w$ since $\sim M_k \sim A_i \in w$. But by assumption, $M_k\ A \in w$; again a contradiction. So $\Gamma \cup \{A\}$ is consistent. So by Lindenbaum's Lemma,

$\Gamma(w)$ is contained in an mc. S-system v, and by lemma 1 w R_k v. Hence by lemma 2, w R_1 v, w R_2 w, ..., w R_k v where A ϵ v, and the consequent of the lemma is satisfied by the pairs (v, α_j), j = 1, 2, ..., k.

By induction on i.

<u>Basis</u> i = 1. Assume w R_{α_1} V_1 and A ϵ V_1. Then M_{α_1} A ϵ w. But $\alpha_1 \geq 1$. So by scheme (1), M_1 A ϵ w.

<u>Inductive Step</u> i = k. Assume there are k distinct pairs (V_j, α_j) st w R_{α_j} V_j and A ϵ V_j, j = 1, 2, ..., k. We distinguish two cases.

(A) All the V_j's are identical. Then clearly there is an $\alpha_j \geq k$. So M_{α_j} A ϵ w and by scheme (1) M_k A ϵ w.

(B) At least two V_j's are distinct. Clearly we may assume $V_1 = V_2 = \ldots = V_h$ and $V_j \neq V_1$ for j = h + 1, h + 2, ..., k. So for j = h + 1, h + 2, ..., k there is a B_j such that B_j ϵ V_j and $\sim B_j$ ϵ V_1. Let B = $\bigvee B_j$. Then by P C, B ϵ V_j for each j and $\sim$ B ϵ V_1. By the I.H., M_{k-h} (A $\wedge$ B) ϵ w and M_h (A $\wedge \sim$ B) ϵ w. So by scheme (2) M_k A ϵ w.

<u>§3. Characteristic Models</u>

Let S be a consistent normal system. In this section, I show how to construct a characteristic model for S from W_s and R_i as raw data.

The intuitive interpretation of w R_i v is that there are i worlds which are accessible from w and which have the same truth-value assignments as v. So if X is an arbitrary non-empty set let us say that a relation R defined on a non-empty subset N of W_s x X is <u>sound</u> if

(i) for (W, n) ϵ N and (V, m_i) = N; (W, n) R (V, M_i), i = 1, 2, ..., k $\Rightarrow$ w R_k v and

(ii) for $w, v \in W_s$ and $(w, n) \in N$; $w R_k v \Rightarrow$ there are k distinct (v, m_i) in N such that $(w, n) R (v, m_i)$, $i = 1, 2, \ldots, k$.

Given R defined on N, we may define a <u>natural</u> structure $N_s = (N, R, \emptyset)$ by letting $\emptyset(P_i, (w, n)) = t$ (f) if $P_i \in w$ ($P_i \notin w$), $(w, n) \in N$ and $i = 1, 2, \ldots$

<u>Theorem 1</u>. If N_s is a natural structure defined as above, with R as sound, then for each $(w, n) \in N$, $\overset{N_s}{\underset{(w,n)}{=}}$ A iff $A \in w$.

<u>Proof</u>. By induction on the length of A. The one tricky case is when A is of the form M_k B.

$\Rightarrow$ Assume $\models^{N_s}_{(w,n)} A$ $(= M_k B)$. Then by the definition of $\models$, there are k pairs (v_j, m_j) in N such that $\models^{N_s}_{(v_j,m_j)} B$ and $(w,n) R (v_j, m_j)$, $j = 1, 2, \ldots, k$. Suppose u_i are the distinct v_j's and that there are n_i pairs with first member u_i, $i = 1, 2, \ldots, l \leq k$. Then since R is sound, $w R_{n_i} u_i$, $i = 1, 2, \ldots, l$. So by lemma 2.2, for each $i = 1, 2, \ldots, l$ there are n_i pairs (u_i, j), $j + 1, 2, \ldots, n_i$, such that $w R_j u_i$. Clearly,

(i) there are k such (u_i, j)'s in all. Also, for $j = 1, 2, \ldots, k$, $\models^{N_s}_{(v_j, m_j)}$ B. So by I.H.,

(ii) $B \in u_i$ for each i.

From (i) and (ii), by lemma 2.3, $M_k A \in w$.

$\Leftarrow$ Assume A $(= M_k B) \in w$. Then by lemma 2.3, there are k pairs (v_j, α_j) such that $w R_{\alpha_j} v_j$ and $B \in v_j$, $j = 1, 2, \ldots, k$. As before, let u_i be the distinct v_j's and n_i the number of pairs with first member u_i, $i = 1, 2, \ldots, l \leq k$. Since R is sound, there is for each i, n_i pairs (u_i, m_j) in N, $j = 1, 2, \ldots, n_i$, such that $(w, n) R (u_i, m_j)$. Clearly,

(i) there are k such pairs (u_i, m_j).

Also, $B \in u_i$ for each i. So by the I.H.,

(ii) $\models^{N_s}_{(u_i, m_j)} B$ for each (u_i, m_j).

From (i) and (ii), by the definition of $\models$, $\models^{N_s}_{(w, n)} M_k B$.

§4. The System Mn

To prove completeness, we shall use the construction of §3. Let $X = \{1, 2, \ldots,\}$. We then define R on $W_{M_n} \times X$ as follows:

for $w, v \in W_{M_n}$ and $n, m \in X$, $(w, n)\ R\ (v, m)$ iff

(i) $w = v$, $n \leq m$ and $w\ R_{m+1-n}\ V$

or (ii) $w \neq v$, $n < m$ and $w\ R_{m-n}\ V$.

