A Cut-free Sequent Calculus for the Logic of Constant Domains with a Limited Amount of Duplications

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Abstract

Cut-free sequent calculi for the predicate intermediate logic **CD** of constant domains have appeared only very recently in literature, even if this logic has been axiomatized since the early seventies. In the present paper we propose a different cut-free sequent calculus for **CD**, in which a great care is devoted in avoiding duplications of formulas.

Keywords: tableau calculus, context-rule, cut-free sequent calculus, duplications

1 Introduction

The class of predicate Kripke models having constant domain function has been proposed by Grzegorczyk in [9] as a semantics for the intuitionistic predicate logic. Actually, this semantics is not adequate, since it validates the non intuitionistic principle $\forall x(A \vee B(x)) \rightarrow A \vee \forall x B(x)$. It was proved by Klemke [12] and, independently, by Görnemann [8] that the correct and complete logic with respect to Grzegorczyk's semantics, known as Grzegorczyk logic or logic **CD** of constant domains, is obtained by adding to intuitionistic calculus all the formulas such as the above as instances of an axiom scheme. The problem of getting cut-free calculi for **CD** has been successively analyzed in [13, 14], in connection with the question of the interpolation of this logic (reported by Ono [16] as an open problem, in contrast with the pretended solution of [7]). According to the results of [14], the problem has a negative answer, if one limits himself to cut-free calculi of bounded grade (corresponding, more or less, to the usual and more natural notion of sequent calculus). Thus, some oddness involved in any cut-free calculus for **CD** is justified and perhaps unavoidable.

In line with the above picture, non standard cut-free calculi for **CD** have been presented in [11], where special devices are introduced to represent relations or implicit contexts within the sequents. These calculi, treated with the traditional proof theoretical tools, give rise to interesting results, having a general logical relevance. However, they disregard recent issues raised in areas oriented towards automatic deduction; in particular, no care is devoted to the problem of duplications [1, 5, 10, 15, 17]. On the other hand, the aim of the present paper is to provide a genuine cut-free calculus for **CD** (as done in [11]), yet devoting a great care to the problem of duplications. In the frame of predicate logic, as pointed out in [15], duplications cannot be completely eliminated; however, we will limit as much as possible

[1] This is one of the last works to which Pierangelo devoted himself, before his sudden death on August 7th, 1999.

their amount, as we will discuss in Section 5, where also a comparison with the work of [11] will be made.

To realize this, we exploit the semantical techniques developed in [1, 2, 15], regarding superintuitionistic logics L having Kripke semantics. The key point is the definition of a tableau calculus for L such that, starting from an unprovable formula A (with respect to L), a counter model for A (that is, a Kripke model for L in which A is not valid) can be built under the control of the rules of the calculus itself. Despite such calculi are inspired by semantical matters, they have relevant importance from a proof-theoretical viewpoint since, via a natural translation, they yield cut-free sequent calculi having the properties mentioned above. The calculus for **CD** here described presents some peculiarities with respect to standard sequent calculi; this corresponds to the particular strategy required in the completeness proof in order to obtain models with constant domain. Generally, the countermodel $\underline{K}$ is constructed bottom-up: one starts with the complete definition (domain and forcing) of the root of $\underline{K}$, then defines the successors of the root (without affecting the root), and so on. This technique does not work in this case; as a matter of fact, when we add to $\underline{K}$ a new element, it may be necessary to enlarge the domain, so the definition of the forcing of all the elements of $\underline{K}$ must be updated. In order to control this situation, we have to adopt rules that allow to act on subformulas inside a "context" which remains unchanged; this corresponds to the idea of specifying the forcing of an element of $\underline{K}$ inside a context already defined. We point out that this kind of rules are also used in the framework of modal logics, where similar problems arise (see for instance [3]).

Nevertheless, we have not yet been able to derive an interpolation lemma starting from this calculus, thus the main question regarding **CD** remains open. We remark that similar difficulties arise when we attempt to prove the interpolation lemma by adapting the syntactical method used for intuitionistic and classical logic to multi-succedent sequent calculi (as this calculus is).

2 Basic definitions

The first order language $\mathcal{L}$ we will consider, i.e. the set of the predicate *(well formed) formulas*, is defined, as usual, starting from the propositional connectives $\wedge$, $\vee$, $\rightarrow$, the quantifiers $\forall$ and $\exists$, the propositional constant $\bot$, a denumerable set $\mathcal{P}^n$ of n-ary predicate variables for each $n \geq 0$ and a denumerable set $\mathcal{V}$ of individual variables. We also use, as an auxiliary symbol, the propositional connective $\neg$ and we write $\neg A$ as an abbreviation for $A \rightarrow \bot$. **Int** and **Cl** denote the set of intuitionistically valid predicate formulas and the set of classically valid predicate formulas respectively.

A *(predicate) Kripke frame* is a triple $\underline{P} = \langle P, \leq, \mathcal{D} \rangle$, where $\langle P, \leq \rangle$ is a partially ordered set and $\mathcal{D}$ is the *domain function* associating, with every element $\alpha \in P$, a nonempty domain $\mathcal{D}(\alpha)$ such that, for each $\alpha, \beta \in P$, $\alpha \leq \beta$ implies $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$. A *(predicate) Kripke model* is a quadruple $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$, where $\langle P, \leq, \mathcal{D} \rangle$ is a Kripke frame and V is the *valuation function*, which satisfies the following conditions for any $\alpha, \beta \in P$:

(i) $V(\alpha, \bot) = \emptyset$;

(ii) for any predicate variable R^0 of arity 0, $V(\alpha, R^0) \subseteq \{t\}$;

(iii) for any predicate variable R^n of arity $n > 0$, $V(\alpha, R^n) \subseteq (\mathcal{D}(\alpha))^n$;

(iv) for any $n \geq 0$, $\alpha \leq \beta$ implies $V(\alpha, R^n) \subseteq V(\beta, R^n)$.

Given an element $\alpha \in P$, an α-*assignment* is a function defined between the set of individual

variables of the language and $\mathcal{D}(\alpha)$. Let $d \in \mathcal{D}(\alpha)$; the *$\alpha$-reassignment* $a(d/x)$ is the α-assignment a' such that $a'(x) = d$ and $a'(y) = a(y)$ for any $y \neq x$.
Let $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ be a Kripke model, let $\alpha \in P$ and let a be an α-assignment; then $\Vdash_a$ denotes the the *forcing* relation (induced by V and a) between α and the predicate formulas defined as usual. We only recall some cases:

- $\alpha \Vdash_a R^0$ iff $V(\alpha, R^0) = t$;
- $\alpha \Vdash_a R^n(x_1, \dots, x_n)$, with $n > 0$, iff $\langle a(x_1), \dots, a(x_n) \rangle \in V(\alpha, R^n)$;
- $\alpha \Vdash_a \exists x A(x)$ iff $\alpha \Vdash_{a(d/x)} A(x)$ for some $d \in \mathcal{D}(\alpha)$;
- $\alpha \Vdash_a \forall x A(x)$ iff $\beta \Vdash_{a(d/x)} A(x)$ for all $\beta \geq \alpha$ and all $d \in \mathcal{D}(\beta)$.

We say that a formula A is *valid* in $\underline{K}$ if and only if $\alpha \Vdash_a A$ for all $\alpha \in P$ and all α-assignments a. In the sequel, we will deal with Kripke models (frames) *with constant domain*, i.e. models (frames) having a constant domain function. In such models the definition of forcing for universally quantified formulas can be formulated as follows:

- $\alpha \Vdash_a \forall x A(x)$ iff $\alpha \Vdash_{a(d/x)} A(x)$ for all $d \in \mathcal{D}(\alpha)$.

3 A tableau calculus for CD

We recall that the *logic of constant domains* (known also as *Grzegorczyk Logic*) is the intermediate (predicate) logic

$$\mathbf{CD} = \mathbf{Int} \ + \ \forall x(R^1(x) \vee Q^0) \rightarrow \forall x R^1(x) \vee Q^0$$

obtained by adding to (predicate) intuitionistic logic the single axiom scheme
$\forall x(R^1(x) \vee Q^0) \rightarrow \forall x R^1(x) \vee Q^0$; that is, **CD** is the smallest set of predicate formulas closed under modus ponens and generalization, including intuitionistic predicate logic and the set of all the predicate formulas obtained by applying a predicate substitution to the predicate formula $\forall x(R^1(x) \vee Q^0) \rightarrow \forall x R^1(x) \vee Q^0$, where R^1 and Q^0 are predicate variables of arity 1 and 0 respectively.

It is well known (see [8, 12]) that $\mathbf{CD} = \mathcal{L}(\mathcal{F}_{CD})$, where $\mathcal{F}_{CD}$ is the class of all predicate Kripke frames having constant domain and $\mathcal{L}(\mathcal{F}_{CD})$ is the set of all the formulas valid in any model $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ based on a frame $\underline{P} = \langle P, \leq, \mathcal{D} \rangle$ of $\mathcal{F}_{CD}$.

The tableau calculus for **CD** we are going to explain uses the signs **T** and **F**. Given a formula A, a *signed (well formed) formula* will be any expression of the kind $\mathbf{s}A$, where $\mathbf{s} \in \{\mathbf{T}, \mathbf{F}\}$. We call **T**-formulaany formula with sign **T**, **F**-formulaany formula with sign **F**. Given a set Δ of signed formulas, with Δ_T we denote the set of all **T**-formulasof Δ.

The meaning of the signs **T** and **F** is explained in terms of *realizability*. Given a Kripke model $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ and given a set of signed formulas Δ, we say that an element $\alpha \in P$ *realizes* Δ (and we write $\alpha \rhd \Delta$) if and only if there is an α-assignment a such that:

(a) $\alpha \Vdash_a A$ for every $\mathbf{T}A \in \Delta$;
(b) $\alpha \nVdash_a B$ for every $\mathbf{F}B \in \Delta$.

We say that $\underline{K}$ realizes Δ if Δ is realized in some $\alpha \in P$.

