Truthmaker Semantics, Ground, and Generality

Kit Fine and Louis deRosset

May 9, 2024

Abstract

Our aim in this paper is to extend the semantics for the kind of logic of ground developed in [deRosset and Fine, 2023]. In that paper, the authors very briefly suggested a way of treating universal and existential quantification over a fixed domain of objects. Here we explore some options for extending the treatment to allow for a variable domain of objects.

KEYWORDS: Truthmaker Semantics; Logic of Ground; Quantification; Variable Domain

Our aim in this paper is to extend the truthmaker semantics for the kind of logic of ground developed in [deRosset and Fine, 2023] to handle quantification. In that paper, the authors offered a semantics for a propositional language with negation, conjunction, and disjunction. They also very briefly suggested a way of treating universal and existential generalizations assuming a fixed domain of quantification. Here we explore the prospects for extending that treatment to allow variation in which objects exist.

We begin with a brief recapitulation of adjustments to deRosset and Fine's system to accommodate the possibility of infinite grounds. We next briefly review the fixed domain semantics for quantification characterized by deRosset and Fine [2023]. We then discuss the additional complications that arise from generalizing that approach to a variable domain semantics and suggest two ways of dealing with those problems - one drawing on the notion, developed below, of *dependent* content and the other drawing on a distinction, characterized below, between what we call *back-grounds* and *fore-grounds.* In reading the present paper, the reader may find it helpful to have the more detailed description of the semantics in [deRosset and Fine, 2023] at hand, especially in regard to the critical distinction between condition and content.

1 Infinitary Grounds

deRosset and Fine's original semantics [2023] assigns to each sentence of a propositional language a *content* (or *bilateral proposition*) comprising a *truth-condition* and a *falsity-condition*.[1] It also appeals to two operations on contents, *choice* and *combination*, described on an analogy due to [Fine, 2017, p. 637-8] with menus. A typical breakfast menu, might offer a choice of either oatmeal with fruit or eggs with toast. Each of the two options is itself a combination, and the toast might itself comprise a choice between whole wheat and white toast. Thus, on typical menus, there is a hierarchical organization of choices and combinations, with the menu itself generally offering, at the highest level, a choice of options. Clearly, the character of choices on a menu is, intuitively, disjunctive, since any of the options on offer may be selected. Likewise, the character of combinations is, intuitively, conjunctive, since any selection includes all of the items combined. The original semantics appeals to two analogous operations on finite sequences of contents, with *choice* providing a semantic analogue of disjunction and *combination* a semantic analogue of conjunction.[2]

Given an assignment of (bilateral) contents to atomic sentences, the contents of molecular sentences are defined inductively.[3] So, for instance, the truth-condition of a disjunction is the choice of the contents of the disjuncts, and the falsity-condition of the disjunction is the combination of the contents of the negations of the disjuncts. Write $[a+b]$ for the choice of the contents a and b and $[a.b]$ for their combination. Then, if ϕ has content a, if $\neg\phi$ has content a', and if ψ and $\neg\psi$ have contents b, b', respectively, then the truth-condition for $(\phi \vee \psi)$ is $[a + b]$ and its falsity-condition is $[a'.b']$. Truth- and falsity-conditions for conjunctions are the obvious dual. Connections of ground are then defined by appeal to relations of what were called *selection* among contents and conditions. So, for instance, if the truth-condition for ϕ is a choice of the content of ψ or

[1]See [deRosset and Fine, 2023, p. 420] (reproduced in ["Approaches", this volume, appendix]) for a specification of the distinction between contents and conditions, in which we adapt a similar idea in standard truthmaker semantics.

[2]Since choice and combination are defined on sequences of contents, rather than on sets, the operations are sensitive to the order of contents. There is no general guarantee that either choice or combination are commutative.

[3]See [deRosset and Fine, 2023, p. 427] (reproduced in ["Approaches", this volume, appendix]) for a formal definition of an *assignment* of contents to sentences. The content of a sentence is indicated using double bars, writing $\overline{\overline{\phi}}$ for the content of ϕ.

something else, then the content of ψ is a selection from that of ϕ, and so ψ grounds ϕ. Similarly, if the truth-condition for ϕ is the combination of the contents of ψ and χ, then the contents of ψ and χ are jointly a selection from the content of ϕ, and the corresponding grounding claim is true [deRosset and Fine, 2023, D2.2, p. 426], ["Approaches", this volume, appendix]. Models for the object language specify the universe of conditions, along with the operations of choice and combination. Each model thereby specifies which contents are selections from a given content. A model will also result in an assignment of a content to every sentence. A grounding claim will then be true in a model just in case the contents of its constituent sentences stand in an appropriate selection relation. A rigorous specification of the notion of truth in a model for grounding claims is reproduced in ["Approaches", this volume, appendix].

