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MODEL THEORY FOR MODAL LOGIC – PART III EXISTENCE AND PREDICATION*

This paper is concerned with the technical implications of a certain view connecting existence to predication. This is the view that in no possible world is there a genuine relation among the non-existents of that world or between the non-existents and the existents.[1] The meaning of the term 'genuine' here may be variously explained. On an extreme interpretation, all relations are 'genuine', so that none of them are to relate non-existents. On a milder interpretation, the genuine relations are those that are simple or primitive in some absolute sense. But even without appeal to an absolute concept of simplicity, we can require that all relations should be analyzable in terms of *some* suitable set of relations, relating only existents to existents.

In order to make our results applicable to the thesis, we shall suppose that the primitive non-logical predicates of our language correspond to the genuine relations, whatever they might be taken to be. Thus, the linguistic formulation of the thesis becomes that the primitive predicates of the language should only be true, in each world, of the existents of that world. Of course, the thesis might have been given a linguistic formulation, without any reference to relations, in the first place.

The thesis is an instance of what has been called *Actualism*. This is the ontological doctrine that ascribes a special status to actual or existent objects. Another form of the doctrine, so-called *World Actualism*, says that the behaviour of non-existents is supervenient upon the behaviour of the existents, that two possible worlds which agree in the latter respect cannot differ in the former respect. The present thesis, by contrast, might be called *Predicate Actualism*. It should be clear that Predicate Actualism implies World Actualism, at least if the predicates used to describe the world are to express 'genuine' relations; for then there are no relationships involving non-existents by which two worlds might be distinguished. On the other hand, World Actualism does not, as it stands, imply Predicate Actualism.

For our purposes, it will be useful to distinguish two versions of the actualist doctrine for predicates. Under one, the primitive predicates do not

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apply to non-existents, they are neither true nor false of them. Under the other, the primitive predicates do apply to the non-existents, but are false of them.

For the first actualist, there is no more to say of a possible world than that there are certain relations among the existents. If the extensions of the predicates are to be extended to non-existents then this must be done by convention. One could adopt the Falsehood Convention, that the predicates are to be false of non-existents; but one could equally well adopt any other convention in its place. For the second actualist, however, it is true to say that the relations fail to hold among non-existents; and that they do not hold is not a matter of convention but of principle. One could not equally well say, for example, that the relations always hold of non-existents.

For someone who holds to the first type of actualism, the standard semantics of modal logic, as presented in [8], is problematic; for ir requires that the extensions of the predicates be defined over all possible objects in each possible world. Now our actualist could simply eschew the standard semantics. This is, in effect, the approach of Prior in his system Q ([10], Chaps. IV–V), who sets up the semantics in such a way that appeal to the extension of a predicate over non-existents is never required. But there is also a way in which our actualist may make use of the standard semantics; for he may arbitrarily extend the extensions of the predicates to existents and non-existents alike, thus making that semantics applicable.

However, in the resulting truth-conditions, not all sentences of the modal language will be on a par. For the truth-value of some of them in a world will depend upon how exactly the extensions of the predicates are extended to non-existents, while the truth-value of others will not so depend. In so far then, as the behaviour of non-existents is a conventional feature of the model, it is only the latter class of sentence that will prove acceptable to the actualist.

Now one simple syntactic device for securing the desired independence from non-existents is to require that each non-logical predicate P always occur in contexts of the form '$Px_1 \ldots x_n$ & $x_1, \ldots, x_n$ all exist', for the extension of the predicate over non-existents can then make no difference to the ensuing truth-conditions. Our first technical result is a sort of converse to this: any sentence whose truth-value is independent of the behaviour of non-existents is logically equivalent to one whose predicates are existentially restricted in this way. Thus, the actualist loses nothing by limiting himself to such sentences in the first place.

Like Theorem 13 of [4] (herafter called 'Part I'), the above result is an instance of a preservation theorem, with equivalence to a member of a syntactically characterized class of sentences being explained in terms of the given sentence preserving its truth-value in the transition from one sort of model to another. The proof, which is rather trivial, is given in Section 1. What is surprising is that the result no longer holds when the underlying logic is replaced with an arbitrary theory. In other words, something may be lost in confining one's attention to existentially restricted sentences when the models to be considered are required to verify a certain class of non-logical axioms. The exceptions to the result and some extensions of the result for different semantic conditions or for different languages are given in Section 2.

The framework of the standard semantics for modal logic is not problematic on the second version of actualism. The predicates will always apply to non-existents in each possible world even if their extensions only include the existents. However, this version, and the other, may raise a problem for analysis; for they require that predicates which are true of non-existents be analyzed in terms of predicates which are not. Now on a certain view, upheld, among others, by Prior ([10], Chaps IV–V), Plantinga ([9], Chaps VII–VIII) and Stalnaker ([11], pp. 333–336), no predicates, simple or complex, are true of non-existents. On this view, then, there would be no problem of analysis, since there would be no predicates that stood in need of analysis. However, it has always seemed clear to me that there are complex predicates, such as the external complement of existence, that are true of non-existents and, moreover, that the analysis of some of these predicates, such as the outer truth-predicate of Section 8 in [7b], is problematic. If this is so, the actualist doctrine requires, for its vindication, a detailed programme of analysis.