We now require two lemmas:

Lemma 1. R as defined above on $W_{M_n} \times X$ is sound.

Proof. (A) Assume $(w,n)\ R\ (v, m_i)$, $i = 1, 2, \ldots, k$.

Clearly we may suppose $m_1 < m_2 < \ldots < m_k$. We distinguish two cases.

(a) $w = v$. Then $n \leq m_i$. So $n + k - 1 \leq m_k$. But $w\ R_{m_{n+1-n}}\ V$. So by lemma 2.2., $w\ R\ (n+k-1)+1-n\ V$. i.e. $w\ R_k\ V$.

(b) $w \neq v$. Then $n < m_1$. So $n + k \leq m_k$. But $w\ R_{m_k-n}\ V$. So by lemma 2.2, $w\ R_{n+k-n}\ V$ i.e. $w\ R_k V$.

(B) Assume $w\ R_k\ V$. Then either

(a) $w = v$ and $(w, n)\ R\ (v,m)$, $m=n, n+1, \ldots, n+k-1$ or

(b) $w \neq v$ and $(w, n)\ R\ (v,m)$, $m+ n+1, \ldots, n+k$.

Lemma 2. R is reflexive.

Proof. (w,n) R (w,n) if w R_{n+1-n} w i.e. if w R_1 w. But by scheme (4) and M, if A ϵ w, then M_1 A ϵ w.

We now have:

Theorem 1. $\vdash_{M_n}$ A iff $\models_{M_n}$ A.

Proof. ⇒ (Consistency). Straightforward.

⇐ Assume not $\vdash_{M_n}$ A. So by PC and the Deduction Theorem, $\{\sim A\}$ is M_n - consistent. So by Lindenbaum's Lemma, there is an mc M_n-system w such that $\sim$ A ϵ w. Let N_{M_n} be the natural structure obtained from W_{M_n} and R as defined above. Then by lemma 1 and theorem 3.1, not $\models^{N_{M_n}}_{(w,1)}$ A.
So by lemma 2, not $\models_{M_n}$ A.

Clearly, a similar argument proves strong completeness. In subsequent proofs of completeness, I shall merely prove the appropriate lemmas and leave the application of theorem 3.1 to the reader.

Call a relation R antisymmetric if for all x and y in its domain, x R y & R x ⇒ x = y. Now R, as defined above, is antisymmetric. For suppose (w, n) R (v,m) and (v,m) R (w,n). Then $n \leq m$ and $m \leq n$; so n = m; and so v = w. Hence $\models^{M}$ A for all M in which R is reflexive iff $\vdash_{M_n}$ A iff $\models^{M}$ for all M in which R is reflexive and antisymmetric. Thus for M it makes no difference whether accessibility is antisymmetric or not.

§5. The System Bn.

The construction of N_{M_n} from W_{B_n} and R_i is a little tricky.

First, we define by induction the relations S_i and sets D_i, i = 1, 2, ...

(i) $S_i = \emptyset$; $D_i = \{(w,1): w \ \epsilon \ W_{B_n}\}$;

(ii) $(w,n)\ S_{p+1}\ (v,m)$ iff $(w,n) \in D_p$,

there are exactly l m's such that $(v, m')\ S_p\ (w,n)$, $m = 2^n \cdot 3^k$ and

either (i) $w = v$ and $w\ R_{k+l+1}\ v$

or (ii) $w \neq v$ and $w R_{k+l}\ v$,

D_{p+1} = the domain of S_{p+1}.

Put $N = \cup D_i$, and for $x, y \in D$, let $x\ S y$ iff for some i, $x\ S_i\ y$.

We now let R be the smallest reflexive and symmetric relation which contains S i.e. for $x, y \in N$ $x\ R y$ iff $x = y$ or $x\ S y$ or $y\ S x$. Thus we see that S is a tree-relation such that if (w,n) has exactly l R-predecessors with first member v then (w,n) has k S-successors with first member v iff $w\ R_{k+l}\ v$.

By definition, R is reflexive and symmetric.

To prove soundness, we first prove:

<u>Lemma 1</u>. If $w\ R_1\ v$, then $v\ R_1\ w$.

<u>Proof</u>. Suppose $A \in w$. Then by scheme (5), $L\ M\ A \in W$. So by scheme (4), $\sim M_1 \sim M_1\ A \in w$. But then by lemma 2.1, $M_1\ A \in v$.

Also, $w\ R_1\ w$ for each $w \in W_{B_n}$, since lemma 2.2 holds for any system which contains M_n. We now have:

<u>Lemma 2</u>. R is sound.

<u>Proof</u>. Routine, given lemma 1 and the remark above.

Consistency and completeness then follow as for M_n.

<u>§6. The System $S4^o_n$</u>

First, we prove consistency. We may easily show $\models_{S4^o_n}$ (6) So it remains to show $\models_{S4^o_n}$ (7). For this we require the following lemma:

<u>Lemma 1</u>. Suppose R is a transitive reflexive relation defined on a

set $Y \cup Z$, that for each $\bar{y}$ in Y there are exactly 1 y's in Y and at least k z's in z such that $\bar{y}$ R y and $\bar{y}$ R z, and that for each z in Z there is a y in Y such that z R y. Then if Y contains at least $m \geq 1$ elements, then Z contains at least n = (qt (m-1), 1)) + 1) k elements.