It is standard to make two orthogonal distinctions among grounding connections: they may be either *strict* or *weak*, and either *partial* or *full*. See [Fine, 2012a,b] for a detailed explanation of these distinctions. deRosset and Fine [2023] use the semantics to interpret an object language in which each of the resulting four types of grounding claims may be expressed. As is standard, we write $<$ for strict, full ground; $\leq$ for weak, full ground; $\prec$ for strict, partial ground; and $\preceq$ for weak, partial ground. The original system GG of [deRosset and Fine, 2023] for deriving grounding claims inductively defines a derivability relation $\Vdash$ between a *set* T of grounding claims from a *set* S of grounding claims by appeal to a battery of natural rules adapted from [Fine, 2012b]. As in the standard sequent calculus, a derivability claim (or *sequent*) $T \Vdash S$ is read conjunctively on the left and disjunctively on the right. GG is finitary: in the full, weak or strict, grounding claims $\Delta \leq A$ and $\Delta < A$, the set of formulas Δ must be finite; and a set of grounding claims is derivable from T just in case there is a finite set $T' \subseteq T$ from which the grounding claim is derivable. Handling quantification in infinite domains requires relaxing the first of these requirements and allowing a universal statement to have infinitely many of its instances as grounds; and once this is done, it will be natural to relax the second of these requirements so as to allow derivability from an infinite set T without necessarily having derivability from any of its finite subsets. Finally, our instantial approach to quantification will require us, in developing a semantics for the system, to generalize the operations of choice and combination so as to allow their application to infinitely many contents.

Although deRosset and Fine did not specify details in [deRosset and Fine, 2023], the necessary adjustments are fairly straightforward. In order to stay within the confines of ZF, we set an upper bound on the number of formulas in the language and hence on both the number of grounding claims and the number of sequents $T \Vdash S$ for T and S sets of grounding claims.[4]

The system of derivation can stay much as before, but with changes to four of its rules. The four rules - THINNING, SNIP, CUT, and REVERSE SUBSUMPTION - then become:[5]

THINNING If $T \Vdash S$, then $T, T' \Vdash S, S'$

GENERALIZED SNIP $T \Vdash S$ if, for some set U of grounding claims, $T, U_1 \Vdash U_2, S$ whenever $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \emptyset$

GENERALIZED CUT $(\Delta_i \leq \phi_i), (\phi_i) \leq \psi \Vdash (\Delta_i) \leq \psi$[6]

GENERALIZED REVERSE SUBSUMPTION $(\phi_i \prec \psi), (\phi_i) \leq \psi \Vdash (\phi_i) < \psi$

The finitary rule SNIP in GG is a cut rule familiar from the standard sequent calculus. Its infinitary generalization GENERALIZED SNIP says, in effect, that the sequent $T \Vdash S$ should be taken to hold if it holds under any assignment of truth-values to the grounding claims of U, where the true grounding claims for a given assignment appear in the left-hand partition U_1 and the false ones in the right-hand partition U_2. Consider, for instance, a language with sentences $\phi, \chi, (\psi_0^i, \psi_1^i)$, for $i \in \omega$. Let S be the set of grounding claims containing $(\phi < (\psi_0^i \vee \psi_1^i))$, together with every grounding claim of the form $(\psi_{k_i}^i) \leq \chi$, where $k_0, k_1, \ldots$ are each either 0 or 1. Then $\phi \leq \chi$ is derivable from S in the infinitary system. Applications of the elimination rule for disjunction allow the derivation of a pair $\phi \leq \psi_0^i; \phi \leq \psi_1^i$, for every i. Then, no matter which member of the cartesian product of all of those pairs of grounding claims turns out to be true, $\phi \leq \chi$ can be derived by an application of CUT.

The finitary rule REVERSE SUBSUMPTION in GG says that a weak full grounding claim can be converted to a strict, full grounding claim when each of the

[4] See [deRosset and Fine, 2023, pp. 428-9] for the original specification of rules of derivation, reproduced for reference in ["Approaches", this volume, appendix].

[5] See [deRosset and Fine, 2023, pp. 428-9] for the original, finitary specification of these rules, reproduced for reference in ["Approaches", this volume, appendix].

[6] As specified in ["Approaches", this volume, appendix] we write (ϕ_i) to indicate the indexed set $\phi_0, \phi_1, \ldots$, and, similarly, for (Δ_i) and $(\phi_i \prec \psi)$.

finitely many weak partial grounds is strict. Its infinitary generalization GENERALIZED REVERSE SUBSUMPTION says that this is so even when there are arbitrarily many weak, partial grounds. Infinitary versions of the original finitary rules are required since they are valid yet cannot be obtained by iterated application of the finitary rules.