In order to define the calculus, the key notion is that of *context formula*. We say that a formula $H[q]$ is a context formula if and only if one of the following inductive conditions is satisfied:

(i) $H[q] = q$
(ii) $H[q] = D_1 \vee \cdots \vee H'[q] \vee \cdots \vee D_l$
(iii) $H[q] = C \to H'[q]$

where $H'[q]$ is a context formula and the 0-ary variable q does not occur in any of the formulas $D_1, \ldots, D_l, C$. Given a context formula $H[q]$, with the notation $H[A]$ we mean the formula obtained from $H[q]$ by replacing q with A. We will define *context rules* that allow us to reduce formulas A within a context H which remains unchanged. The tableau calculus **CDt** for the logic **CD** is expounded in Figure 1. We assume the reader to be familiar with the usual

$$\frac{\Delta, \mathbf{T}A \wedge B}{\Delta, \mathbf{T}A, \mathbf{T}B}\ \mathbf{T}\wedge \qquad \frac{\mathbf{F}H[A \wedge B]}{\mathbf{F}H[A] \ / \ \mathbf{F}H[B]}\ \mathbf{F}[\wedge]$$

$$\frac{\mathbf{F}H[C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h \to D]}{\mathbf{F}H[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D] \ / \ \mathbf{F}H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]}\ \mathbf{F}[\vee\to]$$

$$\frac{\Delta, \mathbf{F}D_1 \vee \cdots \vee D_l}{\Delta, \mathbf{F}D_h}\ \mathbf{F}\vee \text{ with } 1 \leq h \leq l$$

$$\frac{\mathbf{F}H[C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h \to D]}{\mathbf{F}H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D] \ / \ \mathbf{F}H[C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h \to A \vee D]}\ \mathbf{F}[\to\to]$$

$$\frac{\mathbf{F}H[C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h \to H'[K \to D]\,]}{\mathbf{F}H[C \to H'[K \wedge B \to D]\,] \ / \ \mathbf{F}H[C \to H'[K \to A \vee D]\,]}\ \mathbf{F}[\to[\to]] \ \begin{array}{l}\text{where } C \text{ is the formula}\\ C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h\end{array}$$

$$\frac{\Delta, \mathbf{F}A \to B}{\Delta_T, \mathbf{T}A, \mathbf{F}B}\ \mathbf{F}\to$$

$$\frac{\mathbf{F}H[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]}{\mathbf{F}H[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]}\ \mathbf{F}[\exists\to] \text{ with } p \text{ new} \qquad \frac{\mathbf{F}H[\exists x A(x)]}{\mathbf{F}H[A(y) \vee \exists x A(x)]}\ \mathbf{F}[\exists]$$

$$\frac{\mathbf{F}H[C_1 \wedge \cdots \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}{\mathbf{F}H[C_1 \wedge \cdots \wedge A(y) \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}\ \mathbf{F}[\forall\to] \qquad \frac{\mathbf{F}H[\forall x A(x)]}{\mathbf{F}H[A(p)]}\ \mathbf{F}[\forall] \text{ with } p \text{ new}$$

FIG. 1. The tableau calculus **CDt**

definitions and conventions concerning the tableau calculi (see, for instance, [1, 2, 6, 15]). In this framework, a *configuration* is a finite sequence $\Delta_1/\Delta_2/\ldots/\Delta_n$ (with $n \geq 1$) such that every Δ_j $(1 \leq j \leq n)$ is a set of signed formulas; a **CDt**-*proof-table* is a finite sequence of applications of the rules of the calculus **CDt** starting from some configuration. A set of signed formulas Δ is *closed* if it contains either $\mathbf{T}\bot$ or both $\mathbf{T}A$ and $\mathbf{F}A$ for some formula A; obviously a closed set cannot be realized in any Kripke model. A **CDt**-proof-table is *closed* iff all the sets Δ_j of its final configuration are closed; a *proof* of a formula B in **CDt** is a closed **CDt**-proof-table starting from $\{\mathbf{F}B\}$. Let $\Delta_{fin} = \{\mathbf{T}C_1, \ldots, \mathbf{T}C_h, \mathbf{F}D_1, \ldots, \mathbf{F}D_l\}$

be a finite set of signed formulas; we say that Δ_{fin} is **CDt**-*consistent* iff there is no **CDt**-proof-table for $\mathbf{F}C_1 \wedge \cdots \wedge C_h \to D_1 \vee \cdots \vee D_l$ (in case there are no **F**-formulasor **T**-formulas, we have to take the formula $\mathbf{F}\neg(C_1 \wedge \cdots \wedge C_h)$ or $\mathbf{F}D_1 \vee \cdots \vee D_l$ respectively). An infinite set Δ of signed formulas is **CDt**-consistent iff, for every finite $\Delta_{fin} \subseteq \Delta$, Δ_{fin} is **CDt**-consistent. This definition guarantees that sets Δ containing, for instance, the signed formulas $\mathbf{T}A \vee B$, $\mathbf{F}A$, $\mathbf{F}B$, are not **CDt**-consistent, even if there is no closed **CDt**-proof-table starting with Δ. Unless otherwise stated, consistent means **CDt**-consistent and closed tableau means **CDt**-proof-table.

Every rule of Figure 1 is applied to a signed formula of a set Δ_j occurring in a configuration $\Delta_1, \ldots, \Delta_m$; e.g., the notation $\Delta, \mathbf{T}A \wedge B$ points out that the rule $\mathbf{T}\wedge$ is applied to the signed formula $\mathbf{T}A \wedge B$ of the set $\Delta \cup \{\mathbf{T}A \wedge B\}$, where Δ is possibly empty.

We remark that the rule $\mathbf{F}[\to[\to]]$ is necessary for the completeness of the calculus. As a matter of fact, without this rule we are not able to derive the intuitionistically valid formula $(A \to B) \to (A \wedge C \to B)$; indeed, using only the rule $\mathbf{F}[\to\to]$ (together with the rules $\mathbf{F}\to$ and $\mathbf{T}\wedge$), we have no means to combine the A of $A \to B$ with the A of $A \wedge C \to B$, as required in order to obtain a closed tableau. Incidentally, we point out that if we adopt $\mathbf{F}H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to H'[K \to D]]$ as the left hand consequence of the rule $\mathbf{F}[\to[\to]]$ (i.e., if we let the left hand consequence of this rule behave just as the left hand consequence of the rule $\mathbf{F}[\to\to]$), we obtain a stronger calculus, since we are able to derive the formula $(A \to A) \to A \vee \neg A$ (where the whole formula is identified with $\mathbf{F}H[C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h \to H'[K \to D]]$, while the formula $A \vee \neg A = A \vee (A \to \bot)$ is identified with $H'[K \to D]$), which does not belong to **CD**. However such a calculus is not closed under modus ponens since, despite $A \to A$ and $(A \to A) \to A \vee \neg A$ are derivable, we are not able to prove $A \vee \neg A$. We also stress that in both the context rules for implication we have to use the whole formula $A \vee D$ (instead of the formula A alone or the formula D alone, which would not affect the soundness of the rule) in the right hand consequence, since both formulas A and D may be of use. One can test this fact by deriving Markov axiom $\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \to \exists x A(x)$.
Finally we remark that all the rules of the calculus, except for $\mathbf{F}[\to\to]$, $\mathbf{F}[\to[\to]]$, $\mathbf{F}[\exists]$, $\mathbf{F}[\forall\to]$, are duplication-free in the sense explained in [1, 2, 5, 15]. An accurate discussion about this point is deferred in Section 5.

In order to prove the soundness of the calculus, the following lemma assures that the rules preserve the realizability of signed formulas in models with constant domain.

Lemma 3.1 *Let $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ be any Kripke model with constant domain; let R be any rule of the calculus having Δ as the premise and the configuration $\Delta' \,/\, \Delta''$ (including the case $\Delta'' = \emptyset$) as the consequence. Then, for every $\alpha \in P$, $\alpha \rhd \Delta$ implies either $\alpha \rhd \Delta'$ or $\alpha \rhd \Delta''$.*

Proof. We analyze only some cases.

- Rule $\mathbf{F}[\vee\to]$.
We prove, by induction on the complexity of $H[q]$, that, for every $\alpha \in P$ and every α-assignment a, if $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h \to D]$ then either $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]$ or $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]$.
(i) $H[q] = q$.
Since, by hypothesis, $\alpha \nVdash_a C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h \to D$, there is $\beta \geq \alpha$ such that $\beta \Vdash_a C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h$ and $\beta \nVdash_a D$; hence either $\beta \Vdash_a C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h$ or $\beta \Vdash_a C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h$. Suppose, by definiteness, that the former hypothesis holds; then $\alpha \nVdash_a C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D$, that is $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]$. In the

other case we get $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]$.
(ii) $H[q] = D_1 \vee \cdots \vee H'[q] \vee \cdots \vee D_l$.
By hypothesis we have $\alpha \nVdash_a D_1, \ldots, \alpha \nVdash_a D_l$ and $\alpha \nVdash_a H'[C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h \to D]$. By induction hypothesis applied to $H'[q]$ (which is less complex then $H[q]$) either $\alpha \nVdash_a H'[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]$ or $\alpha \nVdash_a H'[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]$; so we can conclude that either $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]$ or $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]$.
(iii) $H[q] = C \to H'[q]$.
This case is proved as the last one.

- Rule $\mathbf{F}[\to\to]$.
We prove, by induction on the complexity of the context formula $H[q]$, that:
(*) for every $\alpha \in P$ and every α-assignment a, if $\alpha \nVdash_a H[C \to H'[K \to D]]$ and $\alpha \Vdash_a H[C \to H'[K \to A \vee D]]$ then $\alpha \nVdash_a H[C \to H'[K \wedge B \to D]]$.
(i) $H[q] = q$.
Since $\alpha \nVdash_a C \to H'[K \to D]$, there is $\beta \geq \alpha$ such that $\beta \Vdash_a C$ and $\beta \nVdash_a H'[K \to D]$; moreover, since $\alpha \Vdash_a C \to H'[K \to A \vee D]$, we also have $\beta \Vdash_a H'[K \to A \vee D]$. It is not difficult to prove, by a secondary induction on the complexity of $H'[q]$, the following fact:
(**) for every $\beta' \geq \beta$, if $\beta' \nVdash_a H'[K \to D]$ and $\beta' \Vdash_a H'[K \to A \vee D]$ then $\beta' \nVdash_a H'[K \wedge B \to D]$.
From (**) we can infer $\beta \nVdash_a H'[K \wedge B \to D]$, hence $\alpha \nVdash_a C \to H'[K \wedge B \to D]$, that is $\alpha \nVdash_a H[C \to H'[K \wedge B \to D]]$.
(ii) $H[q] = D_1 \vee \cdots \vee Z[q] \vee \cdots \vee D_l$.
By hypothesis we have $\alpha \nVdash_a D_1, \ldots, \alpha \nVdash_a D_l$, $\alpha \nVdash_a Z[C \to H'[K \to D]]$ and $\alpha \Vdash_a Z[C \to H'[K \to A \vee D]]$. By induction hypothesis (being $Z[q]$ less complex than $H[q]$) we have $\alpha \nVdash_a Z[C \to H'[K \wedge B \to D]]$, from which $\alpha \nVdash_a H[C \to H'[K \wedge B \to D]]$ follows.
In a similar way we can prove the case
(iii) $H[q] = W \to Z[q]$.