In Section 3, I consider the question of when existentially unrestricted predicates can be analyzed in terms of restricted predicates. It is shown that such an analysis is possible under two separate sufficient conditions. The first is that any two possible objects both exist in some possible world and that each world contain infinitely many actuals. The second is that there be infinitely many necessary existents. It is also shown that such an analysis can always be given so long as the domain of objects is appropriately expanded.

In the proofs, it is only required that the analyses preserve the first-order truths for the respective predicates. In regard to particular predicates, the analyses may not be acceptable – either because they do not preserve all truths or because they do not explain the complex in terms of the more simple. What the results do is to remove certain technical obstacles to the programme of analysis required by this form of actualism.

I should emphasize that I have not, by any means, given a complete account of the technical, let alone philosophical, issues raised by Predicate Actualism. One might consider alternative semantical schemes for the actualist philosophy, their relationship to one another, and to other, more standard, schemes. For example, the semantics for Prior's system Q can be plausibly modified or extended in various respects. On the other hand, one might take the supervaluation route, stipulating that a sentence should be super-true just in case it is true under the standard semantics, no matter what the behaviour of the non-existents. In another direction altogether, one might follow the lead of [5] and lay down conventional postulates which guarantee that any sentence is equivalent to one acceptable to the actualist.

1. THE CHARACTERIZATION OF RESTRICTED FORMULAS

In this section, we show that the truth-value of a sentence does not depend upon the behaviour of non-existents iff that sentence is provably equivalent to one in which all of the predicates are restricted to existents.

Restricted Formulas

Let $\mathscr{L}$ be any language, with or without constants. A formula ϕ of $\mathscr{L}$ is *restricted* if each non-logical n-place predicate R of $\mathscr{L}$ always occurs in ϕ within a context $Rt_1 \ldots t_n \wedge Et_1 \wedge \ldots \wedge Et_n$ where $t_1, \ldots, t_n$ are variables or constants of $\mathscr{L}$. Let $\bar{R}t_1 \ldots t_n$ abbreviate $Rt_1 ,,, t_n \wedge Ex_1 \wedge \ldots \wedge Ex_n$. Then a formula is restricted if it is built up from formulas $\bar{R}t_1 \ldots t_n, Et$ and $t = s$ by means of the usual formation rules.

It is not required that the logical predicates $=$ and E be subject to a similar requirement. For E, the requirement is of no significance. For $=$, it is not; but it does no harm. Each occurrence of $t = s$ can be rewritten as $\Diamond(t = s \wedge Et \wedge Es)$.

The formula ϕ is *implicitly restricted* in the theory T if ϕ is equivalent to a restricted formula in T; and ϕ is *implicitly restricted* if it is implicitly restricted in the quantificational system S5 of Part I. For example, $\exists x \Box(Ex \supset Px)$ is implicitly restricted since it is equivalent in S5 to $\exists x \Box(Ex \supset Px \wedge Ex)$.

As in Section 4 of Part I, we may show that an implicitly restricted *sentence* in T is equivalent in T to a restricted sentence.

Internal Indistinguishability

We now define the model-theoretic counterpart to being implicitly restricted.

Let $\mathfrak{A} = (W, \bar{A}, \nu)$ be a structure and w a world in W. Recall from p. 144 of Part I that $\mathscr{L}^-$ is the language $\mathscr{L}$ without its constants. Then we let $\breve{\mathfrak{A}}_w = (\bar{A}_w, \breve{\nu}_w)$, where $\breve{\nu}_w$ is the function on $\mathscr{L}^- \cup \{c : \nu(c) \in \bar{A}_w\}$ such that:

(i) $\breve{\nu}_w(R) = \{\langle a_1, \ldots, a_n\rangle \in A^n_w : \langle w, a_1, \ldots, a_n\rangle \in \nu(R)\}$ for each n-place predicate R;

(ii) $\breve{\nu}_w(c) = \nu(c)$ for each constant c for which $\nu(c) \in \bar{A}_w$.

If $\mathscr{L}$ contains no constants, then $\breve{\mathfrak{A}}_w$ is the same as the inner structure $\bar{\mathfrak{A}}_w$ of Part I (p. 136). If $\mathscr{L}$ contains constants then $\breve{\mathfrak{A}}_w$ and $\bar{\mathfrak{A}}_w$ may differ, since $\bar{\nu}_w$ is only defined on $\mathscr{L}^-$.