Because infinitary grounds and corresponding rules of inference are allowed, soundness requires that there be no requirement of finitude on the relation $\vDash$ of *semantic* consequence among sets of grounding claims.[7] Further, it is natural to allow the infinitary application of choice and combination, and, in order to stay within the confines of ZF, we set an infinite upper cardinal bound κ on the ordinal length of the sequences to which they may apply. Thus, choice and combination will now be defined on sequences $\langle v_\zeta : \zeta < \alpha \rangle$ of contents v_ζ for $\alpha \leq \kappa$.

The proof of soundness [deRosset and Fine, 2023, T3.1] will go through, essentially unchanged. The proof of completeness [deRosset and Fine, 2023, T8.6] will also go through, but calls for some changes which we loosely specify below:

(i) In the course of giving a completeness proof, deRosset and Fine [2023, §5] need to show that a prime set of grounding claims can be conservatively extended in ways that facilitate the construction of a model witnessing a failure of derivability.[8] For instance, they need to show that we can conservatively extend a given set of grounding claims so that for every partial grounding claim $\phi \preceq \psi$, there is a corresponding full grounding claim $\phi, \Delta \leq \psi$. Some of these syntactic arguments were made simpler by restricting attention to applications of CUT with a single minor premise. With the infinitary strengthening of CUT, these simplifications can no

[7] As specified in ["Approaches", this volume, appendix], $S \vDash T$ iff every model in which every member of S is true is also a model in which some member of T is true. Thus, $\vDash$, like $\Vdash$, should be interpreted conjunctively on the left and disjunctively on the right. Naturally, if we required that $S \vDash T$ only if both S and T were finite, we would have infinitary instances of, say CUT of the form $(\psi_i \leq \phi_i); (\phi_i) \leq \chi \Vdash (\psi_i) \leq \chi$ with no corresponding semantic consequence relation $(\psi_i \leq \phi_i); (\phi_i) \leq \chi \vDash (\psi_i) \leq \chi$.

[8] A set T of grounding claims is *prime* iff whenever T' is derivable from it, T contains some $\tau \in T'$. Intuitively, a prime set of grounding claims is a fully determinate description of what grounds what: whenever it implies some disjunction of grounding claims, it already implies at least one of the disjuncts. Since models are fully determinate, providing a Henkin-style completeness proof requires showing that any set of grounding claims from which T is not derivable can be extended to a prime set from which T is not derivable. A model witnessing this failure of derivability is then constructed from that prime extension.

longer be made. So, the syntactic proofs of that section will be somewhat more complex, though the relevant adjustments are fairly straightforward.

(ii) deRosset and Fine's [2023] construction of a model for the purpose of witnessing a failure of derivability appeals to finitary conjunctions and disjunctions. Roughly, to construct a model in which a given grounding claim $\Delta < \phi$ is true, they create a conjunction of the contents of Δ, treat that conjunction as a combination, and identify combinations and choices in a somewhat indirect way to to ensure that the conjuncts are a selection from ϕ; see [deRosset and Fine, 2023, §4] for an explanation of this proof strategy. To carry this through in an infinitary context, their use of finitary conjunctions and disjunctions must now be replaced with infinitary conjunctions and disjunctions. These infinitary complexes may then be used to construct suitable models to witness failures of derivability. This adjustment is relatively straightforward.

(iii) The proof in [deRosset and Fine, 2023, §8] that every consistent set of grounding claims has a suitable prime extension can now appeal to GENERALIZED SNIP, thereby becoming simpler and more direct.[9]

2 Quantification over a Fixed Domain

We now turn to the more specific question of how to treat the logic of ground for quantified sentences. We begin with the case in which we presuppose given a fixed domain $\mathtt{D}$ of individuals and follow the approach to this problem indicated in [deRosset and Fine, 2023, §9.2]. We suppose that the language contains names for every individual in $\mathtt{D}$. Thus, given that $\mathtt{a}_1, \mathtt{a}_2, \dots$ are the distinct individuals of $\mathtt{D}$, let $D = \{a_1, a_2, \dots\}$ be a set of corresponding distinct names for those individuals. As before, we stay within the confines of ZF by requiring that the cardinality of $\mathtt{D}$ be some $\alpha \leq \kappa$. An interpretation over $\mathtt{D}$ should then assign to every n-place predicate F a function $\mathtt{F}$ taking each n-tuple of individuals from $\mathtt{D}$ into a content; and the content of the atomic sentence $Fa_{k_1}a_{k_2}\dots a_{k_n}$ should then be taken to be $\mathtt{F}(\mathtt{a}_{\mathtt{k}_1}, \mathtt{a}_{\mathtt{k}_2}, \dots, \mathtt{a}_{\mathtt{k}_\mathtt{n}})$. In order to say within ZF, we assume