- Rule $\mathbf{F}[\exists \to]$.
Let a be any α-assignment, let p be any individual variable not occurring in the formula $H[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]$ (so that p can be used as the parameter of the rule). We prove that if $\alpha \nVdash_a H[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]$, then, for some $d \in \mathcal{D}(\alpha)$, $\alpha \nVdash_{a(d/p)} H[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]$.
(i) $H[q] = q$.
By hypothesis $\alpha \nVdash_a C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D$, so there is $\beta \geq \alpha$ such that $\beta \Vdash_a C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h$ and $\beta \nVdash_a D$. Since $\beta \Vdash_a \exists x A(x)$, there is $d \in \mathcal{D}(\beta)$ such that $\beta \Vdash_{a(d/x)} A(x)$, from which, renaming the free variable, $\beta \Vdash_{a(d/p)} A(p)$ follows. Since $\underline{K}$ has constant domain and since p does not occur in any of the formulas $C_1, \ldots, C_h, D$, we can infer $\alpha \nVdash_{a(d/p)} C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D$, that is $\alpha \nVdash_{a(d/p)} H[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]$.
(ii) $H[q] = D_1 \vee \cdots \vee H'[q] \vee \cdots \vee D_l$.
By hypothesis we have $\alpha \nVdash_a D_1, \ldots, \alpha \nVdash_a D_l$ and $\alpha \nVdash_a H'[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]$. By induction hypothesis, $\alpha \nVdash_{a(d/p)} H'[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]$ for some $d \in \mathcal{D}(\alpha)$. Since p does not occur in any of the formulas $D_1, \ldots, D_l$, we can conclude $\alpha \nVdash_{a(d/p)} H[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]$.
In a similar way one also proves the case
(iii) $H[q] = C \to H'[q]$.
We remark that the hypothesis of constant domain is used only in this case and in the similar case of the rule $\mathbf{F}[\forall]$. ∎

The soundness of the calculus can be stated in the following terms.

Theorem 3.2 (Soundness) *Let Δ be any set of signed formulas. If Δ is realized in some Kripke model with constant domain, then Δ is consistent.*

PROOF. Let $\underline{K} = \langle P, \leq, \mathcal{D}, V\rangle$ and $\alpha \in P$ be such that $\alpha \rhd \Delta$; suppose, by absurd, that Δ is not consistent. By definition there is a set:

$$\Delta_{fin} = \{\mathbf{T}C_1, \ldots, \mathbf{T}C_h, \mathbf{F}D_1, \ldots, \mathbf{F}D_l\} \subseteq \Delta$$

and there is a closed tableau $\mathcal{G}$ for $\mathbf{F}C_1 \wedge \cdots \wedge C_h \to D_1 \vee \cdots \vee D_l$. We cannot have $\alpha \rhd \{\mathbf{F}C_1 \wedge \cdots \wedge C_h \to D_1 \vee \cdots \vee D_l\}$, otherwise, by the previous lemma, α should realize some closed set of the final configuration of $\mathcal{G}$. It follows that $\alpha \rhd \Delta_{fin}$ does not hold and, a fortiori, $\alpha \rhd \Delta$ does not hold, contradicting the initial hypothesis. We can therefore conclude that Δ is consistent. ∎

It is worth remarking that our tableau calculus provides a simple and direct proof of the well known fact that **CD** satisfies the disjunction and existential properties. To show this, the following definition is in order.
Given a closed tableau $\mathcal{G}$, the length $\lambda(\mathcal{G})$ of $\mathcal{G}$ is defined inductively as follows:

(a) If $\mathcal{G}$ consists only of one closed set, then $\lambda(\mathcal{G}) = 1$.

(b) Suppose $\mathcal{G}$ begins with

$$\frac{\Gamma}{\Gamma'}\,R$$

and let $\mathcal{G}'$ be the closed tableau for Γ'. Then $\lambda(\mathcal{G}) = \lambda(\mathcal{G}') + 1$.

(c) Suppose $\mathcal{G}$ begins with

$$\frac{\Gamma}{\Gamma' \;/\; \Gamma''}\,R$$

and let $\mathcal{G}'$ and $\mathcal{G}''$ be the closed tableaux for Γ' and Γ'' respectively. Then $\lambda(\mathcal{G}) = max(\lambda(\mathcal{G}'), \lambda(\mathcal{G}'')) + 1$.

Proposition 3.3 *(i) Suppose there is a closed tableau $\mathcal{G}$ for $\mathbf{F}D_1 \vee \cdots \vee D_k$; then there is a closed tableau $\mathcal{G}'$ for $\mathbf{F}D_j$, with $1 \leq j \leq k$, such that $\lambda(\mathcal{G}') < \lambda(\mathcal{G})$.*
(ii) Suppose there is a closed tableau $\mathcal{G}$ for $\mathbf{F}\exists x A(x)$; then, for some y, there is a closed tableau $\mathcal{G}'$ for $\mathbf{F}A(y)$ such that $\lambda(\mathcal{G}') < \lambda(\mathcal{G})$.

PROOF. (i) Let $\mathcal{G}$ be a closed tableau for $\mathbf{F}D_1 \vee \cdots \vee D_k$ and let l be the length of $\mathcal{G}$; we prove (i) by induction on l. Suppose that the first rule R applied in $\mathcal{G}$ is $\mathbf{F}\vee$; then $\mathcal{G}$ begins with

$$\frac{\mathbf{F}D_1 \vee \cdots \vee D_k}{\mathbf{F}D_j}\,\mathbf{F}\vee$$

($1 \leq j \leq k$), and continues with a closed tableau $\mathcal{G}'$ for $\mathbf{F}D_j$; since $\lambda(\mathcal{G}') = l - 1$, (i) immediately follows. Otherwise, let us assume that the rule R has two consequences (in case R has only one consequence, the reasoning is quite similar). Then $\mathcal{G}$ begins with

$$\frac{\mathbf{F}D_1 \vee \cdots \vee D_j \vee \cdots \vee D_k}{\mathbf{F}D_1 \vee \cdots \vee D'_j \vee \cdots \vee D_k \;/\; \mathbf{F}D_1 \vee \cdots \vee D''_j \vee \cdots \vee D_k}\,R$$

where R is a context rule applied to D_j and $D_1, \ldots, D_{j-1}, D_{j+1}, \ldots, D_k$ form the context, and continues with a closed tableau $\mathcal{G}'$ for $\mathbf{F}D_1 \vee \cdots \vee D'_j \vee \cdots \vee D_k$ and a closed tableau $\mathcal{G}''$ for $\mathbf{F}D_1 \vee \cdots \vee D''_j \vee \cdots \vee D_k$. Since $\lambda(\mathcal{G}') \leq l-1$ and $\lambda(\mathcal{G}'') \leq l-1$, by induction hypothesis there is a closed tableau $\tilde{\mathcal{G}}_1$ for $\mathbf{F}\tilde{D}_1$, with $\tilde{D}_1 \in \{D_1, \ldots, D'_j, \ldots, D_k\}$, and a closed tableau $\tilde{\mathcal{G}}_2$ for $\mathbf{F}\tilde{D}_2$, with $\tilde{D}_2 \in \{D_1, \ldots, D''_j, \ldots, D_k\}$, such that $\lambda(\tilde{\mathcal{G}}_1) < l-1$ and $\lambda(\tilde{\mathcal{G}}_2) < l-1$. If either $\tilde{D}_1 \neq D'_j$ or $\tilde{D}_2 \neq D''_j$, (i) is proved. Consider now the case $\tilde{D}_1 = D'_j$ and $\tilde{D}_2 = D''_j$. We can build a closed tableau $\tilde{\mathcal{G}}$ for $\mathbf{F}D_j$, which begins with the application of the rule R to D_j in a narrower context in the following way:

$$\frac{\mathbf{F}D_j}{\mathbf{F}D'_j \;/\; \mathbf{F}D''_j}\;R;$$

$\tilde{\mathcal{G}}$ then continues with the closed tableau $\tilde{\mathcal{G}}_1$ for $\mathbf{F}D'_j$ and $\tilde{\mathcal{G}}_2$ for $\mathbf{F}D''_j$. Since $\lambda(\tilde{\mathcal{G}}) \leq l-1$, (i) is proved also in this case.
(ii) Let $\mathcal{G}$ be a closed tableau for $\mathbf{F}\exists x A(x)$ and suppose l is the length of $\mathcal{G}$; we prove (ii) by induction on l. The tableau $\mathcal{G}$ necessarily begins with an application of the rule $\mathbf{F}[\exists]$

$$\frac{\mathbf{F}\exists x A(x)}{\mathbf{F}A(y) \vee \exists x A(x)}\;\mathbf{F}[\exists]$$

and continues with a closed tableau $\mathcal{G}'$ for $\mathbf{F}A(y) \vee \exists x A(x)$ of length $l-1$. By (i), there is either a closed tableau $\mathcal{G}_1$ for $\mathbf{F}A(y)$ of length less than $l-1$ or a closed tableau $\mathcal{G}_2$ for $\mathbf{F}\exists x A(x)$ of length less than $l-1$. If the former hypothesis holds (ii) is already satisfied; if the latter hypothesis holds we can apply the induction hypothesis to $\mathcal{G}_2$ and (ii) follows also in this case. ∎

4 Completeness

Our aim is to realize finite consistent sets of signed formulas Δ_{fin} in Kripke models with constant domain. In line with standard completeness proofs (see for instance [6]), the starting point is the following definition.

Definition 4.1 Let $\mathcal{C}$ be a collection of sets of signed formulas and let Π be a nonempty set of individual variables. We say that $\mathcal{C}$ is a **CD**-*collection with respect to* Π if and only if, for every $\Gamma \in \mathcal{C}$, all the elements of Γ have the form $\mathbf{s}H$, with $\mathbf{s} \in \{\mathbf{T}, \mathbf{F}\}$ and H belonging to the sublanguage $\mathcal{L}_\Pi$ of $\mathcal{L}$, and the following conditions are satisfied:

- $\mathbf{T}\bot \notin \Gamma$.
- If A is an atomic formula and $\mathbf{T}A \in \Gamma$, then $\mathbf{F}A \notin \Gamma$.
- $\mathbf{T}A \wedge B \in \Gamma$ implies $\mathbf{T}A \in \Gamma$ and $\mathbf{T}B \in \Gamma$.
- $\mathbf{F}A \wedge B \in \Gamma$ implies either $\mathbf{F}A \in \Gamma$ or $\mathbf{F}B \in \Gamma$.
- $\mathbf{T}A \vee B \in \Gamma$ implies either $\mathbf{T}A \in \Gamma$ or $\mathbf{T}B \in \Gamma$.
- $\mathbf{F}A \vee B \in \Gamma$ implies $\mathbf{F}A \in \Gamma$ and $\mathbf{F}B \in \Gamma$.
- $\mathbf{T}A \rightarrow B \in \Gamma$ implies either $\mathbf{F}A \in \Gamma$ or $\mathbf{T}B \in \Gamma$.
- If $\mathbf{F}A \rightarrow B \in \Gamma$ then there is a $\Gamma' \in \mathcal{C}$ such that $\Gamma_T \subseteq \Gamma'$, $\mathbf{T}A \in \Gamma'$ and $\mathbf{F}B \in \Gamma'$.
- $\mathbf{T}\exists x A(x) \in \Gamma$ implies $\mathbf{T}A(p) \in \Gamma$ for some $p \in \Pi$.