Two structures $\mathfrak{A} = (W, \bar{A}, \nu)$ and $\mathfrak{B} = (V, \bar{B}, \mu)$ are *internally indistinguishable* – in symbols, $\mathfrak{A}$ ii $\mathfrak{B}$ – if $W = V$ and $\forall w \in W(\breve{\mathfrak{A}}_w = \breve{\mathfrak{B}}_w)$. Thus internally indistinguishable structures possess the same inner structures; they do not differ in what goes on among the actuals of each world. Two *models* $(\mathfrak{A}, w_0)$ and $(\mathfrak{B}, v_0)$ are *internally indistinguishable* if $w_0 = v_0$ and $\mathfrak{A}$ ii $\mathfrak{B}$.

Relative to a theory T, say that ϕ is preserved under internally indistinguishable models if whenever $\mathfrak{M}$ and $\mathfrak{N}$ are models for T, $\mathfrak{M} \models \phi$ and $\mathfrak{M}$ ii $\mathfrak{N}$ then $\mathfrak{N} \models \phi$. We wish to show that a sentence is implicitly restricted in S5 (*not* arbitrary theory T) iff it is preserved under internally indistinguishable models.

One direction requires:

LEMMA 1. Suppose that $\mathfrak{A}$ ii $\mathfrak{B}$. Let $\phi(x_1, \ldots, x_n)$ be a restricted formula and $a_1, \ldots, a_n$ members of $A = B$. Then for any $w \in W = V$:

$$(\mathfrak{A}, w) \models \phi[a_1, \ldots, a_n] \text{ iff } (\mathfrak{B}, w) \models \phi[a_1, \ldots, a_n].$$

Proof. By formula induction. The crucial case is $\phi = Rt_1 \ldots t_n$. But then ϕ occurs in the context $Rt_1 \ldots t_n \wedge Et_1 \wedge \ldots \wedge Et_n$ and the conclusion follows.

THEOREM 2. Suppose the sentence ϕ is implicitly restricted in T. Then whenever $\mathfrak{M}$ and $\mathfrak{N}$ are models for T, $\mathfrak{M} \models \phi$ and $\mathfrak{M}$ ii $\mathfrak{N}$ then $\mathfrak{N} \models \phi$.

Proof. By lemma 1.

For the other direction we require two definitions. Given a model $\mathfrak{M}$, let the *restriction* $\mathfrak{N} = \tilde{\mathfrak{M}}$ *of* $\mathfrak{M}$ be the model that differs from $\mathfrak{M}$ only in the respect that $\mu_w(R) = \nu_w(R) \cap \bar{A}_w^n$ for each n-place predicate R of $\mathscr{L}$ and world $w \in W$. Thus in $\mathfrak{N}$ the extensions of the predicates are restricted to existents. Given a formula ϕ, let $\bar{\phi}$ be the result of replacing each atomic subformula $Rt_1 \ldots t_n$ in ϕ with $\bar{R}t_1 \ldots t_n$. Then we have the desired result:

THEOREM 3. The sentence ϕ is implicitly restricted (in S5) iff whenever $\mathfrak{M}$ and $\mathfrak{N}$ are models, $\mathfrak{M} \models \phi$, and $\mathfrak{M}$ ii $\mathfrak{N}$ then $\mathfrak{N} \models \phi$.

Proof. The $\Rightarrow$ direction follows from Theorem 2. For the other direction, we shall show that S5 $\vdash \phi \equiv \bar{\phi}$ under the supposition that $\forall \mathfrak{M}, \mathfrak{N}$ ($\mathfrak{M} \models \phi$ & $\mathfrak{M}$ ii $\mathfrak{N} \Rightarrow \mathfrak{N} \models \phi$). $\mathfrak{M}$ ii $\bar{\mathfrak{M}}$. So by the supposition, $\mathfrak{M} \models \phi$ iff $\bar{\mathfrak{M}} \models \phi$. An easy induction shows that $\bar{\mathfrak{M}} \models \phi$ iff $\mathfrak{M} \models \bar{\phi}$. Hence, $\mathfrak{M} \models \phi$ iff $\mathfrak{M} \models \bar{\phi}$; and so, by completeness, S5 $\vdash \phi \equiv \bar{\phi}$. This completes the proof.

Call a theory T *restricted* if it can be given restricted axioms, i.e., if there is a set Δ of restricted sentences such that $T = \{\phi : \phi$ a sentence and $\Delta \vdash \phi\}$. Then the above result can be extended to restricted theories.

COROLLARY 4. The sentence ϕ is implicitly restricted in a restricted theory T iff whenever $\mathfrak{M}$ and $\mathfrak{N}$ are models for T, $\mathfrak{M} \models \phi$, and $\mathfrak{M}$ ii $\mathfrak{N}$ then $\mathfrak{N} \models \phi$.

Proof. As for Theorem 3. We need only verify that $\mathfrak{M}$ is a model for T iff $\tilde{\mathfrak{M}}$ is a model for T. But this follows from the fact that T is restricted.

Corollary 4 can be extended in the obvious way so as to apply to formulas, sets of sentences and necessary equivalence. In each case, the proofs of these results show something stronger. For the restricted equivalents can be effectively obtained by replacing each formula ϕ with $\bar{\phi}$.