[9] Suppose S is consistent, so that $S \nVdash T$, for some T. In particular, no instance of GENERALIZED SNIP allows derivation of T from S. So, there is a partition $\{U_1, U_2\}$ of the set of all grounding claims in the language such that $U_1, S \nVdash T, U_2$. It is straightforward to show that U_1, S is prime and does not allow derivation of T.

that D is a set. This means, of course, that we cannot deal with the case in which the quantifier ranges over a proper class of objects. But this is a general problem in providing a model theory for the quantifier and is not peculiar to our own approach.

When it comes to the quantifiers, we might think of the universal statement $\forall x \phi(x)$, under an instantial approach, as the conjunction $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$ of its instances and of an existential statement $\exists x \phi(x)$ as the disjunction $\phi(a_1) \vee \phi(a_2) \vee \ldots$ of its instances. Since there is an obvious extension of the introduction and elimination rules for binary conjunction and disjunction to conjunctions and disjunctions of arbitrary length, we may read off the introduction and elimination rules for universal and existential quantification from the extended rules for conjunction and disjunction.[10] This leads naturally to some fairly straightforward introduction and elimination rules for quantified sentences and their negations, with instances taking the place of conjuncts and disjuncts. So, for instance, if $\mathtt{D} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, then $Fa, Fb, Fc < (\forall x)Fx$, and any full strict ground Δ for $(\forall x)Fx$ may be exhaustively split into full weak grounds, respectively, for Fa, Fb, and Fc. deRosset and Fine [2023, §9.2] offer a rigorous specification of the rules in question.

There is a corresponding semantic treatment. For, as we have seen, the semantics for binary conjunction and disjunction may be extended to conjunctions and disjunctions of arbitrary length; and we may then let the semantics for these conjunctions and disjunctions of arbitrary length be our guide in providing a semantics for the quantifiers. However, within the present semantic setting, there

[10] Generalizing the rules specified in ["Approaches", this volume, appendix], the introduction rules for conjunctions and disjunctions of arbitrary length will be

$$\Vdash \phi_i < (\phi_0 \vee \phi_1 \vee \ldots) \qquad \text{and} \qquad \Vdash (\phi_i) < (\phi_0 \wedge \phi_1 \wedge \ldots).$$

It is convenient for the statement of elimination rules to introduce some notation due to [Fine, 2012b]. For non-empty (γ_i), let $\Delta \leq (\gamma_i)$ abbreviate

$$\Delta^0_0 \leq \gamma_0; \Delta^0_1 \leq \gamma_1; \ldots \quad | \quad \Delta^1_0 \leq \gamma_0; \Delta^1_1 \leq \gamma_1; \ldots \quad | \quad \ldots$$

where the family of sets $(\{\Delta^j_0, \Delta^j_1, \ldots\})$ contains exactly the sets of appropriate cardinality whose union is Δ. Intuitively, $\Delta \leq (\gamma_i)$ is the disjunction of all the ways of divvying up Δ exhaustively, but not necessarily exclusively, into weak grounds for the (γ_i). Then the elimination rule for conjunctions of arbitrary length is

$$\Delta < (\phi_0 \wedge \phi_1 \wedge \ldots) \Vdash \quad \Delta \leq (\phi_i)$$

and the elimination rule for disjunctions of arbitrary length is

$$\Delta < (\phi_0 \vee \phi_1 \vee \ldots) \Vdash \quad (\quad \Delta \leq \Gamma_0 \quad | \quad \Delta \leq \Gamma_1 \quad | \quad \ldots \quad)$$

where $(\Gamma_i) = \mathfrak{P}(\phi_i) \setminus \{\emptyset\}$.

is a hitch, since the semantics for $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$ or for $\phi(a_1) \vee \phi(a_2) \vee \ldots$ takes account of the order of the conjuncts or of the disjuncts. The truth-condition for $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$, for example, will be the combination of the contents of $\phi(a_1), \phi(a_2), \ldots$ in that very order.[11] Since the combination may vary with the order, this makes it unclear what the content of the universal statement should be. deRosset and Fine [2023, p. 491] proposed to solve this problem by appealing to a well-ordering of the domain (and a corresponding well-ordering of names for individuals), so that, for instance, the truth-condition for the universal generalization $(\forall x)Fx$ is interpreted, in effect, as the combination $[\mathtt{Fa_1}.\mathtt{Fa_2}.\cdots]$ where the conditions combined occur, intuitively, *in order*. Similar remarks apply to the specification of falsity-conditions, and the specification of content for existential generalizations and for the negations of quantified claims may be made in the obvious way.