Proof. First we show that R is symmetric on Y. For suppose that $\bar{y}$, y ϵ Y, $\bar{y}$ R y and not - (y R $\bar{y}$). Now y R y_i, for distinct y_i ϵ Y, i = 1, 2, ..., l. So for each i, $\bar{y} \neq y_i$. But then by reflexivity and transitivity, $\bar{y}$ R $\bar{y}$ and $\bar{y}$ R y_i, contrary to assumption.

Next we show that for $\bar{y}$ ϵ Y, z ϵ Z if $\bar{y}$ R z, then z R $\bar{y}$. For z R y for some y ϵ Y. So by transitivity, $\bar{y}$ R y. So by symmetry on Y, y R $\bar{y}$. So by transitivity again, z R $\bar{y}$.

By the first paragraph, R is an equivalence relation on Y. Suppose Y/R, the partition on Y induced by R, has p elements. Then the reader may easily convince himself that $p \geq n$ = qt ((m-1)l) + 1.

By the second paragraph, if not ($\bar{y}$ R y), then the sets {z ϵ Z: $\bar{y}$ R z} and {z ϵ Z: y R z} are disjoint. So there are at least n.k elements in Z.

We now show that $\models_{S4^o_n}$ (7). Suppose $\models^M_w$ C, where C is the antecedent of (7). Temporarily we say w is an A-world if $\models^M_w$ A. Then:

(i) there are m B-worlds w which have R to $\bar{w}$,

(ii) for each w there are just l B-worlds v which have R to w,

(iii) for each v there are k A-worlds u which have R to v, and

(iv) for each u there are l B-worlds t which have R to v.

By transitivity, each v has R to just l B-worlds and each world t is a v. Letting Y be the set of w's and v's and z be the set of u's, we see that the assumption of lemma 1 is satisfied. So there are n.k A's in Z.

So by transitivity, $\models^{M}_{W} M_{n.k} A$.

So we have:

<u>Theorem</u> 1.(Consistency). If $\vdash_{S4^o_n} A$, then $\models_{S4^o_n} A$.

For completeness we require the following two lemmas:

<u>Lemma 2</u>. If w R v and v R_k u, then w R_k u.

<u>Proof</u>. Assume w R_1 v and v R_k u. Suppose $A \in u$. Then $M_k A \in v$. Sp $M_1 M_k A \in u$. But then by schemes (4) and (6), $M_k A \in u$.

We say: w S_k v, k = 1,2,..., if w R_k v and not - (w R_{k+1} V); and w $S_{\aleph_0}$ v (or w $R_{\aleph_0}$ v) if w R_k v for k = 1,2,... .

<u>Lemma 3</u>. If w R_m v, v R_k u and u S_l v, then w $R_{k.n}$ u, n = qt ((m-1),l) + 1.

<u>Proof</u>. Since u S_l v, there is a B such that $B \in v$, $Q_l B \in u$. By lemma 2, v R_l v. So $M_l B \in v$. If $M_{l+1} B \in v$, then since u R_1 v, $M_1 (M_{l+1} B) \in u$, and so by scheme (6) $M_{l+1} B \in u$. So $Q_l B \in v$ also.

Now suppose $A \in u$. Then $A \wedge M_l B \in u$. So $M_k (A \wedge M_l B) \in v$. So $Q_l (B \wedge M_k (A \wedge M_l B)) \in v$. So $M_m (B \wedge Q_l (B \wedge M_k (A \wedge M_l B))) \in w$.
So by scheme (7), $M_{k.n} A \in w$.

For $w \in W_{S4^o_n}$, let α_w be the cardinal c such that w S_c w. For w, $v \in W_{S4^o_n}$, let $w \sim v$ iff w R_1 v and v R_1 w. We may now define R on $N = \{(w, (n_1, n_2)): w \in W_{S4^o_n}\ n_1 = 1, 2, \ldots, \text{ and } n_2 \lesssim \alpha_w\}$.

For $(w, (n_1, n_2)), (v, (m_1, m_2)) \in N$,

$(w, (n_1, n_2))$ R $(v, (m_1, m_2))$ iff

(i) $w \sim v$ and $n_1 = m_1$ or

(ii) not $(w \sim v)$ and

(a) α_v is finite and w $R_{(m_1-1)\alpha_v + m_2}$ V or

(b) α_v is infinite and for each t such that $v \sim t$, $w \, R \, m_1 \, \alpha_t \, t$.

$(w, (n_1, n_2))$ may be looked upon as the n_2-th pair, with first member w, in the n_1-th "pool" i.e. set of pairs in which each element is related by R to every other element. The pair (n_1, n_2) could, of course, have been replaced by the single number $2^{n_1}.3^{n_2}$.

The next two lemmas show that R has the required properties. We say $(n_1, n_2) \preccurlyeq (m_1, m_2)$ iff $n_1 < m_1$ or $n_1 = m_1$ and $n_2 \leq m_2$. Then:

<u>Lemma 4</u>. R is sound.

<u>Proof</u>. (A). Assume $w \, R_k \, v$. (i) $w \sim v$. Then (by lemma 2) $\alpha_v \geq k$. So $(w, (n_1, n_2)) \, R \, (v, (n_1, m))$, $m = 1, 2, \ldots, k$. (ii) not- $(w \sim v)$ (a) α_v finite. Suppose $k = p \, \alpha_v + q$, $q < \alpha_v$. Then $(w, (n_1, n_2)) \, R \, (v, (m_1, m_2))$ for all $(m_1, m_2) \preccurlyeq (p+1, q)$. (b) α_v infinite. Then $(w, (n_1, n_2)) \, R \, (v, (1, m))$, $m= 1, 2, \ldots, k$.