2. EXCEPTIONS AND EXTENSIONS

In this section we first show that the previous preservation theorem does not hold for arbitrary theories. We then show that the theorem holds when the model-theoretic condition is strengthened or when the language is enriched.

Negative Results

Let ϕ be the sentence $\exists x Ex \wedge \forall x \exists y \Diamond Rxy$, and let T be the theory with axioms ϕ and $\Box\forall x\Box(\phi \supset Ex) \vee \Diamond\exists x\Box(\phi \supset -Ex)$. Let ψ be the sentence $\Box\forall x\Box(\phi \supset Ex)$. Then ψ is preserved under internally indistinguishable models of T. For suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are models for T, $\mathfrak{M}$ ii $\mathfrak{N}$ and $\mathfrak{M} \models \psi$. Then $\mathfrak{N} \models \psi$. For otherwise $\mathfrak{N} \models \Diamond\exists x\Box(\phi \supset -Ex)$. But then $\mathfrak{N} \models -E[a]$ and $\mathfrak{M} \models E[a]$ for some $a \in A$, and so not $\mathfrak{M}$ ii $\mathfrak{N}$ after all.

However, ψ is not implicitly restricted in T. In order to show this, we shall need a definition and a lemma. Say that the one-one map ρ from A onto B is an *internal isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$ – in symbols, $\rho : \mathfrak{A} \cong_i \mathfrak{B}$ – if $\forall w \in W\ \exists v \in V(\rho \upharpoonright \bar{A}_w : \tilde{\mathfrak{A}}_w \cong \tilde{\mathfrak{B}}_v)$ and $\forall v \in V\ \exists w \in W(\rho \upharpoonright \bar{A}_w : \tilde{\mathfrak{A}}_w \cong \tilde{\mathfrak{B}}_v)$. Then we have:

LEMMA 5. Suppose that $\rho : \tilde{\mathfrak{A}}_{w_0} \cong \tilde{\mathfrak{B}}_{v_0}$ and that $\forall$finite $\rho' \subseteq \rho\ \exists \sigma \supseteq \rho'$ $(\sigma : \mathfrak{A} \cong_i \mathfrak{B})$. Then for any restricted formula $\phi(x_1, \ldots, x_n)$ and for any $a_1, \ldots, a_n \in A_{w_0}$.

$$\mathfrak{M} \models \phi[a_1, \ldots, a_n] \text{ iff } \mathfrak{N} \models \phi[\rho(a_1), \ldots, \rho(a_n)].$$

Proof. By induction on $\phi(x_1, \ldots, x_n)$. The crucial case is when $\phi(x_1, \ldots, x_n)$ is of the form $\Diamond\psi(x_1, \ldots, x_n)$. Suppose that $\mathfrak{M} \models \Diamond\psi[a_1, \ldots, a_n]$ for $a_1, \ldots, a_n \in \bar{A}_{w_0}$. Then $(\mathfrak{A}, w) \models \psi[a_1, \ldots, a_n]$ for some $w \in W$. Let ρ' be the restriction of ρ to $\{a_1, \ldots, a_n\}$ and let σ be an internal isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ that contains ρ'. Choose $v \in V$ so that $\sigma \upharpoonright \bar{A}_w : \tilde{\mathfrak{A}}_w \cong_i \tilde{\mathfrak{B}}_v$. Then an easy induction shows that $(\mathfrak{B}, v) \models \psi[\sigma(a_1), \ldots, \sigma(a_n)]$, and so $\mathfrak{N} \models \Diamond\psi[\rho(a_1), \ldots, \rho(a_n)]$. The other direction is established similarly, thus completing the proof.

This result will later be generalized.

We now construct two models $\mathfrak{M}$ and $\mathfrak{N}$. Let w, v, and u be three distinct worlds, let A_0 be a set of distinct individuals $a_1, a_2, \ldots$ and let $A_1 = \{a_2, a_4, \ldots\}$. For each permutation τ on A_0 for which $\{a \in A_0 :$

$\tau(a) \neq a\}$ is finite, choose a new world u_τ distinct from w, v and u. Now let $\mathfrak{M} = (\mathfrak{A}, w_0)$ be a model for $\mathscr{L} = \{R\}$ such that: $w_0 = w$; $A = A_0$, $\bar{A}_w = A$, $\bar{A}_v = \emptyset$, $\bar{A}_u = A_1$ and $\bar{A}_{u_\tau} = \tau(\bar{A}_u)$, i.e., $\{\tau(a) : a \in \bar{A}_u\}$; and $\nu(R) = \{\langle v, a_n, a_{n+1}\rangle : n = 1, 2, \ldots\}$. Let $\mathfrak{N} = (\mathfrak{B}, v_0)$ be the same as $\mathfrak{M}$, but with $v_0 = u$ and $\nu(R) = \{\langle v, a_{2n}, a_{2n+2}\rangle : n = 1, 2, \ldots\}$. $\mathfrak{M}$ and $\mathfrak{N}$ are pictured below in obvious fashion (with the actual worlds w_0 and v_0 circled):