Appeal to a particular well-ordering may seem arbitrary. However, since our interest is in the set of instances and not their order, we may, following [deRosset and Fine, 2023], take the combination (or choice) of a specific sequence of contents of $\phi(a_1), \phi(a_2), \ldots$ to *represent* the combination (or choice) of the corresponding *set* of contents. Appeal to an admittedly arbitrary ordering thereby provides a way of representing operations on sets without requiring us to extend the existing apparatus of combination and choice to include their application to sets as well as to sequences.

Semantic clauses for truth-functional operators are specified as before. The quantificational system of derivation is then readily shown to be sound for the proposed semantics. However, it is not complete since, under the semantics, the contents of $\forall x\phi(x)$ and $\forall y\phi(y)$ (and of $\exists x\phi(x)$ and $\exists y\phi(y)$) will always be the same and so $\forall x\phi(x) \leq \forall y\phi(y)$ and $\exists x\phi(x) \leq \exists y\phi(y)$ should also be theorems. deRosset and Fine [2023, p. 491] therefore suggested that $A \leq A'$ should be a theorem whenever A' is an alphabetic variant on A; and they conjectured that the proposed semantics is complete with respect to the resulting system [deRosset and Fine, 2023, §9.2].

[11] See n. 2.

3 Variable Domain Semantics

Quantifiers over variable domains raise additional complications. In this case, we should take $\mathtt{D}$ to consist of all the *candidate* individuals over which the quantifiers may range. The different domains over which the quantifiers may vary will then be subsets of $\mathtt{D}$; and as before, in order to stay within ZF, we take $\mathtt{D}$ itself to be a set.

We follow [Fine, 2012b, 59 et seq.] in supposing that, for each subset $\mathtt{E}$ of $\mathtt{D}$, there is a totality statement T_E to the effect that the individuals of $\mathtt{E}$ are exactly the individuals that there are.[12] To simplify the statement of rules for the quantifiers, we introduce some further notation. Let $\mathcal{A}(\Delta \leq \Gamma_1 | \Delta \leq \Gamma_2 | \dots) = (\Delta \leq \Sigma_1 | \Delta \leq \Sigma_2 | \dots)$, where (Σ_i) are exactly the unions of one or more of the sets $\Gamma_1\, \Gamma_2$ .... Intuitively, $\mathcal{A}$ handles elimination of strict grounding claims obtained by AMALGAMATION from one or more of $(\Delta \leq \Gamma_i)$.[13] To illustrate, recall that the elimination rule for disjunction says that any strict ground for $(\phi \vee \psi)$ is either a weak ground for ϕ, a weak ground for ψ, or the amalgamation Δ_ϕ, Δ_ψ of weak grounds for ϕ and ψ, respectively. We could thus state the elimination rule for binary disjunction using $\mathcal{A}$ as:

$$\Delta < (\phi \vee \psi) \quad \Vdash \quad \mathcal{A}(\Delta \leq \phi | \Delta \leq \psi).$$

The positive introduction and elimination rules for the universal and existential quantifier now take the form:

∀I $\Vdash T_E, \phi(a_1), \phi(a_2), \dots < \forall x \phi(x)$ (where $\mathtt{E} = \{\mathtt{a_1}, \mathtt{a_2}, \dots\}$)

∀E $\Delta < \forall x \phi(x) \Vdash \mathcal{A}(\Delta \leq T_{E_1}, \phi(a_1^{E_1}), \phi(a_2^{E_1}), \dots, \quad | \quad \Delta \leq T_{E_2}, \phi(a_1^{E_2}), \phi(a_2^{E_2}), \dots, \quad | \quad \dots)$
(where $\{\mathtt{E_1}, \mathtt{E_2}, \dots\} = \mathfrak{P}(\mathtt{D})$ and $\mathtt{E_i} = \{\mathtt{a_j^{E_i}}\}$)

∃I $\Vdash T_E, \phi(a), \phi(b), \dots < \exists x \phi(x)$ for any non-empty subset $\{\mathtt{a}, \mathtt{b}, \dots\}$ of $\mathtt{E} \subseteq \mathtt{D}$

[12] To be strictly accurate, when $\mathtt{E} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \dots\}$, [Fine, 2012b] would use $T(a, b, c, \dots)$ for the totality statement in place of T_E. But the present formulation is preferable in that it takes no account of the order in which the individual names are given. The idea of totality facts or statements appears in [Russell, 1918, pp. 198-9] and [Armstrong, 1997] as part of a solution to related problems.