- $\mathbf{F}\exists x A(x) \in \Gamma$ implies $\mathbf{F}A(p) \in \Gamma$ for all $p \in \Pi$.
- $\mathbf{T}\forall x A(x) \in \Gamma$ implies $\mathbf{T}A(p) \in \Gamma$ for all $p \in \Pi$.
- $\mathbf{F}\forall x A(x) \in \Gamma$ implies $\mathbf{F}A(p) \in \Gamma$ for some $p \in \Pi$.

□

Definition 4.2 Let $\mathcal{C}$ be a **CD**-collection and let $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ be any predicate Kripke model. We say that $\underline{K}$ is a *model for* $\mathcal{C}$ if and only if $\underline{K}$ realizes every $\Gamma \in \mathcal{C}$. □

Proposition 4.3 *Every* **CD***-collection has a model with constant domain.*

Proof. Let $\mathcal{C}$ be any **CD**-collection with respect to some nonempty set Π of individual variables. Let $\underline{K} = \langle P, \leq, \mathcal{D}, V \rangle$ be the Kripke model with constant domain defined as follows:
- $P = \mathcal{C}$;
- for every $\Gamma, \Gamma' \in P$, $\Gamma \leq \Gamma'$ iff $\Gamma_T \subseteq \Gamma'$;
- for every $\Gamma \in P$, $\mathcal{D}(\Gamma) = \Pi$.

Let Γ be any element of P, let R^n be any predicate variable of arity $n \geq 0$ and let $p_1, \ldots, p_n \in \Pi$; then:
- $V(\Gamma, R^0) = t$ iff $\mathbf{T}R^0 \in \Gamma$;
- if $n > 0$, $\langle p_1, \ldots, p_n \rangle \in V(\Gamma, R^n)$ iff $\mathbf{T}R^n(p_1, \ldots, p_n) \in \Gamma$.

Consider now the Γ-assignment a coinciding with the identity function on Π. By a straightforward induction on the complexity of formulas, one can prove that, for every formula B in the language generated by Π:
- $\mathbf{T}B \in \Gamma$ implies $\Gamma \Vdash_a B$;
- $\mathbf{F}B \in \Gamma$ implies $\Gamma \nVdash_a B$

(in the basis step it is used the fact that $\mathbf{T}\bot \notin \Gamma$ and that, for any atomic formula A, at most one between $\mathbf{T}A$ and $\mathbf{F}A$ belongs to Γ). This means that $\Gamma \rhd \Gamma$, consequently $\underline{K}$ is a model for $\mathcal{C}$ with constant domain. ■

As an immediate consequence of the previous proposition and the Soundness Theorem, it follows that every set Γ of a **CD**-collection is consistent.
We now define $Tree_{CD}$as the class of of finite trees $\mathcal{T}$ whose nodes are finite sets of signed formulas such that, denoting with $\mathcal{T} = \{\Gamma_0, \ldots, \Gamma_m\}$ the tree having $\Gamma_0, \ldots, \Gamma_m$ as the nodes and Γ_0 as the root, the following properties (P1) and (P2) are satisfied:

(P1) Γ_0 contains only one **F**-formulaand Γ_0 is consistent.

(P2) Let Γ_m be any node of $\mathcal{T}$ different from the root and let $\Gamma_{m'}$ be such that Γ_m is an immediate successor of $\Gamma_{m'}$ (by definition of tree, $\Gamma_{m'}$ is uniquely determined); then:
- $\Gamma_{m'} = \Delta_T \cup \{\mathbf{F}D_1 \vee \cdots \vee (C \to D) \vee \cdots \vee D_l\}$
- $\Gamma_m = \Delta_T \cup \{\mathbf{T}C, \mathbf{F}D\}$

where Δ_T is a (possibly empty) set of **T**-formulasand the formulas $D_1, \ldots, D_l$ may lack. We say that $\mathbf{T}C$ is the *proper* **T***-formula* of Γ_m and that $C \to D$ is the *reference formula* to Γ_m.

We remark that each node Γ contains exactly one **F**-formulaand the root contains only one signed formula (which is a **F**-formula). We also assume that the successor relation induces in $\mathcal{T}$ a partial ordering $\leq$ defined in the obvious way, so that $\Gamma \leq \Gamma'$ implies $\Gamma_T \subseteq \Gamma'$.
Let A be any formula; with $Clos_\wedge(A)$ (resp. $Clos_\vee(A)$) we denote the closure of A under conjunction (resp. under disjunction), that is the intersection of all the sets $\mathcal{I}$ such that

$A \in \mathcal{I}$ and $B \wedge C \in \mathcal{I}$ (resp. $B \vee C \in \mathcal{I}$) implies $B \in \mathcal{I}$ and $C \in \mathcal{I}$.
Let $\Gamma = \Delta_T \cup \{\mathbf{F}D\}$ be any node; then

$$Sat(\Gamma) = \{\mathbf{T}C' : C' \in Clos_\wedge(C) \quad \text{and } \mathbf{T}C \in \Gamma\} \cup \{\mathbf{F}D' : D' \in Clos_\vee(D)\}.$$

Clearly $\Gamma \subseteq Sat(\Gamma)$.

Lemma 4.4 *Let $\mathcal{T}$ be any tree of $Tree_{CD}$and let Γ be any node of $\mathcal{T}$; then $Sat(\Gamma)$ is not closed.*

Proof. Let Γ be any node of $\mathcal{T}$; we firstly prove, by induction on the distance $h \geq 0$ between Γ and the root Γ_0 of $\mathcal{T}$, that there is no closed tableau starting from Γ. If $h = 0$, then Γ is the root and, by definition, the statement holds. Suppose Γ has distance $h > 0$ from Γ_0 and assume Γ is a successor of the node Γ'. Then:

$$\begin{aligned}\Gamma &= \Delta_T \cup \{\mathbf{T}C, \mathbf{F}D\}\\ \Gamma' &= \Delta_T \cup \{\mathbf{F}D_1 \vee \cdots \vee (C \rightarrow D) \vee \cdots \vee D_l\}.\end{aligned}$$

By induction hypothesis there is no closed tableau for Γ'. Starting from the configuration $\Delta_T, \mathbf{F}D_1 \vee \cdots \vee (C \rightarrow D) \vee \cdots \vee D_l$ and applying the rules $\mathbf{F}\vee$ and $\mathbf{F}\rightarrow$, we can obtain the configuration $\Delta_T, \mathbf{T}C, \mathbf{F}D$. This means that, if there were a closed tableau for Γ, we could get a closed tableau for Γ' as well; hence there is no closed tableau for Γ.
Suppose now that $Sat(\Gamma)$ is closed; then there are some formulas $C_1, \ldots, C_h$, $D_1, \ldots, D_l$ such that either $\mathbf{T}C_1 \wedge \cdots \wedge \bot \wedge \cdots \wedge C_h \in \Gamma$ or $\{\mathbf{T}C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h, \mathbf{F}D_1 \vee \cdots \vee A \vee \cdots \vee D_l\} \subseteq \Gamma$. In both cases, applying only the rules $\mathbf{T}\wedge$ and $\mathbf{F}\vee$, we obtain a closed tableau for Γ, contradicting what has been proved above. ∎

We could also prove the stronger result that $Sat(\Gamma)$ is consistent, but the proof is more demanding, while it becomes trivial after having proved the Completeness Theorem.

Lemma 4.5 *Let $\mathcal{T} \in Tree_{CD}$ and let $\Gamma_m = \Delta_T \cup \{\mathbf{F}D\}$ be any node of $\mathcal{T}$ (possibly $\Delta_T = \emptyset$). Then:*

(i) if $m > 0$, for every $\Gamma_j < \Gamma_m$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[C \rightarrow D]$ is the $\mathbf{F}$-formulaof Γ_j, where $\mathbf{T}C$ is the proper $\mathbf{T}$-formulaof Γ_m.

(ii) If $m \geq 0$, for every $\Gamma_j \leq \Gamma_m$ there is a context formula $H'_j[q]$ such that $\mathbf{F}H'_j[D]$ is the $\mathbf{F}$-formulaof Γ_j.

Proof.
(i) By induction on the distance between Γ_j and Γ_m. Suppose, as basic step, that Γ_m is an immediate successor of Γ_j. By definition, the $\mathbf{F}$-formulaof Γ_j is $\mathbf{F}D_1 \vee \cdots \vee (C \rightarrow D) \vee \cdots \vee D_l$; we can take as $H_j[q]$ the formula $D_1 \vee \cdots \vee q \vee \cdots \vee D_l$ and the lemma is satisfied. If Γ_m is not an immediate successor of Γ_j, there is an immediate successor Γ_s of Γ_j such that $\Gamma_s < \Gamma_m$. Suppose $\Gamma_s = \Delta'_T \cup \{\mathbf{T}K, \mathbf{F}D'\}$ where $\mathbf{T}K$ is the proper $\mathbf{T}$-formulaof Γ_s; by definition, the $\mathbf{F}$-formulaof Γ_j is $\mathbf{F}Z_1 \vee \cdots \vee (K \rightarrow D') \vee \cdots \vee Z_m$. By induction hypothesis applied to Γ_s, there is a context formula $H_s[q]$ such that D' coincides with $H_s[C \rightarrow D]$. Consider the context formula $H_j[q]$ defined as $Z_1 \vee \cdots \vee (K \rightarrow H_s[q]) \vee \cdots \vee Z_m$; then $\mathbf{F}H_j[C \rightarrow D]$ is the $\mathbf{F}$-formulaof Γ_j.
(ii) Is a straightforward consequence of (i). ∎

Let $\mathbf{F}E$ be any consistent $\mathbf{F}$-formulaand assume that the formula E belongs to the sublanguage $\mathcal{L}_E$ of $\mathcal{L}$ generated by a nonempty set $\mathcal{V}_E$ of individual variables. Let Π be a denumerable set of individual variables $p_0, \dots, p_k, \dots$ such that $\Pi \cap \mathcal{V}_E = \emptyset$ and let $\overline{\mathcal{L}}$ be the sublanguage of $\mathcal{L}$ generated by $\overline{\Pi} = \mathcal{V}_E \cup \Pi$. We are going to define a sequence $\mathcal{S}(\mathbf{F}E)$

$$\mathcal{T}^0, \mathcal{T}^1, \dots, \mathcal{T}^k, \dots$$

where each $\mathcal{T}^k$ is a tree of $Tree_{CD}$equipped with a nonempty finite set $\Pi^k \subset \overline{\Pi}$ of parameters such that:

(A) $\mathcal{T}^0$ is the tree having as the unique node the set $\Gamma_0^0 = \{\mathbf{F}E\}$ and $\Pi^0 = \mathcal{V}_E$.

(B) If $\mathcal{T}^k = \{\Gamma_0^k, \dots, \Gamma_m^k\}$, then $\mathcal{T}^{k+1}$ contains at least the nodes $\Gamma_0^{k+1}, \dots, \Gamma_m^{k+1}$ and $\Pi^k \subseteq \Pi^{k+1}$; moreover, if Γ_j^k is a successor of Γ_r^k, then Γ_j^{k+1} is a successor of Γ_r^{k+1}.