(B) Assume $(w, (n_1, n_2)) \, R \, (v, (m_{1i}, m_{2i}))$, $i = 1, 2, \ldots, k$. Clearly we may suppose that $(m_{1i}, m_{2i}) \preccurlyeq (m_{1i+1}, m_{2i+1})$, $i = 1, 2, \ldots, k$. (i) $w \sim v$. Then the m_{1i}'s are equal and $\alpha_v \geq m_{2k} \geq k$. So by lemma 2, $w \, R_k \, v$. (ii) not-$(w \sim v)$. (a) α_v finite. Clearly, $(m_{1k}-1). \ \alpha(V) + m_{2k} >$ K. So $w \, R_k \, v$. (b) α_v infinite. Then by lemma 2, $w \, R_k \, v$.

<u>Lemma 5</u>. R is reflexive and transitive.

<u>Proof</u>. (A) By lemma 4.2, $w \sim w$ and so $(w, (m_1, m_2)) \, R \, (w, (m_1, m_2))$.

(B) Assume $(w, (n_1, n_2)) \, R \, (v, (m_1, m_2))$ and $(v, (m_1, m_2)) \, R \, (u, (l_1, l_2))$. We distinguish four main cases:

(a) $w \sim v$, $v \sim u$. Then $n_1 = m_1 = l_1$, and by lemma 2, $w \sim u$. So $(w, (n_1, n_2))$ R $(u, (l_1, l_2))$.

(b) $w \sim v$, not $(v \sim u)$. (1) α_u finite. Then $v\, R_{(l_1-1)\,\alpha_v + l_2}\, u$. But $w\, R_1\, v$. So by lemma 2, $w\, R_{(l_1-1)\,\alpha_v + l_2}\, u$ as required. (2) α_u infinite. By lemma 2, if $t \sim u$, then $w\, R_{m_1\,\alpha_t}\, t$.

(c) not $(w \sim v)$, $v \sim u$. (1) α_v and α_u finite. $w\, R_{(m_1-1)\,\alpha_v+m_2}\, v$. Now $l_1 = m_1$. So by lemma 3, $w\, R_{(l_1-1)\,\alpha_u+l_2}\, u$. (2) α_v finite, α_u infinite. By lemma 2, if $t \sim u$, then $w\, R_{m_1\alpha_t}\, u$. (3) α_v infinite, α_u finite. Since $v \sim u$, $w\, R_{m_2\,\alpha_u}\, u$. So $w\, R_{(l_1\,-1)\,\alpha_u + l_2}\, u$, since $l_1 = m_2$ and $l_2 \leq \alpha_u$. (4) α_v and α_u infinite. As for (2).

(d) not $(w \sim v)$, not $(v \sim u)$. (1) α_u infinite. As for (c) (2). (2) α_u finite. Then $v\, R_{(l_1-1)\,\alpha_u + l_2}\, u$. So by lemma 2, $w\, R_{(l_1-1)\,\alpha_u + l_2}\, u$. But not $(w \sim u)$, otherwise $u\, R_1\, w$, $w\, R_1\, v$, $u\, R_1 v$, $v\, R_1\, u$ and so $v \sim u$.

Consistency and completeness now follow.

<u>§7. References.</u> The method of mc systems was first used by L. Henkin. Applications to modal logic have been made by D.Scott and D.C. Makinson among others.

Chapter 7

SOME ANTISYMMETRIC SYSTEMS

In the previous chapter, it was not required that the relation R of accessibility be antisymmetric. In this chapter we consider systems which satisfy this requirement.

§1. The Systems.

We shall use the schemes:

(1) $M\,M_k\,A \supset M_k\,A$

(2) $A \wedge M\,(\sim A \wedge M_k\,A) \supset M_{k+1}\,A$

(3) $M_2\,(A \wedge M_k\,A) \supset M_{k+1}\,A$

(4) $M\,L\,A \supset L\,M\,A$

(5) $L\,M\,A \supset \sim M_2\,(A \wedge \sim M_2\,A)$

(6) $\sim M_k\,(A \wedge \sim M_l\,B) \vee \sim M_l\,(B \wedge \sim M_k\,A)$

(7) $L\,(M\,B \supset B \vee A) \wedge A \wedge QA \wedge \sim B \wedge M\,B \supset M_2\,B$,

$m = 3, 4, \ldots, k,\ l = 1, 2, \ldots$.

The systems mentioned on the left are obtained by adding to M_n the axiom-schemes on the right:

$S4'_n$	(1), (2), (3)
$S4.2'_n$	(1), (2), (3), (4), (5),
$S4.3'_n$	(1), (2), (3), (4), (6)
$S4.3^2_n$	(1), (2), (3), (4), (6), (7).

We now come to semantics. A relation R is:

Convergent if for all w and v there is a u such that w R u and v R u;

Total if for all w and v, w R v or v R w; and

<u>Dense</u> if for all distinct w and v there is a u distinct from w and v such that w R u and u R v.

w, v and u range, of course, over the domain of R.

We may now define validity:

$\models_{S4^1_n}$ A iff $\models^M$ A for all structures M (=(W, R, ∅)) such that R is reflexive, antisymmetric and transitive.

$\models_{S4.2^1_n}$ A iff $\models^M$ A for all M such that R is reflexive, antisymmetric, transitive and convergent.

$\models_{S4.3^1_n}$ A iff $\models^M$ A for all M such that R is reflexive, antisymmetric, transitive and total.