	$\mathfrak{M}$		$\mathfrak{N}$
ⓦ	$a_1 a_2 \ldots \mid$	w	$a_1 a_2 \ldots \mid$
v	$\mid R a_1 a_2 \, R a_2 a_3$	v	$\mid R a_2 a_4 \, R a_4 a_6$
u	$a_2 a_4 \ldots \mid$	ⓤ	$a_2 a_4 \ldots \mid$
	$\vdots$		$\vdots$
u_τ	$\tau(\bar{A}_u) : a_{\tau(2)} a_{\tau(4)} \ldots$	u_τ	$\tau(\bar{A}_u) : a_{\tau(2)} a_{\tau(4)} \ldots$
	$\vdots$		$\vdots$

It is easy to verify the following:

(1) $\mathfrak{M}$ and $\mathfrak{N}$ are both models for T;

(2) $\rho = \{\langle a_n, a_{2n}\rangle : a_n \in A\}$, $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the condition of Lemma 5

(for given ρ' defined on $\{a_1, \ldots, a_n\}$ for which $\rho'(a_i) = a_{2i}$, let $\sigma(a_i) = a_{2i}$ for $1 \leqslant i \leqslant n$, $\sigma(a_i) = a_{2n-i}$ for $n < i \leqslant 2n$, and $\sigma(a_i) = a_i$ for $2n < i$).

(3) $\mathfrak{M} \models \psi$ and not $\mathfrak{N} \models \psi$.

But then it follows that ψ is not implicitly restricted in T. For suppose that $T \vdash \psi = \theta$ for θ a restricted sentence. By (1) and (3), $\mathfrak{M} \models \theta$; by (2) and Lemma 5, $\mathfrak{N} \models \theta$; and by (1), $\mathfrak{N} \models \psi$ – contrary to (3).

Let us use Πx for the possibilist quantifier, so that relative to a full structure $\mathfrak{A}$ (as explained on p. 129 of Part I):

$$w \models \Pi x \phi(x) \text{ iff } w \models \phi(c) \text{ for } \textit{all} \text{ constants } c.$$

Then in the particular example above, ψ is equivalent to the restricted sentence $\Pi x E x$ in T. However, it is possible to construct other counterexamples to the preservation theorem for a language with possibilist quantifiers. Indeed, these difficulties run deep; and in [6], it is shown that the Interpolation Lemma also fails for quantified S5.

Positive Results

By weakening the notion of internal indistinguishability, it is possible to find a semantic counterpart to being implicitly restricted in an arbitrary theory. For each $n = 0, 1, 2, \ldots$, let us define an isomorphism relation $\approx_n$ between models as follows:

(i) $\rho : (\mathfrak{A}, w_0) \approx_0 (\mathfrak{B}, v_0)$ iff $\rho : \tilde{\mathfrak{A}}_{w_0} \cong \tilde{\mathfrak{B}}_{v_0}$

(ii) $\rho : (\mathfrak{A}, w_0) \approx_{n+1} (\mathfrak{B}, v_0)$ iff (a) $(\mathfrak{A}, w_0) \approx_0 (\mathfrak{B}, v_0)$, (b) $\forall w \in W \forall$ finite $\rho' \subseteq \rho\ \exists v \in V \exists \sigma \supseteq \rho'(\sigma | \bar{A}_w : (\mathfrak{A}, w) \approx_n (\mathfrak{B}, v))$ and (c) $\forall v \in V \forall$ finite $\rho' \subseteq \rho\ \exists w \in W \exists \sigma \supseteq \rho'(\sigma | \bar{A}_w : (\mathfrak{A}, w) \approx_n (\mathfrak{B}, v))$.

Let us say that $\mathfrak{M} \approx \mathfrak{N}$ if $\forall n \exists \rho(\rho : \mathfrak{M} \approx_n \mathfrak{N})$. Then we may show:

THEOREM 6. The sentence ϕ is implicitly restricted in the theory T iff whenever $\mathfrak{M}$ and $\mathfrak{N}$ are models for T, $\mathfrak{M} \models \phi$ and $\mathfrak{M} \approx \mathfrak{N}$ then $\mathfrak{N} \models \phi$.

There are two other positive results of interest. First the preservation theorem for arbitrary theories holds for weaker modal logics such as T and S4 with the old notion $\cong_i$ used in place of $\approx$. Thus the preservation theorem also holds for theories containing S5 as long as S4-models, say, are permitted. Second, the preservation theorem holds for the classical language $\mathscr{L}^*$ of Section 2 in Part I. In this case, a sentence is restricted if each atomic subformula $R^*wt_1 \ldots t_n$ occurs in the context $R^*wt_1 \ldots t_n \wedge E^*wt_1 \wedge \ldots \wedge E^*wt_n$. These results may be proved by the classical method of diagrams or by the modal method explained in Section 3 of Part I.