(C) All formulas occurring in $\mathcal{T}^k$ belong to the sublanguage of $\mathcal{L}$ generated by Π^k.

Let $\mathcal{T}^k = \{\Gamma_0^k, \dots, \Gamma_s^k\}$ be any tree of the sequence and let Π^k be the associated parameter set, let Γ_m^k be any element of $\mathcal{T}^k$ and let $\mathbf{s}H \in Sat(\Gamma_m^k)$; we now define, by induction on the complexity of H, the *reduction* function $\mathcal{R}$ so that the tree

$$\mathcal{T}^{k+1} = \mathcal{R}(\mathcal{T}^k, \Gamma_m^k, \mathbf{s}H)$$

is a successor of $\mathcal{T}^k$ according to the previous definition. Unless otherwise stated, $\mathcal{T}^{k+1} = \{\Gamma_0^{k+1}, \dots, \Gamma_s^{k+1}\}$ and the parameter set Π^{k+1} associated with $\mathcal{T}^{k+1}$ is the same as Π^k.

- H atomic or $\mathbf{s}H = \mathbf{T}A \wedge B$ or $\mathbf{s}H = \mathbf{F}A \vee B$.
In this case we set $\mathcal{T}^{k+1} = \mathcal{T}^k$ (we mean $\Gamma_j^{k+1} = \Gamma_j^k$ for every $0 \le j \le s$).

- $\mathbf{s}H = \mathbf{F}A \wedge B$.
Suppose that $\Gamma_m^k = \Delta_T \cup \{\mathbf{F}D_1 \vee \dots \vee (A \wedge B) \vee \dots \vee D_l\}$.
By Lemma 4.5, for every $\Gamma_j^k \le \Gamma_m^k$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[A \wedge B]$ is the $\mathbf{F}$-formulaof Γ_j^k. Since $\Gamma_0^k = \{\mathbf{F}H_0[A \wedge B]\}$ is consistent, by definition of the rule $\mathbf{F}[\wedge]$ either $\{\mathbf{F}H_0[A]\}$ or $\{\mathbf{F}H_0[B]\}$ is consistent. Suppose that the former hypothesis holds. Then we define:

$$\begin{array}{l} \Gamma_j^{k+1} = (\Gamma_j^k \setminus \{\mathbf{F}H_j[A \wedge B]\}) \cup \{\mathbf{F}H_j[A]\} \quad \text{if } \Gamma_j^k \le \Gamma_m^k \\ \Gamma_j^{k+1} = \Gamma_j^k \quad \text{otherwise.} \end{array}$$

Note in particular that $\mathbf{F}A \in Sat(\Gamma_m^{k+1})$.
In case $\{\mathbf{F}H_0[B]\}$ is consistent we proceed in an analogous way; thus either $\mathbf{F}A \in Sat(\Gamma_m^{k+1})$ or $\mathbf{F}B \in Sat(\Gamma_m^{k+1})$.

- $\mathbf{s}H = \mathbf{T}A \vee B$.
By definition there are $\Gamma_{\overline{m}}^k \le \Gamma_m^k$ and some formulas $C_1, \dots, C_h, D$ such that

$$\Gamma_{\overline{m}}^k = \Delta_T \cup \{\mathbf{T}C, \mathbf{F}D\} \qquad \text{with } C = C_1 \wedge \dots \wedge (A \vee B) \wedge \dots \wedge C_h$$

$\mathbf{T}C$ is the proper $\mathbf{T}$-formulaof $\Gamma_{\overline{m}}^k$ and, for all $\Gamma_j^k < \Gamma_{\overline{m}}^k$, $\mathbf{T}C$ is not the proper $\mathbf{T}$-formulaof Γ_j^k. By Lemma 4.5, for every $\Gamma_j^k < \Gamma_{\overline{m}}^k$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[C \to D]$ is the $\mathbf{F}$-formulaof Γ_j^k. Since $\Gamma_0^k = \{\mathbf{F}H_0[C_1 \wedge \dots \wedge (A \vee B) \wedge \dots \wedge C_h \to D]\}$ is consistent, by definition of the rule $\mathbf{F}[\vee\to]$ either $\{\mathbf{F}H_0[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]\}$ or $\{\mathbf{F}H_0[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]\}$ is consistent. Suppose that the former hypothesis holds; then we define:

$$\begin{array}{ll}
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{T}C\}) \cup \{\mathbf{T}C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h\} \quad \text{if } \Gamma_j^k \geq \Gamma_{\overline{m}}^k \text{ and} \\
 & (\dagger) \text{ for all } \Gamma_s^k \text{ s.t. } \Gamma_{\overline{m}}^k < \Gamma_s^k \leq \Gamma_j^k,\ \mathbf{T}C \text{ is not the proper } \mathbf{T}\text{-formulaof } \Gamma_s^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \cup \{\mathbf{T}C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h\} \quad \text{if } \Gamma_j^k > \Gamma_{\overline{m}}^k \text{ and } (\dagger) \text{ does not hold} \\
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{F}H_j[C \to D]\}) \cup \{\mathbf{F}H_j[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D]\} \\
 & \quad \text{if } \Gamma_j^k < \Gamma_{\overline{m}}^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \quad \text{otherwise}
\end{array}$$

(note that $\mathbf{T}A \in Sat(\Gamma_{\overline{m}}^{k+1})$). In a similar way we proceed if the latter hypothesis holds. Thus either $\mathbf{T}A \in Sat(\Gamma_{\overline{m}}^{k+1})$ or $\mathbf{T}B \in Sat(\Gamma_{\overline{m}}^{k+1})$.

- $\mathbf{s}H = \mathbf{T}A \to B$.
By definition there are $\Gamma_{\overline{m}}^k \leq \Gamma_m^k$ and some formulas $C_1, \ldots, C_h, D$ such that

$$\Gamma_{\overline{m}}^k = \Delta_T \cup \{\mathbf{T}C, \mathbf{F}D\} \qquad \text{with } C = C_1 \wedge \cdots \wedge (A \to B) \wedge \cdots \wedge C_h$$

$\mathbf{T}C$ is the proper $\mathbf{T}$-formulaof $\Gamma_{\overline{m}}^k$ and, for all $\Gamma_j^k < \Gamma_{\overline{m}}^k$, $\mathbf{T}C$ is not the proper $\mathbf{T}$-formulaof Γ_j^k. Consider the case $\Gamma_{\overline{m}}^k < \Gamma_m^k$; we can assume

$$\Gamma_m^k = \Delta'_T \cup \{\mathbf{T}K, \mathbf{F}Z\}$$

where $\Delta_T \subseteq \Delta'_T$ and $\mathbf{T}K$ is the proper $\mathbf{T}$-formulaof Γ_m^k. By Lemma 4.5, for every $\Gamma_j^k < \Gamma_m^k$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[K \to Z]$ is the $\mathbf{F}$-formulaof Γ_j^k and there is a context formula $H[q]$ such that $\mathbf{F}H[C \to D]$, is the $\mathbf{F}$-formulaof Γ_0^k. Since $\Gamma_0^k = \{\mathbf{F}H[C \to H_{\overline{m}}[K \to Z]]\}$ is consistent, by definition of the rule $\mathbf{F}[\to[\to]]$ one of the following statements holds:
(i) $\{\mathbf{F}H[C \to H_{\overline{m}}[K \wedge B \to Z]]\}$ is consistent;
(ii) $\{\mathbf{F}H[C \to H_{\overline{m}}[K \to A \vee Z]]\}$ is consistent.
Suppose (i) holds; then we set:

$$\begin{array}{ll}
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{T}K\}) \cup \{\mathbf{T}K \wedge B\} \quad \text{if } \Gamma_j^k \geq \Gamma_m^k \text{ and} \\
 & (\dagger) \text{ for all } \Gamma_s^k \text{ s.t. } \Gamma_m^k < \Gamma_s^k \leq \Gamma_j^k,\ \mathbf{T}K \text{ is not the proper } \mathbf{T}\text{-formulaof } \Gamma_s^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \cup \{\mathbf{T}K \wedge B\} \quad \text{if } \Gamma_j^k > \Gamma_m^k \text{ and } (\dagger) \text{ does not hold} \\
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{F}H_j[K \to Z]\}) \cup \{\mathbf{F}H_j[K \wedge B \to Z]\} \quad \text{if } \Gamma_j^k < \Gamma_m^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \quad \text{otherwise}
\end{array}$$

(note that $\Gamma_0^{k+1} = \{\mathbf{F}H[C \to H_{\overline{m}}[K \wedge B \to Z]]\}$).
Suppose (ii) holds; in this case we define:

$$\begin{array}{l}
\Gamma_m^{k+1} = (\Gamma_m^k \setminus \{\mathbf{F}Z\}) \cup \{\mathbf{F}A \vee Z\} \\
\Gamma_j^{k+1} = (\Gamma_j^k \setminus \{\mathbf{F}H_j[K \to Z]\}) \cup \{\mathbf{F}H_j[K \to A \vee Z]\} \quad \text{if } \Gamma_j^k < \Gamma_m^k \\
\Gamma_j^{k+1} = \Gamma_j^k \quad \text{otherwise}
\end{array}$$

(note that $\Gamma_0^{k+1} = \{\mathbf{F}H[C \to H_{\overline{m}}[K \to A \vee Z]]\}$).
In case $\Gamma_{\overline{m}} = \Gamma_m$ we proceed in an analogous way taking in account the rule $\mathbf{F}[\to\to]$ for

consistent issues.
We remark that in both cases we have either $\mathbf{F}A \in Sat(\Gamma_m^{k+1})$ or $\mathbf{T}B \in Sat(\Gamma_m^{k+1})$; moreover, if $\mathbf{T}B$ is not added to $Sat(\Gamma_m^{k+1})$ we still have $\mathbf{T}A \to B \in Sat(\Gamma_m^{k+1})$.

- $\mathbf{s}H = \mathbf{F}A \to B$
If $A \to B$ is a reference formula to some successor $\Gamma_{m'}^k$ of Γ_m^k, then $\mathcal{T}^{k+1} = \mathcal{T}^k$.
Otherwise we add in $\mathcal{T}^{k+1}$ a new node

$$\Gamma_{s+1}^{k+1} = \Delta_T \cup \{\mathbf{T}A, \mathbf{F}B\}$$

such that Γ_{s+1}^{k+1} is a successor of Γ_m^{k+1}, where Δ_T is the set of all $\mathbf{T}$-formulasof Γ_m^k.