$\models_{S4.3^2_n}$ A iff $\models^M$ A for all M such that R is reflexive, antisymmetric, transitive, total and dense.

The superscripts, 0, 1, 2 indicate distinctions for the systems with M_k which need not be made for the systems without M_k.

§2. The System $S4^1_n$

<u>Theorem 1</u>. (Consistency). If $\vdash_{S4^1_n}$ A, then $\models_{S4^1_n}$ A. .

<u>Proof</u>. Straightforward, except perhaps for (2) and (3). For (2), suppose $\models^M_w$ A ∧ M (∼ A ∧ M_k A). Then $\models^M_w$ A and there is a v such that w R v, $\models^M_v$ M_k A. So there are k distinct u_i such that v R u_i and $\models^M_{u_i}$ A, i = 1, 2, ..., k. By antisymmetry $u_i \neq$ w. By transitivity, w R u. So given reflexivity, $\models^M_w$ M_{k+1} A. The validation of (3) likewise depends upon anti-symmetry.

Neither (2) nor (3) are valid for S5n. (Let $W = \{1, 2\}$, $R = \{(1, 2), (2,1), (1, 1), (2, 2)\}$, $\emptyset (P_1, 1) = t$, $\emptyset (P_1, 2) = f$). Since $S4_n^o$ is contained in S5n, this shows that $S4_n^o$ and $S4_n^1$ are distinct.

First we require four lemmas for R_i defined on $W_{S4_n^1}$.

<u>Lemma 1</u>. If $w \; R_2 \; w$, then $w \; R_{\aleph_o} \; w$.

<u>Proof</u>. Assume $w \; R_2 \; w$ and suppose $A \in w$, We shall show that for all $i \geq 2$, $M_i \; A \in w$. Clearly, $M_2 \; A \in w$. Now suppose $M_k \; A \in w$. Then $A \wedge M_k \; A \in w$. So $M_2 \; (A \wedge M_k) \in w$. Hence by (3), $M_{k+1} \; A \in w$.

<u>Lemma 2</u>. If $w \neq v$, $w \; R_1 \; v$ and $v \; R_1 \; w$, then $w \; R_{\aleph_o} \; v$.

<u>Proof</u>. Assume $w \neq v$, $w \; R_1 \; v$ and $v \; R_1 \; w$. Then there is a B such that $B \in w$ and $\sim B \in v$. Suppose $A \in v$. Let

$$C_o = A \wedge \sim B$$

$$\left.\begin{array}{l} C_{2n+1} = \sim (A \wedge \sim B) \wedge M \; C_{2n} \\ C_{2n+2} = (A \wedge \sim B) \wedge M \; C_{2n+1} \end{array}\right\} \quad n \geq 0.$$

We establish by an easy induction that $C_{2n+1} \in w$ and $C_{2n} \in v$. We may also establish by induction that $\vdash_{S4_n^1} \; C_{2n+1} \supset M_{n+1} \; (A \wedge \sim B)$.

For clearly $\vdash_{S4_n^1} \; C_2 \supset M_1 \; (A \wedge \sim B)$.

Now $C_{2n+1} = \sim (A \wedge \sim B) \wedge M \; C_{2n}$

$= \sim (A \wedge \sim B) \wedge M \; ((A \wedge \sim B) \wedge M \; C_{2n-1})$

wchih provably implies $\sim (A \wedge \sim B) \wedge M \; ((A \wedge \sim B) \wedge M \; M_n \; (A \wedge \sim B))$, by the I.H. and (1),

which implies $\sim (A \wedge \sim B) \wedge M \; ((A \wedge \sim B) \wedge M_n \; (A \wedge \sim B))$, by (1),

which implies $M_{n+1} \; (A \wedge \sim B)$ by (2).

So by the system M_n, $M_k \; A \in w$ for all $k \geq 1$.

Hence $w\ R_{\aleph_0}\ v$.

__Lemma 3.__ If $w\ R_1\ v$ and $v\ R_k\ u$, then $w\ R_k\ u$.

__Proof.__ As for lemma 6.6.2.

We say that w __is cyclic to__ v, in symbols $w \sim v$, if either $w = v$ and $w\ R_2\ v$ or $w \neq v$, $w\ R_1\ v$ and $v\ R_1\ w$. Note that this is not quite the $\sim$ of §6.6. The following lemma states the main properties of $\sim$.

__Lemma 4.__

(i) $\sim$ is an equivalence relation on $\{w \in W_{S4^1_n} : w \sim v \text{ for some } v\}$

(ii) $\sim$ is a congruence with respect to R_k on $W_{S4^1_n}$ i.e. if $w_1\ R_k\ v$ and $w_2 \sim w_1$, then $w_2\ R_k\ v$ and if $w\ R_k\ v_1$ and $v_1 \sim v_2$, then $w\ R_k\ v_2$

(iii) if $w \sim v$, then $w\ R_{\aleph_0}\ v$.

__Proof.__ Straightforward with the help of lemmas 1-3.

In order to define the accessibility relation R, we must order the equivalence classes with regard to $\sim$. For this the R_k are of no help. So let us suppose that $\preccurlyeq$ is a ~~well~~ total-ordering of $W_{S4^1_n}$ (This may be done without the axiom of choice: enumerate all wfs in some standard fashion; treat each mc as a subsequence of the enumeration; and order these subsequences lexicographically).