[13] AMALGAMATION is the rule $(\Delta_i \leq \phi) \Vdash (\Delta_i) \leq \phi$. Intuitively, this rule says that weak grounds for ϕ may be amalgamated into a single weak ground: if each of the Δ_i's weakly grounds ϕ separately, then their union also weakly grounds ϕ. Its admissibility in GG follows from instances of CUT of the form $(\Delta_i \leq \phi); (\phi_i) \leq \phi \quad \Vdash \quad (\Delta_i) \leq \phi$, where each ϕ_i is just ϕ. Note that the major premise has the form $(\phi_i) = \{\phi, \phi, \dots\} = \phi \leq \phi$, and so is derivable via REFLEXIVITY [Fine, 2012b].

$\exists$**E** $\Delta < \exists x\phi(x) \Vdash \mathcal{A}(\Delta \leq T_{F_1}, \phi(a_1^{F_1}), \phi(a_2^{F_1}), \ldots, \quad | \quad \Delta \leq T_{F_2}, \phi(b_1^{F_2}), \phi(b_2^{F_2}), \ldots, \quad | \quad \ldots)$
(where $\{\mathtt{F_1}, \mathtt{F_2}, \ldots\} = \mathfrak{P}(\mathtt{D}) \setminus \{\emptyset\}$ and $\mathtt{F_i} = \{\mathtt{b_j^{F_i}}\}$.

Just as $\forall x\phi(x)$ was previously taken to be equivalent to $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$, we might now think of taking $\forall x\phi(x)$ to be equivalent to

$$(T_{E_0} \wedge \phi(a_{0_0}) \wedge \phi(a_{0_1}) \wedge \ldots) \vee (T_{E_1} \wedge \phi(a_{1_0}) \wedge \phi(a_{1_1}) \wedge \ldots) \vee \ldots$$

where the $\mathtt{E_0}, \mathtt{E_1}, \ldots$ range over all of the subsets of $\mathtt{D}$ and where each $\mathtt{E_k}$ is taken to be of the form $\{\mathtt{a_{k_0}}, \mathtt{a_{k_1}}, \ldots\}$. However, the previous inferential rules can no longer be justified on the basis of this equivalence, since it sanctions taking each conjunction $(T_{E_k} \wedge \phi(a_{k_0}) \wedge \phi(a_{k_1}) \wedge \ldots)$ to be a strict ground for $\forall x\phi(x)$, in violation of the Elimination Rule, which requires that the conjuncts, not the conjunction, should be the maximal grounds.

For the same reason, the proposed equivalence of $\forall x\phi(x)$ to

$$(T_{E_0} \wedge \phi(a_{0_0}) \wedge \phi(a_{0_1}) \wedge \ldots) \vee (T_{E_1} \wedge \phi(a_{1_0}) \wedge \phi(a_{1_1}) \wedge \ldots) \vee \ldots$$

can no longer serve as a guide to the semantics, since it is the content of $(T_{E_k} \wedge \phi(a_{k_0}) \wedge \phi(a_{k_1}) \wedge \ldots)$ rather than the contents of its conjuncts which would then serve as an immediate selection from the truth-condition for $\forall x\phi(x)$. Moreover, it would appear to be impossible in general to regard the truth-condition of $\forall x\phi(x)$ either as a combination or a choice, for it is the contents of $T_{E_k}, \phi(a_{k_0}), \phi(a_{k_1}), \ldots$ for each $\mathtt{E_k}$ that will figure as the immediate selections from the truth-condition of $\forall x\phi(x)$ and, given the dependence on the variation in $\mathtt{E_k}$, these are not of the right form to figure as the immediate selections either from a combination or from a choice. So, the approach of [deRosset and Fine, 2023] appears to be unsuitable for representing the logic of ground for variable domain quantification.

A slight variation, however, is promising. What we would like to be able to say is not that $\forall x\phi(x)$ is equivalent to a disjunction of conjunctions but that, *relative* to a specification $\mathtt{E_k}$ of the domain, $\forall x\phi(x)$ should be equivalent to $\phi(a_{k_0}) \wedge \phi(a_{k_1}) \wedge \ldots$. The immediate grounds of $\forall x\phi(x)$ are then given, for each specification $\mathtt{E_k}$ of the domain, by T_{E_k} and each of $\phi(a_{k_0}), \phi(a_{k_1}), \ldots$. We are appealing here to the idea of *dependent* combinations and choices (somewhat akin to dependent types), which are such that what the combination or choice is can vary with the circumstances. Take a weekly menu, for example, telling you what is on the menu for different days of the week. This would correspond to a dependent choice and, to make an immediate selection from such a menu, one would select some day or some days of the week at which one wanted to dine and, for each such day, one would then select some item or items from the menu for that day. From a mathematical rather than a gastronomic point of view, one might think of a dependent combination (or choice) as a function taking indices from a given index set I into regular combinations (or choices). Suppose, for illustrative purposes, that $I = \{i, j\}$ and that f is a dependent combination taking the index i into the combination $[a.b]$ and the index j into the combination $[c.d]$. Then the immediate selections from f would be $\{i; a, b\}$, $\{j; c, d\}$, where the semi-colon after i and j is used to indicate their status as indices.