- $\mathbf{s}H = \mathbf{T}\exists x A(x)$.
By definition there are $\Gamma_{\overline{m}}^k \leq \Gamma_m^k$ and some formulas $C_1, \dots, C_h, D$ such that

$$\Gamma_{\overline{m}}^k = \Delta_T \cup \{\mathbf{T}C, \mathbf{F}D\} \qquad \text{with } C = C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h$$

$\mathbf{T}C$ is the proper $\mathbf{T}$-formulaof $\Gamma_{\overline{m}}^k$ and, for all $\Gamma_j^k < \Gamma_{\overline{m}}^k$, $\mathbf{T}C$ is not the proper $\mathbf{T}$-formulaof Γ_j^k. By Lemma 4.5, for every $\Gamma_j^k < \Gamma_{\overline{m}}^k$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]$ is the $\mathbf{F}$-formulaof Γ_j^k. Let p be the first parameter in $\Pi \setminus \Pi^k$; since $\Gamma_0^k = \{\mathbf{F}H_0[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \to D]\}$ is consistent, by definition of the rule $\mathbf{F}[\exists \to]$, we claim that $\{\mathbf{F}H_0[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]\}$ is consistent; otherwise, applying the rule $\mathbf{F}[\exists \to]$, we should obtain a closed tableau for Γ_0^k (note that the parameters occurring in Γ_0^k are all contained in Π^k, hence are different from p). Then we can define:

$$\begin{array}{ll}
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{T}C\}) \cup \{\mathbf{T}C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h\} \quad \text{if } \Gamma_j^k \geq \Gamma_{\overline{m}}^k \text{ and} \\
& (\dagger) \text{ for all } \Gamma_s^k \text{ s.t. } \Gamma_{\overline{m}}^k < \Gamma_s^k \leq \Gamma_j^k,\ \mathbf{T}C \text{ is not the proper } \mathbf{T}\text{-formulaof } \Gamma_s^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \cup \{\mathbf{T}C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h\} \quad \text{if } \Gamma_j^k > \Gamma_{\overline{m}}^k \text{ and } (\dagger) \text{ does not hold} \\
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{F}H_j[C \to D]\}) \cup \{\mathbf{F}H_j[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \to D]\} \\
& \quad \text{if } \Gamma_j^k < \Gamma_{\overline{m}}^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \quad \text{otherwise}
\end{array}$$

and $\Pi^{k+1} = \Pi^k \cup \{p\}$. Note that $\mathbf{T}A(p) \in Sat(\Gamma_m^{k+1})$.

- $\mathbf{s}H = \mathbf{F}\exists x A(x)$.
Suppose $\Gamma_m^k = \Delta_T \cup \{\mathbf{F}D_1 \vee \cdots \vee \exists x A(x) \vee \cdots \vee D_l\}$ and $\Pi^k = \{p_0, \dots, p_r\}$. By Lemma 4.5, for every $\Gamma_j^k \leq \Gamma_m^k$ there is a context formula $H_j[q]$ such that $\mathbf{F}H_j[\exists x A(x)]$ is the $\mathbf{F}$-formulaof Γ_j^k. We claim that $\{\mathbf{F}H_0[A(p_0) \vee \cdots \vee A(p_r) \vee \exists x A(x)]\}$ is consistent; otherwise, applying $r+1$ times the rule $\mathbf{F}[\exists]$, the set $\Gamma_0^k = \{\mathbf{F}H_0[\exists x A(x)]\}$ would be inconsistent, which is absurd. We can set:

$$\begin{array}{ll}
\Gamma_j^{k+1} & = (\Gamma_j^k \setminus \{\mathbf{F}H_j[\exists x A(x)]\}) \cup \{\mathbf{F}H_j[A(p_0) \vee \cdots \vee A(p_r) \vee \exists x A(x)]\} \\
& \quad \text{if } \Gamma_j^k \leq \Gamma_m^k \\
\Gamma_j^{k+1} & = \Gamma_j^k \quad \text{otherwise.}
\end{array}$$

Note that $\mathbf{F}A(p_0), \dots, \mathbf{F}A(p_r) \in Sat(\Gamma_m^{k+1})$ and, as an effect of the duplication in the rule $\mathbf{F}[\exists]$, we again have $\mathbf{F}\exists x A(x) \in Sat(\Gamma_m^{k+1})$.
The remaining cases $\mathbf{s}H = \mathbf{T}\forall x A(x)$ and $\mathbf{s}H = \mathbf{F}\forall x A(x)$ can be treated as the last ones.

In order to completely define the sequence $\mathcal{S}(\mathbf{F}E)$, we *fix* an enumeration $\mathcal{E}_f$ of the set $\mathbb{N} \times \mathcal{F}$, where $\mathbb{N}$ and $\mathcal{F}$ denote respectively the set of the natural numbers and the set of signed formulas of the language $\mathcal{L}$, such that each pair $\langle n, \mathbf{s}A \rangle$ occurs infinitely many times in $\mathcal{E}_f$ (for each $k \geq 0$, $f(k)$ is the k^{th} element of the enumeration). We then set, for every $k \geq 0$:

$$\begin{array}{l} \mathcal{T}^{k+1} = \mathcal{R}(\mathcal{T}^k, \Gamma^k_m, \mathbf{s}A) \quad \text{if} \quad f(k) = \langle m, \mathbf{s}A \rangle \text{ and } \mathbf{s}A \in Sat(\Gamma^k_m), \\ \mathcal{T}^{k+1} = \mathcal{T}^k \quad \text{otherwise.} \end{array}$$

We now define the *limit* tree $\mathcal{T}^*$ of the succession $\mathcal{S}(\mathbf{F}E)$ and the parameter set Π^*. The root of $\mathcal{T}^*$ is the set

$$\Gamma^*_0 = \bigcup_{k \geq 0} Sat(\Gamma^k_0).$$

Let $m > 0$ and let l be the least integer such that Γ^l_m is a node of $\mathcal{T}^l$; suppose Γ^l_m is a successor of Γ^l_r in $\mathcal{T}^l$ and that Γ^*_r has been defined yet. Then

$$\Gamma^*_m = (\Gamma^*_r)_T \cup \bigcup_{k \geq l} Sat(\Gamma^k_m)$$

and Γ^*_m is an immediate successor of Γ^*_r. Note that all the nodes of $\mathcal{T}^*$ are completely determined in such a way that, for all $r, s, k \geq 0$:
- $\Gamma^k_r \subseteq Sat(\Gamma^k_r) \subseteq \Gamma^*_r$;
- $\Gamma^*_r \leq \Gamma^*_s$ implies $(\Gamma^*_r)_T \subseteq \Gamma^*_s$.
Finally, we set $\Pi^* = \bigcup_{k \geq 0} \Pi^k$.

Proposition 4.6 *Let* $\mathbf{F}E$ *be any consistent* $\mathbf{F}$*-formulaand let* $\mathcal{T}^*$ *be the limit of the sequence* $\mathcal{S}(\mathbf{F}E)$*. Then* $\mathcal{T}^*$ *is a* **CD***-collection with respect to* Π^**.*

PROOF. Firstly we observe that all the formulas occurring in $\mathcal{T}^*$ belong to the language generated by the nonempty set Π^*; moreover, for every formula H, the following facts hold.

(i) If $\mathbf{s}H \in Sat(\Gamma^k_m)$ and H is not a reference formula, then there is $s \geq k$ such that $\mathcal{T}^{s+1} = \mathcal{R}(\mathcal{T}^s, \Gamma^s_m, \mathbf{s}H)$.

As a matter of fact, by the properties of the enumeration $\mathcal{E}_f$, there is an integer $k' \geq k$ such that $f(k') = \langle m, \mathbf{s}H \rangle$. By construction of $\mathcal{S}(\mathbf{F}E)$, it is easy to check that there must be an integer j such that $k \leq j \leq k'$ and $\mathcal{T}^{j+1} = \mathcal{R}(\mathcal{T}^j, \Gamma^j_m, \mathbf{s}H)$ (note that, in case $\mathbf{s}H = \mathbf{T}H$ and $H \neq A \to B$, we have to use the fact that, if $\Gamma^k_r \leq \Gamma^k_m$ in $\mathcal{T}^k$ and $\mathbf{T}H \in \Gamma^k_r$, then $\mathcal{R}(\mathcal{T}^k, \Gamma^k_r, \mathbf{T}H) = \mathcal{R}(\mathcal{T}^k, \Gamma^k_m, \mathbf{T}H)$).
As far as matters of duplication of formulas are concerned, we can observe what follows.

(ii) If either H is atomic or $\mathbf{s}H$ is a signed formula of the kind $\mathbf{F}\exists x A(x)$, $\mathbf{T}\forall x A(x)$, then $\mathbf{s}H \in Sat(\Gamma^k_m)$ implies $\mathbf{s}H \in Sat(\Gamma^j_m)$ for all $j \geq k$.

(iii) If $\mathbf{T}A \to B \in Sat(\Gamma^k_m)$, then either there is $j > k$ such that $\mathbf{T}B \in Sat(\Gamma^j_m)$ or, for all $j > k$, $\mathbf{T}A \to B \in Sat(\Gamma^j_m)$.

Let Γ^*_m be any element of $\mathcal{T}^*$; we prove that all the conditions of Definition 4.1 are satisfied. Suppose A is any atomic formula and $\mathbf{T}A \in \Gamma^*_m$; by definition of Γ^*_m there are some integers $r, k \geq 0$ such that $\Gamma^*_r \leq \Gamma^*_m$ in $\mathcal{T}^*$ and $\mathbf{T}A \in Sat(\Gamma^k_r)$. If $\mathbf{F}A \in \Gamma^*_m$, there is $j \geq 0$ such that