The relation R in $W_{S4^1_n} \times X$, $X = \{1, 2, \ldots\}$, is now defined as follows:

for $w, v \in W_{S4^1_n}$ and $n, m \in X$,

$(w, n)\ R\ (v, m)$ iff

either (i) $w \sim v$ and

(a) $n < m$

or (b) $n = m$ and $w \preccurlyeq v$

or (ii) not - $(w \sim v)$ and

(a) $w \neq v$ and $w\ R_m\ v$

or (b) $w = v$ and $m = n$.

Lemma 5. R as defined above is sound.

Proof. (A). Assume $w\ R_k\ v$. If $w \sim v$, then $(w,n)\ R\ (v,n+1)$, for $i = 1,2,\ldots,k$ by clause (i) (a) of the definition of R. If not- $(w \sim v)$, then either $w = v$ in which case $k = 1$ and $(w, n)\ R\ (v, n)$ or $w \neq v$ in which case $(w, n)\ R\ (v, m)$, $m = 1, 2, \ldots, k$, by clause (ii) (a).

(B). Assume $(w, n)\ R\ (v, m_i)$, $i = 1, 2, \ldots k$. If $w \sim v$, then $w\ R_k\ v$ by (iii) of lemma 4. If not $(w \sim v)$, then either $w \neq v$ and $w\ R_{m_i}\ v$ or $w = v$ and $m_i = n$. In the first case, there is an i such that $m_i \geq k$, so $w\ R_k\ v$. In the second case $k = 1$ and $w\ R_1\ v$ by the system M_n.

Lemma 6. R is reflexive.

Proof. As for lemma 6.4.2.

Lemma 7. R is antisymmetric.

Proof. Assume $(w, n)\ R\ (v, m)$. We distinguish four cases:

(i) (a) $w \sim v$, $n < m$. Suppose $(v, m)\ R\ (w, n)$. Now $v \sim w$ by lemma 4 (i). So $m < n$ by clause (i) (a). A contradiction.

(i) (b) $w \sim v$, $m = n$ and $w \preccurlyeq v$. Suppose $(v, m)\ R\ (w, n)$. Now $v \sim w$ and so $v \preccurlyeq w$ by clause (i) (b). Hence $w = v$ and $(w, m) = (v, n)$.

(ii) (a) not $(w \sim v)$, $w \neq v$ and $w\ R_m\ v$. Suppose $(v, m)\ R\ (w, n)$. Not $(v \sim w)$ and so $v\ R_n\ w$ by (ii) (a). Hence $v \sim w$ after all. A contradiction.

(ii) (b) not $(w \sim v)$, $m = n$. But then $(w, n) = (v, m)$.

Lemma 8. R is transitive.

Proof: Assume (w, n) R (v, m) and (v, m) R (u, 1). We wish to show that (w, n) R (u, 1).

(A) $w \sim u$. Then $w \sim v$ and $v \sim u$. For, if v = w, then $w \sim v$ by lemma 4 (i), and if $v \neq w$, then $v\, R_1\, u$ and $w\, R_1\, v$ by lemma 5, $w \sim u$, so $v\, R_1\, w$ by lemma 4 (ii), and so $w \sim v$. In the same way we show that $v \sim u$. By (i) (a) and (i) (b), $n \leq m$ and $m \leq 1$. If $n < m$ or $m < 1$, then $n < 1$ and (w, n) R (u, 1) by clause (i) (a). If $n = m = 1$, then $w \lesssim v \lesssim u$. So $w \lesssim u$ and (w, n) R (u, 1) by clause (i) (b).

(B) not - $(w \sim u)$.

(a) $w \neq u$. If $v \sim u$, then $v\, R_{\aleph_0}\, u$ by lemma 4 (iii). Now $w\, R_1\, v$. So by lemma 3, $w\, R_1\, u$ and (w, n) R (u, 1) by clause (ii) (a). If not $(v \sim u)$, then either v = u and m = 1, in which case the theorem is trivial, or $v \neq u$, in which case $v\, R_1\, u$ by clause (i) (a), so $w\, R_1\, u$ by lemma 3, and so (w, n) R (u, 1) by clause (ii) (a).

(b) w = u. Then v = w (= u), otherwise $w \sim u$, contrary to supposition. But then n = m = 1 by clause (ii) (b) and so the case is trivial.

We have now proved all the appropriate lemmas and completeness follows in the usual manner.

The required model for $S4^1_n$ might have been obtained by modifying the model for $S4^0_n$ given that $\alpha_w = 1$ or $\alpha_w = \aleph_0$. I omit details.

§3. The System $S4.2^1_n$

Theorem 1. (Consistency). If $\vdash_{S4.2^1_n} A$, then $\models_{S4.2^1_n} A$.

Proof. Straightforward except perhaps for (4) and (5). The validation of (4) is known from S4.2. To validate (5), assume $\models^M_w L\, M\, A$ and $\models^M_w M_2\, (A \wedge \sim M_2\, A)$. So there are distinct w_i such that $\models^M_{w_i} (A \wedge \sim M_2\, A)$, i = 1,2. Clearly if $w_i\, R_v$, then either $v = w_i$ or $\models^M_v \sim A$. Now by convergence there is a v such that $w_i\, R v$, i = 1,2. So v is distinct from the w_i. But $\models^M_v M\, A$. So there is a u such that v R u and $\models^M_u A$. By transitivity, w_i R u, i = 1, 2. But then $w_1 = w_2$. A contradiction.

For completeness, we can no longer deal with the usual natural structure $N_{S4.2^1_n}$ since its accessibility relation R may not be convergent. However, connected substructures of $N_{S4.2^1_n}$ will have convergent R. So we need the following definitions and results.