We might, more generally, take a dependent combination (or choice) to be a function which takes the indices of I into other dependent combinations (or choices). This is in effect a recursive account with the dependent combinations (or choices) of order $n + 1$ understood as functions from the index set into the dependent combinations (or choices) of order n.

To apply these general ideas in the present case, we should suppose that the semantics assigns a content τ_E to each totality statement T_E, $\mathtt{E} \subseteq \mathtt{D}$. Our previous index set I is then taken to consist of the set $T = \{\tau_E : \mathtt{E} \subseteq \mathtt{D}\}$ of totality contents. Thus the combination or choice represented by a universal or existential quantification is taken, in effect, to be dependent upon the choice of domain.

Recall that we write $\overline{\overline{\phi}}$ for the content of the formula ϕ (see n.3). We now have what we need to inductively specify the contents of closed formulae. We define the content $\overline{\overline{\forall x\phi(x)_E}}$ of a closed universal statement $\forall x\phi(x)$ or the content $\overline{\overline{\exists x\phi(x)_E}}$ of a closed existential statement $\exists x\phi(x)$ *relative* to the choice of a domain $\mathtt{E} \subseteq \mathtt{D}$, in terms of the unrelativized contents of their instances (to be defined below) by:

(*) $\overline{\overline{\forall x\phi(x)_E}} = (\Pi\langle\overline{\overline{\phi(a_\xi)}} : \mathtt{a}_\xi \in \mathtt{E}\rangle, \Sigma\langle\overline{\overline{\neg\phi(a_\xi)}} : \mathtt{a}_\xi \in \mathtt{E}\rangle)$,

(*) $\overline{\overline{\exists x\phi(x)_E}} = (\Sigma\langle\overline{\overline{\phi(a_\xi)}} : \mathtt{a}_\xi \in \mathtt{E}\rangle), \Pi\langle\overline{\overline{\neg\phi(a_\xi)}} : \mathtt{a}_\xi \in \mathtt{E}\rangle)$.

where Π and Σ are the combination and choice operations, respectively, and the $\mathtt{a}_\xi$ keep their original order in the formation of a combination or choice but are now restricted to the given domain $\mathtt{E}$. Given the relativized content of a closed formula ϕ (possibly with individual constants), we then define the unrelativized content of ϕ to be the *function* on $T = \{\tau_E : \mathtt{E} \subseteq \mathtt{D}\}$ for which:

(**) $\overline{\overline{\phi}}(\tau_E) = \overline{\overline{\phi_E}}$

Thus, immediate selections from the truth condition for $\overline{\overline{\forall x\phi(x)}}$ will each contain some totality content τ_E, together with the contents of each of $(\phi(a_\xi))$ for $\mathtt{a}_\xi \in \mathtt{E}$. Note that when $\phi(x)$ itself contains quantifiers, the content of $\phi(a_\xi)$ or of $\neg\phi(a_\xi)$ will also involve appeal to dependent combination or choice.

The present approach has what might be regarded as a somewhat untoward consequence. For suppose that the domain $\mathtt{D}$ consists of three objects $\mathtt{a}$, $\mathtt{b}$, and $\mathtt{c}$. Let $\mathtt{E} = \{\mathtt{a}, \mathtt{b}\}$ and $\mathtt{F} = \{\mathtt{b}, \mathtt{c}\}$. Then we have: $T_E, Pa, Pb < \forall x Px$ and $T_F, Pb, Pc < \forall x Px$. So by AMALGAMATION, $T_E, T_F, Pa, Pb, Pc < \forall x Px$. Or even without appealing to AMALGAMATION, we have that $T_E, T_F, Pa, Pb, Pc < \forall x Px \wedge \forall x Px$. But these may be regarded as odd – or, at least, as unintended – results. For one might want to insist that only one specification of the domain can be relevant to the truth of a quantificational statement.