$\mathbf{F}A \in Sat(\Gamma_m^j)$. Let s be the maximum between k and j, then both Γ_r^s and Γ_m^s are defined in $\mathcal{T}^s$ and $\Gamma_r^s \leq \Gamma_m^s$; by (ii) we have $\mathbf{T}A \in Sat(\Gamma_r^s)$ and $\mathbf{F}A \in Sat(\Gamma_m^s)$, hence $\mathbf{T}A \in Sat(\Gamma_m^s)$ and $\mathbf{F}A \in Sat(\Gamma_m^s)$, against Lemma 4.4. We can conclude that $\mathbf{F}A \in \Gamma_m^*$ does not hold. Likewise it is proved that $\mathbf{T}\bot \notin \Gamma_m^*$.
Suppose $\mathbf{T}A \wedge B \in \Gamma_m^*$; let $r, k \geq 0$ be such that $\Gamma_r^* \leq \Gamma_m^*$ in $\mathcal{T}^*$ (hence $(\Gamma_r^*)_T \subseteq \Gamma_m^*$) and $\mathbf{T}A \wedge B \in Sat(\Gamma_r^k)$. Since $Sat(\Gamma_r^k)$ is closed under $\mathbf{T}$-signed conjunctions, we can infer $\mathbf{T}A \in Sat(\Gamma_r^k)$ and $\mathbf{T}B \in Sat(\Gamma_r^k)$. Hence $\mathbf{T}A, \mathbf{T}B \in \Gamma_r^*$, from which $\mathbf{T}A, \mathbf{T}B \in \Gamma_m^*$ follows. In an analogous way we can prove that $\mathbf{F}A \vee B \in \Gamma_m^*$ implies $\mathbf{F}A, \mathbf{F}B \in \Gamma_m^*$.
Suppose that $\mathbf{T}A \to B \in \Gamma_m^*$ and that $\mathbf{T}B \notin \Gamma_m^*$, we prove $\mathbf{F}A \in \Gamma_m^*$. Let $r, k \geq 0$ such that $\Gamma_r^* \leq \Gamma_m^*$ in $\mathcal{T}^*$ and $\mathbf{T}A \to B \in Sat(\Gamma_r^k)$. Since $\mathbf{T}B \notin \Gamma_r^*$ (being $(\Gamma_r^*)_T \subseteq \Gamma_m^*$), by (iii), for each $j > k$ we have $\mathbf{T}A \to B \in Sat(\Gamma_r^j)$. Let us consider some $\overline{\jmath} > k$ such that both $\Gamma_r^{\overline{\jmath}}$ and $\Gamma_m^{\overline{\jmath}}$ are defined in $\mathcal{T}^{\overline{\jmath}}$. Then, from the fact that $\Gamma_r^{\overline{\jmath}} \leq \Gamma_m^{\overline{\jmath}}$ in $\mathcal{T}^{\overline{\jmath}}$ and $\mathbf{T}A \to B \in Sat(\Gamma_r^{\overline{\jmath}})$, it follows that $\mathbf{T}A \to B \in Sat(\Gamma_m^{\overline{\jmath}})$. Let, by (i), $s \geq \overline{\jmath}$ be such that $\mathcal{T}^{s+1} = \mathcal{R}(\mathcal{T}^s, \Gamma_m^s, \mathbf{T}A \to B)$; we cannot have $\mathbf{T}B \in Sat(\Gamma_m^{s+1})$ (otherwise $\mathbf{T}B \in \Gamma_m^*$), so $\mathbf{F}A \in Sat(\Gamma_m^{s+1})$, hence $\mathbf{F}A \in \Gamma_m^*$.
Suppose $\mathbf{F}A \to B \in \Gamma_m^*$; let k be such that $\mathbf{F}A \to B \in Sat(\Gamma_m^k)$. If $A \to B$ is a reference formula to some successor $\Gamma_{m'}^k$ of Γ_m^k, then $\mathbf{T}A \in Sat(\Gamma_{m'}^k)$ and $\mathbf{F}B \in Sat(\Gamma_{m'}^k)$, hence $(\Gamma_m^*)_T \subseteq \Gamma_{m'}^*$ and $\mathbf{T}A, \mathbf{F}B \in \Gamma_{m'}^*$. Otherwise, by (i), there is $s \geq k$ such that $\mathcal{T}^{s+1} = \mathcal{R}(\mathcal{T}^s, \Gamma_m^s, \mathbf{F}A \to B)$; thus $A \to B$ becomes in $\mathcal{T}^{s+1}$ a reference formula to some immediate successor $\Gamma_{m'}^{s+1}$ of Γ_m^{s+1} and, as before, the definition is satisfied.
Suppose $\mathbf{F}\exists x A(x) \in \Gamma_m^*$ and let p be any parameter in Π^*; we have to prove $\mathbf{F}A(p) \in \Gamma_m^*$. Arguing as above and taking in account (ii), we can assume that there is an integer $s \geq 0$ such that $\mathbf{F}\exists x A(x) \in Sat(\Gamma_m^s)$, $\mathcal{T}^{s+1} = \mathcal{R}(\mathcal{T}^s, \Gamma_m^s, \mathbf{F}\exists x A(x))$ and $p \in \Pi^s$. It follows that $\mathbf{F}A(p) \in Sat(\Gamma_m^{s+1})$, hence $\mathbf{F}A(p) \in \Gamma_m^*$ as required.
The remaining cases can be treated in a similar way. ∎

Theorem 4.7 (Completeness) *Let Δ_{fin} be any finite set of signed formulas. If Δ_{fin} is consistent, then Δ_{fin} is realized in some Kripke model with constant domain.*

Proof. Let $\Delta_{fin} = \{\mathbf{T}C_1, \ldots, \mathbf{T}C_h, \mathbf{F}D_1, \ldots, \mathbf{F}D_l\}$ and let A be the formula $C_1 \wedge \cdots \wedge C_h \to D_1 \vee \cdots \vee D_l$. Suppose Δ_{fin} is consistent; this means that $\mathbf{F}A$ is consistent, hence, by the previous proposition, the limit $\mathcal{T}^*$ of the sequence $\mathcal{S}(\mathbf{F}A)$ is a $\mathbf{CD}$-collection. Since $\mathbf{F}A \in \Gamma_0^*$, by Definition 4.1 there is $\Gamma_m^* \in \mathcal{T}^*$ such that $\Delta_{fin} \subseteq \Gamma_m^*$. By Proposition 4.3, Γ_m^*, and a fortiori Δ_{fin}, is realized in some Kripke model with constant domain, proving the theorem. ∎

In virtue of Soundness and Completeness theorems we can conclude that a formula A is provable in $\mathbf{CDt}$ if and only if $A \in \mathcal{L}(\mathcal{F}_{\mathbf{CD}})$ if and only if $A \in \mathbf{CD}$.

We remark that, as usually happens in completeness proofs of this kind, the construction of $\mathcal{S}(\mathbf{F}A)$ provides us a counter model for A if $\mathbf{F}A$ is consistent, otherwise a closed tableau for $\mathbf{F}A$ is built in finitely many steps. We informally outline a proof of this fact. Suppose $\mathbf{F}A$ is not consistent and define $\mathcal{T}^0$ and Π^0 as above. Let $\{\mathbf{F}A_k\}$ be the non consistent root of $\mathcal{T}^k$; we show how to build a closed tableau $\mathcal{G}^k$ for $\mathbf{F}A_k$.

(1) Suppose $\mathcal{T}^k$ is *closed*, i.e. it contains a node Γ such that $Sat(\Gamma)$ is closed. Then $\mathcal{G}^k$ is immediately definable using only $\mathbf{T}\wedge$, $\mathbf{F}\vee$, $\mathbf{F}\to$ rules (see the proof of Lemma 4.4).

(2) Suppose now that $\mathcal{T}^k$ is not closed. Choose Γ_m^k in $\mathcal{T}^k$ and $\mathbf{s}H$ in $Sat(\Gamma_m^k)$.
If $\mathbf{s}H$ is a signed formula of the kind $\mathbf{T}\exists x B(x)$, $\mathbf{F}\exists x B(x)$, $\mathbf{T}\forall x B(x)$, $\mathbf{F}\forall x B(x)$, then $\mathcal{T}^{k+1}$ is uniquely defined by $\mathcal{R}$. Moreover, we can observe that $\mathcal{T}^{k+1}$ is *less distant* from some closed tree $\mathcal{T}$ (otherwise $\mathbf{F}A_k$ should be consistent), so we can assume that a closed tableau $\mathcal{G}^{k+1}$ for $\mathbf{F}A_{k+1}$ has been already defined. We can therefore build $\mathcal{G}^k$ starting

from $\mathbf{F}A_k$ and $\mathcal{G}^{k+1}$ by applying one of the rules $[\mathbf{F}\exists\to]$, $[\mathbf{F}\exists]$, $[\mathbf{F}\forall\to]$, $[\mathbf{F}\forall]$ respectively. On the other hand, if $\mathbf{s}H$ is one of the formulas $\mathbf{F}B \wedge C$, $\mathbf{T}B \vee C$, $\mathbf{T}B \to C$, then $\mathcal{T}^k$ has two possible successors $\mathcal{T}'$ and $\mathcal{T}''$ having as roots the inconsistent sets $\{\mathbf{F}A'\}$ and $\{\mathbf{F}A''\}$ respectively. Reasoning as above, we can assume that the closed tableaux for $\mathbf{F}A'$ and $\mathbf{F}A''$ have been already given; hence, applying one of the rules $[\mathbf{F}\wedge]$, $[\mathbf{F}\vee\to]$, $[\mathbf{F}\to\to]$, $[\mathbf{F}\to[\to]]$, we can obtain a closed tableau for $\mathbf{F}A^k$ as well.
In all the other cases, $A_k = A_{k+1}$, hence $\mathcal{G}^k = \mathcal{G}^{k+1}$.

We point out that the choice of Γ^k_m and $\mathbf{s}H$ in step (2) is not relevant in order to obtain the closed tableau, provided that some care is taken in order to avoid infinite loops. At this aim it may be sufficient to assure that, for every $\Gamma^k \in \mathcal{T}^k$ and every $\mathbf{s}H \in Sat(\Gamma^k)$, the formula $\mathbf{s}H$ will be eventually chosen, one or more times (see the role of $\mathcal{E}_f$ in the proof of Proposition 4.6).

5 A cut-free sequent calculus for CD

As anticipated, our tableau calculus for **CD** can be translated into a cut-free sequent calculus, we call it **CDs**. The translation can be done as follows (see also [1, 2, 17]):

(1) One has to reverse the rules, i.e. the configuration above the line is to be put below, and the configuration below the line is to be put above.
(2) Each set of signed formulas in a configuration is translated into a sequent, where:
  (a) In the left hand part of the sequent (which is seen as a *set*, rather than a multiset or a sequence of formulas) one has to put the **T**-formulas(of course without the sign **T**);
  (b) In the right hand part of the sequent one has to put the **F**-formulas(of course without the sign **F**). We can assume that in the right hand parts of the sequents there is *at most one* formula.

According to these principles, we give the literal translation of the calculus **CDt**.