Given a structure M = (w, R, ∅) the connected substructure of M generated by $w \in W$ is the structure $M^w = (W^w, R^w, \emptyset^w)$, where W^w is the smallest set S such that $w \in S$ and if $v \in S$ and v R u then $u \in S$, R^w is R restricted to W^w, and $\emptyset^w$ is ∅ restricted to W^w.

The following well-known theorem carries over to the present case:

Theorem 2. (On Connected Substructures). If M and M^w are defined as above and $v \in W^w$, then $\models^{M^w}_v A$ iff $\models^M_v A$.

Proof. As in the standard proof.

In view of theorem 1 we need only show that $N^{(w,1)}_{S4.2^1_n}$ has the required properties in order to prove completeness. In a transitive connected substructure all elements are, of course, related to the generating element.

In the following lemmas, I shall write '$x_1, \dots, x_n$ R y' for 'x_1 R y &... & x_n R y' and 'x R $y_1, \dots, y_n$' for 'x R y_1 &...& x R y_n'.

Lemma 1. If w R_1 v, u, then ($\exists$ t) v, u R_1 t.

Proof. Assume w R_1 v, u and not- ($\exists$ t) v, u R t. Then by familiar reasoning $\Gamma = \{A: L A \in v\} \cup \{B: L B \in u\}$ is inconsistent. So there are $A_1, \dots, A_m$, $B_1, \dots, B_n$ such that L $A_1, \dots,$ L $A_m \in$ v, L $B_1, \dots,$ L $B_n \in$ u and $\vdash_{S4.2^1_n} A_1 \wedge \dots \wedge A_m \supset \sim (B_1 \wedge \dots \wedge B_n)$. Put $A = A_1 \wedge \dots \wedge A_m$ and $B = B_1 \wedge \dots \wedge B_n$. Then L A $\in$ v, L B $\in$ u and (i) $\vdash_{S4.2^1_n} A \supset \sim B$. Since w R v and w R u, M L A, M L B $\in$ w. So by (4), L M A $\in$ u. But by (i), L M A $\supset$ L M $\sim$ B $\in$ w. So L M $\sim$ B $\in$ w. Hence $\sim$ M L B $\in$ w. A contradiction.

Lemma 2. If w R v, u and ($\forall$ t) (v, u R t $\Rightarrow$ t = v), then either not (w R_2 v) or v R_2 v.

Proof. Assume w R v, u, ($\forall$ t) (v, u R t $\Rightarrow$ t = v), w R_2 v and not (v R_2 v). Let $\Gamma = \{A: L A \in v\} \cup \{B: L B \in u\}$. Then, familiarly, for any C $\in$ v, $\Gamma \vdash$ C, and so for some A, B, L A $\in$ v, L B $\in$ u and (i) $\vdash A \wedge B \supset C$. Since not (v R_2 v), there is a C such that C, $\sim M_2$ C $\in$ v. So suppose $\vdash A \wedge B \supset C$, where L A $\in$ v and L B $\in$ u. By S4, L L M B $\in$ u. By lemma 1, u R_1 v. So L M B $\in$ v. By (i) and S4, $\vdash$ L A $\supset$ (L M B $\supset$ L M C). So L M C $\in$ v. So M L M C $\in$ w. But then by (4), L M C $\in$ w. Since w R_2 v, M_2 (C $\wedge \sim M_2$ C) $\in$ w. So by (5), $\sim$ L M C $\in$ w. A contradiction.

Corollary 1. If w R_2 v, then either v R_2 v or there is a u $\neq$ v such that

$v\ R_1\ u$.

Proof. Put v = u in lemma 2.

We define R on $(W_{s4.2_n^1} \times X)^{(w,1)}$ as in §2. Proofs of soundness, reflexivity, anti-symmetry and asymmetry are as in §2. So it remains to prove convergence only.

Lemma 3. R is convergent.

Proof: Assume (w, 1) R (v, m) and (w, 1) R (u, 1). We wish to show $\exists$ (t, k) such that (v, m), (u, 1) R (t, k).

(A). $v \sim u$. Then put (t, k) = (v, m + 1) by clause (i) (a) of the definition of R.

(B) not ($v \sim u$). By lemma 1 there are just the two cases (a) and (b).

(a) $\exists$ r such that $r \not\sim v$, $r \not\sim u$ and v, u R r. If $v \sim r$, then $u\ R_o\ r$: so put (t, k) = (r, m + 1). Similarly, if $u \sim r$. So suppose not ($v \sim r$) and not ($u \sim r$). By clause (ii) (a), put (t, k) = (r, 1).

(b) $(r)\ (u, v\ R\ r \Rightarrow r = v)$ (or the case for r = u which is similar). Then by lemma 2, either not ($w\ R_2\ v$) or $v\ R_2\ v$. In the first case, m = 1 by soundness, so put (t, k) = (v, 1). In the second case, $u\ R_o\ v$, so put (t, k) = (v, m).

Completeness now follows.

§4. The System $S4.3_n^1$

To show that scheme (6) is valid, we need the following result:

Lemma 1. If R is a reflexive, transitive and total relation defined on a set Y with n elements, $n \geq 1$, then there is a first-element in Y i.e. a x in Y such that for all y in Y, x R y.