3. THE FALSEHOOD PRINCIPLE

This section investigates the consequences of the Falsehood Principle for analysis. This principle states that each primitive predicate should be existentially restricted, i.e., false of the non-existents in each world. This means that any unrestricted predicate should have an analysis in terms of restricted predicates. So if the analysis is formulated in a first-order language, the theory for the original predicates should be interpretable within a theory whose predicates satisfy the axiom $(\Box)(Rx_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n)$. (See Part I, p. 128, for an explanation of the notation $(\Box)$.)

This section considers when it is possible, in principle, to give such interpretations. First, two sufficient conditions are given. Then it is shown that there is always such an interpretation so long as new entities are added to the domain. The upshot for analysis is this: there are no technical obstacles to providing first-order analyses that are in conformity with the Falsehood Principle; but in certain cases it may be necessary to expand the domain of existents.

Interpretations

Let $\mathscr{L}$ and $\mathscr{L}'$ be two languages with the same individual constants. Suppose that for each n-place predicate R of $\mathscr{L}$ there is given a formula $\phi_R(x_1, \ldots, x_n)$ of $\mathscr{L}'$. Now suppose that for each formula ϕ of $\mathscr{L}$, $\tau(\phi)$ is the result of replacing any subformula $Rt_1 \ldots t_n$ of ϕ with $\phi_R(t_1, \ldots, t_n)$ (re-writing bound variables of $\phi_R(t_1, \ldots, t_n)$ if necessary). Thus:

(i) (a) $\tau(Rt_1 \ldots t_n) = \phi_R(t_1, \ldots, t_n)$

(b) $\tau(t = s) = t = s$

(c) $\tau(Et) = Et$

(ii) $\tau(-\phi) = -(\tau(\phi))$

(iii) $\tau(\phi \vee \psi) = (\tau(\phi) \vee \tau(\psi))$

(iv) $\tau(\forall x\phi) = \forall x\tau(\phi)$

(v) $\tau(\Box\phi) = \Box\tau(\phi)$.

Then τ is said to be a *translation from* $\mathscr{L}$ into $\mathscr{L}'$.

Let T and T' be theories in $\mathscr{L}$ and $\mathscr{L}'$ respectively, and let τ be a translation from $\mathscr{L}$ into $\mathscr{L}'$. Then τ is an *interpretation of T in T'* if $T \vdash \phi$ iff $T' \vdash \tau(\phi)$ for each sentence ϕ of $\mathscr{L}$; and T is *interpretable in T'* if there exists an interpretation τ of T in T'.

Call a theory T *restrictive* if $(\Box)(Rx_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n)$ is a theorem of T for each n-place predicate R. Then our original question takes the form: when is a theory interpretable within a restrictive theory?

There are some simple examples of such interpretations. Let us use $x \equiv y$ for $x = y \wedge Ex \wedge Ey$ and $x \mathrel{\bar{\in}} y$ for $x \in y \wedge Ex \wedge Ey$. Then treating = as a non-logical predicate, $x = y$ may be defined as $\Diamond(x \equiv y)$ and, under the assumptions of [7a], $x \in y$ may be defined as $\Diamond(x \mathrel{\bar{\in}} y)$; and so the theory of the unrestricted predicate may thereby be interpreted within an appropriate theory of the restricted predicate. However not all theories are thus interpretable. For example, consider the theory T whose axioms are $\exists x_1 \ldots x_n [\Diamond(Fx_1 \wedge - \exists xEx) \wedge \Diamond(-Fx_1 \wedge Fx_2 \wedge - \exists xEx) \wedge \ldots \wedge \Diamond(-Fx_1 \wedge \ldots \wedge -Fx_{n-1} \wedge Fx_n \wedge - \exists xEx)]$ for each $n = 2, 3, \ldots$. Then it may be shown that T is not interpretable within a restrictive theory.

Let us now give the sufficient conditions.

Condition One. We say that a theory T *satisfies the first condition* if the following sentences are theorems of T:

(i) $(\Box) \Diamond (Ex \wedge Ey)$;

(ii) $\Box \exists x_1 \ldots \exists x_n(Dif(x_1, \ldots, x_n))$ for each $n = 1, 2, \ldots$. ($Dif(x_1, \ldots, x_n)$ is the formula $\bigwedge_{1 \leqslant i < j \leqslant n} x_i \neq x_j$.)

Note that the sentences under (i) and (ii) contain logical predicates only. Thus if T satisfies the first condition then so does any extension of T.

(i) says that any two possible objects possibly co-exist. I call it the *Principle of Possible Co-existence.* For some domains the principle does not hold. Assume, for example, that the fact-that-p exists iff p is the case. Then for contingent p, the possible fact-that-p and the possible-fact-that-not-p cannot co-exist (see [7c]). (ii) simply says that each world contains an infinite number of existents.