Axioms:

$$\frac{}{\Gamma, A \vdash A} \qquad\qquad \frac{}{\Gamma, \bot \vdash D}$$

Rules for $\wedge$:

$$\frac{\Gamma, A, B \vdash D}{\Gamma', A \wedge B \vdash D}\, L\wedge \quad \text{with } \Gamma \subseteq \Gamma' \subseteq \Gamma \cup \{A, B\} \qquad\qquad \frac{\vdash H[A] \qquad \vdash H[B]}{\vdash H[A \wedge B]}\, R[\wedge]$$

Rules for $\vee$:

$$\frac{\Gamma \vdash A}{\Gamma \vdash D_1 \vee \cdots \vee A \vee \cdots \vee D_l}\, R\vee$$

$$\frac{\vdash H[C_1 \wedge \cdots \wedge A \wedge \cdots \wedge C_h \to D] \qquad \vdash H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \to D]}{\vdash H[C_1 \wedge \cdots \wedge (A \vee B) \wedge \cdots \wedge C_h \to D]}\, R[\vee\to]$$

Rules for $\rightarrow$:

$$\frac{\vdash H[C_1 \wedge \cdots \wedge B \wedge \cdots \wedge C_h \rightarrow D] \quad \vdash H[C_1 \wedge \cdots \wedge (A \rightarrow B) \wedge \cdots \wedge C_h \rightarrow A \vee D]}{\vdash H[C_1 \wedge \cdots \wedge (A \rightarrow B) \wedge \cdots \wedge C_h \rightarrow D]}\, R[\rightarrow\rightarrow]$$

If $C = C_1 \wedge \cdots \wedge (A \rightarrow B) \wedge \cdots \wedge C_h$,

$$\frac{\vdash H[C \rightarrow H'[K \wedge B \rightarrow D]] \qquad \vdash H[C \rightarrow H'[K \rightarrow A \vee D]]}{\vdash H[C \rightarrow H'[K \rightarrow D]]}\, R[\rightarrow[\rightarrow]]$$

$$\frac{\Gamma, A \vdash B}{\Gamma' \vdash A \rightarrow B}\, R\rightarrow \quad \text{with } \Gamma \subseteq \Gamma' \subseteq \Gamma \cup \{A\}$$

Rules for $\exists$:

$$\frac{\vdash H[C_1 \wedge \cdots \wedge A(p) \wedge \cdots \wedge C_h \rightarrow D]}{\vdash H[C_1 \wedge \cdots \wedge \exists x A(x) \wedge \cdots \wedge C_h \rightarrow D]}\, R[\exists\rightarrow] \qquad \text{with } p \text{ new}$$

$$\frac{\vdash H[A(y) \vee \exists x A(x)]}{\vdash H[\exists x A(x)]}\, R[\exists]$$

Rules for $\forall$:

$$\frac{\vdash H[C_1 \wedge \cdots \wedge A(y) \wedge \forall x A(x) \wedge \cdots \wedge C_h \rightarrow D]}{\vdash H[C_1 \wedge \cdots \wedge \forall x A(x) \wedge \cdots \wedge C_h \rightarrow D]}\, R[\forall\rightarrow]$$

$$\frac{\vdash H[A(p)]}{\vdash H[\forall x A(x)]}\, R[\forall] \qquad \text{with } p \text{ new}$$

We remark that Γ has to be understood as a *set* of formulas, thus Γ, A denotes the set $\Gamma \cup \{A\}$. This means that the order of the formulas of Γ is irrelevant; similarly, the order of the formulas inside a context (for instance, the order of $C_1, \ldots, C_m, D_1, \ldots, D_n$ in $H[C_1 \wedge \cdots \wedge C_m \rightarrow D_1 \vee \cdots \vee D_n]$) is irrelevant, and this is the same as saying that we admit permutations between formulas when needed.

As said in the introduction, according to the results of [14], it is not possible to define standard cut-free sequent calculi for the logic **CD**, thus the calculi known so far (which amount to the ones studied in [11]) present some oddness (incidentally, we point out that our calculus does not match the specifications of [14], since the rules acting on context formulas have not *bounded grade*). We now show that the calculus **CDs**, even if it arises from a semantical background (whereas the calculi in [11] are developed in a syntactical framework), has remarkable features also from a proof-theoretical viewpoint. First of all, we observe that the calculus **CDs** can be considered a system of type $G3$, according to the classification in [17]; indeed, differently from Gentzen-like presentations (type $G1$) and in accordance with Dragalin's formalism (see [4]), the *structural rules* are not explicitly given, but are "absorbed" in the logical rules and in the axioms. As pointed out in [17], "this has advantages in an upside down search procedure for proofs of a given sequent". As far as *exchange* rule is concerned, which actually is unproblematic, we have already discussed its implicit use. Differently, *contraction* rule plays a crucial role; indeed, reading the rule upward, the *duplication* of the main formula (which can be seen as an hidden application of contraction) leads to an increasing of the non-determinism involved in the process of proof-searching (see, for instance, [1, 5] for a comprehensive discussion). On the other hand, it is well known that, at the predicate level, explicit or implicit applications of contraction rule cannot be completely eliminated, otherwise the corresponding calculus would allow to recursively decide the provability of formulas. In **CDs** contractions are presented (in an hidden form) *only* in four cases, that is in the rules $R[\to\to]$, $R[\to[\to]]$, $R[\exists]$ and $R[\forall\to]$. Suppose, for instance, to give the rule $R[\forall\to]$ in a standard form:

$$\frac{\vdash H[C_1 \wedge \cdots \wedge A(y) \wedge \cdots \wedge C_h \to D]}{\vdash H[C_1 \wedge \cdots \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}\ R'[\forall\to]$$

To recover the rule $R[\forall\to]$ (and thus guarantee the completeness of the calculus), we need a contraction rule $C[\forall\to]$ (which acts inside the contexts) in order to obtain:

$$\frac{\dfrac{\vdash H[C_1 \wedge \cdots \wedge A(y) \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}{\vdash H[C_1 \wedge \cdots \wedge \forall x A(x) \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}\ R'[\forall\to]}{\vdash H[C_1 \wedge \cdots \wedge \forall x A(x) \wedge \cdots \wedge C_h \to D]}\ C[\forall\to]$$

To sum up, the use of contraction rule is limited to well-defined cases, thus there is no need of giving it as a general rule (as done in [11]).

Also for *weakness* we have only hidden applications, precisely in the axioms (presented in the form of *general axioms*) and in the rules $L\wedge$ and $R\to$. For instance, if we give $L\wedge$ in a Gentzen-like form:

$$\frac{\Gamma, A, B \vdash D}{\Gamma, A \wedge B \vdash D}\ L'\wedge$$

we need a left-weakening to possibly add the formulas A and B in the antecedent. We point out that, as an immediate consequence of the completeness of **CDs** with respect to the class of Kripke models with constant domain, we get that the structural rules, such as the cut-rule and any rule which is semantically sound, are *admissible* in the calculus.

A remarkable feature of **CDs** is that the rules can be divided into two classes, according to the following remarks.

(1) The rules $L\wedge$, $R\vee$ and $R\to$ act on sequents of then kind $\Gamma \vdash D$, where Γ is nonempty; we call such rules *external-rules*.

(2) All the other rules act on sequents of the kind $\vdash H$ (generally on subformulas of H); we call these rules *context-rules*.

Note that in the latter class we have only right-rules, since they refer to one-side sequents. One may think that there is an asymmetry in the explanation of the rules, due to an excess of right-rules. This is not true; indeed, rules such as $R[\vee\to]$, $R[\to\to]$, $R[\to[\to]]$, $R[\exists\to]$,$R[\forall\to]$, even if formally are right-rules, work as left-rules. It is worth noting that the rules belonging to different classes are *mutually exclusive*, in the sense that, at each step, only the rules of one class can be activated. Since the initial sequents have the form $\Gamma \vdash D$, with Γ nonempty, the derivations of the sequents of the form $\vdash A$ have a typical figure. More precisely, we can identify two well-distinct parts:

(1) An *upper-part*, which begins with an axiom and ends with an application of $R\to$, where only external-rules are applied;

(2) A *lower-part*, which ends with the sequent $\vdash A$, where only context-rules are applied.

Finally, we point out that, among the calculi in [11], **CDs** has some resemblance with $\mathbf{LDS}'$, where something similar to context-rules are used (note however that the treatment of "right-implication" is quite different, since in $\mathbf{LDS}'$ implications are not directly decomposed inside the contexts in the sense of the rule $R[\to[\to]]$).
As an example, we give a derivation π of the sequent

$$\vdash \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x A(x) \to \exists x A(x)$$

which corresponds to an instance of Markov axiom. We start with constructing the following proof π_1, which uses only external-rules.

$$\dfrac{\dfrac{\dfrac{A(p), \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \vdash A(p)}{A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \vdash A(p)}\,L\wedge}{A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \vdash \neg A(p) \vee A(p) \vee \exists x A(x)}\,R\vee}{\vdash A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to \neg A(p) \vee A(p) \vee \exists x A(x)}\,R\to$$

Likewise, we can build a proof π_2 of the sequent

$$\vdash \neg A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to \neg A(p) \vee A(p) \vee \exists x A(x)$$

using only external-rules. Now, we can combine the two proofs π_1 and π_2 by means of the context rule $R[\vee \to]$ (note that, from this point downward, we will only use context-rules). Let Z be the formula $\neg A(p) \vee A(p) \vee \exists x A(x)$ and consider the following derivation π_3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to Z\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \neg A(p) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to Z\end{array}}{\vdash (A(p) \vee \neg A(p)) \wedge \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to Z}\,R[\vee \to]}{\vdash \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to Z}\,R[\forall \to]}{\vdash \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to \neg A(p) \vee \exists x A(x)}\,R[\exists]}{\vdash \forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists A(x) \to \neg\exists x A(x) \vee \exists x A(x)}\,R[\exists \to]$$

Note that, in the last step, we have applied the rule $R[\exists \to]$ to the subformula $A(p)$ of $\neg A(p) \equiv A(p) \to \bot$ (this is the only point of the whole proof π which is not intuitionistically sound). To continue our proof, we need the following proof π_4 (in which only external-rules are used):

$$\dfrac{\dfrac{\forall x(A(x) \vee \neg A(x)), \bot \vdash \exists x A(x)}{\forall x(A(x) \vee \neg A(x)) \wedge \bot \vdash \exists x A(x)}\,L\wedge}{\vdash \forall x(A(x) \vee \neg A(x)) \wedge \bot \to \exists x A(x)}\,R\to$$

Then, the proof π looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_4\\ \vdash \forall x(A(x) \vee \neg A(x)) \wedge \bot \to \exists x A(x)\end{array} \qquad \begin{array}{c}\vdots\,\pi_3\\ \vdash \forall x(A(x) \vee \neg A(x)) \wedge (\neg\exists x A(x) \to \bot) \to \neg\exists x A(x) \vee \exists x A(x)\end{array}}{\vdash \forall x(A(x) \vee \neg A(x)) \wedge (\neg\exists x A(x) \to \bot) \to \exists x A(x)}\,R[\to\to]$$

6 Conclusion

Among the advantages of this kind of calculi, we think that they constitute a good ground for defining cut-free calculi for logics which extend **CD** and are characterizable by means of Kripke semantics; for instance, the logic $\mathbf{CD} + \mathbf{Kur}$, characterized by the class of Kripke frames with constant domain and enough final states (i.e., each state of a model $\underline{K}$ can see at least a final state), the logics $\mathbf{CD} + \mathbf{Lc}$, characterized by the class of linear Kripke frames with constant domain, and so on. Usually such extensions arise rather naturally in a semantical framework, while it seems to be hard to perform them using only syntactical techniques.

Finally, we briefly recall the main open question regarding **CD**, that is the interpolation of such a logic. Clearly, the propositional part of **CDs** becomes a cut-free sequent calculus for intuitionistic propositional logic, which is well known to be interpolable. Now, in the attempt of proving the interpolability of **CD** using such a calculus, we have found difficulties *just in treating the propositional rules of implication*. Thus, an interpolation proof for intuitionistic propositional logic using our calculus should naturally yield an interpolation proof for **CD**. We think that, possibly without passing through the calculus here defined, the semantical tools developed in this paper could be a good base for further investigations about the problem.

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