Proof. If $Y = \{y_1\}$, use reflexivity. If $Y = \{y_1, y_2\}$, use the fact that Y is total and reflexive. Now suppose $Y = \{y_1, y_2, \ldots, y_{n+1}\}$, $n > 1$. R restricted to $\{y_1, y_2, \ldots, y_{n+1}\}$ is reflexive, transitive and total. So by IH, there is an element, y_1 say, such that $y_1 R y_i$, $i = 1, 2, \ldots, n$. Similarly, for $\{y_2, \ldots, y_{n+1}\}$, there is an element, y_2 say, such that $y_2 R y_i$, $i = 2, \ldots, n+1$. But $y_1 R y_2$. So $y_1 R y_i$, $i = 1, 2, \ldots, n+1$.

Theorem 1. (Consistency). If $\vdash_{S4.3^1_n} A$, then $\models_{S4.3^1_n} A$.

Proof. Straightforward except for (6).

Assume $\models^M_w M_k (A \wedge \sim M_l B)$ and $\models^M_w M_l (B \wedge \sim M_k A)$. Then there are distinct $w_1, w_2, \ldots, w_k$ and distinct $v_1, v_2, \ldots, v_l$ such that $w R w_i$, $w R v_j$, $\models^M_{w_i} A \wedge \sim M_k A$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$.

Since $\models^M_{w_i} \sim M_l B$, for each w_1 there is a v_j such that not - $(w_i R v_j)$. Since $\models^M_{v_j} \sim M_k A$, for each v_j there is a w_i such that not- $(v_j R w_i)$. But then R as restricted to $\{w_1, \ldots, w_k, v_1, \ldots, v_l\}$ fails to satisfy lemma 1.

For completeness we require the following lemma.

Lemma 2. If $w R_k v$ and $w R_l u$, then either $v R_l u$ or $u R_k v$.

Proof. Assume $w R_k v$, $w R_l u$, not- $v R_l u$ and not- $u R_k v$. Then there is an A and a B such that $A \in u$, $\sim M_l A \in v$, $B \in v$, and $\sim M_k B \in u$. So $A \wedge \sim M_k B \in u$ and $B \wedge \sim M_l A \in v$. Hence $M_l (A \wedge \sim M_k B)$, $M_k (B \wedge \sim M_l A) \in w$, which by (6) is a contradiction.

We define R on $(W_{S4.3^1_n} \times X)^{(w,1)}$ as in §2. It is only at this stage that use is made of $\lesssim$ as a total ordering. Given the previous results, we need only prove that R is total:

Lemma 3. R is total.

Proof. Assume (w, 1) R (v, m), (u, 1). We wish to show (v,m) R (u, 1) or (u, 1) R (v, m).

(A) $v \sim u$. If $m > 1$, then (v, m) R (u, 1) by clause (i) (a). Similarly if $1 > m$. If $m = 1$, then either $v \lesssim u$ or $u \lesssim v$. Suppose $v \lesssim u$. Then (v, m) R (u, 1) by clause (i) (b). Similarly if $u \lesssim v$.

(B) not $(v \sim u)$. (a) $w \sim v$. Then not $(w \sim v)$. So either (w, 1) = (u, 1) and the case is trivial or w R_1 u. But then v R_1 u and so (v, m) R (u, 1) by clause (ii) (a).

(b) $w \sim u$. Similar to (a).

(c) not $(w \sim v)$, not $(w \sim u)$.

Then w R_m v and w R_1 v. But then by lemma 2, either v R_1 u or u R_k v. So if $w \neq v$, either (v, m) R (u, 1) or (u, 1) R (v, m). If $w = v$, the case is trivial.

§5. The System $S4.3^2_n$

Theorem 1 (Consistency). If $\vdash_{S4.3^2_n}$ A, then $\models_{S4.3^2_n}$ A.

Proof. Straightforward. I leave the proof that $\models_{S4.3^2_n}$ (7) to the reader.

For completeness we require the following lemma:

Lemma 1. If w S_1 w, w S_1 v and $w \neq v$, then there is a u distinct from w and v such that w R_1 u and u R_1 v.

Proof. Assume otherwise. Then since w S_1 w, there is an A such that A, Q_1 A ϵ w. Since $w \neq v$ and w S_1 v, there is a B such that B ϵ v, and $\sim$ B, Q B ϵ w. Now suppose M $[M\ B \wedge \sim B \wedge \sim A]$ ϵ w. Then for some t, w R_1 t and M B, $\sim$ B, $\sim$ A ϵ t. So for some t', t R t' and B ϵ t'. We may easily show that $v\ R\ t'$ and $v \not\leq t'$. So there is a C such that $C \in v$ and $\sim C \in t'$. But then $M\ (B \wedge C)$, $M\ (B \wedge \sim C) \in w$, and so $M_2\ B \in w$, contrary to $Q\ B \in w$. Hence $L\ (M\ B \supset B \vee A) \in w$. But A, $Q\ A$, $\sim B$, $M\ B \in w$. So by scheme (7), $M_2\ B \in w$, again contrary to $Q\ B \in w$.

Let I be the set of rationals r such that $1 < r < 2$. We define R on $N^{(w,1)}$ where $N = \{(w, n) : w \in W_{S4.3^2_n}\ \&\ ((w \sim w\ \&\ n \in I) \text{ or } (\text{not } (w \sim w)\ \&\ n = 1))\} \cup \{(w, 1)\}$ as follows:

$(w, n)\ R\ (v, m)$ iff

either (i) $w \sim v$ and

(a) $w \lessgtr v$

or (b) $w = v$ and $n \leq m$

or (ii) not $(w \sim v)$ and $w\ R_1\ v$.

Given the above lemma and previous lemmas, we may show that R has the required properties. I omit details.

§6. References.

For information on the systems S4.2 and S4.3 see Prior's Past, Present and Future, chapter 2.

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