We shall show that any theory T satisfying the first conditions is interpretable within a restrictive theory T'. The language $\mathscr{L}'$ of T' contains the same constants as $\mathscr{L}$, a one-place predicate F, a three-place predicate P, and a one-place predicate R' for each predicate R in the language $\mathscr{L}$ of T. Intuitively speaking, F is true of a in world w if a *is* w; P is true of a, b and c in w if a, b and c exist in w and c is the pair $\langle a, b\rangle$; and R' is true of a in w if a exists in w, a is the $(n + 1)$-tuple $\langle v, a_1, \ldots, a_n\rangle$ and R is true of $\langle a_1, \ldots, a_n\rangle$ in v. This interpretation presupposes, of course, that the worlds are individuals and that the domain of individuals is closed under a pairing function.

For each $n = 2, 3, \ldots$ define a formula $P_g^n(x_1, \ldots, x_n, x_{n+1})$ with $n + 1$ free variables $x_1, \ldots, x_{n+1}$:

(i) $P_g^2(x_1, x_2, x_3)$ is $\Diamond Px_1x_2x_3$;

(ii) $P_g^{n+1}(x_1, \ldots, x_{n+1}x_{n+2})$ is $\Diamond\exists y(P_g^n(x_1, \ldots, x_n, y) \wedge P_g^2(y, x_{n+1}, x_{n+2}))$, where $n \geqslant 2$ and where y is a new variable.

Let τ be the translation from $\mathscr{L}$ into $\mathscr{L}'$ in which $\phi_R(x_1, \ldots, x_n)$ is $\exists y(Fy \wedge \Diamond\exists z(P_g^{n+1}(y, x_1, \ldots, x_n, z) \wedge \Diamond R'z)$ for each n-place predicate R of $\mathscr{L}$. Let T' be the theory whose axioms are $\tau(\phi)$ for $\phi \in T$ and $(\Box)(Rx_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n)$ for each n-place predicate R of $\mathscr{L}'$.

LEMMA 7. T is interpretable in T' as defined above.

Proof. That $\phi \in T$ implies $\tau(\phi) \in T'$ follows immediately from the definitions. For the other direction, suppose that the sentence ϕ of $\mathscr{L}$ is not a theorem of T. By the strong completeness theorem and the Skolem–Löwenheim theorem (see the remarks after Theorem 3.5 of Part I), ϕ is false in a model for T in which W and A are countable. By the theorems under (ii) above, each $\bar{A}_w$ is infinite.

Let $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \ldots$ be an enumeration, without repetitions, of all pairs of individuals in A. Now define a sequence $c_1, c_2, \ldots$ as follows:

c_1 is such that $a_1, b_1, c_1 \in \bar{A}_w$ for some world w_1;

c_{n+1} is such that $c_1, \ldots, c_{n+1}$ are all distinct and $a_{n+1}, b_{n+1}, c_{n+1} \in \bar{A}_{w_{n+1}}$ for some world w_{n+1}.

Since $\mathfrak{M}$ satisfies the conditions (i) and (ii), such a sequence exists. For each $n \geqslant 2$, define a n-tpling function p^n as follows:

$$p^2(e, f) = c_m \text{ if } e = a_m \text{ and } f = b_m;$$

$$p^{n+1}(e_1, \ldots, e_{n+1}) = p^2(p^n(e_1, \ldots, e_n), e_{n+1}).$$

Also, for each world $w \in W$, choose a distinct individual $a_w \in A_w$.

Let $\mathfrak{M}'$ be the model in which: $W' = W$; $\bar{A}'_w = \bar{A}_w$ for each $w \in W$; $v'(a) = v(a)$ for each constant a of $\mathscr{L}$; $v'(P) = \{\langle w_n, a_n, b_n, c_n\rangle : n = 1, 2, \ldots\}$, $v'(F) = \{\langle w, a_w\rangle : w \in W\}$ and $v'(R') = \{\langle w, f\rangle : f \in \bar{A}_w$ and, for some $v, e_1, \ldots, e_n, \langle v, e_1, \ldots, e_n\rangle \in v(R)$ and $f = p^{n+1}(a_v, e_1, \ldots, e_n)\}$.

An easy induction establishes that $\mathfrak{M} \models \psi\,[e_1, \ldots, e_n]$ iff $\mathfrak{M} \models \tau(\psi)$ $[e_1, \ldots, e_n]$ for each formula $\psi(x_1, \ldots, x_n)$ of $\mathscr{L}$. Hence, $\tau(\phi)$ is false in $\mathfrak{M}'$ and each $\tau(\psi)$ for $\psi \in T$ is true in $\mathfrak{M}'$. But $(\Box)(Rx_1 \ldots x_n \supset Ex_1 \wedge \ldots \wedge Ex_n)$ for R an n-place predicate of $\mathscr{L}'$ is also true in $\mathfrak{M}'$, and so $\tau(\phi)$ is not a theorem of T'. Thus, Lemma 7 is proved.

Since T' is a restrictive theory and since T was an arbitrary theory satisfying the first conditions, we obtain:

THEOREM 8. Any theory satisfying the first condition is interpretable within a restrictive theory.