The difficulty here lies not just in the semantics but also in the logic; and perhaps the only plausible way in which it might be removed is to build into the syntax of ground a distinction between what one might call "back-grounds" and "fore-grounds". Thus $\Delta : \Gamma < A$ (or $\Gamma <_\Delta A$) is taken to mean that, relative to the back-grounds in Δ, Γ constitute fore-grounds for A.[14] Within the logic, the back-grounds should be regarded as a "fixed" parameter to the other rules, which therefore, in contrast to the fore-grounds, should not be subject to Amalgamation. The rules for the universal quantifier will now take the following simpler form:

∀I $\quad \Vdash T_E : \phi(a_1), \phi(a_2), \dots < \forall x\phi(x)$ (where $\mathtt{E} = \{\mathtt{a}_1, \mathtt{a}_2, \dots\}$)

∀E $\quad T_E : \Delta < \forall x\phi(x) \Vdash T_E : \Delta \leq \phi(a_1), \phi(a_2), \dots$ (where $\mathtt{E} = \{\mathtt{a}_1, \mathtt{a}_2, \dots\}$).

And similarly for the existential quantifier.

Corresponding changes should also be made to the other rules. The introduction and elimination rules for double negation, for example, should now take the respective forms:

¬¬I $\quad \Vdash T_E : \phi < \neg\neg\phi$ and

[14] A related distinction between background conditions and genuinely causal conditions is familiar from the literature on cause. But we are here suggesting that it has a certain logical and semantic significance within the theory of ground. The distinction might also be relevant to the question of whether grounding is an internal relation. For we may allow grounding by the fore-grounds to be contingent on which back-grounds hold even though grounding by the fore- and the back-grounds together is not. For further discussion of these issues, see [Baron-Schmitt, 2021],[Litland, 2015],[Poggiolesi, 2018],[Skiles, 2015],[Trogdon, 2013]).

$\neg\neg$**E** $\quad T_E : \Delta < \neg\neg\phi \ \Vdash \ T_E : \Delta \leq \phi$,

and similarly for the introduction and elimination rules for the other connectives.

On the semantic side, we can dispense with the conception of dependent combination and choice and take the selection function itself to be relative to the specification of the underlying domain. Clauses (*) above would then take the form:

$$(*)' \ \overline{\overline{\forall x \phi(x)_E}} = (\Pi\langle\overline{\overline{\phi(a_\xi)_E}} : \mathtt{a}_\xi \in \mathtt{E}\rangle, \Sigma\langle\overline{\overline{\neg\phi(a_\xi)_E}} : \mathtt{a}_\xi \in \mathtt{E}\rangle),$$

$$(*)' \ \overline{\overline{\exists x \phi(x)_E}} = (\Sigma\langle\overline{\overline{\phi(a_\xi)_E}} : \mathtt{a}_\xi \in \mathtt{E}\rangle), \Pi\langle\overline{\overline{\neg\phi(a_\xi)_E}} : \mathtt{a}_\xi \in \mathtt{E}\rangle).$$

and the content of a formula would in general be taken to be relative to the specification $\mathtt{E}$ of the domain.

The resulting system can readily be shown to be sound for the given semantics; and we conjecture that, upon the addition of the previous rule of alphabetic variance, it will also be complete.

References

D.M. Armstrong. *A World of States of Affairs*. Cambridge University Press, Cambridge, UK, 1997.

N. Baron-Schmitt. Contingent grounding. *Synthese*, January 2021. doi: 10.1007/s11229-020-02991-8.

L. deRosset and K. Fine. A Semantics for the Impure Logic of Ground. *Journal of Philosophical Logic*, 52(2):415–93, 2023. doi: 10.1007/s10992-022-09676-2.

K. Fine. The pure logic of ground. *Review of Symbolic Logic*, 5(1):1–25, 2012a.

K. Fine. Guide to Ground. In Benjamin Schnieder and Fabrice Correia, editors, *Metaphysical Grounding: Understanding the Structure of Reality*, pages 37–80. Cambridge University Press, 2012b. reprinted online in 'Philosophers Annual' for 2012 (eds. P. Grim, C. Armstrong, P. Shirreff, N-H Stear).

K. Fine. A Theory of Truthmaker Content I: Conjunction, Disjunction, and Negation. *Journal of Philosophical Logic*, 46(6):625–74, 2017. doi: 10.1007/s10992-016-9413-y.

J.E. Litland. Grounding, Explanation, and the Limit of Internality. *The Philosophical Review*, 124(4):481–533, 2015.

F. Poggiolesi. On constructing a logic for the notion of complete and immediate formal grounding. *Synthese*, 195(3):1231–54, 2018.

Bertrand Russell. The philosophy of logical atomism. *The Monist*, 29(2):190–222, April 1918.

A. Skiles. Against Grounding Necessitarianism. *Erkenntnis*, 80(4):717–751, August 2015.

K. Trogdon. Grounding: Necessary or Contingent? *Pacific Philosophical Quarterly*, 94(4):465–85, December 2013.