Condition Two. Let us say that a theory T *satisfies the second condition* if $\exists x_1 \ldots \exists x_n(Dif(x_1, \ldots, x_n) \wedge \Box(Ex_1 \wedge \ldots \wedge Ex_n))$ is a theorem for $n = 2, 3, \ldots$. Thus, a theory satisfying the second condition is one which implies that there are infinitely many necessary existents.

To show that such a theory T is interpretable within a restrictive theory T', let the language $\mathscr{L}'$ of T' contain the same constants as $\mathscr{L}$, a two-place predicate C, and for each n-place predicate R of $\mathscr{L}$ an n-place predicate R'. Intuitively, we may think of C as a counterpart relation that associates with each possible individual an abstract counterpart (the individual's essence, say). The relation R' then holds of $b_1, \ldots, b_n$ if R holds of the entities $a_1, \ldots, a_n$ of which $b_1, \ldots, b_n$ are counterparts. The translation τ is the one in which $\phi_R(x_1, \ldots, x_n)$ is $\exists y_1 \ldots y_n(\bigwedge^n Cx_iy_i \wedge R'y_1 \ldots y_n)$. T' is then defined as before, but with the new τ in place of the old.

Now let $\mathfrak{M}$ be a model for T in which A is countably infinite. Let $a_1, a_2, \ldots$ be an enumeration without repetitions of A, and for each $i = 1, 2, \ldots$, let w_i be a world such that $a_i \in \bar{A}_{w_i}$. Let $b_1, b_2, \ldots$ be an enumeration without repetitions of b for which $b \in$ each $\bar{A}_w$. The model $\mathfrak{M}'$ for $\mathscr{L}'$ is defined as follows: $W' = W$; $\bar{A}'_w = \bar{A}_w$ for each world $w \in W$; $\nu'(a) = \nu(a)$ for each constant a of $\mathscr{L}$; $\nu'(C) = \{\langle w_i, a_i, b_i\rangle : i = 1, 2, \ldots\}$, and $\nu'(R') = \{\langle w, b_{i_1}, \ldots, b_{i_n}\rangle : \langle w, a_{i_1}, \ldots, a_{i_n}\rangle \in \nu(R)\}$. $\mathfrak{M}'$ may then be used as before to establish:

THEOREM 9. Any theory satisfying the second condition is interpretable within a restrictive theory.

Further Results. Say that the translation τ is *weak* if for some formula $\phi_D(x)$ with one free variable x the translation clause for the universal quantifier is:

$$\tau(\forall x\phi(x)) = \forall x(\phi_D(x) \supset \tau(\phi(x)).$$

Thus, in a weak translation, the original domain may be a restriction of the new domain. Correspondingly, say that T is *weakly interpretable in* T' if for some weak translation τ from $\mathscr{L}$ into $\mathscr{L}'$, $T \vdash \phi$ iff $T \vdash \tau(\phi)$ for each sentence ϕ of $\mathscr{L}$.

Now introduce a new one-place predicate G into the previous language $\mathscr{L}'$ and modify the previous translation τ so that $\phi_D(x)$ is Gx. Then the new τ is a weak interpretation of any theory T into the corresponding restrictive theory. For in obtaining the model $\mathfrak{M}'$ from $\mathfrak{M}$ we may expand the domain so that $\mathfrak{M}'$ contains infinitely many necessary existents. Thus, this gives:

THEOREM 10. Any theory is weakly interpretable within a restrictive theory.

Some open problems remain. First, necessary and sufficient conditions have not been given for interpretability within a restrictive theory. I doubt that there are any perspicuous conditions of this sort. Second, no conditions have been given for a theory T to be *mutually* interpretable within a restrictive theory T', where this last notion requires that there be a translation τ from T into T' and a translation τ' from T' into T, possibly with the added proviso that $T \vdash \tau'(\tau(\phi)) \equiv \phi$ and $T' \vdash \tau(\tau'(\psi)) \equiv \psi$ for each formula ϕ of $\mathscr{L}$ and each formula ψ of $\mathscr{L}'$. Finally, we should like to know the scope of particular translations. For any translation τ, we should like to determine the theories T for which τ is an interpretation of T within the restrictive theory that has the axioms $\tau(\phi)$ for $\phi \in T$.

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NOTES

* This paper is the third and final part of a series (see the references below). It was completed and submitted to the *Journal of Philosophical Logic* in 1977, at about the same time as the other parts. But because of some mishap in the mail, its publication was delayed. The present part is independent from the other parts in its results, but draws upon the terminology of Section 2 of Part I.

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[1] I have briefly discussed this thesis elsewhere. The reader may like to consult Section 7 of [1], pp. 151 and 156–160 of [2], p. 564 of [3], and Section 8 of [7b]. There has been a fair amount of recent literature on the topic. I cannot give a complete survey, but the reader may like to consult Chapters IV–V of [9], p. 86 of [8], Chapters VII–VIII of [10], and pp. 333–336 of [11].

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